Macroscopic states and operations: a review of recent results

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Abstract

To understand the emergence of macroscopic irreversibility from microscopic reversible dynamics, the idea of coarse-graining plays a fundamental role. In this work, we focus on the concept of macroscopic states, i.e. coarse representations of microscopic details, defined as states that can be inferred solely from the outcomes of macroscopic measurements. Building on the theories of quantum statistical sufficiency and quantum Bayesian retrodiction, we characterize macroscopic states through several equivalent formulations, ranging from algebraic to explicitly constructive. We introduce a hierarchy of macroscopicity-non-decreasing operations and develop a resource theory of microscopicity that unifies and generalizes existing resource theories of coherence, athermality, purity, and asymmetry. Finally, we introduce the concept of inferential reference frames and reinterpret macroscopic entropy as a measure of inferential asymmetry, i.e., irretrodictability.

Macroscopic states and operations: a generalized resource theory of coherence

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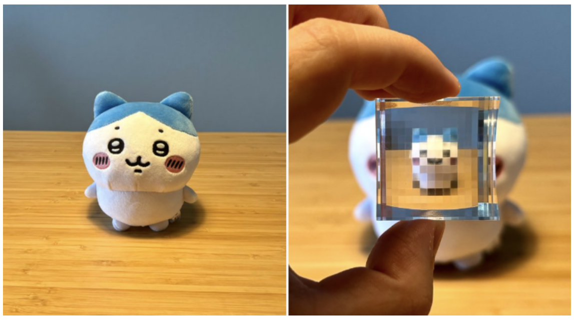

collaborators on this journey

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a growing list:
The Observational Entropy Appreciation Club
(www.observationalentropy.com)

von Neumann entropy

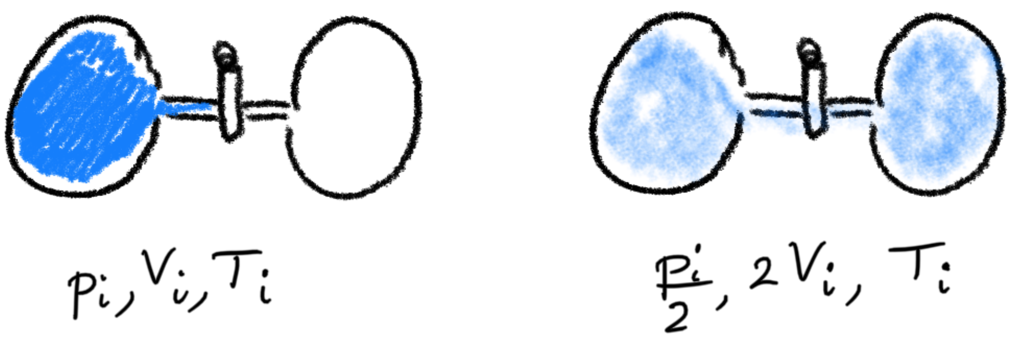
For $\varrho = \sum_{x=1}^{d} \lambda_x |\varphi_x\rangle\langle\varphi_x|$ d-dimensional density matrix ($\lambda_x \geqslant 0$, $\sum_x \lambda_x = 1$),

$$\boxed{S(\varrho) := -\operatorname{Tr}[\varrho \log \varrho] = -\sum_{x=1}^{d} \lambda_x \log \lambda_x}$$

with the convention $0 \log 0 := 0$.

Fact: von Neumann entropy is invariant under unitary evolution.

the problem: free expansion of an ideal gas



$$\boxed{\Delta S(\text{universe}) = nR \log 2 > 0}$$

$\Longrightarrow$ there is net entropy increase in an isolated system's evolution, but von Neumann entropy remains constant

von Neumann's insight (inspired by Szilard's)

> *"For a classical observer, who knows all coordinates and momenta, the entropy is constant. [...]*
>
> *The time variations of the entropy are then based on the fact that the observer does not know everything—that he cannot find out (measure) everything which is measurable in principle."* von Neumann, 1932 (transl. 1955)

von Neumann's proposal: macroscopic entropy

von Neumann recognizes that thermodynamic entropy should be a quantity *relative* to the observer's knowledge

Modern version: observational entropy (OE)

For a density matrix ϱ and a positive operator-valued measure (POVM) $\mathbf{P} = \{P_i\}_i$

$$S_{\mathbf{P}}(\varrho) := -\sum_i p(i) \log \frac{p(i)}{V(i)} ,$$

where $p(i) := \mathrm{Tr}[\varrho\, P_i]$ and $V(i) := \mathrm{Tr}[P_i]$.

What is the meaning of OE?

the fundamental bound

Umegaki relative entropy

For density matrices $\varrho \geqslant 0$ and $\gamma > 0$, the Umegaki relative entropy $D(\varrho\|\gamma)$ is defined as $\mathrm{Tr}[\varrho(\log\varrho - \log\gamma)]$. We can thus write

$$S_{\mathbf{P}}(\varrho) = \log d - D(\mathcal{P}(\varrho)\|\mathcal{P}(u)) \ ,$$

where $\mathcal{P}(\bullet) := \sum_i \mathrm{Tr}[\bullet\ P_i]\,|i\rangle\langle i|$, and $u := d^{-1}\mathbb{1}$.

Theorem (NJP, 2023)

For any d-dimensional density matrix ϱ and any POVM $\mathbf{P} = \{P_i\}_i$,

$$S(\tilde{\varrho}_{\mathbf{P}}) - S(\varrho) \geqslant \boxed{S_{\mathbf{P}}(\varrho) - S(\varrho) \equiv D(\varrho\|u) - D(\mathcal{P}(\varrho)\|\mathcal{P}(u))} \geqslant D(\varrho\|\tilde{\varrho}_{\mathbf{P}}) \ ,$$

where $\tilde{\varrho}_{\mathbf{P}} := \sum_i \mathrm{Tr}[\varrho\ P_i]\,\frac{P_i}{V_i}$.

In particular, $\log d \geqslant S(\tilde{\varrho}_{\mathbf{P}}) \geqslant S_{\mathbf{P}}(\varrho) \geqslant S(\varrho)$.

OE tells us something about how much ϱ and $\tilde{\varrho}_{\mathbf{P}}$ "differ" from each other.

Hence, the question: what is the meaning of $\tilde{\varrho}_{\mathbf{P}}$?

Petz's transpose/recovery map

Definition

Given a channel $\mathcal{E}$ and a prior state γ, the corresponding *transpose* or *recovery channel* is defined as

$$\mathcal{R}^{\gamma}_{\mathcal{E}}(\bullet) := \sqrt{\gamma}\ \mathcal{E}^{\dagger}\Big[\ \mathcal{E}(\gamma)^{-1/2}\ (\bullet)\ \mathcal{E}(\gamma)^{-1/2}\ \Big]\sqrt{\gamma}\ .$$

Fact: $\tilde{\varrho}_{\mathbf{P}}$ is the "recovered" state

In terms of the measurement channel $\mathcal{P}(\bullet) := \sum_i \mathrm{Tr}[P_i\ \bullet]\ |i\rangle\langle i|$, it turns out that

$$\tilde{\varrho}_{\mathbf{P}} = [\mathcal{R}^{u}_{\mathcal{P}} \circ \mathcal{P}](\varrho) = \frac{1}{d}\ \mathcal{P}^{\dagger}\Big[\ \mathcal{P}(u)^{-1/2}\ \mathcal{P}(\varrho)\ \mathcal{P}(u)^{-1/2}\ \Big]\ .$$

(Note that in this case $\gamma = u = d^{-1}\mathbb{1}$.)

So, the real question is: what is the meaning of Petz's transpose map?

statistical sufficiency and exact recovery

Petz (1986, 1988)

Given two density matrices $\varrho \geqslant 0$ and $\gamma > 0$ and a channel $\mathcal{E}$, while in general we have $D(\varrho\|\gamma) \geqslant D(\mathcal{E}(\varrho)\|\mathcal{E}(\gamma))$, the equality

$$D(\varrho\|\gamma) = D(\mathcal{E}(\varrho)\|\mathcal{E}(\gamma))$$

holds if and only if $[\mathcal{R}^{\gamma}_{\mathcal{E}} \circ \mathcal{E}](\varrho) = \varrho$.

Note that the other equality $[\mathcal{R}^{\gamma}_{\mathcal{E}} \circ \mathcal{E}](\gamma) = \gamma$ is always satisfied *by construction*.

Question: does Petz's transpose map have a clear operational interpretation also when $D(\varrho\|\gamma) > D(\mathcal{E}(\varrho)\|\mathcal{E}(\gamma))$?

classically, it does: Bayesian update

- consider a classical discrete noisy channel $P(i|x)$ and a prior $\gamma(x)$ on the input
- when the receiver reads a definite value i_0, (vanilla) Bayes' rule says that their posterior should be updated to $R^\gamma_P(x|i_0) := \dfrac{\gamma(x)P(i_0|x)}{[P\gamma](i_0)}$
- but what if the observation is noisy and returns some p.d. $\sigma(i)$ instead?

Theorem (Bayes–Jeffrey–Pearl update)

Given a channel $P(i|x)$ and a prior $\gamma(x)$, the result of a noisy observation $\sigma(i)$ is updated to

$$\widetilde{\sigma}(x) := \sum_i \boxed{R^\gamma_P(x|i)}\, \sigma(i)\ .$$

Note that the usual Bayes' rule is recovered for $\sigma(i) = \delta_{i,i_0}$.

It is easy to see that, when everything commutes, Petz's transpose map coincides with Bayes's update rule.

But is this just a coincidence, or is there something deeper?

the principle of minimum change (idea, classical case)

"To avoid unwarranted bias and remain maximally non-committal, the updated belief should be consistent with the new information (the result of the observation), while deviating as little as possible from the initial belief."

How Bayes' rule has been derived from this principle:

- given channel and prior, construct the forward process $F_P^\gamma(x,i) = \gamma(x)P(i|x)$; note that $\sum_i F_P^\gamma(x,i) = \gamma(x)$
- given the new data as $\sigma(i)$, consider the optimization

$$\min_R \mathbb{D}(F_P^\gamma, R) \ ,$$

  where $\mathbb{D}(\bullet,\bullet)$ is an information divergence, and the minimum is taken over all joint probability distributions $R \equiv R(x,i)$ such that $\sum_x R(x,i) = \sigma(i)$
- for many choices of $\mathbb{D}$, it turns out that $\boxed{\arg\min \mathbb{D}(F_P^\gamma, R) = R_P^\gamma(x|i)\sigma(i)}$

the principle of minimum change (formal, quantum)

Theorem (arXiv:2410.00319)

Given a qc-channel $\mathcal{P}(\bullet_{\text{in}}) = \sum_i \mathrm{Tr}[P_i \bullet] \, |i\rangle\langle i|_{\text{out}}$ *and a prior state* $\gamma_{\text{in}} > 0$ *such that* $\mathcal{P}(\gamma) > 0$, *let* $F_\mathcal{P}^\gamma := \sum_i |i\rangle\langle i|_{\text{out}} \otimes \left(\sqrt{\gamma^T} P_i^T \sqrt{\gamma^T}\right)_{\text{in}}$, *with* $\mathrm{Tr}_{\text{in}}[F_\mathcal{P}^\gamma] = \mathcal{P}(\gamma)$ *and* $\mathrm{Tr}_{\text{out}}[F_\mathcal{P}^\gamma] = \gamma^T$, *represent the "quantum forward process".*

Then, given any observation result $\sigma(i)$, *represented as* $\sigma_{\text{out}} = \sum_i \sigma(i)|i\rangle\langle i|_{\text{out}}$, *the optimization problem* $\max_{R\geqslant 0, \mathrm{Tr}_{\text{in}}[R]=\sigma_{\text{out}}} \mathtt{Fidelity}(F_\mathcal{P}^\gamma, R)$ *has a unique solution* R_o, *i.e., the optimal "reverse process", which satisfies* $\mathrm{Tr}_{\text{out}}[R_o] = [\mathcal{R}_\mathcal{P}^\gamma(\sigma_{\text{out}})]^T$.

In words: in the case of quantum measurements, Petz's map is **the quantum analog** of Bayes' rule.

We saw that $\tilde{\varrho}_{\mathsf{P}}$ is nothing but the state retrodicted by the observer according to the quantum Bayes rule.

Hence, the bound $S_{\mathsf{P}}(\varrho) - S(\varrho) \geqslant D(\varrho \| \tilde{\varrho}_{\mathsf{P}})$ tells us that, the larger the difference between OE and von Neumann entropy, the less retrodictable ϱ is.

entropy increase = lack of retrodictability (Watanabe's thesis)

"The phenomenological onewayness of temporal developments in physics is due to irretrodictability, and not due to irreversibility." Satosi Watanabe (1965)

macroscopic = retrodictable

Definition (macroscopic states)

Recalling the fundamental bound $S_{\mathbf{P}}(\varrho) - S(\varrho) \geqslant D(\varrho \| \tilde{\varrho}_{\mathbf{P}})$ with $\tilde{\varrho}_{\mathbf{P}} = [\mathcal{R}^u_{\mathcal{P}} \circ \mathcal{P}](\varrho)$, we say that a state ϱ is macroscopic w.r.t. measurement $\mathbf{P}$ and uniform prior u whenever $\varrho = \tilde{\varrho}_{\mathbf{P}}$.

More generally, for non-uniform prior γ, we denote the set of macroscopic states as $\mathfrak{M}^\gamma_{\mathbf{P}} := \{\varrho : \varrho = [\mathcal{R}^\gamma_{\mathcal{P}} \circ \mathcal{P}](\varrho)\}$.

Theorem (arXiv:2504.12738)

A state ϱ is in $\mathfrak{M}^\gamma_{\mathbf{P}}$ if and only if there exists a PVM $\mathbf{\Pi} = \{\Pi_j\}_j$, with $\Pi_j = \sum_i \mu(j|i) P_i$, such that $[\Pi_i, \gamma] = 0$, together with coefficients $c_j \geqslant 0$, such that $\boxed{\varrho = \sum_j c_j \Pi_j \gamma}$.

Note that $\gamma \in \mathfrak{M}^\gamma_{\mathbf{P}}$ by construction.

resolving the problem of entropy increase in closed systems

- let the initial state of the system at time $t = t_0$ be a macrostate $\mathfrak{M}^u_{\mathbf{P}} \ni \varrho^{t_0} \neq u$
- its initial OE satisfies $S_{\mathbf{P}}(\varrho^{t_0}) = S(\varrho^{t_0})$; let's see how it changes in time
- the system evolves unitarily, i.e., $\varrho^{t_0} \mapsto \varrho^{t_1} = U \varrho^{t_0} U^\dagger$, so that $S(\varrho^{t_1}) = S(\varrho^{t_0})$; however,

$$\begin{aligned} S_{\mathbf{P}}(\varrho^{t_1}) &= -\sum_i \mathrm{Tr}\left[P_i \, (U \varrho^{t_0} U^\dagger)\right] \log \frac{\mathrm{Tr}\left[P_i \, (U \varrho^{t_0} U^\dagger)\right]}{\mathrm{Tr}[P_i]} \\ &= -\sum_i \mathrm{Tr}\left[(U^\dagger P_i U) \, \varrho^{t_0}\right] \log \frac{\mathrm{Tr}\left[(U^\dagger P_i U) \, \varrho^{t_0}\right]}{\mathrm{Tr}[U^\dagger P_i U]} \\ &= S_{U^\dagger \mathbf{P} U}(\varrho^{t_0}) \\ &\geqslant S(\varrho^{t_0}) = S_{\mathbf{P}}(\varrho^{t_0}) \end{aligned}$$

- summarizing: in general, $S_{\mathbf{P}}(\varrho^{t_1}) \geqslant S_{\mathbf{P}}(\varrho^{t_0})$ even in closed systems, with equality if and only if $U \varrho^{t_0} U^\dagger \in \mathfrak{M}^u_{\mathbf{P}}$

What does it mean that $U\varrho^{t_0}U^\dagger \in \mathfrak{M}^u_{\mathsf{P}}$?

More generally: which evolutions map macrostates onto macrostates?

macroscopic operations (idea)

Resource destroying map (RDM)

Recalling the form of macroscopic states $\varrho = \sum_j c_j \Pi_j \gamma$, the map

$$\Delta^\gamma_{\mathsf{P}}(\bullet) := \sum_j \mathrm{Tr}[\Pi_j \,\bullet]\, \frac{\Pi_j \gamma}{\mathrm{Tr}[\Pi_j \gamma]}$$

is such that $\Delta^\gamma_{\mathsf{P}}(\sigma) \in \mathfrak{M}^\gamma$ for all σ, while $\varrho \in \mathfrak{M}^\gamma_{\mathsf{P}} \implies \Delta^\gamma_{\mathsf{P}}(\varrho) = \varrho$.

Macroscopic (RDM-covariant) operations

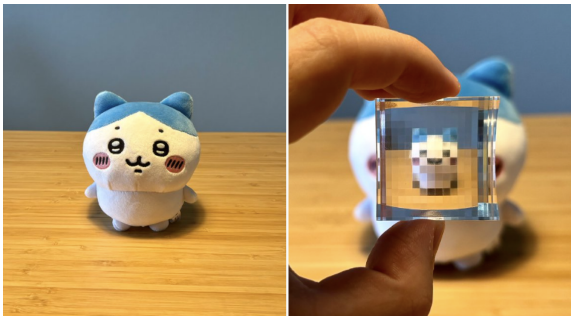
A CPTP linear map $\mathcal{N}$ is macroscopic (in the sense of RDM-covariant) whenever

$$\mathcal{N} \circ \Delta^\gamma_{\mathsf{P}} = \Delta^\gamma_{\mathsf{P}} \circ \mathcal{N}\ .$$

The above framework contains the case of coherence, i.e., $\mathfrak{M}^\gamma_{\mathsf{P}} = \{\text{diagonal states}\}$, or athermality, i.e., $\mathfrak{M}^\gamma_{\mathsf{P}} = \{\gamma\}$.

Conclusions

take-home messages



1. observational entropy quantifies the gap between microscopic states and macroscopic knowledge, and the second law is a statement about the generic loss of retrodictability in time
2. Petz's transpose map is the quantum analog of Bayesian update
3. microscopicity (i.e., "unobservability") can be framed as a resource theory, generalizing those of coherence and athermality

The End: Thank You!

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