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A theory of quantum local asymptotic normality
Part I: Quantum Contiguity

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Joint Work with Koichi Yamagata

[Bernoulli, 26 (2020) 2105; Annals of Statistics, 51 (2023) 1159]

Plan of talk

- Chap. I (Allegro ma non troppo): Review of classical LAN
- Chap. II (Moderato assai): Quantum likelihood ratio
- Chap. III (Allegro con moto): Quantum contiguity
- Chap. IV (Prestissimo):
  Preliminary application to quantum LAN
- → to be continued in Part II by Koichi Yamagata

Chap. I: Classical LAN

Local Asymptotic Normality

A sequence of models $\{P_\theta^{(n)} : \theta \in \Theta \subset \mathbb{R}^d\}$

is called LAN at $\theta_0 \in \Theta$ if

$$\log \frac{dP^{(n)}_{\theta_0 + h/\sqrt{n}}}{dP^{(n)}_{\theta_0}} = h^i \Delta_i^{(n)} - \frac{1}{2} h^i h^j J_{ij} + o_{P_{\theta_0}}(1)$$

where

$$\boldsymbol{\Delta}^{(n)} \overset{P_{\theta_0}^{(n)}}{\rightsquigarrow} N(\mathbf{0}, J)$$

Prototype of LAN: $p_\theta^{(n)} = p_\theta^{\otimes n}$

$$\log \frac{p^{\otimes n}_{\theta_0 + h/\sqrt{n}}}{p^{\otimes n}_{\theta_0}}(X_1, \ldots, X_n)$$

$$= h^i \underbrace{\left\{ \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \partial_i \log p_{\theta_0}(X_k) \right\}}_{\Delta_i^{(n)}(X_1,\ldots,X_n)} - \frac{1}{2} h^i h^j \underbrace{\left\{ -\frac{1}{n} \sum_{k=1}^{n} \partial_i \partial_j \log p_{\theta_0}(X_k) \right\}}_{J_{ij} + o_{p_{\theta_0}}(1)} + o(\frac{1}{n})$$

where

$$\Delta^{(n)} \overset{p^{\otimes n}_{\theta_0}}{\rightsquigarrow} N(0, J)$$

with J being the Fisher information matrix

Similarity to Gaussian Shift model

LAN

$$\log \frac{dP^{(n)}_{\theta_0 + h/\sqrt{n}}}{dP^{(n)}_{\theta_0}} = h^i \Delta^{(n)}_i - \frac{1}{2} h^i h^j J_{ij} + o_{P_{\theta_0}}(1)$$

Gaussian shift model

$$\log \frac{dN(Jh, J)}{dN(0, J)}(X^1, \ldots, X^d) = h^i X_i - \frac{1}{2} h^i h^j J_{ij}$$

Contiguity

A sequence $Q^{(n)}$ of probability measures is called contiguous to another sequence $P^{(n)}$ of probability measures, denoted $Q^{(n)} \triangleleft P^{(n)}$, if

$$P^{(n)}(A^{(n)}) \to 0 \implies Q^{(n)}(A^{(n)}) \to 0$$

Le Cam's third lemma

Weak convergence analogue of the Radon-Nikodym theorem: $Q \ll P \implies dQ = \frac{dQ}{dP} dP$

<u>Theorem (Le Cam)</u>

If $Q^{(n)} \lhd P^{(n)}$ and $\left(X^{(n)}, \frac{dQ^{(n)}}{dP^{(n)}} \right) \overset{P^{(n)}}{\rightsquigarrow} (X, V)$, then

$$X^{(n)} \overset{Q^{(n)}}{\rightsquigarrow} L \qquad (L(B) := E[1_B(X)V])$$

Le Cam's third lemma

(Gaussian version)

Theorem

If

$$\begin{pmatrix} X^{(n)} \\ \log \frac{dQ^{(n)}}{dP^{(n)}} \end{pmatrix} \overset{P^{(n)}}{\rightsquigarrow} N\left(\begin{pmatrix} 0 \\ -\frac{1}{2}\sigma^2 \end{pmatrix}, \begin{pmatrix} \Sigma & \tau \\ \tau^\top & \sigma^2 \end{pmatrix} \right)$$

then ($Q^{(n)} \lhd P^{(n)}$ and)

$$X^{(n)} \overset{Q^{(n)}}{\rightsquigarrow} N(\tau, \Sigma)$$

Third lemma under LAN

Suppose

$$\log \frac{dP^{(n)}_{\theta_0+h/\sqrt{n}}}{dP^{(n)}_{\theta_0}} = h^i \Delta^{(n)}_i - \frac{1}{2} h^i h^j J_{ij} + o_{p_{\theta_0}}(1)$$

and

$$\begin{pmatrix} X^{(n)} \\ \Delta^{(n)} \end{pmatrix} \overset{P^{(n)}_{\theta_0}}{\rightsquigarrow} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma & \tau \\ \tau^\top & J \end{pmatrix} \right)$$

Then

$$X^{(n)} \overset{P^{(n)}_{\theta_0+h/\sqrt{n}}}{\rightsquigarrow} N(\tau h, \Sigma)$$

The moral:
LAN model is locally asymptotically similar to Gaussian shift model

History of quantum LAN

- Guta and Kahn's strong q-LAN (2006, 2009)

$$\lim_{n\to\infty} \sup_{h\in K^{(n)}} \left\| \sigma_h - \Gamma^{(n)}(\rho^{\otimes n}_{\theta_0+h/\sqrt{n}}) \right\|_1 = 0$$

and

$$\lim_{n\to\infty} \sup_{h\in K^{(n)}} \left\| \Lambda^{(n)}(\sigma_h) - \rho^{\otimes n}_{\theta_0+h/\sqrt{n}} \right\|_1 = 0$$

Drawbacks

- iid
- nondegeracy

- Guta and Jencova's weak q-LAN (2007)

Quantum Radon-Nikodym?

Difficulties in extending LAN to the quantum domain

LAN: $$\log \frac{dP^{(n)}_{\theta_0+h/\sqrt{n}}}{dP^{(n)}_{\theta_0}} = h^i \Delta^{(n)}_i - \frac{1}{2} h^i h^j J_{ij} + o_{P_{\theta_0}}(1)$$

$$\left(\Delta^{(n)} \overset{\theta_0}{\rightsquigarrow} N(0, J) \right)$$

What are the quantum counterparts of

1) Radon-Nikodym density?
2) contiguity and third lemma?
3) weak convergence?

Chap. II: Quantum likelihood ratio

Singularity & absolute continuity

1) ρ is singular to σ, denoted $\rho \perp \sigma$, if

$$\operatorname{supp} \rho \perp \operatorname{supp} \sigma$$

2) ρ is absolutely continuous to σ, denoted $\rho \ll \sigma$, if there is a positive operator $R \,(\geq 0)$ that satisfies $\rho = R \sigma R$

Remark

For pure states $\rho = |\xi\rangle\langle\xi|$ and $\sigma = |\eta\rangle\langle\eta|$

1) $\operatorname{supp}\rho \subset \operatorname{supp}\sigma \iff \rho = \sigma$

2) $\rho \ll \sigma \iff \langle\xi|\eta\rangle \neq 0$

Quantum Lebesgue decomposition

<u>Theorem.</u> Given quantum states $\rho, \sigma \in \mathcal{S}(\mathcal{H})$, the decomposition

$$\sigma = \underbrace{R\rho R}_{\sigma^{ac}} + \underbrace{\tau}_{\sigma^{\perp}} \qquad (R \geq 0,\ \tau \geq 0,\ \tau \perp \rho)$$

uniquely exists.

Quantum likelihood ratio

A positive operator R that satisfies

$$\sigma = R\rho R + \tau \qquad (R \geq 0,\ \tau \geq 0,\ \tau \perp \rho)$$

is called a square-root likelihood ratio,

and is denoted as

$$\mathcal{R}(\sigma|\rho)$$

Chap. III: Quantum contiguity

Classical contiguity

Recall that $Q^{(n)} \triangleleft P^{(n)}$ if

$$P^{(n)}(A^{(n)}) \to 0 \implies Q^{(n)}(A^{(n)}) \to 0$$

This is equivalent to saying that

i) $$\lim_{n\to\infty} E_{P^{(n)}}\left[\frac{dQ^{(n)}}{dP^{(n)}}\right] = 1$$

ii) $\dfrac{dQ^{(n)}}{dP^{(n)}}$ is uniformly integrable under $P^{(n)}$

Quantum contiguity

$\sigma^{(n)} \lhd \rho^{(n)}$ if

i) $\lim_{n\to\infty} \mathrm{Tr}\,\rho^{(n)} \mathcal{R}(\sigma^{(n)}|\rho^{(n)})^2 = 1$

ii) $(\overline{R}^{(n)})^2$ is uniformly integrable under $\rho^{(n)}$

where $\overline{R}^{(n)} := \mathcal{R}(\sigma^{(n)}|\rho^{(n)}) + O^{(n)} \geq 0$ with

$O^{(n)} = o_{L^2}(\rho^{(n)})$ being an L^2-infinitesimal term

We also denote as $\sigma^{(n)} \lhd_{O^{(n)}} \rho^{(n)}$

Properties

1) Suppose $\rho^{(n)}, \sigma^{(n)} \in \mathcal{S}(\mathcal{H})$ have limiting states

$$\lim_{n\to\infty} \rho^{(n)} = \rho^{(\infty)} \quad \text{and} \quad \lim_{n\to\infty} \sigma^{(n)} = \sigma^{(\infty)}$$

Then $\sigma^{(n)} \lhd \rho^{(n)} \iff \sigma^{(\infty)} \ll \rho^{(\infty)}$

2) Suppose $\rho_i, \sigma_i \in \mathcal{S}(\mathcal{H}_i)$ satisfy $\sigma_i \ll \rho_i$. Let

$$\rho^{(n)} := \bigotimes_{i=1}^{n} \rho_i \quad \text{and} \quad \sigma^{(n)} := \bigotimes_{i=1}^{n} \sigma_i$$

Then $\sigma^{(n)} \lhd \rho^{(n)} \iff \sum_{i=1}^{\infty} \left(1 - \mathrm{Tr}\sqrt{\sqrt{\sigma_i}\rho_i\sqrt{\sigma_i}}\right) < \infty$

Quantum weak convergence

$\rho^{(n)}$: state, $X^{(n)} = \left(X_1^{(n)}, \ldots, X_d^{(n)}\right)$ obs. on $\mathcal{H}^{(n)}$

ϕ : state, $X^{(\infty)} = \left(X_1^{(\infty)}, \ldots, X_d^{(\infty)}\right)$ obs. on $\mathcal{H}^{(\infty)}$

We say $(X^{(n)}, \rho^{(n)}) \rightsquigarrow \left(X^{(\infty)}, \phi\right)$ if

$$\lim_{n\to\infty} \mathrm{Tr}\,\rho^{(n)} \left(\prod_{t=1}^{r} e^{\sqrt{-1}\xi_t^i X_i^{(n)}}\right) = \phi\left(\prod_{t=1}^{r} e^{\sqrt{-1}\xi_t^i X_i^{(\infty)}}\right)$$

When $\left(X^{(\infty)}, \phi\right) \sim N(h, J)$, we denote $X^{(n)} \overset{\rho^{(n)}}{\rightsquigarrow} N(h, J)$

Quantum Gaussian state

A state ϕ on a CCR(S) with

$$e^{\sqrt{-1}\xi^i X_i} e^{\sqrt{-1}\eta^j X_j} = e^{\sqrt{-1}\xi^i \eta^j S_{ij}} e^{\sqrt{-1}(\xi+\eta)^i X_i}$$

is called a quantum Gaussian state if

$$\phi(e^{\sqrt{-1}\xi^i X_i}) = e^{\sqrt{-1}\xi^i h_i - \frac{1}{2}\xi^i \xi^j V_{ij}}$$

where

$$J := V + \sqrt{-1}S \geq 0$$

We denote it as $\phi \sim N(h, J)$

A variant: "Sandwiched" weak convergence

If

$$\lim_{n\to\infty} \operatorname{Tr} \rho^{(n)} e^{\sqrt{-1}\eta_1 \boldsymbol{Y}^{(n)}} \left\{ \prod_{t=1}^{r} e^{\sqrt{-1}\xi_t^i \boldsymbol{X}_i^{(n)}} \right\} e^{\sqrt{-1}\eta_2 \boldsymbol{Y}^{(n)}}$$

$$= \phi\left(e^{\sqrt{-1}\eta_1 \boldsymbol{Y}^{(\infty)}} \left\{ \prod_{t=1}^{r} e^{\sqrt{-1}\xi_t^i \boldsymbol{X}_i^{(\infty)}} \right\} e^{\sqrt{-1}\eta_2 \boldsymbol{Y}^{(\infty)}} \right)$$

we denote

$$\left(\left\langle \boldsymbol{Y}^{(n)}, \boldsymbol{X}^{(n)}, \boldsymbol{Y}^{(n)} \right\rangle, \rho^{(n)} \right) \rightsquigarrow \left(\left\langle \boldsymbol{Y}^{(\infty)}, \boldsymbol{X}^{(\infty)}, \boldsymbol{Y}^{(\infty)} \right\rangle, \phi \right)$$

or

$$\left\langle \boldsymbol{Y}^{(n)}, \boldsymbol{X}^{(n)}, \boldsymbol{Y}^{(n)} \right\rangle_{\rho^{(n)}} \rightsquigarrow \left\langle \boldsymbol{Y}^{(\infty)}, \boldsymbol{X}^{(\infty)}, \boldsymbol{Y}^{(\infty)} \right\rangle_{\phi}$$

Quantum Le Cam third lemma

If $\sigma^{(n)} \lhd_{O^{(n)}} \rho^{(n)}$ and $R^{(n)} = \mathcal{R}(\sigma^{(n)}|\rho^{(n)})$ enjoys

$$\left\langle R^{(n)} + O^{(n)}, X^{(n)}, R^{(n)} + O^{(n)} \right\rangle_{\rho^{(n)}} \rightsquigarrow \left\langle R^{(\infty)}, X^{(\infty)}, R^{(\infty)} \right\rangle_{\phi}$$

Then

$$\left(X^{(n)}, \sigma^{(n)} \right) \rightsquigarrow \left(X^{(\infty)}, \psi \right)$$

where

$$\psi(A) := \phi\left(R^{(\infty)} A R^{(\infty)} \right)$$

This gives a complete characterization of alternative state ψ in terms of reference state Φ and limit likelihood ratio

Quantum Le Cam third lemma

(q-Gaussian version)

If

$$\begin{pmatrix} X^{(n)} \\ 2\log(R^{(n)} + O^{(n)}) + \tilde{O}^{(n)} \end{pmatrix} \overset{\rho^{(n)}}{\rightsquigarrow} N\left(\begin{pmatrix} \mu \\ -\frac{1}{2}s^2 \end{pmatrix}, \begin{pmatrix} \Sigma & \kappa \\ \kappa^* & s^2 \end{pmatrix} \right)$$

Then ($\sigma^{(n)} \triangleleft_{O^{(n)}} \rho^{(n)}$ and)

$$X^{(n)} \overset{\sigma^{(n)}}{\rightsquigarrow} N(\mu + \mathrm{Re}(\kappa), \Sigma)$$

Chap. IV: Quantum LAN

Quantum LAN

$\mathcal{S}^{(n)} = \{\rho_\theta^{(n)} : \theta \in \Theta \subset \mathbb{R}^d\}$ is called q-LAN at $\theta_0 \in \Theta$ if

$R_h^{(n)} := \mathcal{R}\left(\rho^{(n)}_{\theta_0 + h/\sqrt{n}} \middle| \rho^{(n)}_{\theta_0}\right)$ is expanded in h as

$$\log\left(R_h^{(n)} + o_{L^2}(\rho^{(n)}_{\theta_0})\right)^2 = h^i \Delta_i^{(n)} - \frac{1}{2}(J_{ij} h^i h^j)\, I^{(n)} + o_D(h^i \Delta_i^{(n)}, \rho^{(n)}_{\theta_0})$$

where $\Delta^{(n)} \overset{\rho^{(n)}_{\theta_0}}{\rightsquigarrow} N(0, J)$

Third lemma under q-LAN

Suppose $\mathcal{S}^{(n)} = \{\rho_\theta^{(n)} : \theta \in \Theta \subset \mathbb{R}^d\}$ is q-LAN at $\theta_0 \in \Theta$ and

$$\begin{pmatrix} X^{(n)} \\ \Delta^{(n)} \end{pmatrix} \overset{\rho_{\theta_0}^{(n)}}{\rightsquigarrow} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma & \tau \\ \tau^* & J \end{pmatrix}\right)$$

Then ($\rho_{\theta_0+h/\sqrt{n}}^{(n)} \triangleleft \rho_{\theta_0}^{(n)}$ and)

$$X^{(n)} \overset{\rho_{\theta_0+h/\sqrt{n}}^{(n)}}{\rightsquigarrow} N((\operatorname{Re}\tau)h, \Sigma)$$

The moral:
q-LAN model is locally asymptotically similar to q-Gaussian shift model

Embedding $\rho_\theta^{\otimes n}$ into Gaussian shift models

q-LAN & 3rd lemma

$X^{(n)}$ ↙ $\quad$ ↓ $D^{(n)}$ $\quad$ ↘ $Y^{(n)}$

$N((\mathrm{Re}\tau_X)h, \Sigma_X)$ $\qquad$ $N((\mathrm{Re}\tau_Y)h, \Sigma_Y)$

$N((\mathrm{Re}\tau_D)h, \Sigma_D)$

Choose $D^{(n)}$ suitably so that the Holevo bound of the resulting q-Gaussian is identical to that of ρ_θ

Theorem

For any quantum statistical model that fulfills some mild regularity conditions, the Holevo bound is asymptotically achievable for all h.

Summary

- quantum Lebesgue decomposition
- quantum contiguity and quantum Le Cam third lemma
- applications of quantum local asymptotic normality
  - → achievability of Holevo bound

Question

- Is there a sequence of estimators that breaks the Holevo bound? (Issues of superefficiency)
- Beyond i.i.d.?
- These questions are resolved in Part II by establishing an "asymptotic quantum representation theorem"

"Thank you for your attention"

- Akio Fujiwara