Compact convex structure and simulability of measurements

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Motivation

- The notion of the measurement in quantum theory (or in more general theories) is an important constituent of the theory for it connects the abstract theory to empirical events.
- However, little is known about the global structure of the totality of measurements on a given physical system.

Motivation

- How should we understand the continuous-outcome measurements (e.g. homodyne/heterodyne measurements or position measurement)?
- It is impossible to handle true continuous data by experimental devices with finite memories and the mathematical descriptions of continuous measurements might be considered as an approximation of the real measurement.
- Then what does this approximation mean?

Summary of results

- This work studies general structure of the measurement space $\mathfrak{M}(\Omega)$, which is the set of post-processing equivalence classes on a GPT Ω.
- Specifically we consider order and compact convex structures of $\mathfrak{M}(\Omega)$.
- We also consider simulability of measurements based on this formalism.

Outline

GPTs

Measurements

Measurement space and compact convex structure

Simulability

Outline

GPT

- A generalized probabilistic theory (GPT) (with the no-restriction hypothesis) is described by the notion of the base-norm Banach space.

- We may derive the notion of the base-norm Banach space from operationally natural requirements (e.g. Ludwig's embedding theorem).

Base-norm Banach space

A triple (V, V_+, Ω) is called a base-norm Banach space $:\overset{\text{def.}}{\Leftrightarrow}$

1. V is a real vector space.
2. V_+ is a positive cone of V, i.e. $\lambda V_+ \subseteq V_+$ $(\forall \lambda \in [0, \infty))$, $V_+ + V_+ \subseteq V_+$, and $V_+ \cap (-V_+) = \{0\}$ hold. We define the linear order on V induced from V_+ by
$$x \leq y :\overset{\text{def.}}{\Leftrightarrow} y - x \in V_+ \quad (x, y \in V).$$
3. V_+ is generating, i.e. $V = V_+ + (-V_+)$.
4. Ω is a base of V_+, i.e. Ω is a convex subset of V_+ and for every $x \in V_+$ there exists a unique $\lambda \in [0, \infty)$ such that $x \in \lambda\Omega$.
5. We define the base-norm on V by
$$\|x\| := \inf \{ \alpha + \beta \mid x = \alpha\omega_1 - \beta\omega_2;\ \alpha, \beta \in [0, \infty);\ \omega_1, \omega_2 \in \Omega \}$$
We require that the base-norm $\| \cdot \|$ is a complete norm on V.

Base-norm Banach space

- The structure of V and V_+ are uniquely up to isomorphism determined by the structure of Ω as a convex set and thus we write as $V = V(\Omega)$, $V_+ = V_+(\Omega)$.

- The base Ω is occasionally called a state space or a GPT.

Dual space

Let Ω be a GPT.

- The continuous dual space $V^*(\Omega)$ is equipped with the dual positive cone

$$V_+^*(\Omega) := \{ f \in V^* \mid \langle f, x \rangle \geq 0 \, (\forall x \in V_+) \}$$

and the dual linear order

$$f \leq g :\overset{\text{def.}}{\Leftrightarrow} g - f \in V_+^* \Leftrightarrow [f(x) \leq g(x) \quad (\forall x \in V_+)].$$

- We occasionally write as

$$\langle f, x \rangle := f(x) \quad (f \in V^*(\Omega),\, x \in V(\Omega)).$$

- It can be shown that there exists a unique positive element, called the unit element, $1_\Omega \in V_+^*(\Omega)$ such that $\langle 1_\Omega, \omega \rangle = 1$ for all $\omega \in \Omega$.

Dual space and order unit Banach space

- The dual norm on $V^*(\Omega)$

$$\|f\| := \sup_{x \in V,\, \|x\| \leq 1} |\langle f, x \rangle| \quad (f \in V^*(\Omega))$$

coincides with the order unit norm with respect to 1_Ω:

$$\|f\| = \inf \{ \lambda \in [0, \infty) \mid -\lambda 1_\Omega \leq f \leq \lambda 1_\Omega \} \quad (f \in V^*(\Omega)).$$

- An ordered linear space equipped with an order unit and complete order norm is called a order unit Banach space.
- The dual space $(V(\Omega)^*, V_+^*(\Omega), 1_\Omega)$ is an order unit Banach space with a predual $V(\Omega)$.

Example: quantum theory

Let $\mathcal{H}$ be a complex Hilbert space.

- The set $\Omega = \mathbf{D}(\mathcal{H})$ of density operators on $\mathcal{H}$ is a state space.
- $V(\Omega) = \mathbf{T}(\mathcal{H})_{\mathrm{sa}}$: the set of self-adjoint trace-class operators on $\mathcal{H}$.
- $V(\Omega) = \mathbf{T}(\mathcal{H})_{+}$: the set of positive semidefinite trace-class operators on $\mathcal{H}$.
- The dual space $V^*(\Omega)$ is identified with the set $\mathbf{B}(\mathcal{H})_{\mathrm{sa}}$ of self-adjoint bounded operators on $\mathcal{H}$ by the duality

$$\langle a, b\rangle := \mathrm{tr}(ab) \quad (a \in \mathbf{B}(\mathcal{H})_{\mathrm{sa}},\, b \in \mathbf{T}(\mathcal{H})_{\mathrm{sa}}).$$

- The base norm on $\mathbf{T}(\mathcal{H})_{\mathrm{sa}}$ is the trace norm.
- The dual norm on $\mathbf{B}(\mathcal{H})_{\mathrm{sa}}$ is the uniform norm.

Example: operator algebraic theory

Let $\mathcal{M}$ be a von Neumann algebra (i.e. ultraweakly closed $*$-subalgebra of $\mathbf{B}(\mathcal{H})$) acting on a Hilbert space $\mathcal{H}$.

- A linear functional $\psi\colon \mathcal{M} \to \mathbb{C}$ is called a state if ψ is nonnegative ($a \geq 0 \implies \psi(a) \geq 0$) and $\psi(\mathbb{1}) = 1$.
- A positive linear functional ψ on $\mathcal{M}$ is called normal if $\psi(\sup_i a_i) = \sup_i \psi(a_i)$ for any upper-bouded monotone net a_i.
  This condition is equivalent to the ultraweak continuity of ψ.
- $\Omega = \mathcal{S}_\sigma(\mathcal{M})$: the set of normal states on $\mathcal{M}$.
- $V(\Omega) = \mathcal{M}_{*,\mathrm{sa}}$: the set of self-adjoint ultraweakly continuous linear functionals on $\mathcal{M}$.
- $V_+(\Omega) = \mathcal{M}_{*,\mathrm{sa}}$: the set of normal positive linear functionals on $\mathcal{M}$.
- $V^*(\Omega) = \mathcal{M}_{\mathrm{sa}}$: the set of self-adjoint elements of $\mathcal{M}$.
  The duality is given by

$$\langle a, \psi \rangle = \psi(a) \quad (a \in \mathcal{M}_{\mathrm{sa}},\ \psi \in \mathcal{M}_{*,\mathrm{sa}}).$$

Classical theory

A GPT Ω is called classical if Ω is affinely isomorphic to $\mathcal{S}_\sigma(\mathcal{M})$ for some abelian vN algebra $\mathcal{M}$. Some equivalent characterizations of the classicality are known:

Theorem 1

Let Ω be a GPT. Then the following conditions are equivalent.

(i) Ω is classical.

(ii) The set $S(V^*(\Omega))$ of states on $V^*(\Omega)$ (normalized positive linear functionals on $V^*(\Omega)$) is a Bauer simplex, i.e. the set $\mathrm{ext}(S(V(\Omega)^*))$ of extremal points of $S(V^*(\Omega))$ is weakly$*$ compact and any element $\psi \in S(V^*(\Omega))$ is represented as a unique boundary integral.

Classical theory

(iii) (Special case of the no-broadcasting theorem). There exists a bilinear map $B\colon V^*(\Omega) \times V^*(\Omega) \to V^*(\Omega)$ such that

$$f, g \geq 0 \implies B(f,g) \geq 0,$$
$$B(f, 1_\Omega) = B(1_\Omega, f) = f$$

$(\forall f, g \in V^*(\Omega))$.

(iii) $(V(\Omega), \leq)$ is a lattice (i.e. any two elements $x, y \in V(\Omega)$ have the least upper bound $x \vee y \in V(\Omega)$).

(iv) $(V(\Omega)^*, \leq)$ is a lattice.

(v) Any pair of two-outcome measurements is compatible (i.e. simultaneously measurable).

(Plávala (2016), YK (2020)).

Moreover, the bilinear map B is, if it exists, unique and satisfies

$$B(f,g) = B(g,f), \quad B(f, B(g,h)) = B(B(f,g),h) \quad (f,g,h \in V^*(\Omega)).$$

(B is the multiplication if $V(\Omega)^*$ is an abelian vN algebra.)

Example: finite-dimensional classical space

- $\mathcal{S}_n := \{ (p_j)_{j=1}^n \in \mathbb{R}^n \mid p_j \geq 0, \sum_{j=1}^n p_j = 1 \}$: the simplex consisting of n-outcome probabilities.
- $V(\mathcal{S}_n) = V^*(\mathcal{S}_n) = \mathbb{R}^n$.
- $V_+(\mathcal{S}_n) = V_+^*(\mathcal{S}_n) = [0, \infty)^n$.
- The duality is given by $\langle (a_j)_{j=1}^n, (b_j)_{j=1}^n \rangle = \sum_{j=1}^n a_j b_j$.
- $\mathcal{S}_n$ is a finite-dimensional classical GPT.
- Conversely, any d-dimensional classical GPT Ω is affinely isomorphic to $\mathcal{S}_{d+1}$.

Example: the classical space of probability measures

- (X, Σ): measurable space.
- $\mathcal{P}(X, \Sigma)$: the set of probability measures defined on Σ.
- $\mathrm{ca}(X, \Sigma)$: the set of signed measures defined on Σ.
- $\mathrm{ca}(X, \Sigma)_+ := \{ \mu \in \mathrm{ca}(X, \Sigma) \mid \mu(A) \geq 0 \, (\forall A \in \Sigma) \}$.
- Then $(\mathrm{ca}(X, \Sigma), \mathrm{ca}(X, \Sigma)_+, \mathcal{P}(X, \Sigma))$ is a base-norm Banach space.
- The base-norm on $\mathrm{ca}(X, \Sigma)$ is the total variation norm.
- The GPT $\mathcal{P}(X, \Sigma)$ is classical because $\mathrm{ca}(X, \Sigma)$ is a lattice.
- (The dual space $\mathrm{ca}(X, \Sigma)^*$ is the self-adjoint part of an abelian vN algebra $\mathcal{M}$, while we have no intuitive characterization of $\mathcal{M}$.)

Outline

Measurement map and EVM

We fix a GPT Ω.

- General measurement can be described by an affine map

$$\Psi\colon \Omega \to \mathcal{P}(X,\Sigma)$$
$$\Psi(t\omega + (1-t)\omega') = t\Psi(\omega) + (1-t)\Psi(\omega').$$

- For such Ψ, there exists a unique map $\mathsf{M}\colon \Sigma \to V^*(\Omega)$ s.t.
  (i) $\mathsf{M}(A) \geq 0$ $(A \in \Sigma)$;
  (ii) $\mathsf{M}(\emptyset) = 0$, $\mathsf{M}(X) = 1_\Omega$;
  (iii) for disjoint sequence $\{A_n\} \subseteq \Sigma$, $\mathsf{M}(\bigcup_n A_n) = \sum_n \mathsf{M}(A_n)$,

  and

$$\Psi(\omega)(A) = \langle \mathsf{M}(A), \omega\rangle \quad (A \in \Sigma,\ \omega \in \Omega).$$

- A map M satisfying (i)-(iii) is called an effect-valued measure (EVM).

Measurement map and EVM

- Conversely, any EVM $\mathsf{M}\colon \Sigma \to V^*(\Omega)$ defines an affine map $\Psi^{\mathsf{M}}\colon \Omega \to \mathcal{P}(X, \Sigma)$ by

$$\Psi^{\mathsf{M}}(\omega)(A) = \langle \mathsf{M}(A), \omega \rangle \quad (A \in \Sigma,\, \omega \in \Omega).$$

General channel and measurement map

- For any output GPT $\mathcal{S}$, an affine map $\Psi\colon \Omega \to \mathcal{S}$ is called a channel.
- If the outcome space $\mathcal{S}$ is classical, Ψ is called a measurement map.
- A channel $\Psi\colon \Omega \to \mathcal{S}$ uniquely extends to a positive linear map $\Psi\colon V(\Omega) \to V(\mathcal{S})$.
- The dual map (channel in the Heisenberg picture) $\Psi^*\colon V^*(\mathcal{S}) \to V^*(\Omega)$ is a weakly$*$ continuous positive linear map that is unital
$$\Psi^*(1_{\mathcal{S}}) = 1_\Omega.$$

Post-processing relations for channels

Let $\Psi_1 \colon \Omega \to \mathcal{S}_1$ be $\Psi_2 \colon \Omega \to \mathcal{S}_2$ channels.

- $\Psi_1 \preceq_{\mathrm{post}} \Psi_2$ (Ψ_1 is a post-processing of Ψ_2) $:\overset{\mathrm{def.}}{\Leftrightarrow} \exists \Phi \colon \mathcal{S}_2 \to \mathcal{S}_1$: channel s.t. $\Psi_1 = \Phi \circ \Psi_2$.
- $\preceq_{\mathrm{post}}$ is a preorder.
- $\Psi_1 \sim_{\mathrm{post}} \Psi_2$ (Ψ_1 and Ψ_2 are post-processing equivalent) $:\overset{\mathrm{def.}}{\Leftrightarrow} \Psi_1 \preceq_{\mathrm{post}} \Psi_2$ and $\Psi_2 \preceq_{\mathrm{post}} \Psi_1$
- $\sim_{\mathrm{post}}$ is an equivalence relation.

Measurement map is essentially an EVM

A general classical space $\mathcal{S}$ is not always isomorphic to some $\mathcal{P}(X, \Sigma)$.
We can still regard each measurement map $\Psi\colon \Omega \to \mathcal{S}$ as an EVM in the following sense:

Proposition 1

For any measurement map $\Psi\colon \Omega \to \mathcal{S}$ there exists an EVM s.t. $\Psi \sim_{\mathrm{post}} \Psi^{\mathsf{M}}$.

Finite-outcome EVM

- For any measurement map $\Psi\colon \Omega \to \mathcal{S}_n$ there exists a sequence $\mathsf{M} = (\mathsf{M}_j)_{j=1}^n \in V^*(\Omega)^n$ s.t.

$$\Psi(\omega) = \Psi^{\mathsf{M}}(\omega) := (\langle \mathsf{M}_j, \omega \rangle)_{j=1}^n.$$

- M satisfies (i) $\mathsf{M}_j \geq 0$ $(\forall j)$ and (ii) $\sum_{j=1}^n \mathsf{M}_j = 1_\Omega$.
- Such M is called an n-outcome (finite-outcome) EVM.
- $\mathbf{EVM}(\Omega; n)$: the set of n-outcome EVMs on Ω.

Outline

Ensemble and state discrimination probability functional

- A finite sequence $\mathcal{E} = (\varphi_j)_{j=1}^n \in V_+(\Omega)^n$ is called an ensemble $:\overset{\text{def.}}{\Leftrightarrow} \sum_{j=1}^n \langle 1_\Omega, \varphi_j \rangle = 1$.
- $\mathcal{E}$ corresponds to the situation where the state $\omega_j = \langle 1_\Omega, \varphi_j \rangle^{-1} \varphi_j$ is prepared in the probability $\langle 1_\Omega, \varphi_j \rangle$.
- For a measurement map $\Psi \colon \Omega \to \mathcal{S}$, the state discrimination probability functional is defined by

$$P_{\mathrm{g}}(\mathcal{E}; \Psi) := \sup_{\mathsf{M} \in \mathbf{EVM}(\mathcal{S}; n)} \sum_{j=1}^n \langle \mathsf{M}_j, \Psi(\varphi_j) \rangle$$

- $P_{\mathrm{g}}(\mathcal{E}; \Psi)$ is the optimal probability of guessing the index j when we are given the measurement outcome Ψ.
The EVM M on the RHS corresponds the guessing strategy.

Blackwell-Sherman-Stein theorem for EVMs

The post-processing relation $\preceq_{\mathrm{post}}$ is characterized by the state discrimination probability functionals:

Theorem 2 (BSS theorem for measurements)

Let $\Psi_1 \colon \Omega \to \mathcal{S}_1$ and $\Psi_2 \colon \Omega \to \mathcal{S}_2$ be measurement maps. Then $\Psi_1 \preceq_{\mathrm{post}} \Psi_2 \iff P_{\mathrm{g}}(\mathcal{E};\Psi_1) \leq P_{\mathrm{g}}(\mathcal{E};\Psi_2)$ ($\forall \mathcal{E}$: ensemble).

Quantum case: Buscemi (2012); Skrzypczyk and Linden (2019)

Sketch of the proof

- $\Psi_1 \preceq_{\mathrm{post}} \Psi_2 \implies P_{\mathrm{g}}(\mathcal{E}; \Psi_1) \leq P_{\mathrm{g}}(\mathcal{E}; \Psi_2)$ ($\forall \mathcal{E}$: ensemble) is easy.
- Converse implication when Ψ_1 is finite-outcome: application of the Hahn-Banach theorem.
- Converse implication for general Ψ_1 reduces to the finite-outcome case by approximating Ψ_1 by finite-outcome measurement maps.

Measurement space

For a given GPT Ω, the class $\mathbf{Meas}(\Omega)$ of measurement maps (or EVMs) on Ω is a proper class (a class larger than any set).
The set of post-processing equivalence classes of measurement maps is well-defined.

Proposition 2

For any GPT Ω, there exists a set $\mathfrak{M}(\Omega)$ and class-to-set surjection

$$\mathbf{Meas}(\Omega) \ni \Psi \mapsto [\Psi] \in \mathfrak{M}(\Omega)$$

s.t. $\Psi_1 \sim_{\mathrm{post}} \Psi_2 \iff [\Psi_1] = [\Psi_2]$. We fix such $\mathfrak{M}(\Omega)$ (and $[\cdot]$) and call $\mathfrak{M}(\Omega)$ the measurement space of Ω.

Each element $[\Psi] \in \mathfrak{M}(\Omega)$ is just called a measurement.

Proof

Define $[\cdot]$ by

$$[\Psi] := (P_{\mathrm{g}}(\mathcal{E};\Psi))_{\mathcal{E}\in\mathbf{Ens}(\Omega)} \in \mathbb{R}^{\mathbf{Ens}(\Omega)},$$

where $\mathbf{Ens}(\Omega)$ is the set of ensembles on Ω, and $\mathfrak{M}(\Omega)$ as the image of the map $\mathbf{Meas}(\Omega) \ni \Psi \to [\Psi]$.
Then the claim follows from the BSS theorem.

Post-processing partial order on $\mathfrak{M}(\Omega)$

- For measurements $[\Psi], [\Phi] \in \mathfrak{M}(\Omega)$,
  $[\Psi] \preceq_{\mathrm{post}} [\Phi] :\overset{\text{def.}}{\Leftrightarrow} \Psi \preceq_{\mathrm{post}} \Phi$.
- $\preceq_{\mathrm{post}}$ defined on $\mathfrak{M}(\Omega)$ is a partial order.
- We may also define the state discrimination probability functional on $\mathfrak{M}(\Omega)$ by

$$P_{\mathrm{g}}(\mathcal{E}; [\Psi]) := P_{\mathrm{g}}(\mathcal{E}; \Psi).$$

Weak topology on $\mathfrak{M}(\Omega)$

- The weak topology on $\mathfrak{M}(\Omega)$ is the coarsest topology such that
$$\mathfrak{M}(\Omega) \ni \mu \mapsto P_{\mathrm{g}}(\mathcal{E};\mu) \in \mathbb{R}$$
is continuous for every ensemble $\mathcal{E}$.
- $\mu_i \xrightarrow{\text{weakly}} \mu \iff P_{\mathrm{g}}(\mathcal{E};\mu_i) \to P_{\mathrm{g}}(\mathcal{E};\mu)$ ($\forall \mathcal{E}$: ensemble).

Theorem 3

The weak topology on $\mathfrak{M}(\Omega)$ is a compact Hausdorff topology.

The proof is an application of the compactness of the pointwise convergence topology of the weak$*$ topology on the set of unital positive maps (Tychonoff's theorem).

Density of finite-outcome measurements

Theorem 4

The set of finite-outcome measurements is a dense subset of $\mathfrak{M}(\Omega)$ in the weak topology.

Direct sum of GPTs

- For GPTs Ω_1 and Ω_2, the direct sum base-norm Banach space $(V(\Omega_1) \oplus V(\Omega_2), V_+(\Omega_1) \oplus V_+(\Omega_2), \Omega_1 \oplus \Omega_2)$ is defined by

$$\Omega_1 \oplus \Omega_2 = \{ t\omega_1 \oplus (1-t)\omega_2 \mid t \in [0,1], \, \omega_1 \in \Omega_1, \, \omega_2 \in \Omega_2 \} .$$

- If $\mathcal{S}_1$ and $\mathcal{S}_2$ are classical, so is $\mathcal{S}_1 \oplus \mathcal{S}_2$.

Direct mixture of measurements

- For measurements $\Psi_1 \colon \Omega \to \mathcal{S}_1$ and $\Psi_2 \colon \Omega \to \mathcal{S}_2$ and $t \in [0,1]$, we define the direct mixture measurement map $t\Psi_1 \oplus (1-t)\Psi_2 \colon \Omega \to \mathcal{S}_1 \oplus \mathcal{S}_2$ by

$$t\Psi_1 \oplus (1-t)\Psi_2(\omega) := t\Psi_1(\omega) \oplus (1-t)\Psi_2(\omega).$$

- $P_{\mathrm{g}}(\mathcal{E};\cdot)$ is affine w.r.t. the direct mixture:

$$P_{\mathrm{g}}(\mathcal{E}; t\Psi_1 \oplus (1-t)\Psi_2) = tP_{\mathrm{g}}(\mathcal{E};\Psi_1) + (1-t)P_{\mathrm{g}}(\mathcal{E};\Psi_2)$$

General convex structure

- A set S equipped with a map

$$[0,1] \times S \times S \ni (p, s_1, s_2) \mapsto \langle p; s_1, s_2 \rangle \in S$$

  is called a convex prestructure (Gudder 1973).
- For a convex prestructure $(S, \langle \cdot; \cdot, \cdot \rangle)$ a function $f \colon S \to \mathbb{R}$ satisfying

$$f(\langle p; s_1, s_2 \rangle) = pf(s_1) + (1-p)f(s_2)$$

  is said to be affine.
- If S is further a topological space, we denote by $A_{\mathrm{c}}(S)$ the set of continuous affine functionals on S.

Compact convex structure

A convex prestructure $(S, \langle\cdot;\cdot,\cdot\rangle)$ equipped with a topology τ on S is called a compact convex structure if τ is a compact topology and $A_\mathrm{c}(S)$ separates points of S, i.e. for any $s_1, s_2 \in S$

$$s_1 \neq s_2 \implies \exists f \in A_\mathrm{c}(S) \text{ s.t. } f(s_1) \neq f(s_2).$$

Proposition 3

A compact convex structure $(S, \langle\cdot;\cdot,\cdot\rangle, \tau)$ is always continuously and affinely isomorphic to a compact convex set $S(A_\mathrm{c}(S))$ of normalized positive linear functionals on $A_\mathrm{c}(S)$ equipped with the weak$*$ topology.

Compact convex structure of measurements

Theorem 5

The measurement space $\mathfrak{M}(\Omega)$ equipped with the convex operation

$$(t, [\Psi_1], [\Psi_2]) \mapsto \langle t; [\Psi_1], [\Psi_2] \rangle := [t\Psi_1 \oplus (1-t)\Psi_2]$$

and the weak topology is a compact convex structure.

This follows from the affinity of P_{g} and the BSS theorem.

Thus we can and do identify $\mathfrak{M}(\Omega)$ with the concrete compact convex set $S(A_{\mathrm{c}}(\mathfrak{M}(\Omega)))$, where the direct mixture $[t\Psi_1 \oplus (1-t)\Psi_2]$ is the ordinary convex mixture in the linear space $A_{\mathrm{c}}(\mathfrak{M}(\Omega))^*$.

Outline

Simulability

Let $\mathfrak{L} \subseteq \mathfrak{M}(\Omega)$ be an arbitrary subset.

- $\mu \in \mathfrak{M}(\Omega)$ is simulable by $\mathfrak{L} :\overset{\text{def.}}{\Leftrightarrow} \exists \nu \in \mathrm{conv}(\mathfrak{L})$ s.t. $\mu \preceq_{\mathrm{post}} \nu$. ($\mathrm{conv}(\cdot)$ denotes the convex hull.)
- $\mathfrak{sim}(\mathfrak{L})$: the set of measurements simulable by $\mathfrak{L}$.
- $\mu \in \mathfrak{M}(\Omega)$ is weakly simulable by $\mathfrak{L} :\overset{\text{def.}}{\Leftrightarrow} \exists \nu \in \overline{\mathrm{conv}}(\mathfrak{L})$ s.t. $\mu \preceq_{\mathrm{post}} \nu$.
  ($\overline{\mathrm{conv}}(\cdot)$ denotes the closed convex hull w.r.t. the weak topology.)
- $\overline{\mathfrak{sim}}(\mathfrak{L})$: the set of measurements weakly simulable by $\mathfrak{L}$.
- It can be shown that $\overline{\mathfrak{sim}}(\mathfrak{L})$ is the closure of $\mathfrak{sim}(\mathfrak{L})$ w.r.t. the weak topology.

Characterization of the weak simulabity

For a subset $\mathfrak{L} \subseteq \mathfrak{M}(\Omega)$ and an ensemble $\mathcal{E}$, we define

$$P_{\mathrm{g}}(\mathcal{E}; \mathfrak{L}) := \sup_{\nu \in \mathfrak{L}} P_{\mathrm{g}}(\mathcal{E}; \nu).$$

Theorem 6

$\mu \in \overline{\mathfrak{sim}}(\mathfrak{L}) \iff P_{\mathrm{g}}(\mathcal{E}; \mu) \leq P_{\mathrm{g}}(\mathcal{E}; \mathfrak{L})$ ($\forall \mathcal{E}$: ensemble).

Quantum case: Skrzypczyk (2019)

Simulation irreducibility

- $\mu \in \mathfrak{M}(\Omega)$ is maximal $:\overset{\text{def.}}{\Leftrightarrow}$ $\mu \preceq_{\text{post}} \nu \in \mathfrak{M}(\Omega)$ implies $\mu = \nu$.
- $\mu \in \mathfrak{M}(\Omega)$ is extremal in $\mathfrak{M}(\Omega) \iff$ there exists a representative $\Psi \colon \Omega \to \mathcal{S}$ of μ ($\mu = [\Psi]$) s.t. $\Psi^* V^*(\mathcal{S}) \to V^*(\Omega)$ is injective.
- $\mu \in \mathfrak{M}(\Omega)$ is simulation irreducible $:\overset{\text{def.}}{\Leftrightarrow}$ for any subset $\mathfrak{L} \subseteq \mathfrak{M}(\Omega)$, $\mu \in \mathfrak{sim}(\mathfrak{L})$ implies $\mu \in \mathfrak{L}$. $\mathfrak{M}_{\text{irr}}(\Omega)$: the set of simulation irreducible measurements.
- $\mu \in \mathfrak{M}(\Omega)$ is simulation irreducible iff μ is maximal and extremal.

Krein-Milman-type theorem for measurement simulability

Theorem 7
For any GPT Ω, $\mathfrak{M}_{\mathrm{irr}}(\Omega) \neq \emptyset$ and $\mathfrak{M}(\Omega) = \overline{\mathfrak{sim}}(\mathfrak{M}_{\mathrm{irr}}(\Omega))$.

The proof is almost parallel to that of the Krein-Milman theorem and based on Theorem 6 and the following lemma.

Lemma 1
Any weakly continuous and post-processing monotone affine functional $f \colon \mathfrak{M}(\Omega) \to \mathbb{R}$ attains its maximum value at a point $\mu_0 \in \mathfrak{M}_{\mathrm{irr}}(\Omega)$.