Some characterizations of sufficient quantum channels

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Classical sufficient statistics

- Classical statistical model:

$$\mathcal{P} = \{P_\theta \in \mathcal{P}(X, \Sigma),\ \theta \in \Theta\}.$$

- Sufficient statistic: a transformation

$$T : (X, \Sigma) \to (Y, \Xi)$$

  preserving all information on θ.
- Definition: there exists a common version of conditional expectation:

$$E_{P_\theta}(\cdot|T) = E(\cdot|T), \qquad \theta \in \Theta$$

Quantum sufficiency

Let $\mathcal{S}$ be a family of quantum states, Φ a quantum channel.

Possible notions of sufficiency (equivalent in the classical case):

(a) definitions based on conditional expectations - too restrictive

(b) reversibility of Φ on $\mathcal{S}$

(c) preserving quantum divergences (relative entropy, Rényi divergences)

(d) preserving optimal errors in hypothesis testing

Quantum sufficiency

Relations between the conditions:

(b) reversibility of Φ on $\mathcal{S}$

(c) preserving quantum divergences (relative entropy, Rényi divergences)

(d) preserving optimal errors in hypothesis testing

- clearly (b) $\implies$ (c), (d)
- (b) $\iff$ (c) with relative entropy or standard Rényi divergences (Petz, 1986, 1988)
- this talk: (b) - (d) are equivalent, with sandwiched Rényi divergences in (c).

Quantum sufficiency

We say that Φ is sufficient with respect to $\mathcal{S}$ if there exists some channel Ψ (recovery channel) such that

$$\Psi \circ \Phi(\rho) = \rho, \qquad \rho \in \mathcal{S}.$$

D. Petz, Commun. Math. Phys., 1986

D. Petz, The Quarterly J. of Math., 1988

The setting and assumptions

$B(\mathcal{H})$ - bounded operators on a Hilbert space $\mathcal{H}$

- $L_p(\mathcal{H})$ - Schatten class, with norm $\|\cdot\|_p$, $p > 1$
- a set of states $\mathcal{S} \subset \mathcal{S}(\mathcal{H}) = \{\rho \in L_1(\mathcal{H}),\ \rho \geq 0,\ \mathrm{Tr}\ \rho = 1\}$
- a channel $\Phi : L_1(\mathcal{H}) \to L_1(\mathcal{K})$ - completely positive and trace preserving
- the adjoint map $\Phi^* : B(\mathcal{K}) \to B(\mathcal{H})$,

$$\mathrm{Tr}\ \Phi^*(A)\rho = \mathrm{Tr}\ A\Phi(\rho), \qquad \forall \rho \in L_1(\mathcal{H}), A \in B(\mathcal{K})$$

  is a coarse-graining - completely positive, unital and normal.

Assumptions:

There is a faithful state $\sigma \in \mathcal{S}$, its image $\Phi(\sigma)$ is also faithful.

Rényi divergences and relative entropy

For $\alpha \geq 0$ and $\rho, \sigma \in \mathcal{S}(\mathcal{H})$:

$$D_\alpha(\rho\|\sigma) = \begin{cases} \dfrac{1}{\alpha - 1} \log \operatorname{Tr}\left[\rho^\alpha \sigma^{1-\alpha}\right], & \alpha \in [0,1) \text{ or } \alpha \neq 1,\ s(\rho) \leq s(\sigma) \\ \operatorname{Tr}\left[\rho(\log(\rho) - \log(\sigma))\right], & \alpha = 1 \text{ and } s(\rho) \leq s(\sigma) \\ \infty, & \text{otherwise.} \end{cases}$$

Data processing inequality: for a channel Φ and $\alpha \in [0,2]$,

$$D_\alpha(\Phi(\rho)\|\Phi(\sigma)) \leq D_\alpha(\rho\|\sigma).$$

The Petz recovery map

Introduce an inner product $\langle \cdot, \cdot \rangle_\sigma$ in $B(\mathcal{H})$ as

$$\langle A, B\rangle_\sigma = \mathrm{Tr}\,[A^*\sigma^{1/2}B\sigma^{1/2}], \qquad A, B \in B(\mathcal{H}).$$

Let Φ_σ^* be determined as the adjoint to Φ^*:

$$\langle B, \Phi_\sigma^*(A)\rangle_\sigma = \langle \Phi^*(B), A\rangle_{\Phi(\sigma)}, \quad A \in B(\mathcal{H}),\ B \in B(\mathcal{K})$$

The Petz recovery map is given by: $\Phi_\sigma := (\Phi_\sigma^*)_*$.

- Φ_σ is a channel.
- $\Phi_\sigma \circ \Phi(\sigma) = \sigma$.
- In finite dimensions:

$$\Phi_\sigma(\cdot) = \sigma^{1/2}\Phi^*(\Phi(\sigma)^{-1/2} \cdot \Phi(\sigma)^{-1/2})\sigma^{1/2}$$

D. Petz, The Quarterly J. of Math., 1988

The Petz theorem

The following are equivalent.

(i) For some $\alpha \in (0,2)$ we have

$$D_\alpha(\Phi(\rho)\|\Phi(\sigma)) = D_\alpha(\rho\|\sigma), \quad \rho \in \mathcal{S}.$$

(ii) Connes cocycles:

$$\Phi^*(\Phi(\rho)^{is}\Phi(\sigma)^{-is}) = \rho^{is}\sigma^{-is}, \quad s \in \mathbb{R},\, \rho \in \mathcal{S}.$$

(iii) Universal recovery map:

$$\Phi_\sigma \circ \Phi(\rho) = \rho, \qquad \rho \in \mathcal{S}.$$

(iv) Φ is sufficient with respect to $\mathcal{S}$.

D. Petz, Commun. Math. Phys., 1986

D. Petz, The Quarterly J. of Math., 1988

Semigroup of channels preserving $\mathcal{S}$

Let us consider the set of channels

$$\mathcal{C}_{\mathcal{S}} := \{\Theta : L_1(\mathcal{H}) \to L_1(\mathcal{H}),\ \Theta(\rho) = \rho,\ \forall \rho \in \mathcal{S}\}$$

- convex and closed semigroup (in the point-weak topology)
- has a faithful fixed state: $\sigma \in \mathcal{S}$.

By the mean ergodic theorem, there is some $\mathcal{E}_{\mathcal{S}} \in \mathcal{C}_{\mathcal{S}}$ such that

$$\mathcal{E}_{\mathcal{S}} \circ \Theta = \Theta \circ \mathcal{E}_{\mathcal{S}} = \mathcal{E}_{\mathcal{S}}, \qquad \forall \Theta \in \mathcal{C}_{\mathcal{S}}.$$

B. Kümmerer, R. Nagel, Acta Sci. Math., 1979

We see that such $\mathcal{E}_{\mathcal{S}}$ is unique and

$$\mathcal{E}_{\mathcal{S}}^2 = \mathcal{E}_{\mathcal{S}}, \qquad \mathcal{E}_{\mathcal{S}}(\rho) = \rho, \quad \forall \rho \in \mathcal{S}.$$

The minimal sufficient subalgebra

The adjoint $\mathcal{E}_{\mathcal{S}}^*$ is a faithful normal conditional expectation:

- the range $\mathcal{M}_{\mathcal{S}} := \mathcal{E}_{\mathcal{S}}^*(B(\mathcal{H}))$ is a subalgebra
- $\mathcal{M}_{\mathcal{S}}$ is atomic: there is a decomposition $\mathcal{H} \equiv \oplus_n \mathcal{H}_n^{\mathcal{S},L} \otimes \mathcal{H}_n^{\mathcal{S},R}$ such that

$$\mathcal{M}_{\mathcal{S}} \equiv \bigoplus_n B(\mathcal{H}_n^{\mathcal{S},L}) \otimes I_{\mathcal{H}_n^{\mathcal{S},R}}$$

- Consequently,

$$\mathcal{E}_{\mathcal{S}}(L_1(\mathcal{H})) \equiv \bigoplus_n L_1(\mathcal{H}_n^{\mathcal{S},L}) \otimes \sigma_n$$

for some fixed $\sigma_n \in \mathcal{S}(\mathcal{H}_n^{\mathcal{S},R})$.

The Koashi-Imoto decomposition

Since $\mathcal{S} \subseteq \mathcal{E}_{\mathcal{S}}(L_1(\mathcal{H}))$, we must have

$$\rho \equiv \bigoplus_n \mu_n(\rho)\rho_n \otimes \sigma_n, \qquad \forall \rho \in \mathcal{S},$$

where

- $\{\mu_n(\rho)\}$ is a probability distribution (classical part of $\mathcal{S}$)
- $\rho_n \in \mathcal{S}(\mathcal{H}_n^{\mathcal{S},L})$ are states depending on ρ
- $\sigma_n \in \mathcal{S}(\mathcal{H}_n^{\mathcal{S},R})$ are fixed.

M. Koashi, N. Imoto, Phys. Rev. A, 2002
P. Hayden, R. Jósza, D. Petz, A. Winter, Commun. Math. Phys., 2004
A. Łuczak, Int. J. Theor. Phys., 2014
Y. Kuramochi, J. Math. Phys., 2018

Properties of $\mathcal{M}_\mathcal{S}$

- $\mathcal{M}_\mathcal{S}$ is the set of fixed points of $\mathcal{C}_\mathcal{S}$:

$$\mathcal{M}_\mathcal{S} = \{A \in B(\mathcal{H}),\ \Theta^*(A) = A,\ \forall\Theta \in \mathcal{C}_\mathcal{S}\}$$

- $\mathcal{M}_\mathcal{S}$ is invariant under the modular group for all $\rho \in \mathcal{S}$:

$$\rho^{it}\mathcal{M}_\mathcal{S}\rho^{-it} = \mathcal{M}_\mathcal{S}, \qquad \forall t \in \mathbb{R},\ \rho \in \mathcal{S}$$

- $\mathcal{M}_\mathcal{S}$ is generated by Connes cocycles:

$$\rho^{it}\sigma^{-it}, \qquad \rho \in \mathcal{S},\ t \in \mathbb{R}.$$

Sufficient channels with respect to $\mathcal{S}$

Assume that Φ is sufficient, with a recovery channel Ψ.

- Then $\Psi \circ \Phi \in \mathcal{C}_{\mathcal{S}}$, so that

$$\mathcal{E}_{\mathcal{S}} \circ (\Psi \circ \Phi) = (\Psi \circ \Phi) \circ \mathcal{E}_{\mathcal{S}} = \mathcal{E}_{\mathcal{S}}.$$

- Replacing Ψ by $\mathcal{E}_{\mathcal{S}} \circ \Psi$, we obtain

$$\Psi \circ \Phi = \mathcal{E}_{\mathcal{S}}, \qquad \Phi \circ \Psi = \mathcal{E}_{\mathcal{S}_0},$$

where

$$\mathcal{S}_0 := \{\Phi(\rho),\ \rho \in \mathcal{S}\}.$$

Sufficient channels with respect to $\mathcal{S}$

Φ is reversible with respect to $\mathcal{S}$ iff

$$\Phi^*|_{\mathcal{M}_{\mathcal{S}_0}} : \mathcal{M}_{\mathcal{S}_0} \xrightarrow{iso} \mathcal{M}_{\mathcal{S}}.$$

Equivalently, there is

- a decomposition $\mathcal{K} \equiv \oplus_n \mathcal{K}_n^L \otimes \mathcal{K}_n^R$
- unitaries $U_n : \mathcal{H}_n^{\mathcal{S},L} \to \mathcal{K}_n^L$
- channels $\Phi_n : L_1(\mathcal{H}_n^{\mathcal{S},R}) \to L_1(\mathcal{K}_n^R)$

such that

$$\Phi|_{L_1(\mathcal{H}_n^{\mathcal{S},L} \otimes \mathcal{H}_n^{\mathcal{S},R})} \equiv U_n^* \cdot U_n \otimes \Phi_n.$$

Universal recovery map and Connes cocycles

- From $\Psi^* \circ \Phi^* = \mathcal{E}^*_{\mathcal{S}_0}$, we get for $A \in \mathcal{M}_{\mathcal{S}_0}$:

$$\Phi^*(\Phi(\sigma)^{it} A \Phi(\sigma)^{-it}) = \sigma^{it} \Phi^*(A) \sigma^{-it}, \qquad t \in \mathbb{R}$$

- This implies that $\Phi^*_\sigma \circ \Phi^*(A) = A$ (Petz, 1988), so that

$$\Phi \circ \Phi_\sigma = \mathcal{E}_{\mathcal{S}_0}, \qquad \Phi_\sigma \circ \Phi = \mathcal{E}_{\mathcal{S}}.$$

  Hence Φ_σ is a recovery map.
- The condition $\Phi^*(\Phi(\rho)^{it}\Phi(\sigma^{-it})) = \rho^{it}\sigma^{-it}$ for all t and ρ also follows, by the properties of the cocycles.

Conditions on $\mathcal{S}$

Given a channel Φ, what are the conditions for states in $\mathcal{S}$?

We fix a faithful state $\sigma \in \mathcal{S}$. Then we must have

$$\mathcal{S} \subset \mathrm{Fix}(\Phi_\sigma \circ \Phi) := \{\rho,\ \Phi_\sigma \circ \Phi(\rho) = \rho\}.$$

Put

$$\mathcal{F} := \lim_n \frac{1}{n} \sum_{k=1}^{n} (\Phi_\sigma \circ \Phi)^k,$$

then $\mathcal{F}^*$ is a conditional expectation and

$$\mathcal{F}(B(\mathcal{H})) = \mathrm{Fix}(\Phi_\sigma \circ \Phi).$$

Conditions on $\mathcal{S}$

There is

- a decomposition $\mathcal{H} \equiv \oplus_n \mathcal{H}_n^{\Phi,\sigma,L} \otimes \mathcal{H}_n^{\Phi,\sigma,R}$
- and states $\omega_n \in \mathcal{S}(\mathcal{H}_n^{\Phi,\sigma,R})$

such that Φ is reversible with respect to $\mathcal{S}$ if and only if all $\rho \in \mathcal{S}$ have the form

$$\rho \equiv \bigoplus_n \lambda_n(\rho)\rho_n \otimes \omega_n$$

for some probability distribution $\{\lambda_n(\rho)\}$ and states $\rho_n \in \mathcal{S}(\mathcal{H}^{\Phi,\sigma,L})$.

This decomposition can be different from the Koashi-Imoto decomposition.

Sandwiched Rényi divergences

For $\alpha > 0$, $\alpha \neq 1$, we set

$$\tilde{D}_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1} \log \tilde{Q}_\alpha(\rho\|\sigma)$$

where (for $\dim(\mathcal{H}) < \infty$)

$$\tilde{Q}_\alpha(\rho\|\sigma) = \mathrm{Tr}\left(\sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}}\right)^\alpha .$$

- satisfy data processing inequality for $\alpha \in [1/2, \infty]$
- for $\alpha > 1$, we will need the Kosaki L_p-spaces for a proper definition when $\dim(\mathcal{H}) = \infty$.

The interpolation L_p-spaces with respect to σ

- a continuous embedding

$$B(\mathcal{H}) \subseteq L_1(\mathcal{H}), \qquad X \mapsto \sigma^{1/2} X \sigma^{1/2}$$

- interpolation spaces: for $1 \leq p \leq \infty$

$$L_p(\mathcal{H}, \sigma) := C_{1/p}(B(\mathcal{H}), L_1(\mathcal{H})) \subseteq L_1(\mathcal{H})$$

- for $1/p + 1/q = 1$, we have

$$L_p(\mathcal{H}, \sigma) = \{\sigma^{1/2q} X \sigma^{1/2q},\ X \in L_p(\mathcal{H})\},$$

the norm: $\|\sigma^{1/2q} X \sigma^{1/2q}\|_{p,\sigma} = \|X\|_p$

Hadamard three lines theorem

For any function on $S = \{z \in \mathbb{C},\ \mathrm{Re}(z) \in [0,1]\}$,

$$f : S \to L_1(\mathcal{H}), \quad \text{bounded, continuous, analytic in } \mathrm{int}(S)$$

we have:

- for any $p > 1$,

$$\|f(1/p)\|_{p,\sigma} \leq \max_{t\in\mathbb{R}} \|f(it)\|_{\infty,\sigma} \max_{t\in\mathbb{R}} \|f(1+it)\|_1$$

- If equality holds for some $p > 1$, then it holds for all

Hadamard three lines theorem

For any $\rho = \sigma^{1/2q}\tau^{1/p}\sigma^{1/2q}$, $\tau \in L_1(\mathcal{H})^+$ we define a function

$$f_{\rho,p}(z) = \|\rho\|_{p,\sigma}^{1-zp}\sigma^{\frac{1-z}{2}}\tau^z\sigma^{\frac{1-z}{2}}, \qquad z \in S$$

Then

- $f_{\rho,p}(1/p) = \rho$,
- The equality in Hadamard three lines theorem is attained:

$$\|f_{\rho,p}(1/p)\|_{p,\sigma} = \max_{t\in\mathbb{R}} \|f_{\rho,p}(it)\|_{\infty,\sigma} \max_{t\in\mathbb{R}} \|f_{\rho,p}(1+it)\|_1$$

Positive trace preserving maps are contractions

Let $\Phi : L_1(\mathcal{H}) \to L_1(\mathcal{K})$ be a positive trace preserving linear map:

- For $p = 1$,

$$\|\Phi(X)\|_1 \leq \|X\|_1, \qquad X \in L_1(\mathcal{H})$$

- For $p = \infty$, $X \in B(\mathcal{H})$,

$$\|\Phi(\sigma^{1/2} X \sigma^{1/2})\|_{\infty,\Phi(\sigma)} = \|\Phi_\sigma^*(X)\|_\infty \leq \|X\|_\infty = \|\sigma^{1/2} X \sigma^{1/2}\|_{\infty,\sigma}$$

- For $p > 1$, by Riesz-Thorin (complex interpolation)

$$\|\Phi(X)\|_{p,\Phi(\sigma)} \leq \|X\|_{p,\sigma}, \qquad X \in L_p(\mathcal{H}, \sigma).$$

Data processing inequality

Now we can define for $\alpha > 1$:

$$\tilde{Q}_\alpha(\rho\|\sigma) = \begin{cases} \|\rho\|_{\alpha,\sigma}, & \rho \in L_p(\mathcal{H},\sigma) \\ \infty, & \text{otherwise.} \end{cases}$$

For any positive trace preserving map, $\alpha > 1$, we have the DPI:

$$\tilde{D}_\alpha(\Phi(\rho)\|\Phi(\sigma)) \leq \tilde{D}_\alpha(\rho\|\sigma).$$

We next want to prove that for a channel Φ, equality implies sufficiency of the channel.

Preservation and sufficiency

Let $\alpha = 2$.

- $\|\cdot\|_{2,\sigma}$ is a Hilbert space norm, with the inner product

$$\langle \sigma^{1/4} X \sigma^{1/2}, \sigma^{1/4} Y \sigma^{1/4} \rangle_\sigma = \mathrm{Tr}\, X^* Y, \qquad X, Y \in L_2(\mathcal{H}).$$

- A positive trace preserving map Φ defines a contraction $L_2(\mathcal{H}, \sigma) \to L_2(\mathcal{K}, \sigma)$, with adjoint given by Φ_σ:

$$\langle A, \Phi(B) \rangle_{\Phi(\sigma)} = \langle \Phi_\sigma(A), B \rangle_\sigma, \quad A \in L_2(\mathcal{K}, \sigma),\ B \in L_2(\mathcal{H}, \sigma)$$

- Since Φ is a contraction,

$$\|\Phi(\rho)\|_{2,\Phi(\sigma)} = \|\rho\|_{2,\sigma} \iff \Phi_\sigma \circ \Phi(\rho) = \rho.$$

Preservation and reversibility

For $\alpha = \bar{\alpha} > 1$: Let

$$\rho = \sigma^{\frac{\bar{\alpha}-1}{2\bar{\alpha}}} \tau^{1/\bar{\alpha}} \sigma^{\frac{\bar{\alpha}-1}{2\bar{\alpha}}} \in L_{\bar{\alpha}}(\mathcal{H}, \sigma), \quad \tau \in L_1(\mathcal{H})^+$$

and assume $\|\Phi(\rho)\|_{\bar{\alpha},\Phi(\sigma)} = \|\rho\|_{\bar{\alpha},\sigma}$. Put

$$f(z) := f_{\rho,\bar{\alpha}}(z) = \|\rho\|_{\bar{\alpha},\sigma}^{1-z\bar{\alpha}} \sigma^{\frac{1-z}{2}} \tau^z \sigma^{\frac{1-z}{2}}, \qquad z \in S$$

Then

$$\begin{aligned} \|\rho\|_{\bar{\alpha},\sigma} &= \|f(1/\bar{\alpha})\|_{\bar{\alpha},\sigma} = \|\Phi(f(1/\bar{\alpha}))\|_{\bar{\alpha},\Phi(\sigma)} \\ &\leq \max_{t\in\mathbb{R}} \|\Phi(f(it))\|_{\infty,\Phi(\sigma)} \max_{t\in\mathbb{R}} \|\Phi(f(1+it))\|_1 \\ &\leq \max_{t\in\mathbb{R}} \|f(it)\|_{\infty,\sigma} \max_{t\in\mathbb{R}} \|f(1+it)\|_1 = \|\rho\|_{\bar{\alpha},\sigma} \end{aligned}$$

Preservation and reversibility

We have equalities, for any $\alpha > 1$. This implies

$$\|\Phi(f(1/\alpha))\|_{\alpha,\Phi(\sigma)} = \|f(1/\alpha)\|_{\alpha,\sigma}, \qquad \alpha > 1.$$

In particular,

$$\|\Phi(\tau)\|_{2,\Phi(\sigma)} = \|\tau\|_{2,\sigma}, \text{ so that } \Phi_\sigma \circ \Phi(\omega) = \omega,$$

for

$$\omega := f(1/2) = \sigma^{1/4}\tau^{1/2}\sigma^{1/4}.$$

We know that $\Phi_\sigma \circ \Phi(\rho) = \rho$ iff ρ is of the form

$$\rho \equiv \bigoplus_n \rho_n \otimes \omega_n \qquad \text{(with fixed faithful states } \omega_n\text{)}$$

Since $\Phi_\sigma \circ \Phi(\sigma) = \sigma$, $\Phi_\sigma \circ \Phi(\omega) = \omega$, this must be true.

A variational formula for $\alpha \in [1/2, 1)$

For $\alpha \in [1/2, 1)$, we have

$$\tilde{Q}_\alpha(\rho\|\sigma) = \inf_{X \in B(\mathcal{H})^{++}} \alpha \operatorname{Tr} \rho X + (1-\alpha) \operatorname{Tr} \left(\sigma^{\frac{1-\alpha}{2\alpha}} X^{-1} \sigma^{\frac{1-\alpha}{2\alpha}}\right)^{\frac{\alpha}{1-\alpha}}$$

With $\gamma := \frac{\alpha}{1-\alpha} > 1$, this can be written as

$$\tilde{Q}_\alpha(\rho\|\sigma) = \inf_{X \in B(\mathcal{H})^{++}} \alpha \operatorname{Tr} \rho X + (1-\alpha) \|\sigma^{1/2} X^{-1} \sigma^{1/2}\|_{\gamma,\sigma}^{\gamma}.$$

If ρ is also faithful, attained at the unique element $\bar{X}$ such that

$$\sigma^{1/2} \bar{X}^{-1} \sigma^{1/2} = \sigma^{1/2\gamma^*} \mu^{1/\gamma} \sigma^{1/2\gamma}, \qquad \mu = |\sigma^{\frac{1-\alpha}{2\alpha}} \rho^{1/2}|^{2\alpha}$$

R. L. Frank, E. H. Lieb, J. Math. Phys., 2013

F. Hiai, Quantum f-Divergences in von Neumann Algebras: Reversibility of Quantum Operations, 2021

Positive trace preserving maps

Let $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$ be a positive trace preserving map.

For $Y \in B(\mathcal{K})^{++}$, we have

$$\begin{aligned}
\|\sigma^{1/2}\Phi^*(Y)^{-1}\sigma^{1/2}\|_{\gamma,\sigma}^{\gamma} &\leq \|\sigma^{1/2}\Phi^*(Y^{-1})\sigma^{1/2}\|_{\gamma,\sigma}^{\gamma} \\
&= \|\Phi_\sigma(\Phi(\sigma)^{1/2}Y^{-1}\Phi(\sigma)^{1/2})\|_{\gamma,\sigma}^{\gamma} \\
&\leq \|\Phi(\sigma)^{1/2}Y^{-1}\Phi(\sigma)^{1/2}\|_{\gamma,\Phi(\sigma)}^{\gamma}
\end{aligned}$$

We used the Choi inequality $\Phi^*(Y)^{-1} \leq \Phi^*(Y^{-1})$, definition of Φ_σ and monotonicity of $\tilde{Q}_\gamma$, $\gamma > 1$.

Positive trace preserving maps

We get, for $Y \in B(\mathcal{K})^{++}$,

$$\begin{aligned}\tilde{Q}_\alpha(\rho\|\sigma) &\le \alpha \operatorname{Tr} \rho\Phi^*(Y) + (1-\alpha)\|\sigma^{1/2}\Phi^*(Y)^{-1}\sigma^{1/2}\|_{\gamma,\sigma}^{\gamma} \\ &\le \alpha \operatorname{Tr} \Phi(\rho)Y + (1-\alpha)\|\Phi(\sigma)^{1/2}Y^{-1}\Phi(\sigma)^{1/2}\|_{\gamma,\Phi(\sigma)}^{\gamma}\end{aligned}$$

Taking the inf,

$$\tilde{Q}_\alpha(\rho\|\sigma) \le \tilde{Q}_\alpha(\Phi(\rho)\|\Phi(\sigma)),$$

so that

$$\tilde{D}_\alpha(\rho\|\sigma) \ge \tilde{D}_\alpha(\Phi(\rho)\|\Phi(\sigma)).$$

Preservation and reversibility

Let $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$ be a channel such that

$$\tilde{Q}_\alpha(\rho\|\sigma) = \tilde{Q}_\alpha(\Phi(\rho)\|\Phi(\sigma)).$$

If ρ is faithful, then the infima in the variational formulas are attained at unique $\bar{X} \in B(\mathcal{H})^{++}$ resp. $\bar{Y} \in B(\mathcal{K})$ and

$$\bar{X} = \Phi^*(\bar{Y}).$$

We also infer that

$$\begin{aligned} \|\sigma^{1/2}\Phi^*(Y)^{-1}\sigma^{1/2}\|_{\gamma,\sigma}^{\gamma} &= \|\sigma^{1/2}\Phi^*(Y^{-1})\sigma^{1/2}\|_{\gamma,\sigma}^{\gamma} \\ &= \|\Phi_\sigma(\Phi(\sigma)^{1/2}Y^{-1}\Phi(\sigma)^{1/2})\|_{\gamma,\sigma}^{\gamma} \\ &= \|\Phi(\sigma)^{1/2}Y^{-1}\Phi(\sigma)^{1/2}\|_{\gamma,\Phi(\sigma)}^{\gamma} \end{aligned}$$

Preservation and reversibility

From this, we can obtain that $\Phi_\sigma \circ \Phi(\mu) = \mu$, where

$$\mu = |\sigma^{\frac{1-\alpha}{2\alpha}} \rho^{1/2}|^{2\alpha}$$

We get $\Phi_\sigma \circ \Phi(\rho) = \rho$ as before, from the decomposition of fixed points of $\Phi_\sigma \circ \Phi$.

Quantum hypothesis testing

Suppose $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ are given, one of them is the true state:

- we test the hypothesis $H_0 = \sigma$ against $H_1 = \rho$
- a test: an effect $0 \leq T \leq I$,

$$\mathrm{Tr}\,[T\omega] - \text{probability of rejecting } H_0 \text{ in the state } \omega$$

- error probabilities:

$$\alpha(T) = \mathrm{Tr}\,[\sigma T], \qquad \beta(T) = \mathrm{Tr}\,[\rho(I - T)]$$

- Bayes error probabilities for $\lambda \in [0, 1]$:

$$P_e(\lambda, \rho, \sigma, T) := \lambda\alpha(T) + (1 - \lambda)\beta(T)$$

Quantum Neyman-Pearson lemma

Put $P_{s,\pm} := \operatorname{supp}((\rho - s\sigma)_\pm)$, $P_{s,0} := I - P_{s,+} - P_{s,-}$.

A test T is Bayes optimal for $\lambda \in (0,1)$ if and only if

$$T = P_{s,+} + X, \quad 0 \le X \le P_{s,0}, \qquad s = \frac{\lambda}{1-\lambda}$$

and then

$$\begin{aligned} P_\lambda(\rho\|\sigma) &:= \min_{0 \le T \le I} P_e(\lambda, \rho, \sigma, T) \\ &= (1-\lambda)(1 - \operatorname{Tr}[(\rho - s\sigma)_+]) \\ &= (1-\lambda)(s - \operatorname{Tr}[(\rho - s\sigma)_-]) \\ &= \frac{1}{2}(1 - (1-\lambda)\|\rho - s\sigma\|_1). \end{aligned}$$

Asymmetric hypothesis testing

We may also fix $\epsilon \in [0,1]$ and put

$$d_\epsilon(\rho\|\sigma) = \inf\{\beta(M) : \ 0 \leq M \leq I, \ \alpha(M) \leq \epsilon\}.$$

We then have

$$P_\lambda(\rho\|\sigma) = \inf_{0<\epsilon<1} \lambda\epsilon + (1-\lambda) d_\epsilon(\rho\|\sigma)$$

and

$$d_\epsilon(\rho\|\sigma) = \sup_{0<\lambda<1} \frac{1}{1-\lambda}(P_\lambda(\rho\|\sigma) - \lambda\epsilon).$$

Data processing inequalities

We clearly have for any quantum channel Φ and $\lambda \in [0, 1]$:

$$P_\lambda(\Phi(\rho) \| \Phi(\sigma)) \geq P_\lambda(\rho \| \sigma)$$

and

$$d_\epsilon(\Phi(\rho), \Phi(\sigma)) \geq d_\epsilon(\rho, \sigma)$$

or equivalently, for any $s \in \mathbb{R}$:

$$\|\Phi(\rho) - s\Phi(\sigma)\|_1 \leq \|\rho - s\sigma\|_1;$$
$$\mathrm{Tr}\left[(\Phi(\rho) - s\Phi(\sigma))_+\right] \leq \mathrm{Tr}\left[(\rho - s\sigma)_+\right];$$
$$\mathrm{Tr}\left[(\Phi(\rho) - s\Phi(\sigma))_-\right] \leq \mathrm{Tr}\left[(\rho - s\sigma)_-\right].$$

Equality in DPI

The following are equivalent:

- $P_\lambda(\Phi(\rho)\|\Phi(\sigma)) = P_\lambda(\rho\|\sigma)$, $\lambda \in [0,1]$;
- $\|\Phi(\rho) - s\Phi(\sigma)\|_1 = \|\rho - s\sigma\|_1$, $s \in \mathbb{R}$;
- $\mathrm{Tr}\left[(\Phi(\rho) - s\Phi(\sigma))_+\right] = \mathrm{Tr}\left[(\rho - s\sigma)_+\right]$, $s \in \mathbb{R}$;
- $\mathrm{Tr}\left[(\Phi(\rho) - s\Phi(\sigma))_-\right] = \mathrm{Tr}\left[(\rho - s\sigma)_-\right]$, $s \in \mathbb{R}$;
- $d_\epsilon(\Phi(\rho)\|\Phi(\sigma)) = d_\epsilon(\rho\|\sigma)$, $\epsilon \in [0,1]$.

Can we get sufficiency?

An integral formula for relative entropy

For any pair if states ρ, σ:

$$D(\rho\|\sigma) = \int_{-\infty}^{\infty} \frac{dt}{|t|(1-t)^2} \operatorname{Tr}\left[((1-t)\rho + t\sigma)_-\right]$$

For $\lambda \geq 0$ such that $\rho \leq \lambda\sigma$:

$$D(\rho\|\sigma) = \int_{0}^{\lambda} \frac{ds}{s} \operatorname{Tr}\left[(\rho - s\sigma)_-\right] + \log(\lambda) + 1 - \lambda$$

P. Frenkel, arxiv:2208.12194

Reversibility via hypothesis testing

Let $\rho, \sigma \in B(\mathcal{H})$ be any states, $\Phi : L_1(\mathcal{H}) \to L_1(\mathcal{K})$ a channel.

Assume that

$$P_\lambda(\Phi(\rho) \| \Phi(\sigma)) = P_\lambda(\rho \| \sigma), \lambda \in [0, 1]$$

Equivalently,

$$\mathrm{Tr}\,(\Phi(\rho) - s\Phi(\sigma))_- = \mathrm{Tr}\,(\rho - s\sigma)_-, s \in \mathbb{R},$$

the same is true with σ replaced by $\sigma_0 := \frac{1}{2}(\rho + \sigma)$.

Reversibility via hypothesis testing

We have

$$\rho \leq 2\sigma_0, \qquad \Phi(\rho) \leq 2\Phi(\sigma_0).$$

By the integral representation,

$$\begin{aligned} D(\rho\|\sigma_0) &= \int_0^2 \frac{ds}{s} \operatorname{Tr}\left[(\rho - s\sigma)_-\right] + \log(2) - 1 \\ &= \int_0^2 \frac{ds}{s} \operatorname{Tr}\left[(\Phi(\rho) - s\Phi(\sigma))_-\right] + \log(2) - 1 \\ &= D(\Phi(\rho)\|\Phi(\sigma_0)) \end{aligned}$$

It follows that

$$\Phi_{\sigma_0} \circ \Phi(\rho) = \rho, \quad \Phi_{\sigma_0} \circ \Phi(\sigma) = \sigma.$$