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A Theory of quantum Local asymptotic normality

Part I: Quantum Contiguity

Akio Fujiwara (Osaka University)

Joint Work with Koichi Yamagata

[Bernoulli, 26 (2020) 2105; Annals of Statistics, 51 (2023) 1159]

Plan of talk

- Chap. I (Allegro ma non troppo): Review of classical LAN
- Chap. II (Moderato assai): Quantum likelihood ratio
- Chap. III (Allegro con moto): Quantum contiguity
- Chap. IV (Prestissimo):
 Preliminary application to quantum LAN
- → to be continued in Part II by Koichi Yamagata

Chap. I: Classical LAN

Local Asymptotic Normality

A sequence of models $\{P_\theta^{(n)} : \theta \in \Theta \subset \mathbb{R}^d\}$

is called LAN at $\theta_0 \in \Theta$ if

$$\log \frac{dP_{\theta_0+h/\sqrt{n}}^{(n)}}{dP_{\theta_0}^{(n)}} = h^i \Delta_i^{(n)} - \frac{1}{2} h^i h^j J_{ij} + o_{P_{\theta_0}}(1)$$

where

$$\Delta^{(n)} \xrightarrow{P_{\theta_0}^{(n)}} N(0, J)$$

Prototype of LAN: $p_{\theta}^{(n)} = p_{\theta}^{\otimes n}$

$$\log \frac{p_{\theta_0 + h/\sqrt{n}}^{\otimes n}(X_1, \dots, X_n)}{p_{\theta_0}^{\otimes n}} = h^i \underbrace{\left\{ \frac{1}{\sqrt{n}} \sum_{k=1}^n \partial_i \log p_{\theta_0}(X_k) \right\}}_{\Delta_i^{(n)}(X_1, \dots, X_n)} - \frac{1}{2} h^i h^j \underbrace{\left\{ -\frac{1}{n} \sum_{k=1}^n \partial_i \partial_j \log p_{\theta_0}(X_k) \right\}}_{J_{ij} + o_{p_{\theta_0}}(1)} + o\left(\frac{1}{n}\right)$$

where

$$\Delta^{(n)} \stackrel{p_{\theta_0}^{\otimes n}}{\rightsquigarrow} N(0, J)$$

with J being the Fisher information matrix

Similarity to Gaussian Shift model

LAN

$$\log \frac{dP_{\theta_0+h/\sqrt{n}}^{(n)}}{dP_{\theta_0}^{(n)}} = h^i \Delta_i^{(n)} - \frac{1}{2} h^i h^j J_{ij} + o_{P_{\theta_0}}(1)$$

Gaussian shift model

$$\log \frac{dN(Jh, J)}{dN(0, J)}(X^1, \dots, X^d) = h^i X_i - \frac{1}{2} h^i h^j J_{ij}$$

Contiguity

A sequence $Q^{(n)}$ of probability measures is called contiguous to another sequence $P^{(n)}$ of probability measures, denoted $Q^{(n)} \triangleleft P^{(n)}$, if

$$P^{(n)}(A^{(n)}) \rightarrow 0 \implies Q^{(n)}(A^{(n)}) \rightarrow 0$$

Le Cam's Third Lemma

Weak convergence analogue of the Radon-Nikodym

theorem: $Q \ll P \implies dQ = \frac{dQ}{dP} dP$

Theorem (Le Cam)

If $Q^{(n)} \ll P^{(n)}$ and $\left(X^{(n)}, \frac{dQ^{(n)}}{dP^{(n)}} \right) \xrightarrow{P^{(n)}} (X, V)$, then

$$X^{(n)} \xrightarrow{Q^{(n)}} L \quad (L(B) := E[1_B(X)V])$$

Le Cam's third lemma (Gaussian version)

Theorem

If

$$\begin{pmatrix} X^{(n)} \\ \log \frac{dQ^{(n)}}{dP^{(n)}} \end{pmatrix} \xrightarrow{P^{(n)}} N\left(\begin{pmatrix} 0 \\ -\frac{1}{2}\sigma^2 \end{pmatrix}, \begin{pmatrix} \Sigma & \tau^\top \\ \tau & \sigma^2 \end{pmatrix}\right)$$

then $(Q^{(n)} \ll P^{(n)}$ and)

$$X^{(n)} \xrightarrow{Q^{(n)}} N(\tau, \Sigma)$$

Third lemma under LAN

Suppose

$$\log \frac{dP_{\theta_0+h/\sqrt{n}}^{(n)}}{dP_{\theta_0}^{(n)}} = h^i \Delta_i^{(n)} - \frac{1}{2} h^i h^j J_{ij} + o_{p_{\theta_0}}(1)$$

and

$$\begin{pmatrix} X^{(n)} \\ \Delta^{(n)} \end{pmatrix} \stackrel{P_{\theta_0}^{(n)}}{\rightsquigarrow} N \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma & \tau \\ \tau^\top & J \end{pmatrix} \right)$$

Then

$$X^{(n)} \stackrel{P_{\theta_0+h/\sqrt{n}}^{(n)}}{\rightsquigarrow} N(\tau h, \Sigma)$$

The moral:
LAN model is
locally
asymptotically
similar to
Gaussian shift
model

History of quantum LAN

- Guta and Káhn's strong q-LAN (2006, 2009)

$$\lim_{n \rightarrow \infty} \sup_{h \in K^{(n)}} \left\| \sigma_h - \Gamma^{(n)}(\rho_{\theta_0+h/\sqrt{n}}^{\otimes n}) \right\|_1 = 0$$

and

$$\lim_{n \rightarrow \infty} \sup_{h \in K^{(n)}} \left\| \Lambda^{(n)}(\sigma_h) - \rho_{\theta_0+h/\sqrt{n}}^{\otimes n} \right\|_1 = 0$$

Drawbacks

- iid
- nondegeneracy

- Guta and Jenčová's weak q-LAN (2007)

Quantum
Radon-
Nikodym?

Difficulties in extending LAN to the quantum domain

LAN: $\log \frac{dP_{\theta_0+h/\sqrt{n}}^{(n)}}{dP_{\theta_0}^{(n)}} = h^i \Delta_i^{(n)} - \frac{1}{2} h^i h^j J_{ij} + o_{P_{\theta_0}}(1)$

$(\Delta^{(n)} \xrightarrow{\theta_0} N(0, J))$

What are the quantum counterparts of

- ① Radon-Nikodym density?
- ② contiguity and third Lemma?
- ③ weak convergence?

Chap. II: Quantum Likelihood
ratio

Singularity & absolute continuity

1) ρ is singular to σ , denoted $\rho \perp \sigma$, if

$$\text{supp } \rho \perp \text{supp } \sigma$$

2) ρ is absolutely continuous to σ , denoted

$\rho \ll \sigma$, if there is a positive operator

$R (\geq 0)$ that satisfies $\rho = R\sigma R$

Remark

For pure states $\rho = |\xi\rangle\langle\xi|$ and $\sigma = |\eta\rangle\langle\eta|$

$$1) \quad \text{supp } \rho \subset \text{supp } \sigma \iff \rho = \sigma$$

$$2) \quad \rho \ll \sigma \iff \langle \xi | \eta \rangle \neq 0$$

Quantum Lebesgue decomposition

Theorem. Given quantum states $\rho, \sigma \in \mathcal{S}(\mathcal{H})$,
the decomposition

$$\sigma = \underbrace{R\rho R}_{\sigma^{ac}} + \underbrace{\tau}_{\sigma^\perp} \quad (R \geq 0, \tau \geq 0, \tau \perp \rho)$$

uniquely exists.

Quantum Likelihood ratio

A positive operator R that satisfies

$$\sigma = R\rho R + \tau \quad (R \geq 0, \tau \geq 0, \tau \perp \rho)$$

is called a square-root Likelihood ratio,

and is denoted as

$$\mathcal{R}(\sigma | \rho)$$

Chap. III: Quantum contiguity

Classical contiguity

Recall that $Q^{(n)} \triangleleft P^{(n)}$ if

$$P^{(n)}(A^{(n)}) \rightarrow 0 \implies Q^{(n)}(A^{(n)}) \rightarrow 0$$

This is equivalent to saying that

i) $\lim_{n \rightarrow \infty} E_{P^{(n)}} \left[\frac{dQ^{(n)}}{dP^{(n)}} \right] = 1$

ii) $\frac{dQ^{(n)}}{dP^{(n)}}$ is uniformly integrable under $P^{(n)}$

Quantum contiguity

$\sigma^{(n)} \triangleleft \rho^{(n)}$ if

i) $\lim_{n \rightarrow \infty} \text{Tr } \rho^{(n)} \mathcal{R}(\sigma^{(n)} | \rho^{(n)})^2 = 1$

ii) $(\bar{R}^{(n)})^2$ is uniformly integrable under $\rho^{(n)}$

where $\bar{R}^{(n)} := \mathcal{R}(\sigma^{(n)} | \rho^{(n)}) + O^{(n)} \geq 0$ with

$O^{(n)} = o_{L^2}(\rho^{(n)})$ being an L^2 -infinitesimal term

We also denote as $\sigma^{(n)} \triangleleft_{O^{(n)}} \rho^{(n)}$

Properties

1) Suppose $\rho^{(n)}, \sigma^{(n)} \in \mathcal{S}(\mathcal{H})$ have limiting states

$$\lim_{n \rightarrow \infty} \rho^{(n)} = \rho^{(\infty)} \quad \text{and} \quad \lim_{n \rightarrow \infty} \sigma^{(n)} = \sigma^{(\infty)}$$

Then $\sigma^{(n)} \triangleleft \rho^{(n)} \iff \sigma^{(\infty)} \ll \rho^{(\infty)}$

2) Suppose $\rho_i, \sigma_i \in \mathcal{S}(\mathcal{H}_i)$ satisfy $\sigma_i \ll \rho_i$. Let

$$\rho^{(n)} := \bigotimes_{i=1}^n \rho_i \quad \text{and} \quad \sigma^{(n)} := \bigotimes_{i=1}^n \sigma_i$$

Then $\sigma^{(n)} \triangleleft \rho^{(n)} \iff \sum_{i=1}^{\infty} \left(1 - \text{Tr} \sqrt{\sqrt{\sigma_i} \rho_i \sqrt{\sigma_i}} \right) < \infty$



Quantum weak convergence

$\rho^{(n)}$: state, $X^{(n)} = (X_1^{(n)}, \dots, X_d^{(n)})$ obs. on $\mathcal{H}^{(n)}$

ϕ : state, $X^{(\infty)} = (X_1^{(\infty)}, \dots, X_d^{(\infty)})$ obs. on $\mathcal{H}^{(\infty)}$

We say $(X^{(n)}, \rho^{(n)}) \rightsquigarrow (X^{(\infty)}, \phi)$ if

$$\lim_{n \rightarrow \infty} \text{Tr } \rho^{(n)} \left(\prod_{t=1}^r e^{\sqrt{-1}\xi_t^i X_i^{(n)}} \right) = \phi \left(\prod_{t=1}^r e^{\sqrt{-1}\xi_t^i X_i^{(\infty)}} \right)$$

When $(X^{(\infty)}, \phi) \sim N(h, J)$, we denote $X^{(n)} \xrightarrow{\rho^{(n)}} N(h, J)$



Quantum Gaussian state

A state ϕ on a CCR(S) with

$$e^{\sqrt{-1}\xi^i X_i} e^{\sqrt{-1}\eta^j X_j} = e^{\sqrt{-1}\xi^i \eta^j S_{ij}} e^{\sqrt{-1}(\xi + \eta)^i X_i}$$

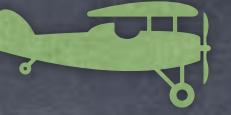
is called a quantum Gaussian state if

$$\phi(e^{\sqrt{-1}\xi^i X_i}) = e^{\sqrt{-1}\xi^i h_i - \frac{1}{2}\xi^i \xi^j V_{ij}}$$

where

$$J := V + \sqrt{-1}S \geq 0$$

We denote it as $\phi \sim N(h, J)$



A variant: "Sandwiched" weak convergence

If

$$\lim_{n \rightarrow \infty} \text{Tr } \rho^{(n)} e^{\sqrt{-1}\eta_1 Y^{(n)}} \left\{ \prod_{t=1}^r e^{\sqrt{-1}\xi_t^i X_i^{(n)}} \right\} e^{\sqrt{-1}\eta_2 Y^{(n)}} \\ = \phi \left(e^{\sqrt{-1}\eta_1 Y^{(\infty)}} \left\{ \prod_{t=1}^r e^{\sqrt{-1}\xi_t^i X_i^{(\infty)}} \right\} e^{\sqrt{-1}\eta_2 Y^{(\infty)}} \right)$$

we denote

$$(\langle Y^{(n)}, X^{(n)}, Y^{(n)} \rangle, \rho^{(n)}) \rightsquigarrow (\langle Y^{(\infty)}, X^{(\infty)}, Y^{(\infty)} \rangle, \phi)$$

or

$$\langle Y^{(n)}, X^{(n)}, Y^{(n)} \rangle_{\rho^{(n)}} \rightsquigarrow \langle Y^{(\infty)}, X^{(\infty)}, Y^{(\infty)} \rangle_{\phi}$$

Quantum Le Cam Third Lemma

If $\sigma^{(n)} \triangleleft_{O^{(n)}} \rho^{(n)}$ and $R^{(n)} = \mathcal{R}(\sigma^{(n)} | \rho^{(n)})$ enjoys

$$\left\langle R^{(n)} + O^{(n)}, X^{(n)}, R^{(n)} + O^{(n)} \right\rangle_{\rho^{(n)}} \rightsquigarrow \left\langle R^{(\infty)}, X^{(\infty)}, R^{(\infty)} \right\rangle_{\phi}$$

Then

$$(X^{(n)}, \sigma^{(n)}) \rightsquigarrow (X^{(\infty)}, \psi)$$

where

$$\psi(A) := \phi(R^{(\infty)} A R^{(\infty)})$$

This gives a complete characterization of alternative state ψ in terms of reference state ϕ and limit likelihood ratio

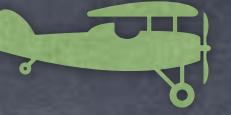
Quantum Le Cam Third Lemma (q-Gaussian version)

If

$$\begin{pmatrix} X^{(n)} \\ 2 \log(R^{(n)} + O^{(n)}) + \tilde{O}^{(n)} \end{pmatrix} \xrightarrow{\rho^{(n)}} N\left(\begin{pmatrix} \mu \\ -\frac{1}{2}s^2 \end{pmatrix}, \begin{pmatrix} \Sigma & \kappa \\ \kappa^* & s^2 \end{pmatrix}\right)$$

Then ($\sigma^{(n)} \triangleleft_{O^{(n)}} \rho^{(n)}$ and)

$$X^{(n)} \xrightarrow{\sigma^{(n)}} N(\mu + \text{Re}(\kappa), \Sigma)$$



Chap. IV: Quantum LAN



Quantum LAN

$\mathcal{S}^{(n)} = \{\rho_{\theta}^{(n)} : \theta \in \Theta \subset \mathbb{R}^d\}$ is called q-LAN at $\theta_0 \in \Theta$ if

$R_h^{(n)} := \mathcal{R}\left(\rho_{\theta_0+h/\sqrt{n}}^{(n)} \middle| \rho_{\theta_0}^{(n)}\right)$ is expanded in h as

$$\log \left(R_h^{(n)} + o_{L^2}(\rho_{\theta_0}^{(n)}) \right)^2 = h^i \Delta_i^{(n)} - \frac{1}{2} (J_{ij} h^i h^j) I^{(n)} + o_D(h^i \Delta_i^{(n)}, \rho_{\theta_0}^{(n)})$$

where $\Delta^{(n)} \stackrel{\rho_{\theta_0}^{(n)}}{\rightsquigarrow} N(0, J)$



Third Lemma under q-LAN

Suppose $\mathcal{S}^{(n)} = \{\rho_{\theta}^{(n)} : \theta \in \Theta \subset \mathbb{R}^d\}$ is q-LAN at $\theta_0 \in \Theta$ and

$$\begin{pmatrix} X^{(n)} \\ \Delta^{(n)} \end{pmatrix} \xrightarrow[\sim]{\rho_{\theta_0}^{(n)}} N \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma & \tau \\ \tau^* & J \end{pmatrix} \right)$$

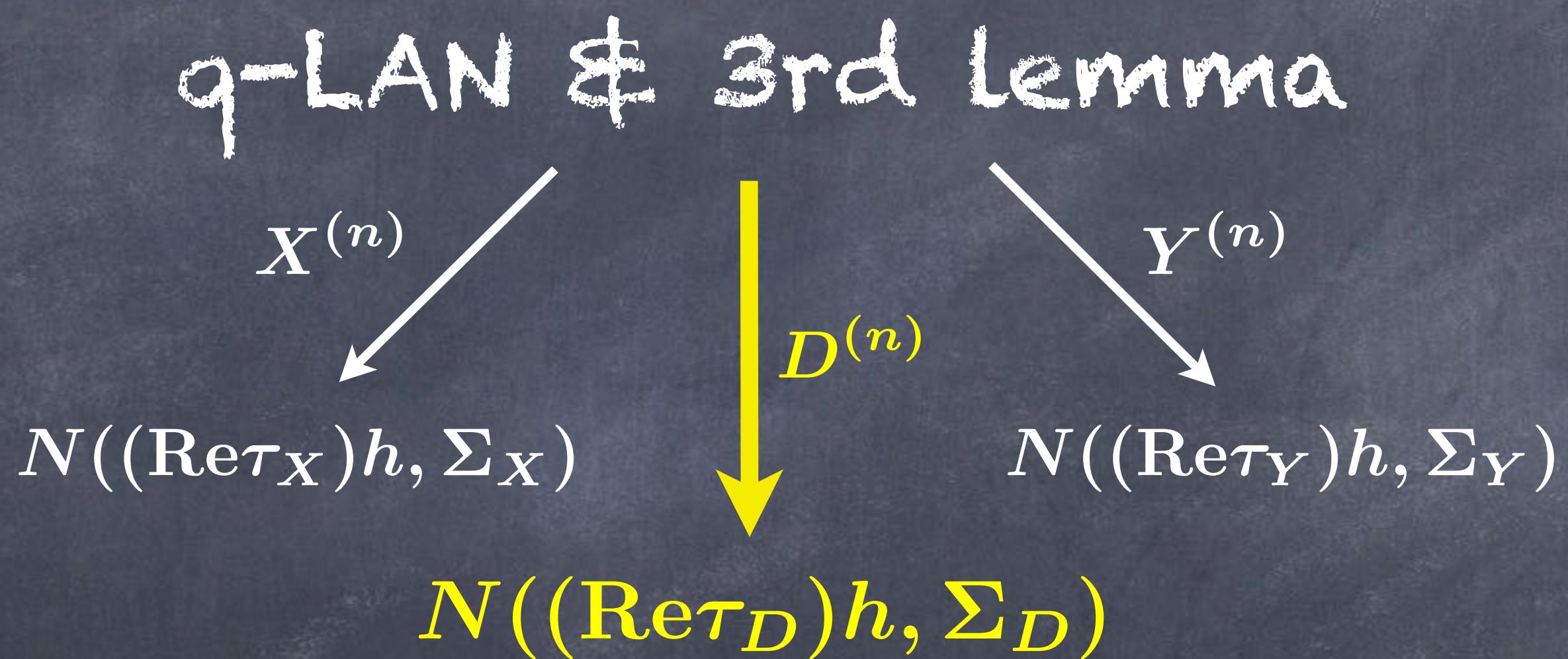
Then ($\rho_{\theta_0+h/\sqrt{n}}^{(n)} \triangleleft \rho_{\theta_0}^{(n)}$ and)

$$X^{(n)} \xrightarrow[\sim]{\rho_{\theta_0+h/\sqrt{n}}^{(n)}} N((\text{Re } \tau)h, \Sigma)$$

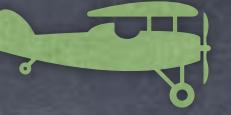
The moral:
q-LAN model is
locally
asymptotically
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Gaussian shift
model



Embedding $\rho_\theta^{\otimes n}$ into Gaussian shift models



Choose $D^{(n)}$ suitably so that the Holevo bound of the resulting q-Gaussian is identical to that of ρ_θ



Theorem

For any quantum statistical model that fulfills some mild regularity conditions, the Holevo bound is asymptotically achievable for all h .

Summary

- quantum Lebesgue decomposition
- quantum contiguity and quantum Le Cam third lemma
- applications of quantum local asymptotic normality
 - achievability of Holevo bound

Question

- Is there a sequence of estimators that breaks the Holevo bound? (Issues of superefficiency)
- Beyond i.i.d.?
- These questions are resolved in Part II by establishing an "asymptotic quantum representation theorem"

"Thank you for your attention"



- Akio Fujiwara