A theory of quantum local asymptotic normality

Part II: Asymptotic representation theorem

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LAN (Local asymptotic normality)

A sequence of parametric statistical models

$$\left\{p_\theta^{(n)}: \theta \in \Theta \subset \mathbb{R}^d\right\}$$

is said to be LAN at $\theta_0 \in \Theta$ if

$$\log \frac{p_{\theta_0 + h/\sqrt{n}}^{(n)}}{p_{\theta_0}^{(n)}} = h^i \Delta_i^{(n)} - \frac{1}{2} h^\top J\, h + o_h(n)$$

where

- $h \in \mathbb{R}^d$: local parameter
- $\Delta^{(\mathrm{n})} \rightsquigarrow N(0, J)$: converges in distribution
- $o_h(n) \to 0$: converges in probability
- $J > 0$: $d \times d$ positive real matrix

※ in i.i.d. case:

- $\Delta_i^{(n)}$ is a logarithmic derivative
- J is a Fisher information matrix

Similarity to Gaussian shift model

■ Gaussian shift model

$$\{N(Jh, J):\ h \in \mathbb{R}^d\}$$

$$\downarrow$$

$$\log \frac{dN(Jh, J)}{dN(0, J)}(X_1, \dots, X_d) = h^i X_i - \frac{1}{2} h^\top J\, h$$

■ LAN

$$\left\{p^{(n)}_{\theta_0 + h/\sqrt{n}}:\ h \in \mathbb{R}^d\right\}$$

$$\downarrow$$

$$\log \frac{p^{(n)}_{\theta_0 + h/\sqrt{n}}}{p^{(n)}_{\theta_0}} = h^i \Delta_i^{(n)} - \frac{1}{2} h^\top J\, h + o_h(n)$$

Local asymptotic representation thm

[Thm1] Suppose $\left\{p_\theta^{(n)}: \theta \in \Theta \subset \mathbb{R}^d\right\}$ is LAN at $\theta_0 \in \Theta$

then

$\forall\, T^{(n)}$:statistics s.t.

$$\left(T^{(n)}, P_{\theta_0+h/\sqrt{n}}^{(n)}\right) \rightsquigarrow \mathcal{L}_h$$

$\exists T$: randomized statistics s.t.

$$\left(T, N(Jh, J)\right) \sim \mathcal{L}_h$$

where $\mathcal{L}_h$ is a distribution depending on $h \in \mathbb{R}^d$

This theorem tell us any estimator of $p_\theta^{(n)}$ can be mimicked by $N(Jh, J)$.

$\rightarrow$ asymptotic local minimax theorem

quantum LAN (q-LAN)

A sequence of parametric quantum statistical models

$$\left\{\rho_\theta^{(n)} : \theta \in \Theta \subset \mathbb{R}^d\right\}$$

is q-LAN at $\theta_0 \in \Theta$ if

$$\log\left\{\mathcal{R}\left(\rho_{\theta_0+h/\sqrt{n}}^{(n)} | \rho_{\theta_0}^{(n)}\right) + o_h^{L^2}(n)\right\}^2 = h^i \Delta_i^{(n)} - \frac{1}{2} h^\top J\, h + o_h(n)$$

where

- $\mathcal{R}(\cdot|\cdot)$: square-root likelihood ratio
- $J \geq 0$: $d \times d$ non-negative <u>complex</u> matrix
- $\left(\Delta^{(\mathrm{n})}, \rho_{\theta_0}^{(n)}\right) \rightsquigarrow N(0, J)$: quantum convergence in distribution
- $N(0, J)$: <u>quantum</u> Gaussian state
- $o_h^{L^2}(n)$, $o_h(n)$: infinitesimal terms

quantum likelihood ratio

$$\log\left\{\mathcal{R}\left(\rho^{(n)}_{\theta_0+h/\sqrt{n}}|\rho^{(n)}_{\theta_0}\right)+o_h^{L^2}(n)\right\}^2 = h^i\Delta_i^{(n)} - \frac{1}{2}h^\top J\,h + o_h(n)$$

Square-root of likelihood ratio $R = \mathcal{R}(\sigma|\rho) \geq 0$ of density operators ρ, σ is defined by quantum Lebesgue decomposition

$$\sigma = R\rho R + \sigma^\perp$$

with a singular part $\sigma^\perp \geq 0$ s.t.

$$\mathrm{Tr}\,\rho\sigma^\perp = 0$$

※ $\sigma > 0$ and $\rho > 0$ are not required!!

Quantum convergence in distribution

$$\log\left\{\mathcal{R}\left(\rho_{\theta_0+h/\sqrt{n}}^{(n)}|\rho_{\theta_0}^{(n)}\right)+o_h^{L^2}(n)\right\}^2=h^i\Delta_i^{(n)}-\frac{1}{2}h^\top J\,h+o_h(n)$$

$\left(\Delta^{(\mathrm{n})},\rho_{\theta_0}^{(n)}\right)\rightsquigarrow N(0,J)$ means

$$\lim_{n\to\infty}\mathrm{Tr}\rho_{\theta_0}^{(n)}\left(\prod_{t=1}^{r}\mathrm{e}^{\sqrt{-1}\xi_t^i\,\Delta_i^{(n)}}\right)=\phi\left(\prod_{t=1}^{r}\mathrm{e}^{\sqrt{-1}\xi_t^i\,\Delta_i}\right)$$

where ϕ is a quantum Gaussian state defined by a quasi-characteristic function

$$\phi\left(\prod_{t=1}^{r}\mathrm{e}^{\sqrt{-1}\xi_t^i\,\Delta_i}\right)=\exp\left(-\frac{1}{2}\sum_{t=1}^{r}\xi_t^i\xi_t^jJ_{ji}-\sum_{t=1}^{r}\sum_{u=t+1}^{r}\xi_t^i\xi_u^jJ_{ji}\right)$$

with canonical observables $\{\Delta_i\}_{i=1}^d$ and $\{\xi_t\}_{t=1}^r\subset\mathbb{R}^d$.

Infinitesimal terms

$$\log\left\{\mathcal{R}\left(\rho^{(n)}_{\theta_0+h/\sqrt{n}}|\rho^{(n)}_{\theta_0}\right)+o_h^{L^2}(n)\right\}^2=h^i\Delta_i^{(n)}-\frac{1}{2}h^\top J\,h+o_h(n)$$

- $o_h^{L^2}(n)$:

$$\lim_{n\to\infty}\mathrm{Tr}\rho^{(n)}_{\theta_0}\left\{o_h^{L^2}(n)\right\}^2=0$$

- $o_h(n)$:

$$\lim_{n\to\infty}\mathrm{Tr}\rho^{(n)}_{\theta_0}e^{\sqrt{-1}\left(\xi^i\Delta_i^{(n)}+\eta o_h(n)\right)}e^{-\sqrt{-1}\xi_t^i\,\Delta_i^{(n)}}=1$$

for any $\xi\in\mathbb{R}^d$ and $\eta\in\mathbb{R}$

Quantum Local asymptotic representation (conjecture)

[conjecture] Suppose $\left\{\rho_\theta^{(n)}\colon \theta \in \Theta \subset \mathbb{R}^d\right\}$ is q-LAN at $\theta_0 \in \Theta$

then

$\forall\, M^{(n)}$: POVM s.t.

$$\left(M^{(n)}, \rho_{\theta_0 + h/\sqrt{n}}^{(n)}\right) \rightsquigarrow \mathcal{L}_h$$

$\exists M$: POVM s.t.

$$\left(M, N\big((ReJ)h, J\big)\right) \sim \mathcal{L}_h$$

where $\mathcal{L}_h$ is a classical distribution depending on $h \in \mathbb{R}^d$

→ This conjecture has counter examples.

D-extension

■List of observables $X^{(n)} = \left(X_1^{(n)}, \ldots, X_r^{(n)}\right)$ is asymptotically D-invariant with respect to $\rho_{\theta_0}^{(n)}$ if

- $\left(X^{(n)}, \rho_{\theta_0}^{(n)}\right) \rightsquigarrow N(0, \Sigma)$ with $r \times r$ complex matrix $\Sigma \geq 0$.
- $$\lim_{n\to\infty} \mathrm{Tr} \sqrt{\rho_{\theta_0}^{(n)}} e^{\sqrt{-1}\xi^i X_i^{(n)}} \sqrt{\rho_{\theta_0}^{(n)}} e^{\sqrt{-1}\eta^i X_i^{(n)}} = e^{-\frac{1}{2}\begin{pmatrix}\xi \\ \eta\end{pmatrix}^\top \begin{pmatrix}\Sigma & \Sigma \# \Sigma^\top \\ \Sigma \# \Sigma^\top & \Sigma^\top\end{pmatrix}\begin{pmatrix}\xi \\ \eta\end{pmatrix}} \quad \forall \xi, \eta \in \mathbb{R}^r$$

  where # is the operator geometric mean

■ $X^{(n)}$ is D-extension of $\Delta^{(n)} = \left(\Delta_1^{(n)}, \ldots, \Delta_d^{(n)}\right)$ if $\Delta_k^{(n)} = F_k^i X_i^{(n)}$ with an $r \times d$ real matrix F.

I.I.D. model

If $\{\rho_\theta : \theta \in \Theta \subset \mathbb{R}^d\}$ is smooth at $\theta_0 \in \Theta$ then
$\left\{\rho_\theta^{\otimes n} : \theta \in \Theta \subset \mathbb{R}^d\right\}$ is q-LAN and D-extendible

- $\Delta_i^{(1)}$ is SLD at ρ_{θ_0}

- $\left\{X_k^{(1)}\right\}_{k=1}^{r}$ is $\mathcal{D}$ invariant extension of $\left\{\Delta_i^{(1)}\right\}_{i=1}^{d}$ (always exist)

- $\Delta_i^{(n)}$ and $X_k^{(n)}$ are I.I.D. extension of $\Delta_i^{(1)}$ and $X_k^{(1)}$

※ $\rho_{\theta_0} > 0$ and non-degeneracy are not necessary.

Quantum local asymptotic representation thm

[Thm] Suppose $\left\{\rho_\theta^{(n)}: \theta \in \Theta \subset \mathbb{R}^d\right\}$ is q-LAN and D-extensible at $\theta_0 \in \Theta$ then

$\forall\, M^{(n)}$: POVM s.t.

$$\left(M^{(n)}, \quad \rho^{(n)}_{\theta_0 + h/\sqrt{n}}\right) \rightsquigarrow \mathcal{L}_h$$

$\exists M$: POVM s.t.

$$\left(M, \quad N\left((\mathrm{Re}\,\Sigma F)h, \Sigma\right)\right) \sim \mathcal{L}_h$$

where $\mathcal{L}_h$ is a classical distribution depending on $h \in \mathbb{R}^d$

This theorem tell us any POVM and estimator of $\rho_\theta^{(n)}$ can be mimicked by quantum Gauss shift model $N\left((\mathrm{Re}\,\Sigma F)h, \Sigma\right)$

Estimation of quantum Gauss shift model

For a quantum Gauss shift model

$$\left\{N\left((\mathrm{Re}\,\Sigma F)h, \Sigma\right) : h \in \mathbb{R}^d\right\},$$

its unbiased estimator M satisfies

$$\mathrm{Tr}\, G\, V[M] \geq c_G^{(H)}$$

where

- $V[M]$ is covariance matrix,
- $G > 0$ is any $d \times d$ real positive matrix
- $c_G^{(H)}$ is Holevo bound defined by

$$c_G^{(H)} = \min_K \left\{\mathrm{Tr}\, GZ + \mathrm{Tr}\left|\sqrt{G}\,\mathrm{Im}\, Z\sqrt{G}\right| : \; Z = K^\top \Sigma K\right\},$$

K is a $r \times d$ real matrix s.t. $K^\top(\mathrm{Re}\,\Sigma F) = I.$

Asymptotic representation bound

- For D-extendible q-LAN model, **asymptotic representation bound**

$$c_G^{(rep)} := c_G^{(H)}$$

  is expected to be the bound of estimators because of the asymptotic representation theorem.

- In i.i.d. case $\{\rho_\theta^{\otimes n}\}_\theta$, $c_G^{(rep)}$ is same as Holevo bound of $\{\rho_\theta\}_\theta$.

- In non-i.i.d. case $\left\{\rho_\theta^{(n)}\right\}_\theta$, $c_G^{(rep)}$ is new bound

Quantum regular estimator

In classical statistics,
regular estimator was considered to exclude pathological estimator like the Hodges superefficient estimator.

For $\left\{\rho_\theta^{(n)}: \theta \in \Theta \subset \mathbb{R}^d\right\}$,
sequence of estimators (POVMs) $M^{(n)}$ is **regular** at θ_0 if

$$\left(\sqrt{n}\left\{M^{(n)} - \left(\theta_0 + \frac{h}{\sqrt{n}}\right)\right\}, \quad \rho_{\theta_0 + h/\sqrt{n}}^{(n)}\right) \rightsquigarrow \mathcal{L}$$

where $\mathcal{L}$ is a classical distribution independent of h

Quantum regular estimator

If $\left\{\rho_\theta^{(n)}: \theta \in \Theta \subset \mathbb{R}^d\right\}$ is D-extendible and q-LAN,
by using quantum local asymptotic representation th,
we can obtain

[Thm] For any $d \times d$ real positive matrix $G > 0$

$$\int_{\mathbb{R}^d} G_{ij} x^i x^j \mathcal{L}(dx) \geq c_G^{(rep)}$$

This inequality is sharp.

Quantum asymptotic minimax theorem

By using quantum local asymptotic representation th,
we can obtain

[Thm] Let $\mathcal{L}_h^{(n)} \sim (M^{(n)}, \rho_{\theta_0+h/\sqrt{n}}^{(n)})$ be a classical distribution with respect to estimator $M^{(n)}$. Then

$$\lim_{\delta\to 0} \liminf_{n\to\infty} \sup_{||h||\leq\delta\sqrt{n}} \int G_{ij}(x-h)^i(x-h)^j \mathcal{L}_h^{(n)}(dx)$$

$$\geq \sup_{H,L>0} \liminf_{n\to\infty} \sup_{h\in H} \int_{\mathbb{R}^d} L \wedge \{G_{ij}(x-h)^i(x-h)^j\} \mathcal{L}_h^{(n)}(dx)$$

$$\geq c_G^{(rep)}$$

The last inequality is sharp.

$H \subset \mathbb{R}^d$ is a finite subset.

Quantum Hodges superefficient estimator

- Let $\rho_\theta = \frac{1}{2}\left(I + \theta^1\sigma_1 + \theta^2\sigma_2 + \sqrt{1-|\theta|^2}\sigma_3\right)$ be a pure state model on a Hilbert space $\mathbb{C}^2$.
- $\{\rho_\theta^{\otimes n}\}_\theta$ is q-LAN and D-extendible at any θ.
  We can see $c_G^{(rep)} = 4$ when G is SLD Fisher information matrix $J_\theta^{(S)}$ at any θ.
- We can construct estimator $M^{(n)}$ s.t.

$$\left(\sqrt{n}\left(M^{(n)} - \theta\right), \rho_\theta^{\otimes n}\right) \rightsquigarrow N(0, 2\, {J_\theta^{(S)}}^{-1})$$

  $M^{(n)}$ can achieve $c_G^{(rep)} = 4$.

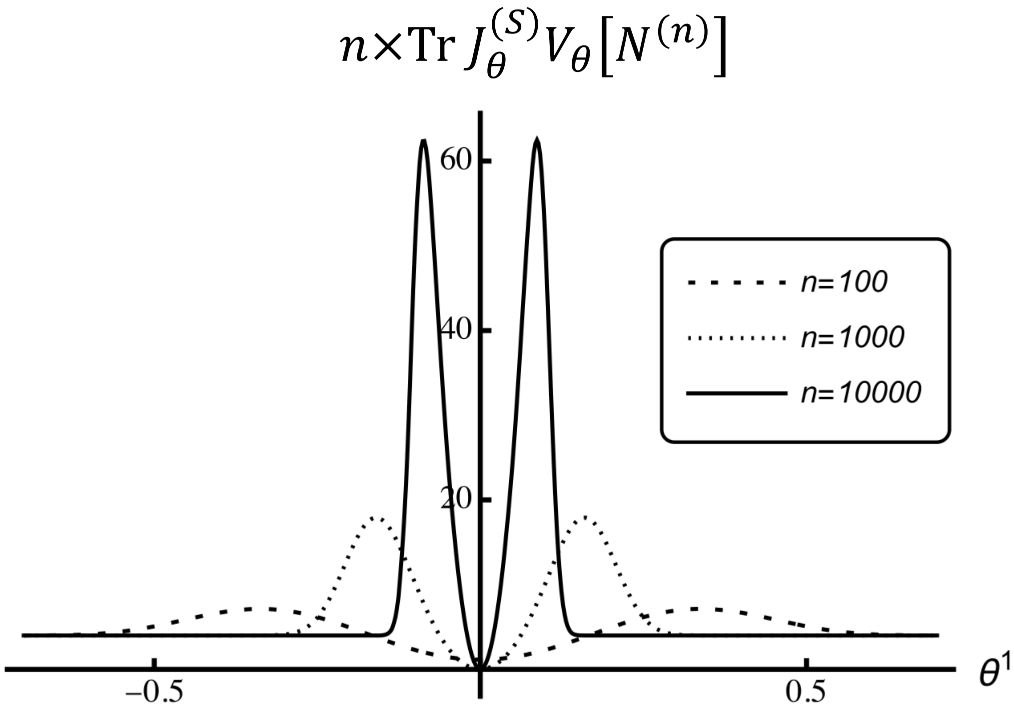
- Another estimator $N^{(n)} \coloneqq h_n\left(M^{(n)}\right)$ has superefficiency,
  where $h_n(x) = \begin{cases} x \;\; if\; |x| \geq n^{-1/4} \\ 0 \;\; if\; |x| < n^{-1/4} \end{cases}$

Quantum Hodges superefficient estimator

- The estimator $N^{(n)} \coloneqq h_n\left(M^{(n)}\right)$ has superefficiency:

$$\left(\sqrt{n}\left(N^{(n)} - \theta\right), \rho_\theta^{\otimes n}\right) \rightsquigarrow \begin{cases} 0 & if \quad \theta = 0 \\ N\left(0, 2\, {J_\theta^{(S)}}^{-1}\right) & if \quad \theta \neq 0 \end{cases}$$

- However, $N^{(n)}$ is not regular, and has bad behavior:



James-Stein superefficient estimator

For a classical Gauss shift model $\{N(h, I) : h \in \mathbb{R}^3\}$,
any unbiased estimator $\hat{h}$ satisfies Cramer Rao inequality

$$\mathrm{Tr}\, V\left[\hat{h}\right] \geq \mathrm{Tr}\, I = 3.$$

James-Stein estimator

$$\hat{h}^{(JS)}(x) = \left(1 - \frac{1}{\|x\|}\right) x$$

$(x \in \mathbb{R}^3)$ has superefficiency

$$\int_{\mathbb{R}^3} \left\|\hat{h}^{(JS)}(x) - h\right\|^2 N(h, I)(dx) < 3 \quad (\forall h \in \mathbb{R}^3)$$

※ $\hat{h}^{(JS)}$ is not unbiased, and minimax cost is 3.

quantum James-Stein superefficient estimator

Let $\rho_\theta = \frac{1}{2}(I + \theta^1\sigma_1 + \theta^2\sigma_2 + \theta^3\sigma_3)$ be a model on a Hilbert space $\mathbb{C}^2$.

I.I.D. model $\rho_\theta^{\otimes n}$ is q-LAN and D-extendible at any θ.

At $\theta = 0$, the limit quantum Gauss shift model is $N(h, I)$.

Since I is 3×3 real identity matrix, $N(h, I)$ is classical, and

$c_I^{(rep)} = 3$.

JS estimator can be mimicked by $\left\{\rho_{h/\sqrt{n}}^{\otimes n}\right\}_h$.

This can break $c_I^{(rep)}$ uniformly.

However, it is not regular,
and minimax cost doesn't break $c_I^{(rep)}$.

Conclusions

- If a sequence of quantum statistical mode is qLAN and D extendible, quantum asymptotic representation theorem can be proved.
- The quantum representation theorem tell us any sequence of POVMs of qLAN model can be mimicked by quantum Gauss shift model.
- The quantum representation theorem give us the lower bound of weighted MSE of regular estimator.
- The quantum representation theorem let us prove the asymptotic minimax theorem
- This theory is applicable to non-iid model
- This theory doesn't require $\rho_{\theta_0} > 0$ and non-degeneracy
- This theory is almost parallel to classical LAN theory

Thank you