Spontaneous Processes as Decrease of Information(s)

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Jaynes' "MAXENT Principle"

"In making inferences on the basis of partial information we must use that probability distribution which has maximum entropy subject to whatever is known. This is the only unbiased assignment we can make; to use any other would amount to arbitrary assumption of information which by hypothesis we do not have."

Jaynes, circa 1995

What about processes?

Jaynes' MAXENT provides an information-theoretic "justification" for the Gibbs distribution (as the "minimally committing" inference possible, given the observations)

Question

Can we "justify" in the same information-theoretic way also spontaneous processes, namely, spontaneous transitions between different microstate distributions?

Thermodynamics vs Infodynamics

- **thermodynamical viewpoint**: spontaneous processes are exactly those associated with a decrease of the free energy contained in the system
- **information-theoretic viewpoint**: spontaneous processes are exactly those associated with a decrease of the information contained in the system

In search of a rigorous link

Can we *infer* spontaneous thermal processes as exactly those that never increase "information"? And what is "information" about in this case?

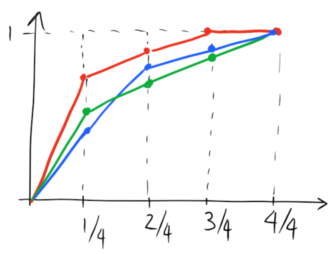
Information: one or many?

- Jaynes clearly considers information to be a totally ordered quantity, i.e., he measures it by the (neg-)entropy
- however, when looking at (microstate) distributions, information often takes on a multi-faceted form, its “content” depending on its “use”
- indeed, in mathematical statistics, information is only a partially ordered quantity and there exist many **incomparable information contents** associated with the same probability distribution

Vishvarupa (Vishnu)

Example: majorization

- given are two probability distributions, $\boldsymbol{p}$ and $\boldsymbol{q}$, of the same dimension d
- consider the truncated sums $P(k) = \sum_{i=1}^{k} p_i^{\downarrow}$ and $Q(k) = \sum_{i=1}^{k} q_i^{\downarrow}$, for all $k = 1, \ldots, d$
- we say that $\boldsymbol{p}$ majorizes $\boldsymbol{q}$, in formula, $\boldsymbol{p} \succ \boldsymbol{q}$, whenever $P(k) \geq Q(k)$, for all k
- remark: $\boldsymbol{p} \succ \boldsymbol{q} \underset{\not\Longleftarrow}{\Longrightarrow} H(\boldsymbol{p}) \geq H(\boldsymbol{q})$
- Hardy-Littlewood-Pólya (1929): $\boldsymbol{p} \succ \boldsymbol{q} \iff \boldsymbol{q} = M\boldsymbol{p}$, for some bistochastic matrix M

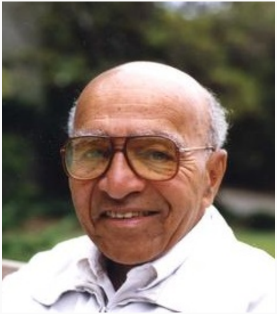


The order is only partial: neither

blue$\succ$green, nor green$\succ$blue.

Information ordering in mathematical statistics

- given are *two pairs* of probability distributions, $(\boldsymbol{p}_1, \boldsymbol{p}_2)$ and $(\boldsymbol{q}_1, \boldsymbol{q}_2)$, of possibly different dimensions
- reorder their entries such that $p_1^i / p_2^i \geq p_1^j / p_2^j$, whenever $i \leq j$; do the same for $(\boldsymbol{q}_1, \boldsymbol{q}_2)$
- construct the linear spline $(x_k, y_k) = \left(\sum_{i=1}^k p_2^i, \sum_{i=1}^k p_1^i \right)$ joining $(0, 0)$ with $(1, 1)$; do the same for $(\boldsymbol{q}_1, \boldsymbol{q}_2)$
- we say that $(\boldsymbol{p}_1, \boldsymbol{p}_2)$ is more informative than $(\boldsymbol{q}_1, \boldsymbol{q}_2)$, in formula $(\boldsymbol{p}_1, \boldsymbol{p}_2) \succ (\boldsymbol{q}_1, \boldsymbol{q}_2)$, whenever the spline corresponding to the former is never below that corresponding to the latter
- Blackwell's theorem for dichotomies (1953): $(\boldsymbol{p}_1, \boldsymbol{p}_2) \succ (\boldsymbol{q}_1, \boldsymbol{q}_2) \iff \boldsymbol{q}_i = M \boldsymbol{p}_i$, for some stochastic matrix M



Blackwell in 1999

Information ordering solves the classical case

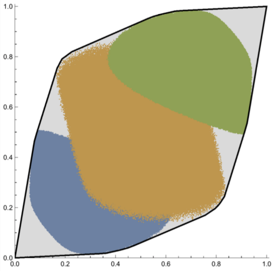
- when $\boldsymbol{p}_2 = \boldsymbol{q}_2 = \boldsymbol{\gamma}$, i.e., the Gibbs distribution, then the information ordering $(\boldsymbol{p}, \boldsymbol{\gamma}) \succ (\boldsymbol{q}, \boldsymbol{\gamma})$ coincides with the order of thermo-majorization
- when everything commutes with the Hamiltonian, thermo-majorization is known to be equivalent to the phenomenological order induced by thermal processes (Horodecki-Oppenheim, 2013)
- information ordering can be expressed in terms of a family $\{F_\alpha\}_\alpha$ of state functions (or "free energies"): $(\boldsymbol{p}, \boldsymbol{\gamma}) \succ (\boldsymbol{q}, \boldsymbol{\gamma}) \iff F_\alpha(\boldsymbol{p}) \geq F_\alpha(\boldsymbol{q})$ for all α

What about the non-commuting case?

- first problem: to define a "quantum information ordering"
- second problem: to define "quantum thermal processes"
- third problem: given that the above two concepts play well with each other, to find the "quantum free energies"
- fourth problem: to provide a compelling physical interpretation of the mathematical results

Tackling the first and the second problem

- given are two pairs of *density matrices*, (ρ_1, ρ_2) and (σ_1, σ_2), of possibly different dimensions
- a "quantum information ordering" is introduced using the concept of quantum statistical comparison (FB, Comm. Math. Phys., 2012) and quantum relative Lorenz curves (FB, Gour; Phys. Rev. A, 2017)
- the thermodynamically relevant case occurs for pairs (ρ, γ) and (σ, γ), where γ is the thermal distribution (FB, arXiv:1505.00535)
- "quantum thermal processes" are introduced as thermal operations (TO) (Janzing et al., Int. J. Th. Ph., 2000; Brandao et al., Phys. Rev. Lett., 2013) or generalized thermal processes (GTP) (Gour et al., arXiv:1708.04302)



Quantum relative testing region

TO vs GTP

Thermal Operations	Generalized Thermal Processes
$\mathcal{E}(\rho_Q) = \mathrm{Tr}_E[V(\rho_Q \otimes \gamma_E)V^\dagger]$	$\mathcal{E}(\rho_Q) = \mathrm{Tr}_E[V(\rho_Q \otimes \delta_E)V^\dagger]$
$[V, H_Q \otimes I_E + I_Q \otimes H_E] = 0$	$[V, H_Q \otimes I_E + I_Q \otimes H_E] = 0$
γ_E: thermal state	δ_E: state commuting with H_E
	$\mathcal{E}(\gamma_Q) = \gamma_Q$

Remark: the final condition (Gibbs-preserving condition) is automatic for Thermal Operations, se we do not need to enforce it explicitly.

Third problem: second laws for GTP

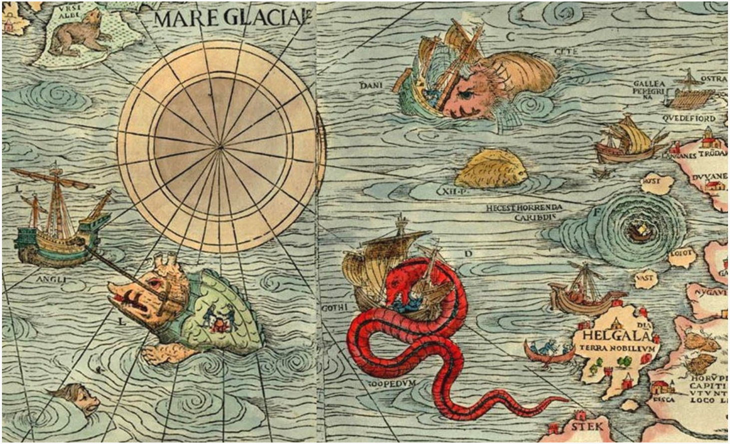
The problem is to understand when the transition $\rho \to \sigma$ is possible with a GTP. Proceed as follows:

- take any third pair of "reference" states (η_1, η_2)
- construct the bipartite states $\Omega = \frac{1}{2}(\eta_1 \otimes \rho + \eta_2 \otimes \gamma)$ and $\Omega' = \frac{1}{2}(\eta_1 \otimes \sigma + \eta_2 \otimes \gamma)$
- take their "twirling time-averages," i.e., $\langle \Omega \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} (\overline{e^{-itH}} \otimes e^{-itH}) \Omega (\cdots)^\dagger$ and $\langle \Omega' \rangle$
- construct the functions (here $|\varphi^+\rangle$ is the max. entangled state) $f_{(\eta_1,\eta_2)}(\rho) = \max_{\mathcal{D}:\text{cov.CPTP}} \langle \varphi^+ | (\text{id} \otimes \mathcal{D}) \langle \Omega \rangle | \varphi^+ \rangle$

Then, $\rho \to \sigma$ is possible with a GTP if and only if $f_{(\eta_1,\eta_2)}(\rho) \geq f_{(\eta_1,\eta_2)}(\sigma)$, for all choices of (η_1, η_2).

Fourth problem: physical interpretation

Hic Sunt Leones!



Fourth problem: physical interpretation

$\Omega = \frac{1}{2}(\eta_1 \otimes \rho + \eta_2 \otimes \gamma)$	$\langle\Omega\rangle = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}(\overline{e^{-itH}} \otimes e^{-itH})\Omega(\cdots)^\dagger$
$f_{(\eta_1,\eta_2)}(\rho) \geq f_{(\eta_1,\eta_2)}(\sigma)$	$f_{(\eta_1,\eta_2)}(\rho) = \max_{\mathcal{D}:\text{cov.CPTP}}\langle\varphi^+\|(\mathrm{id}\otimes\mathcal{D})\langle\Omega\rangle\|\varphi^+\rangle$

- with respect to the time-twirling chosen (i.e., $\overline{U_t} \otimes U_t$), the maximally entangled reference-system state $|\varphi^+\rangle$ is static, though perfectly correlated: it serves as a perfect **Pa**ge-**W**ootters time-energy relative state
- any other static bipartite state, like $\langle\Omega\rangle$, can be understood as a "noisy" PaW state
- hence, the functions $f_{(\eta_1,\eta_2)}(\rho)$ say how well the noisy PaW state $\langle\Omega\rangle$ can be "locally covariantly adjusted" to simulate a perfect PaW state
- in other words, each function $f_{(\eta_1,\eta_2)}(\rho)$ measures **a different time-energy information content** of state ρ relative to γ
- in the classical case, we can replace $|\varphi^+\rangle$ with a classical state perfectly correlated in energy: time-information is irrelevant in this case

Physical interpretation, part two

Summarizing, the family of inequalities

$$f_{(\eta_1,\eta_2)}(\rho) \geq f_{(\eta_1,\eta_2)}(\sigma), \quad \forall(\eta_1, \eta_2),$$

says that the initial state ρ is always more informative than the final state σ, with respect to all time-energy information contents.

Take-home message

Quantum GTPs are exactly those associated with a decrease of all the system's time-energy information contents (possibly infinitely many).

Remark 1. Classically, time-information is irrelevant: only energy-information matters.
Remark 2. Don't worry: GTPs can be decided efficiently (using SDP).

thermodynamics $\equiv$ time-energy information dynamics

A very incomplete and biased list of references

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