The Petz Map in Maths, Information Theory, and Physics: an Overview

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Conventions

- finite dimensions (operators \equiv matrices)
- states, viz., density matrices: $\rho \ge 0$, $\operatorname{Tr} \rho = 1 \ (\rho, \sigma, \dots)$
- pure states, e.g, $|\psi_Q\rangle$ and $|\psi\rangle\langle\psi|_Q=\psi_Q$
- we denote $\operatorname{supp} \rho := (\ker \rho)^{\perp}$
- completely positive trace-preserving linear maps, viz.
 quantum channels, are denoted by *E*, *F*, ...
- trace-dual map \mathcal{E}^{\dagger} is defined by $\operatorname{Tr}[\mathcal{E}(X) \ Y] = \operatorname{Tr}[X \ \mathcal{E}^{\dagger}(Y)]$ for all X, Y
- fidelity (sometimes, squared fidelity) $F(\rho, \sigma) := \|\sqrt{\rho}\sqrt{\sigma}\|_1^2$

Umegaki's relative entropy

Definition (Relative entropy)

or
$$A, B \ge 0, A \ne 0$$
,
$$D(A || B) := \begin{cases} \operatorname{Tr}[A(\log A - \log B)], & \text{if } \operatorname{supp} A \subseteq \operatorname{supp} B, \\ +\infty, & \text{otherwise} \end{cases}$$

Useful properties:

F

- Klein's inequality: $\operatorname{Tr} A \geq \operatorname{Tr} B \implies D(A \| B) \geq 0$
- $B \le B' \implies D(A \| B) \ge D(A \| B')$
- $-D(\rho \| I) = S(\rho)$ $(= -\operatorname{Tr} \rho \log \rho)$
- monotonicity: $D(\rho \| \sigma) \ge D(\mathcal{E}(\rho) \| \mathcal{E}(\sigma))$ for all channels \mathcal{E} and all states ρ, σ

Origin of the transpose map

Question. For which triples $(\rho, \sigma, \mathcal{E})$, $D(\rho \| \sigma) = D(\mathcal{E}(\rho) \| \mathcal{E}(\sigma))$? **Petz (1986,1988)**

If and only if
$$\tilde{\mathcal{E}}_{\sigma}(\bullet) := \sqrt{\sigma} \mathcal{E}^{\dagger} \left[\frac{1}{\sqrt{\mathcal{E}(\sigma)}} \bullet \frac{1}{\sqrt{\mathcal{E}(\sigma)}} \right] \sqrt{\sigma}$$
 satisfies
 $\tilde{\mathcal{E}}_{\sigma} \circ \mathcal{E}(\rho) = \rho$.

(The other equality $\tilde{\mathcal{E}}_{\sigma} \circ \mathcal{E}(\sigma) = \sigma$ is satisfied by construction.)

Remark. The map $\tilde{\mathcal{E}}_{\sigma}$ is already CPTP on supp $[\mathcal{E}(\sigma)]$, but it can always be extended to a linear map CPTP *everywhere*.

Remark. Notice that $\tilde{\mathcal{E}}_{\sigma}$ in general is *not* the linear inverse of \mathcal{E} !

2/29

Re-discovered in quantum error correction

- state ho_Q purified by $|\psi_{RQ}
 angle$, that is, ${
 m Tr}_R[\psi_{RQ}]=
 ho_Q$
- given a state ρ_Q and a channel $\mathcal{E}: Q \to Q$, the entanglement fidelity is $F_e(\rho, \mathcal{E}) := \langle \psi_{RQ} | (\mathrm{id}_R \otimes \mathcal{E}_Q)(\psi_{RQ}) | \psi_{RQ} \rangle$

Barnum and Knill (2002)

Given a state ρ_Q and a channel $\mathcal{E}: Q \to Q'$,

$$\left[\max_{\mathcal{R}:Q'\to Q} F_e(\rho, \mathcal{R} \circ \mathcal{E})\right]^2 \leq F_e(\rho, \tilde{\mathcal{E}}_{\rho} \circ \mathcal{E}) \; .$$

Petz's transpose map can be used as decoder to achieve the quantum capacity (Beigi–Datta–Leditzky 2017). Belavkin's "pretty good measurement" (1975) can also be rederived as a special case. 4/29

Re-discovered in quantum statistical mechanics

Crooks (PRA, 2008) rediscovers Petz's transpose map on the basis of physical reasoning:

- starts from equilibrium, that is, $\mathcal{E}(\omega) = \omega$
- picks a Kraus representation $\mathcal{E}(\bullet) = \sum_k E_k(\bullet) E_k^{\dagger}$
- defines a stochastic "trajectory" over the Kraus representation index: $p(\alpha, \beta | \omega) = \text{Tr}[E_{\beta} (E_{\alpha} \omega E_{\alpha}^{\dagger}) E_{\beta}^{\dagger}]$
- assumes that the "reverse" process, with Kraus operators $\{\tilde{E}_k\}_k$, at equilibrium satisfies microscopic reversibility: $\tilde{p}(\beta, \alpha | \omega) := \operatorname{Tr}[\tilde{E}_{\alpha} (\tilde{E}_{\beta} \omega \tilde{E}_{\beta}^{\dagger}) \tilde{E}_{\alpha}^{\dagger}] \stackrel{!}{=} p(\alpha, \beta | \omega)$
- the above is satisfied if $\tilde{E}_k = \omega^{1/2} E_k^{\dagger} \omega^{-1/2}$, for all indices k
- \implies Crooks' reverse process coincides with $\tilde{\mathcal{E}}_{\omega}$

Extension beyond equality: approximate reversibility

Junge-Renner-Sutter-Wilde-Winter (2018)

$$D(\rho \| \sigma) - D(\mathcal{E}(\rho) \| \mathcal{E}(\sigma)) \ge -\int_{-\infty}^{+\infty} \mathrm{d}t \ p(t) \log F(\rho, \tilde{\mathcal{E}}_{\sigma}^{t/2} \circ \mathcal{E}(\rho))$$
$$\ge -\log F(\rho, \mathcal{R} \circ \mathcal{E}(\rho)) ,$$

where

- $p(t) := \frac{\pi}{2(\cosh(\pi t)+1)}$ is a probability density
- $\tilde{\mathcal{E}}_{\sigma}^{t}(\bullet) := \sigma^{-it} \tilde{\mathcal{E}}_{\sigma}[\mathcal{E}(\sigma)^{it} (\bullet) \mathcal{E}(\sigma)^{-it}] \sigma^{it}$ are "rotated" Petz's maps

•
$$\mathcal{R} := \int_{-\infty}^{+\infty} \mathrm{d}t \ p(t) \tilde{\mathcal{E}}_{\sigma}^{t/2}$$

6/29

What is the Petz transpose map and how to implement it?

The problems with "reversal"



- What is it? A (*the*?) time-reverse? Other symmetry reverse?
- How to achieve it?

The quest for a "physical implementation"

The underlying philosophy is that a channel represents a process that "actually happens".

Problem. Given a circuit implementation of a channel, *algorithmically* construct a circuit implementing its Petz transpose.

Results. Having a realization of the forward process does not mean that its reversal is also available: there is no simple "reversal button"! See e.g.: Quintino *et al.* (2019) and Gilyén *et al.* (2020).

Remark. Any channel's realization involves unobservable degrees of freedom (the inside of the black-box). Should the reverse depend on those?

However, some special cases are "easy". (But beware building your intuition based on them!)

7/29

Hamiltonian (i.e., unitary or one-to-one) dynamics

The following are equivalent:

- the channel \mathcal{E} is unitary (that is, $\mathcal{E}(\bullet) = U \bullet U^{\dagger}$)
- the channel ${\mathcal E}$ is such that $\tilde{{\mathcal E}}_{\sigma}$ does not depend on the choice of σ
- the channel \mathcal{E} is such that its linear inverse \mathcal{E}^{-1} coincides with $\tilde{\mathcal{E}}_{\sigma}$ for some choice of σ

Moreover, for unitary channels all rotated Petz maps coincide.

Interpretation

The Petz transpose channel corresponds to "the movie shown backwards" (intuitive notion of "time-reversal") if and only if the channel is unitary.

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9/29
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Thermal operations (and a little more)

Alhambra–Wehner–Wilde–Woods (2018)

Consider a channel $\mathcal{E}:Q\to Q$ possessing a realization of the form

$$\mathcal{E}(\bullet_Q) = \operatorname{Tr}_E[U_{QE} (\bullet_Q \otimes \tau_E) \ U_{QE}^{\dagger}],$$

such that

$$U_{QE} (\omega_Q \otimes \tau_E) U_{QE}^{\dagger} = \omega_Q \otimes \tau'_E ,$$

for some steady state $\omega_Q > 0$. Then

$$\widetilde{\mathcal{E}}_{\omega}(\bullet_Q) = \operatorname{Tr}_E[U_{QE}^{\dagger}(\bullet_Q \otimes \tau'_E) U_{QE}].$$

Remark. A thermal operation has ω_Q and $\tau_E = \tau'_E$ as Gibbs states of the system's and bath's Hamiltonians, respectively. 10/29

In general, what is the relation between a channel and its Petz transpose?

Petz's transpose map as Bayesian retrodiction

The Bayes–Laplace Rule



Inverse Probability Formula

 ${\cal P}(H|D) \propto {\cal P}(D|H)$ $\mathcal{P}(H)$

inv. prob.

likelihood/model prior

where H is a hypothesis, D is the result of observation (i.e., data or evidence)

11/29

Meanings of the inverse probability

- it is the main *tool* of Bayesian statistics for problems like:
 - estimation (e.g.: how many red balls are in an urn?)
 - decision (e.g.: is ACME's stock a good investment? should I buy some? how much?)
 - inference and learning: predictive inference (e.g.: weather forecasts) and retrodictive inference (e.g.: what kind of stellar event possibly caused the Crab Nebula?)
- it measures the degree of belief that a rational agent should have in one hypothesis, among other mutually exclusive ones, given the data

Inference with noisy data or uncertain evidence

BUT! Bayes-Laplace Rule *does not* tell us how to update the prior in the face of uncertain data...

suppose that a noisy observation suggests a probability distribution Q(D) for the data (e.g., the license plate no.)



 how should we update our prior *P*(*H*) given *uncertain* evidence in the from *Q*(*D*)?

13/29

Jeffrey's rule of probability kinematics

Vanilla Bayes:Extended Bayes:
$$\mathcal{P}(H|D) = \mathcal{P}(D|H)\mathcal{P}(H)/\mathcal{P}(D)$$
 $\mathcal{P}(H|\mathcal{Q}(D)) = ?$ Jeffrey's conditioning* (1965) $\mathcal{P}(H|\mathcal{Q}(D)) = \sum_{D} \underbrace{\mathcal{P}(H|D)}_{D} \mathcal{Q}(D)$
 $= \sum_{D} \underbrace{\mathcal{P}(D|H)\mathcal{P}(H)}_{D} \mathcal{Q}(D)$ * Jeffrey's rule was introduced *ad hoc*, but it can be proved from Bayes-Laplace Rule and

Pearl's method of virtual evidence (1988)

Jeffrey's rule "promotes" Bayes' posterior distribution to a fully fledged channel.

Petz transpose in the classical case

- state $\rho \rightsquigarrow$ probability distribution p(x)
- channel $\mathcal{E} \rightsquigarrow$ discrete noisy channel $\varphi(y|x)$
- $\mathcal{E}(\rho) \rightsquigarrow [\varphi \circ p](y) = \sum_{x} \varphi(y|x)p(x)$
- Petz transpose $\tilde{\mathcal{E}}_{\rho} \rightsquigarrow \tilde{\varphi}_{p}(x|y) = \frac{1}{[\varphi \circ p](y)} \varphi(y|x) p(x)$
- hence, Petz's transpose map coincides with Jeffrey's rule!
- moreover, in the classical case there is only one Jeffrey–Petz reverse (i.e., all rotated maps coincide)

Approximate reversibility in the classical case

Li–Winter (2018)

In the classical case,

$$D(p||q) - D(\varphi[p]||\varphi[q]) \ge D(p||[\tilde{\varphi}_q \circ \varphi]p)$$
.

Remark. Notice that $D(p||q) \ge -\log F(p,q)$, so the above is stronger than the best general quantum bounds we know.

Only for *(sub-)unital* CPTP maps \mathcal{E} , we have a similar bound (Buscemi–Das–Wilde 2016): $S(\mathcal{E}(\rho)) - S(\rho) \ge D(\rho || (\mathcal{E}^{\dagger} \circ \mathcal{E})\rho).$

Open question. What about a different relative entropy, like Belavkin–Staszewski's?

Update 1

A simple calculation shows that the Li–Winter relation can be strengthened to an equality as follows

 $D(p||q) - D(\varphi[p]||\varphi[q]) = D(\varphi \star p||\tilde{\varphi}_q \star \varphi[p]) ,$

where $\varphi \star p \sim \varphi(y|x)p(x)$ and $\tilde{\varphi}_q \star \varphi[p] \sim \tilde{\varphi}_q(x|y)[\varphi \circ p](y)$ are two joint probability distributions, corresponding to the direct and reverse processes, respectively. The Li–Winter relation is then obtained by summing over the y index and applying the monotonicity property of classical relative entropy.

Something similar happens in the quantum case for the Belavkin–Staszewski's relative entropy: see [Bai *et al.*, Quantum 8, 1524 (2024)].

Update 2

While in 2021 (when this slides were first presented) the connection between Petz's transpose map and Bayesian inversion was only at the level of a formal analogy, now we know that a much more compelling connection can be made through the principle of minimum update. See: [Bai, Buscemi, Scarani; arXiv:2410.00319]

Case study: application in statistical mechanics

Satosi Watanabe



"The phenomenological one-wayness of temporal developments in physics is due to irretrodictability, and not due to irreversibility." S. Watanabe (1965)

Ed Jaynes



"To understand and like thermo we need to see it, not as an example of the *n*-body equations of motion, but as an example of the logic of scientific inference."

E.T. Jaynes (1984)

More concretely: to derive fluctuation relations with the reverse process as Bayesian retrodiction

Construction of the reverse process as retrodiction

- physical setup:
 - \circ a stochastic transition rule: $\varphi(y|x)$
 - \circ a steady (viz. invariant) state: $\sum_{x} \varphi(y|x) s(x) = s(y)$
- Bayes–Jeffrey inversion at the steady state:

$$s(y)\tilde{\varphi}(x|y) := s(x)\varphi(y|x) \iff \frac{\varphi(y|x)}{\tilde{\varphi}(x|y)} = \frac{s(y)}{s(x)}$$

- two priors:
 - predictor's prior: p(x)
 - \circ retrodictor's prior q(y)
- two processes:
 - forward process (prediction): $\mathcal{P}_F(x,y) = \varphi(y|x)p(x)$
 - reverse process (retrodiction): $\mathcal{P}_R(x,y) = \tilde{\varphi}(x|y)q(y)$

A picture

$$S \rightarrow \varphi(||^{2}) \rightarrow = \left\{ - \left(\hat{\varphi}(||_{y}) \right) - S \right\}$$
$$P \rightarrow \varphi(||^{2}) \rightarrow \neq \left\{ - \left(\hat{\varphi}(||_{y}) \right) - Q \right\}$$

- at the steady state: prediction = retrodiction
- otherwise: asymmetry (irreversibility, *irretrodictability*)

Quantifying irretrodictability

Idea: fluctuation relations as measures of divergence between prediction and retrodiction

• relative entropy: $D(\mathcal{P}_F || \mathcal{P}_R) := \left\langle -\ln \frac{\mathcal{P}_R(x,y)}{\mathcal{P}_F(x,y)} \right\rangle_F =: \left\langle -\ln r(x,y) \right\rangle_F$

 \rightsquigarrow more generally, one can use $D_f(\mathcal{P}_R \| \mathcal{P}_F) := \langle f(r(x,y)) \rangle_F$

• introduce probability density functions

Examples of known results recovered by retrodiction

Example: driven closed system evolution



- driving protocol: $H(0) \rightarrow H(t) \rightarrow H(\tau)$
- $H(0) = (\epsilon_x)_x$, $H(\tau) = (\eta_y)_y$
- $\varphi(y|x) = \delta_{y,y(x)}$, i.e., one-to-one
- Hamiltonian $\implies \tilde{\varphi}(x|y) \equiv \varphi(y|x)$
- $p_0(x) = e^{\beta(F \epsilon_x)}, \ q_\tau(y) = e^{\beta(F' \eta_y)}$

In this case,

$$\Omega(x,y) = \ln \frac{\mathcal{P}_F(x,y)}{\mathcal{P}_R(x,y)} = \ln \frac{\varphi(y|x)p(x)}{\tilde{\varphi}(x|y)q(y)} = \ln \frac{p(x)}{q(y)}$$
$$= \beta(F - \epsilon_x + F' + \eta_y) = \beta(W - \Delta F)$$
$$\implies \frac{\mu_F(W)}{\mu_R(W)} = e^{\beta(W - \Delta F)} \implies \langle W \rangle \ge \Delta F$$
$$24/29$$

Example: nonequilibrium steady states

- stochastic process $\varphi(y|x)$ with non-thermal steady state s(x)
- thermal equilibrium priors: $p(x) = q(x) \propto e^{-\beta \epsilon_x}$
- fluctuation variable: $\omega = \ln \frac{\mathcal{P}_F(x,y)}{\mathcal{P}_R(x,y)} = \ln \frac{p(x)}{q(y)} \frac{s(y)}{s(x)} = \beta(\epsilon_y - \epsilon_x) + (\ln s(y) - \ln s(x))$
- nonequilibrium potential: $V(x) := -\frac{1}{\beta} \ln s(x)$ (e.g., Manzano&al 2015)
- nonequilibrium potentials (usually introduced *ad hoc*) are understood here as remnants of Bayesian inversion
- $\implies \left\langle e^{\beta(\Delta E \Delta V)} \right\rangle_F = 1 \implies D(p\|s) D(\varphi[p]\|s) \ge 0$

25/29

But why known relations are compatible with Bayesian inversion?

Is that a necessity?

Sketch argument

•
$$D(\boldsymbol{\mathcal{P}}_F \| \boldsymbol{\mathcal{P}}_R) = \left\langle \ln \frac{\boldsymbol{\mathcal{P}}_F(x,y)}{\boldsymbol{\mathcal{P}}_R(x,y)} \right\rangle_F$$

• let us impose that the fluctuation variable is **local**:

$$\ln \frac{\mathcal{P}_F(x,y)}{\mathcal{P}_R(x,y)} = \Omega(x,y) \stackrel{!}{=} g_2(y) - g_1(x)$$

$$\mathcal{P}_F(y|x) = h_2(y)$$

$$\implies \frac{\mathcal{F}_{I}(y|x)}{\mathcal{P}_{R}(x|y)} = \frac{h_{2}(y)}{h_{1}(x)}$$

•
$$\implies$$
 $h_1(x)\mathcal{P}_F(y|x) = h_2(y)\mathcal{P}_R(x|y)$

• sum over
$$x \implies h_2(y) = \sum_x h_1(x) \mathcal{P}_F(y|x)$$

•
$$\Longrightarrow \mathcal{P}_R(x|y) = \frac{1}{\sum_x h_1(x)\mathcal{P}_F(y|x)} h_1(x)\mathcal{P}_F(y|x)$$

Hence, a Bayesian inverse-like form for the reverse process is **inevitable** if we want the fluctuating variable to have a local form!

26/29

Finally, what about the quantum case?

Quantum retrodiction and the Petz map



- assume $\varphi(y|x) = \operatorname{Tr}[\Pi_y \ \mathcal{E}(\rho_x)]$
- let s(x) be invariant distribution
- according to the formalism of *quantum retrodiction*:

- $\circ \Sigma := \sum_{x} s(x) \rho_{x}$ $\circ \tilde{\rho}_{y} := \frac{1}{s(y)} \sqrt{\mathcal{E}(\Sigma)} \Pi_{y} \sqrt{\mathcal{E}(\Sigma)}$
- $\circ \quad \tilde{\Pi}_x := s(x) \frac{1}{\sqrt{\Sigma}} \rho_x \frac{1}{\sqrt{\Sigma}} \\ \circ \quad \tilde{\mathcal{E}}_{\Sigma}(\bullet) := \sqrt{\Sigma} \left\{ \mathcal{E}^{\dagger} \left[\frac{1}{\sqrt{\mathcal{E}(\Sigma)}} (\bullet) \frac{1}{\sqrt{\mathcal{E}(\Sigma)}} \right] \right\} \sqrt{\Sigma}$
- Bayesian inversion works seamlessly $\tilde{\varphi}(x|y) = \operatorname{Tr}[\tilde{\Pi}_x \ \tilde{\mathcal{E}}_{\Sigma}(\tilde{\rho}_y)]$

27/29

Conclusion

Conceptual take-home messages

- 1. the reversal in general depends on a choice of prior (exceptions: Hamiltonian processes)
- 2. classically, Petz's map coincides with Jeffrey's retrodiction
- 3. in quantum theory, it agrees with previous proposals for "quantum retrodiction"
- 4. however, we still lack of a thorough mathematical theory of quantum inference (Petz map is not unique in general!)
- 5. reverse processes in statistical mechanics better seen as "reverse inferences" rather than "time-reverses"
- 6. the inferential approach circumvents the problem of "realization"
 thank you 28/29

About these ideas

Two papers:

- with V. Scarani. Fluctuation relations from Bayesian retrodiction. Phys. Rev. E (2021). arXiv:2009.02849 [quant-ph]
- with C.C. Aw and V. Scarani. Fluctuation Theorems with Retrodiction rather than Reverse Processes. AVS Quantum Science (to appear). arXiv:2106.08589 [cond-mat.stat-mech]