Using data-driven inference to bootstrap quantum tomography

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An unknown quantum device:

how can we infer anything about it?

The usual solution: quantum tomography

- assumption: $\{\rho_i\}$ and $\{\Pi_j\}$ are perfectly known
- data: $p(j|i) = \mathrm{Tr}[\mathcal{X}(\rho_i)\ \Pi_j]$
- reconstruction of $\mathcal{X}$: by linear (pseudo-)inversion
- reconstruction is unique iff $\{\rho_i\}$ and $\{\Pi_j\}$ are *informationally complete* (IC)

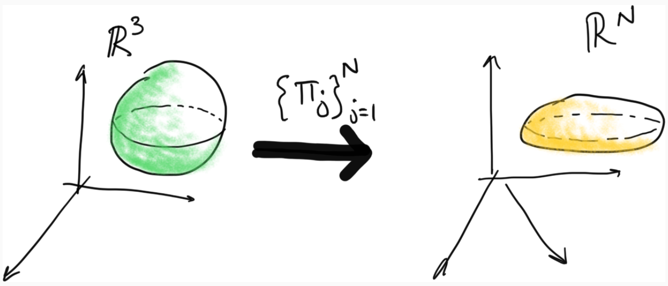
The problem: Wigner's (other) chain

From [E.P. Wigner, Lecture at the Conference on the Foundations of Quantum Mechanics, Xavier University, Cincinnati, 1962]:

> *[...] the experimentalist uses certain apparatus to measure the position, let us say, or the momentum, or the angular momentum. Now, how does the experimentalist know that this apparatus will measure for him the position? [...] Well that means that he carried out a measurement on it. How did he know that the apparatus with which he carried out that measurement will tell him the properties of the apparatus? Fundamentally, this is again a chain which has no beginning.*

Data-driven inference of $\{\Pi_j\}$

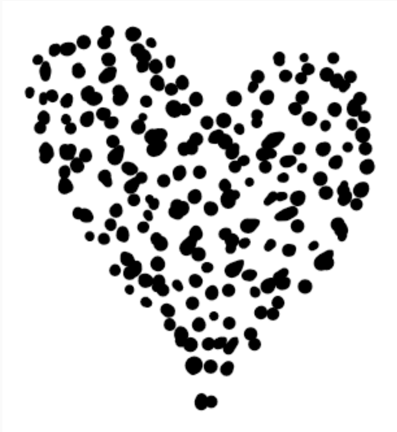
What is a measurement?



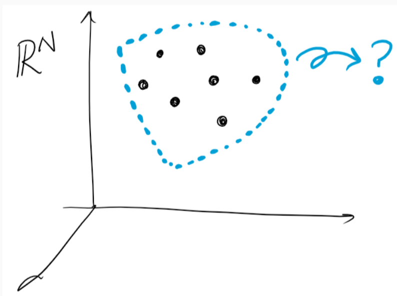
- a qubit measurement $\{\Pi_j\}_{j=1}^N$ is a linear mapping from $\mathbb{R}^3$ to $\mathbb{R}^N$
- the Bloch sphere $\mathbb{S}$ is mapped into an ellipsoid $(\mathbf{y}-\mathbf{c})^T\mathbf{Q}(\mathbf{y}-\mathbf{c}) \leq 1$
- up to gauge symmetries (unitary and antiunitary transformations), $\Pi_j = c_j\mathbf{I} + \sum_{k=1}^3 T_{j,k}\boldsymbol{\sigma}_k$, where $T_{j,k}$ are the matrix elements of $\mathbf{T}\mathbf{T}^T = \mathbf{Q}$

Problem

How to find the ellipsoid without knowing the input states?



The inference algorithm



Data-driven inference (DDI)

Given the data points $\mathbf{p}_i = [p(1|i), \cdots, p(N|i)]^T$, infer the measurement $\{\Pi_j\}_{j=1}^N$ from the ellipsoid which:

1. contains the convex hull of all $\mathbf{p}_i$
2. is of minimum euclidean volume

Some comments

- why volume? because we want the inference to be independent of *linear transformations* (and these are all that matter for a linear theory)
- why minimum? because we want the inference to be the *least committal* one explaining *all the data*

Minimum-volume enclosing ellispoid (MVEE)

The MVEE:

- exists unique for any data-set
- can be efficiently computed
- could fail to correspond to a valid measurement: in this case a failure is announced

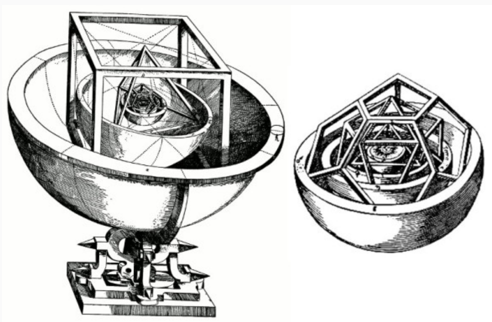
See, e.g, F. John (1948), S. Boyd and L. Vandeberghe (2004), M. J. Todd (2016).

Sufficient data-sets

- how to produce data points $\mathbf{p}_i$ such that **DDI** finds the "right" measurement $\Pi = \{\Pi_j\}_{j=1}^{N}$?
- *all* pure states? unfeasible
- definition: a set $S \subseteq \mathbb{S}$ is said to be *observationally complete* (OC) for measurement Π whenever **DDI**$[\Pi(\mathcal{S})] = \Pi(\mathbb{S})$
- fact: S is OC *for any measurement* iff **DDI**$[S] = \mathbb{S}$

Observationally complete sets for qubits

- in any real dimension ℓ, the MVEE of $\ell + 1$ points is a hypersphere *iff* the points form a regular simplex



- for qubits, the tetrahedron-ensemble is good for measurement DDI

Example: OC vs IC

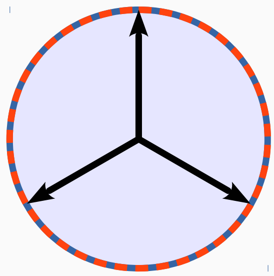


Figure 1: A regular simplex is *observationally complete*.

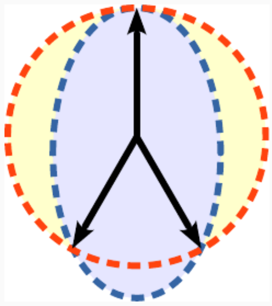


Figure 2: An irregular simplex can be *informationally* complete but is not *observationally* complete.

Example: DDI in action

- suppose that the data-set is given by three points in $\mathbb{R}^4$: $\mathbf{p}_1 = [\frac{1}{2}, 0, \frac{1}{4}, \frac{1}{4}]^T$, $\mathbf{p}_2 = [\frac{1}{8}, \frac{3}{8}, \frac{2+\sqrt{3}}{8}, \frac{2-\sqrt{3}}{8}]^T$, and $\mathbf{p}_3 = [\frac{1}{8}, \frac{3}{8}, \frac{2-\sqrt{3}}{8}, \frac{2+\sqrt{3}}{8}]^T$
- a priori, we can only say that the data have been obtained from three states and a measurement with four outcomes
- let us apply **DDI**: assuming that the system is a real qubit, the four elements of the measurement $\{\Pi_j\}_{j=1}^4$ must be
  1. $\rightsquigarrow$ rank-one
  2. $\rightsquigarrow$ lying on the same plane
  3. $\rightsquigarrow$ arranged in a perfect square
- *cfr* self-testing of non-signaling correlations

Conclusion: how to "bootstrap" quantum tomography

- first: run **DDI** on an extremal IC POVM (e.g., a tetrahedron POVM)
- second: arbitrarily fix a symmetry (i.e., a labeling of the POVM's elements)
- using the above, perform tomography of a complete set of quantum states
- perform gate tomography as usual