A device-independent approach to quantum simulability

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Introduction

Motivation

Quantum statistical models (i.e., families of normalized density matrices) can be regarded as linear maps from the space of effects to the space of probabilities.

Quantum measurements (i.e., positive operator-valued measures) can be regarded as linear maps from the space of states to the space of probability distributions.

The images of such linear maps are called the testing regions of the corresponding model or measurement.

Testing regions are notoriously impractical to treat analytically in the quantum case.

Moreover, testing regions are by definition regions in the probability space parameterized in terms of the state/effect space.

Main results

We provide techniques to construct implicit approximations of the testing region of arbitrary quantum statistical models and measurements that are:

- **Conic,** that is, given in terms of quadratic forms, and hence practical to treat analytically.
- **Implicit,** that is, they are parameterized in the space of probability vetors.
- **Optimal** among all such approximations, that is, we prove that they are the minimal outer and the maximal inner conic approximations.
- **Close,** in the sense that the minimal outer approximation becomes the maximal inner approximation up to a constant scaling factor.

Applications

Our approximation techniques generalize the bounding recently provided[1] by Xu, Schwonnek, and Winter:

- the extension is from Pauli strings to arbitrary measurements;
- the optimization is not restricted to the radius of fixed-axis ellipsoids, but it is a *global* optimization over all the parameters of the ellispoid.

As an application, we utilize our approximation formulas to characterize, in a device independent way, the ability to transform one quantum measurement into another, or one quantum statistical model into another.

[1] Z.-P. Xu, R. Schwonnek, A. Winter, *Bounding the joint numerical range of pauli strings by graph parameters*, arXiv:2308.00753.

Quantum measurements

Testing region of quantum measurements

Given a d-dimensional quantum measurement $\boldsymbol{\pi} = \{\pi_i : 1 \leq i \leq n\}$, $\pi_i \geq 0$, $\sum_i \pi_i = \mathbb{1}$, its testing region is defined as the image $\boldsymbol{\pi}(\mathbb{S}_d)$ of the set $\mathbb{S}_d$ of d-dimensional states through $\boldsymbol{\pi}$.

By definition, this is given in parametric form, that is, it is a body in the probability space parameterized by states in the state space.

Ideally, one would aim at implicitizing it, that is, writing it in the form $f(p) \leq 1$, for probability distributions p.

However, due to intractability of the strucuture of the state space, we resort here to providing inclusion conditions in terms of implicit bodies.

Hyper-ellipsoids for quantum measurements

Definition

For any d-dimensional, n-outcome measurement $\boldsymbol{\pi} = \{\pi_i\}_{i=1}^{n}$, we define the family $\{\mathcal{E}_r(\boldsymbol{\pi})\}_{r\in\mathbb{R}}$ of hyper-ellipsoids given by:

$$\mathcal{E}_r(\boldsymbol{\pi}) := \left\{ \mathbf{p} \in \boldsymbol{\pi}\left(\mathbb{C}^d\right) \ \ s.t. \ \left| \sqrt{Q^+}\left(\mathbf{p} - \mathbf{t}\right) \right|_2^2 \leq \frac{1}{r^2} \right\},$$

where $Q \in \mathbb{R}^{n\times n}$ is the symmetric positive semi-definite covariance matrix given by

$$Q_{ij} = \frac{d-1}{d}\left(\mathrm{Tr}\left[\pi_i \pi_j\right] - \frac{\mathrm{Tr}\left[\pi_i\right]\mathrm{Tr}[\pi_j]}{d} \right),$$

for any $0 \leq i, j \leq n$, and $\mathbf{t} \in \mathbb{R}^n$ is the vector

$$t_i = \frac{1}{d}\,\mathrm{Tr}\left[\pi_i\right], \qquad 1 \leq i \leq n.$$

Hyper-ellipsoidal approximation of the testing region of quantum measurements

Theorem ([2])

For any d-dimensional, n-outcome informationally complete measurement $\boldsymbol{\pi}$, one has that $\mathcal{E}_{d-1}(\boldsymbol{\pi})$ is the maximum volume ellipsoid enclosed in $\boldsymbol{\pi}(\mathbb{S}_d)$ and $\mathcal{E}_1(\boldsymbol{\pi})$ is the minimum volume ellipsoid enclosing $\boldsymbol{\pi}(\mathbb{S}_d)$.

If measurement $\boldsymbol{\pi}$ is not informationally complete, ellipsoids $\mathcal{E}_{d-1}(\boldsymbol{\pi})$ and $\mathcal{E}_1(\boldsymbol{\pi})$ still are inner and outer approximations of $\boldsymbol{\pi}(\mathbb{S}_d)$, although not necessarily maximal and minimal in volume, respectively.

[2] M. Dall'Arno, F. Buscemi, *Tight conic approximation of testing regions for quantum statistical models and measurements*, arXiv:2309.16153

Example: symmetric, informationally complete measurements

A d-dimensional measurement $\boldsymbol{\pi}$ is SIC if and only if it has $n = d^2$ effects satisfying the condition $\operatorname{Tr} \pi_i \pi_j = (d\delta_{i,j} + 1)/(d^2(d+1))$.

By explicit computation one has

$$Q = \frac{d-1}{d^2\,(d+1)} \left(\mathbb{1}_{d^2} - \hat{\mathbf{u}}\hat{\mathbf{u}}^T \right),$$

where $\mathbf{u}$ is the vector with all entries equal to one.

As expected, Q is a $d^2 \times d^2$ matrix of rank $d^2 - 1$, and it is proportional to a projector.

Its pseudo-inverse is then given by

$$Q^+ = \frac{d^2\,(d+1)}{d-1} \left(\mathbb{1}_{d^2} - \hat{\mathbf{u}}\hat{\mathbf{u}}^T \right).$$

Example: mutually unbiased basis measurements

A d-dimensional measurement $\boldsymbol{\pi}$ is a complete MUB if and only if it has $n = d(d+1)$ effects satisfying the condition $\operatorname{Tr} \pi_{i,j}\pi_{k,l} = (\delta_{i,k}\delta j, l + (1-\delta_{i,k})/d)/(d+1)^2$, where indices i, k denote the basis and indices j, l denote the effect within the basis.

By explicit computation one has

$$Q = \frac{d-1}{d\,(d+1)^2}\left(\mathbb{1}_{d(d+1)} - \oplus_{i=1}^{d+1} \hat{\mathbf{u}}_d^i \hat{\mathbf{u}}_d^{iT}\right),$$

where $\mathbf{u}_d^i$ is the vector with ones for the entries corresponding to basis i and zero otherwise.

As expected, Q is a $d(d+1) \times d(d+1)$ matrix of rank $d^2 - 1$, and it is proportional to a projector.

Its pseudo-inverse is then given by

$$Q^{+} = \frac{d\,(d+1)^2}{d-1}\left(\mathbb{1}_{d(d+1)} - \oplus_{i=1}^{d+1} \hat{\mathbf{u}}_d^i \hat{\mathbf{u}}_d^{iT}\right).$$

Simulability of quantum measurements

We say that a d_1-dimensional, n-outcome measurement $\boldsymbol{\pi}_1$ simulates a d_0-dimensional, n-outcome measurement $\boldsymbol{\pi}_0$, and we write

$$\boldsymbol{\pi}_0 \preceq \boldsymbol{\pi}_1,$$

if and only if there exists a completely positive map $\mathcal{C} : \mathcal{L}(\mathbb{C}^{d_0}) \to \mathcal{L}(\mathbb{C}^{d_1})$ such that

$$\boldsymbol{\pi}_1 \circ \mathcal{C} = \boldsymbol{\pi}_0.$$

A Blackwell theorem for quantum measurements

Theorem ([3])

For any qubit or qutrit measurement $\boldsymbol{\pi}_0$ *and any real qubit measurement* $\boldsymbol{\pi}_1$ *(that is, one whose elements are all real in some basis), the following are equivalent:*

- $\boldsymbol{\pi}_0 \preceq \boldsymbol{\pi}_1$.
- $\boldsymbol{\pi}_0(\mathbb{S}_d) \subseteq \boldsymbol{\pi}_1(\mathbb{S}_d)$.

[3] M. Dall'Arno, F. Buscemi, *Extension of the Alberti-Ulhmann criterion beyond qubit dichotomies*, Quantum 4, 233 (2020)

Device independent simulability of quantum measurements

Corollary

Given a set $\mathcal{P}$ of n-element probability distributions generated by a d_1-dimensional (otherwise unspecified) measurement $\boldsymbol{\pi}_1$, for any d_0 and for any d_0-dimensional n-outcome measurement $\boldsymbol{\pi}_0$ such that

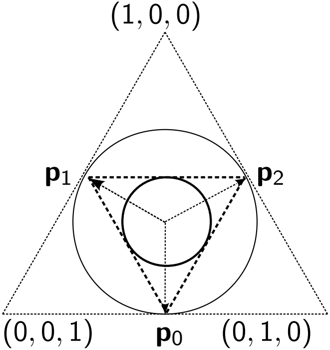
$$\mathcal{E}_1(\boldsymbol{\pi}_0) \subseteq \operatorname{conv} \mathcal{P},$$

there exists a trace preserving map $\mathcal{C}$ that is positive on the support of $\boldsymbol{\pi}_0$ such that $\boldsymbol{\pi}_1 \circ \mathcal{C} = \boldsymbol{\pi}_0$. Moreover, if $d_1 = 2$, $n \leq 3$, and $d_0 \leq 3$, map $\mathcal{C}$ is completely positive, that is, measurement $\boldsymbol{\pi}_1$ simulates measurement $\boldsymbol{\pi}_0$.

Example

Suppose the following distributions are observed for $\theta_x := 2\pi x/m$:

$$\mathbf{p}_x = \begin{pmatrix} 2 - 2\cos\theta_x \\ 2 + cos\theta_x - \sqrt{3}\sin\theta_x \\ 2 + cos\theta_x + \sqrt{3}\sin\theta_x \end{pmatrix},$$



The maximum volume ellipse enclosed in conv($\{\mathbf{p}_x\}$) corresponds to any $[\cos(\pi/m)]$-depolarized trine measurement.

Quantum statistical models

Testing region of statistical models

Given a d-dimensional quantum statistical model $\boldsymbol{\rho} = \{\rho_i : 1 \leq i \leq n\}$, $\rho_i \geq 0$, $\mathrm{Tr}[\rho_i] = 1$, its testing region is defined as the image $\boldsymbol{\rho}(\mathbb{E})$ of the cone $\mathbb{E}$ of effects through $\boldsymbol{\rho}$, seen as a classical-quantum (c-q for short) channel.

By definition, the testing region $\boldsymbol{\rho}(\mathbb{E})$ is given in parametric form, that is, it is a body in the probability space parameterized by effects in the effect space.

Ideally, one would aim at implicitizing it, that is, write it in the form $f(q) \leq 1$, for vectors of probabilities q.

However, due to the intractability of the structure of the effect space, we resort here to providing inclusion conditions in terms of implicit bodies.

Hyper-ellipsoids for statistical models

Definition

For any d-dimensional family $\boldsymbol{\rho} = \{\rho_i\}_{i=1}^{n}$ *of n states, let* $\{\mathcal{E}_r^k(\boldsymbol{\rho})\}_{r\in\mathbb{R}}^{k=0,\ldots d}$ *be the following family of hyper-ellipsoids:*

$$\mathcal{E}_r^k(\boldsymbol{\rho}) = \left\{ \mathbf{q} \in \boldsymbol{\rho}\left(\mathbb{C}^d\right) \;\; s.t. \;\; \left| \sqrt{Q_k^+}\left(\mathbf{q} - \frac{k}{d}\mathbf{u}\right)\right|_2^2 \le \frac{1}{r^2} \right\},$$

where $Q_k \in \mathbb{R}^{n\times n}$ *is the symmetric positive semi-definite covariance matrix given by*

$$(Q_k)_{ij} = \left(k - \frac{k^2}{d}\right)\left(\mathrm{Tr}\left[\rho_i\rho_j\right] - \frac{1}{d}\right),$$

for any $0 \le i, j \le n$, *and* $\mathbf{u} \in \mathbb{R}^n$ *is the vector with all unit entries.*

d-cones for statistical models

We introduce a d-cone as a generalization of the bicone.

A d-cone in $\mathbb{R}^n$ is the convex hull of the origin and d arbitrary $(n-1)$-balls with aligned centers lying on hyperplanes orthogonal to the line of the centers.

Let $r(x)$ be the radius of the ball at distance x from the origin and L be the distance of the furthest ball.

If $r(x)$ is symmetric, that is $r(x) = r(L - x)$ for any $0 \leq x \leq L$, then we say that the d-cone is symmetric.

The usual bicone is recovered as the symmetric 2-cone.

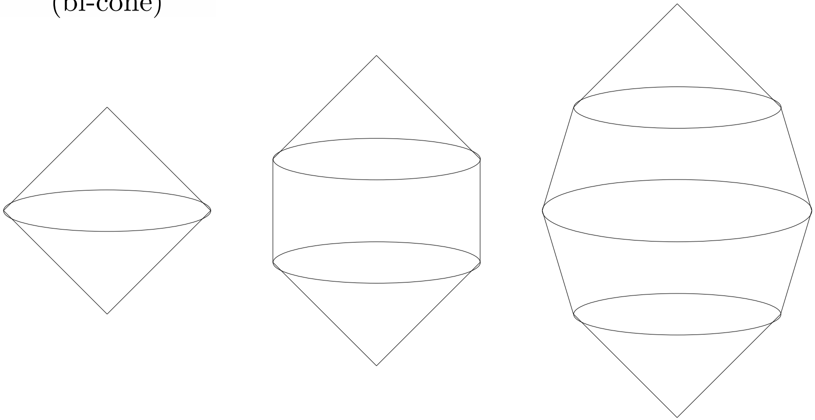
An elliptical d-cone is the image of a d-cone through a linear transformation that preserves the line joining the centers of the balls.

d-cones for statistical models

$d = 2$
symmetric 2-cone
(bi-cone)

$d = 3$
symmetric 3-cone

$d = 4$
symmetric 4-cone



d-conical approximation of the testing region of statistical models

Theorem ([4])

For any d-dimensional, n-outcome informationally complete family $\boldsymbol{\rho}$ of states, one has that $\mathrm{conv} \cup_{k=0}^{d} \mathcal{E}_{d-1}^{k}(\boldsymbol{\rho})$ *is the maximum volume elliptical d-cone enclosed in* $\boldsymbol{\rho}(\mathbb{E})$ *and* $\mathrm{conv} \cup_{k=0}^{d} \mathcal{E}_{1}^{k}(\boldsymbol{\rho})$ *is the minimum volume elliptical d-cone enclosing* $\boldsymbol{\rho}(\mathbb{E})$.

If family $\boldsymbol{\rho}$ of states is not informationally complete, elliptical d-cones $\mathrm{conv} \cup_{k=0}^{d} \mathcal{E}_{d-1}^{k}(\boldsymbol{\rho})$ and $\mathrm{conv} \cup_{k=0}^{d} \mathcal{E}_{1}^{k}(\boldsymbol{\rho})$ still are inner and outer approximations of $\boldsymbol{\rho}(\mathbb{E})$, although not necessarily maximal and minimal in volume, respectively.

[4] M. Dall'Arno, F. Buscemi, *Tight conic approximation of testing regions for quantum statistical models and measurements*, arXiv:2309.16153

Example: symmetric, informationally complete model

A d-dimensional family $\boldsymbol{\rho}$ of states is SIC if and only if it has $n = d^2$ states satisfying the condition $\operatorname{Tr} \rho_i \rho_j = (d\delta_{i,j} + 1)/(d + 1)$.

By explicit computation one has

$$Q_k = \frac{kd - k^2}{d^2\,(d+1)} \left(\mathbb{1}_{d^2} - \hat{\mathbf{u}}\hat{\mathbf{u}}^T \right),$$

where $\mathbf{u}$ is the vector with all entries equal to one.

As expected, Q_k are $d^2 \times d^2$ matrix of rank $d^2 - 1$, and they are proportional to a projector.

Their pseudo-inverses are then given by

$$Q_k^+ = \frac{d^2\,(d+1)}{kd - k^2} \left(\mathbb{1}_{d^2} - \hat{\mathbf{u}}\hat{\mathbf{u}}^T \right).$$

Example: mutually unbiased basis model

A d-dimensional family $\boldsymbol{\rho}$ of states is a complete MUB if and only if it has $n = d(d+1)$ states satisfying the condition $\operatorname{Tr} \rho_{i,j}\rho_{k,l} = (\delta_{i,k}\delta j, l + (1-\delta_{i,k})/d)$, where indices i, k denote the basis and indices j, l denote the effect within the basis.

By explicit copmputation one has

$$Q_k = \frac{kd - k^2}{d\left(d+1\right)^2}\left(\mathbb{1}_{d(d+1)} - \oplus_{i=1}^{d+1} \hat{\mathbf{u}}_d^i \hat{\mathbf{u}}_d^{iT}\right),$$

where $\mathbf{u}_d^i$ is the vector with ones for the entries corresponding to basis i and zero otherwise.

As expected, Q_k are $d(d+1) \times d(d+1)$ matrces of rank $d^2 - 1$, and they are proportional to a projector.

Their pseudo-inverses are then given by

$$Q_k^+ = \frac{d\left(d+1\right)^2}{kd - k^2}\left(\mathbb{1}_{d(d+1)} - \oplus_{i=1}^{d+1} \hat{\mathbf{u}}_d^i \hat{\mathbf{u}}_d^{iT}\right).$$

Simulability of statistical models

We say that a d_1-dimensional, n-outcome model $\boldsymbol{\rho}_1$ simulates a d_0-dimensional, n-outcome model $\boldsymbol{\rho}_0$, and we write

$$\boldsymbol{\rho}_0 \preceq \boldsymbol{\rho}_1,$$

if and only if there exists a completely positive trace preserving map (a quantum channel) $\mathcal{C} : \mathcal{L}(\mathbb{C}^{d_1}) \to \mathcal{L}(\mathbb{C}^{d_0})$ such that

$$\mathcal{C} \circ \boldsymbol{\rho}_1 = \boldsymbol{\rho}_0.$$

A Blackwell theorem for quantum statistical models

Theorem ([5])

For any qubit statistical model $\boldsymbol{\rho}_0$ and any real qubit statistical model $\boldsymbol{\rho}_1$ of qubit states (that is, states that have only real entries in some basis), the following are equivalent:

- $\boldsymbol{\rho}_0 \preceq \boldsymbol{\rho}_1$.
- $\boldsymbol{\rho}_0(\mathbb{E}_d) \subseteq \boldsymbol{\rho}_1(\mathbb{E}_d)$.

If $\boldsymbol{\rho}_1$ contains the identity operator $\mathbb{1}$ in its linear span, the statement holds even if $\boldsymbol{\rho}_0$ is a qutrit model.

[5] M. Dall'Arno, F. Buscemi, *Extension of the Alberti-Ulhmann criterion beyond qubit dichotomies*, Quantum 4, 233 (2020)

Device independent simulability test

Corollary

Given a set $\mathcal{Q}$ of n-element vectors of probabilities generated by a d_1-dimensional (otherwise unspecified) family of n states $\boldsymbol{\rho}_1$, for any d_0 and for any d_0-dimensional family of n states $\boldsymbol{\rho}_0$ such that

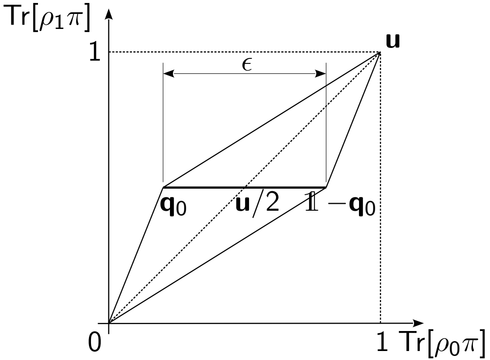
$$\operatorname{conv} \cup_{k=0}^{d} \mathcal{E}_1^k(\boldsymbol{\rho}_0) \subseteq \operatorname{conv} \mathcal{Q},$$

there exists a (not necessarily trace preserving) map $\mathcal{C}$ that is positive on the support of $\boldsymbol{\rho}_0$ such that $\mathcal{C} \circ \boldsymbol{\rho}_1 = \boldsymbol{\rho}_0$. Moreover, if $d_1 = 2$, $n = 2$, and $d_0 = 2$, map $\mathcal{C}$ is completely positive trace preserving, that is, family $\boldsymbol{\rho}_1$ of states simulates family $\boldsymbol{\rho}_0$ of states.

Example

Suppose the following probability vectors are observed for some ϵ:

$$\mathbf{q}_0 = \frac{1}{2}\begin{pmatrix} 1-\epsilon \\ 1 \end{pmatrix}, \qquad \mathbf{u}-\mathbf{q}_0 = \frac{1}{2}\begin{pmatrix} 1+\epsilon \\ 1 \end{pmatrix}.$$



The maximum volume range enclosed in $\mathrm{conv}(0, \mathbf{u}, \mathbf{q}_0, \mathbf{u}-\mathbf{q}_0)$ corresponds to any ϵ-depolarized dichotomy $\{\mathcal{D}_\epsilon(\phi), \mathbb{1}/2\}$, for any pure state ϕ.

Conclusion

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We provided an implicit outer approximation of the image of any given quantum measurement in any finite dimension.

We generalized a recent result[6] by Xu, Schwonnek, and Winter on the image of Pauli strings.

The outer approximation we constructed is

- minimal among all such outer approximations, and
- close, in the sense that it becomes the *maximal inner* approximation up to a constant scaling factor.

We also obtained a similar result for the dual problem of implicitizing the image of the set of effects through a family of quantum states.

We applied our approximations to characterize, in a semi-device independent way, the ability to transform one quantum measurement into another, or one quantum statistical model into another.

[6] Z.-P. Xu, R. Schwonnek, A. Winter, *Bounding the joint numerical range of pauli strings by graph parameters*, arXiv:2308.00753.