Modified phase estimation algorithm and its application

GEnKO Workshop at Nagoya

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2023-12-08

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Contents

1. About Me

Yonghae Lee

Kyung Hee University(2002~2021)

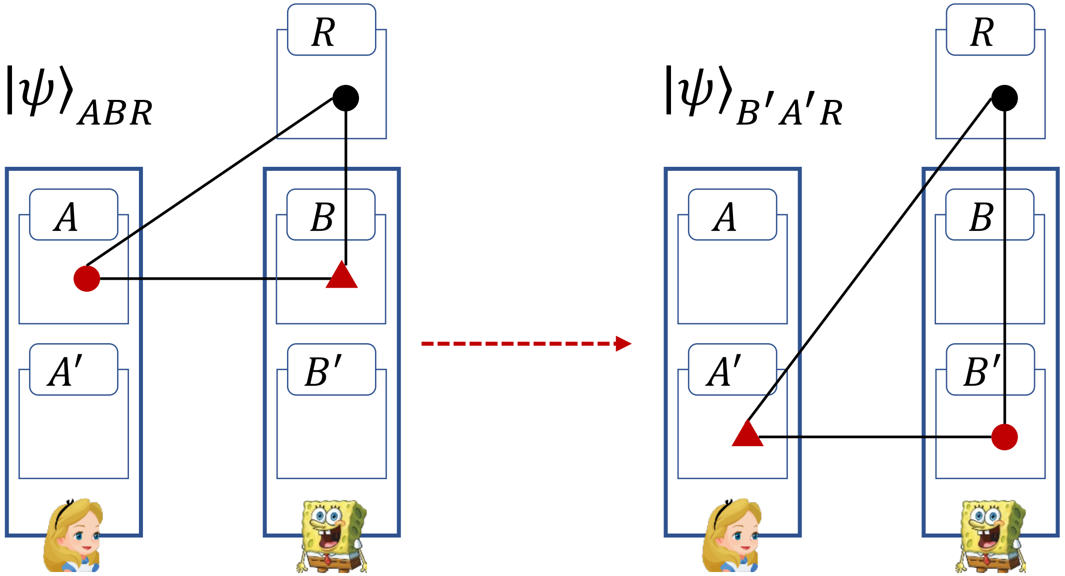
- BSc in Math(2002~2009)
- MSc in Math(2009~2011)
- PhD in Math(2012~2019)
- Research Fellow(2019~2021)

KAIST(2021~2022)

- Research Fellow

Kangwon National University(2022~present)

Research topic



Quantum state exchange

- Optimal entanglement cost for quantum state exchange task

[J. Oppenheim and A. Winter, arXiv:quant-ph/0511082 (2005)]
[Y. Lee, R. Takagi, H. Yamasaki, G. Adesso, and S. Lee, PRL **122**, 010502 (2019)]
[Y. Lee, H. Yamasaki, G. Adesso, and S. Lee, PRA **100**, 042306 (2019)]
[Y. Lee, H. Yamasaki, and S. Lee, PRA **103**, 062613 (2021)]

2. Quantum phase estimation algorithm

Estimation

$$\mathbf{1/\sqrt{2} = 0.10110101000001\cdots}$$

Estimate!

$$1/\sqrt{2} \cong \mathbf{0.1011}$$

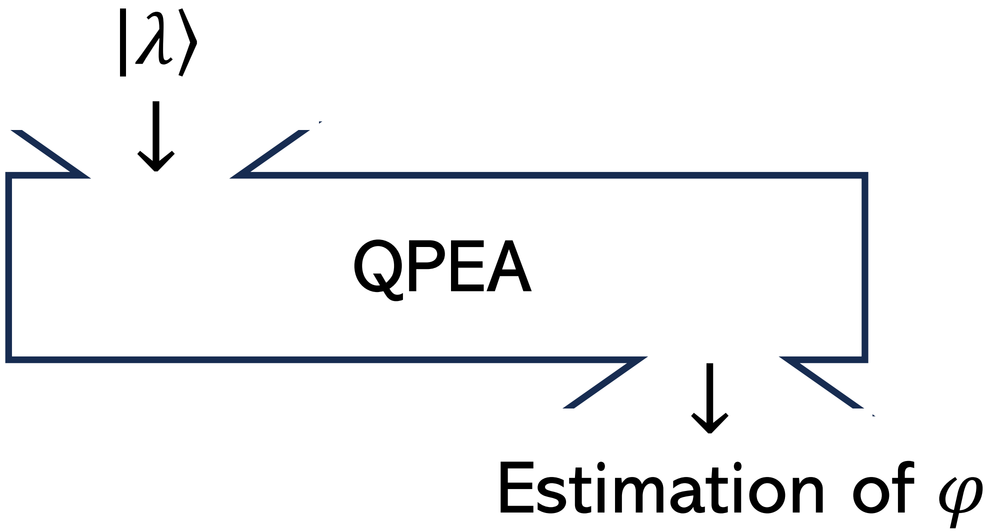
$$\left|\frac{1}{\sqrt{2}} - 0.1011\right| \leq \frac{1}{2^5}$$

Error!

Quantum phase estimation algorithm (QPEA)

$$U|\lambda\rangle = e^{2\pi i\varphi}|\lambda\rangle, \qquad \varphi \in [0,1]$$

QPEA estimates the phase corresponding to an eigenvalue of a given unitary operator.

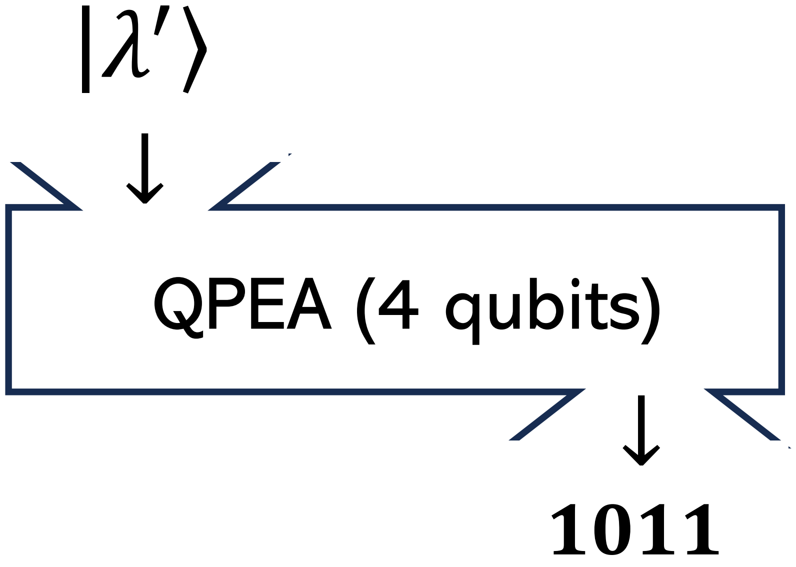


[A. Yu. Kitaev, arXiv:quant-ph/9511026 (1995)]

[Nielsen and Chuang, "Quantum computation and quantum information" (2001)]

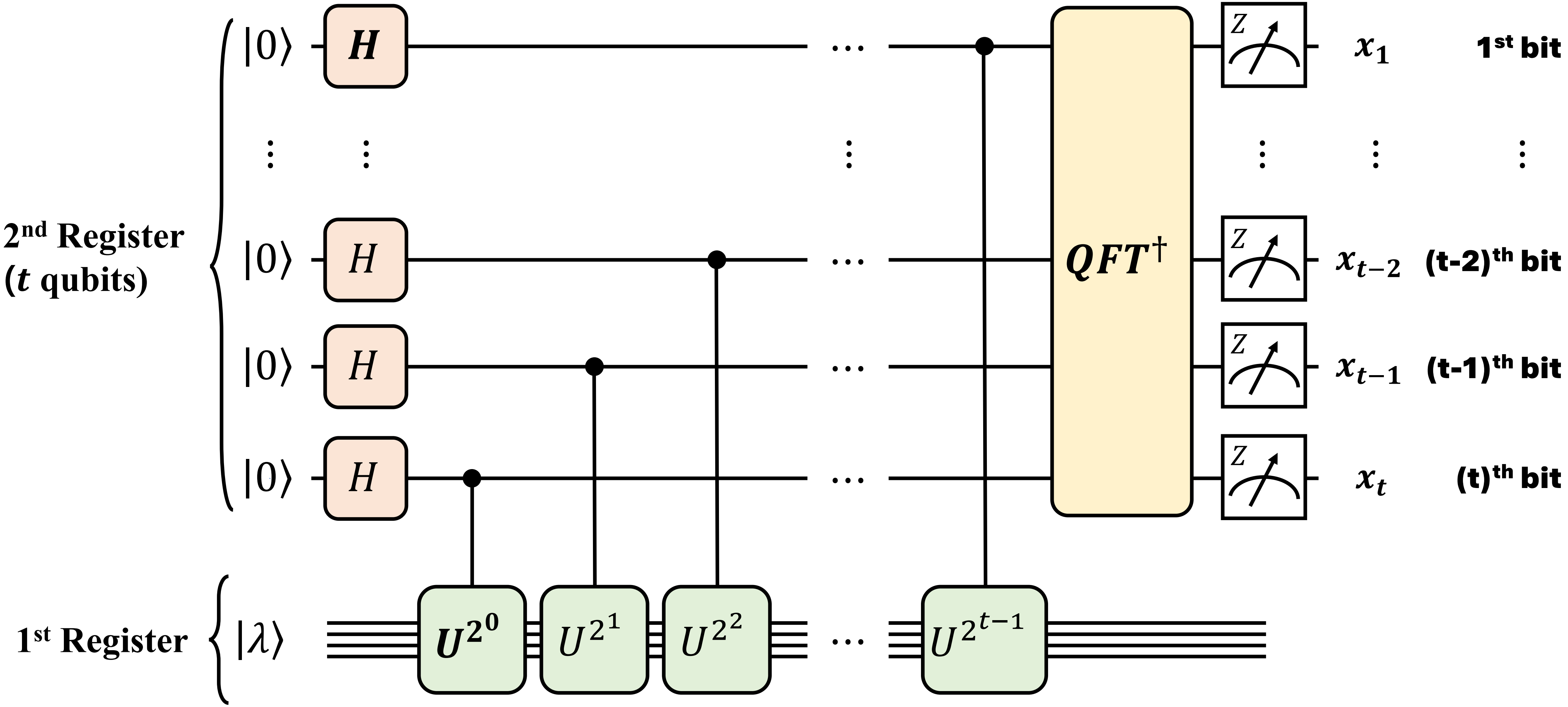
Example: Phase $\frac{1}{\sqrt{2}} = 0.10110\cdots$

$$V|\lambda'\rangle = e^{2\pi i\left(\frac{1}{\sqrt{2}}\right)}|\lambda'\rangle,$$

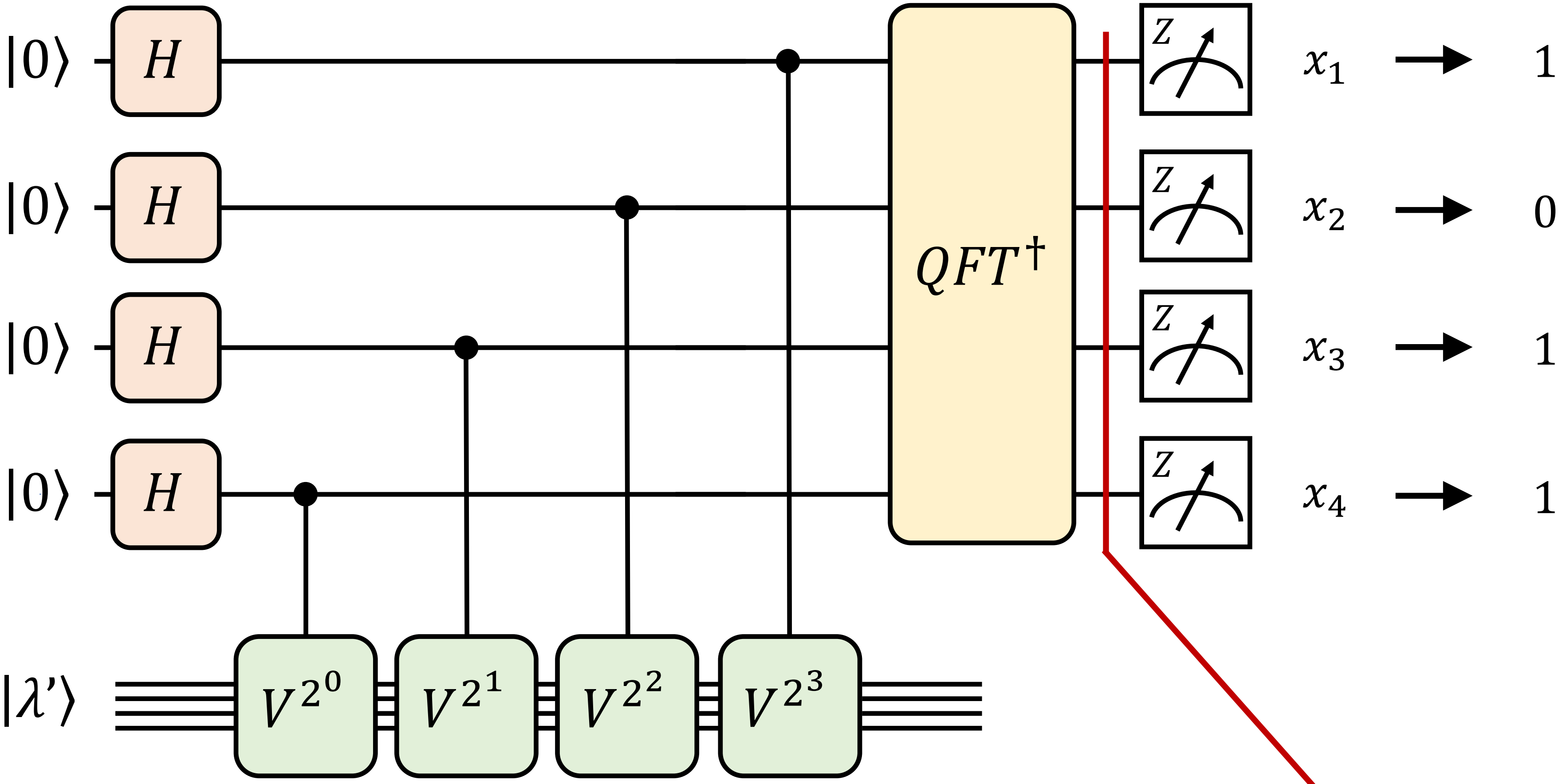


If you use 4 qubits for estimation, QPEA outputs
measurement outcomes 1011

Circuit for QPEA



Example: Phase $\frac{1}{\sqrt{2}}$=0.10110⋯



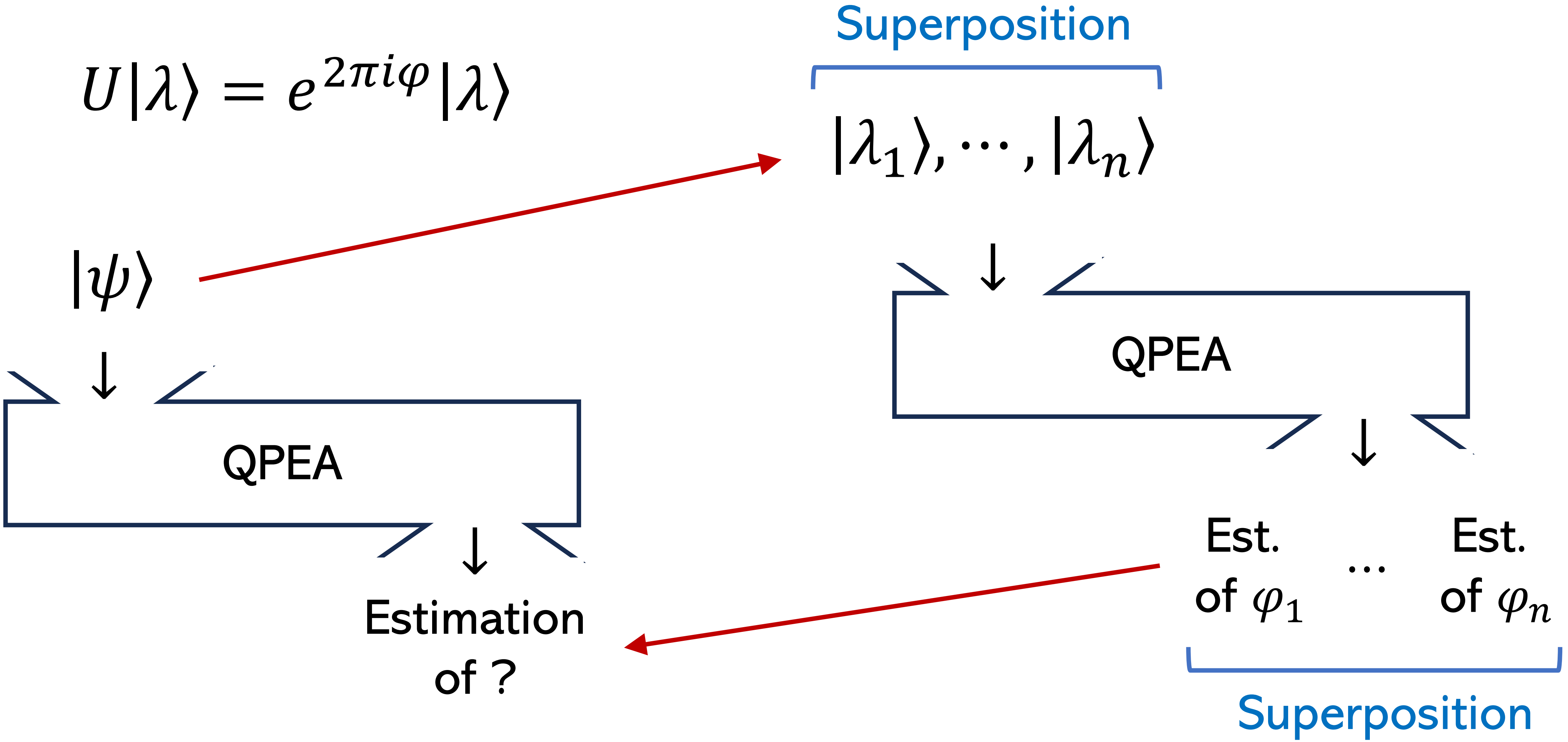
$$\left(\frac{1}{\sqrt{2}} - \frac{x_1 x_2 x_3 x_4}{16}\right)$$

$$V|\lambda'\rangle = e^{2\pi i\left(\frac{1}{\sqrt{2}}\right)}|\lambda'\rangle,$$

$$\frac{1}{16}\sum_{x_1=0}^{1}\sum_{x_2=0}^{1}\sum_{x_3=0}^{1}\sum_{x_4=0}^{1}\sum_{y=0}^{15} e^{2\pi i y\left(\frac{1}{\sqrt{2}} - \frac{x_1 x_2 x_3 x_4}{16}\right)}|x_1\rangle|x_2\rangle|x_3\rangle|x_4\rangle|\lambda'\rangle$$

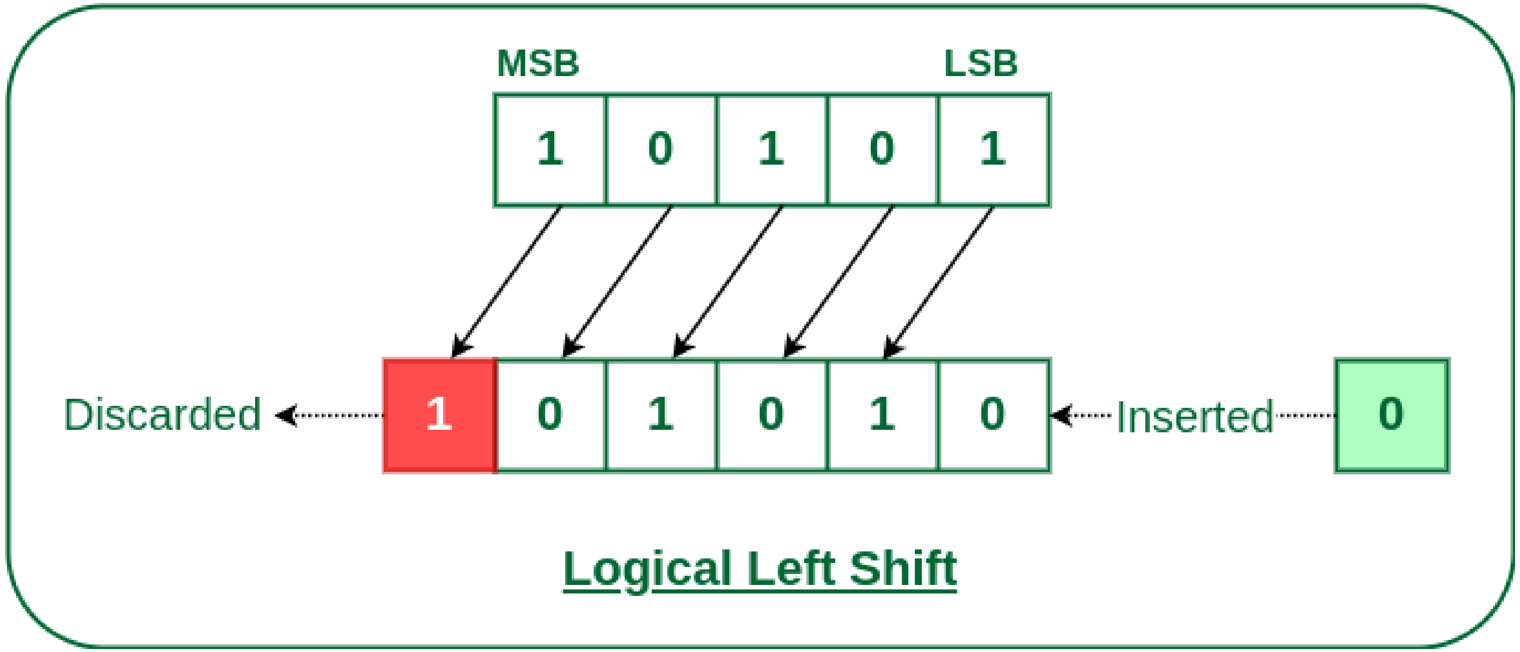
For any input state?

$$U|\lambda\rangle = e^{2\pi i\varphi}|\lambda\rangle$$



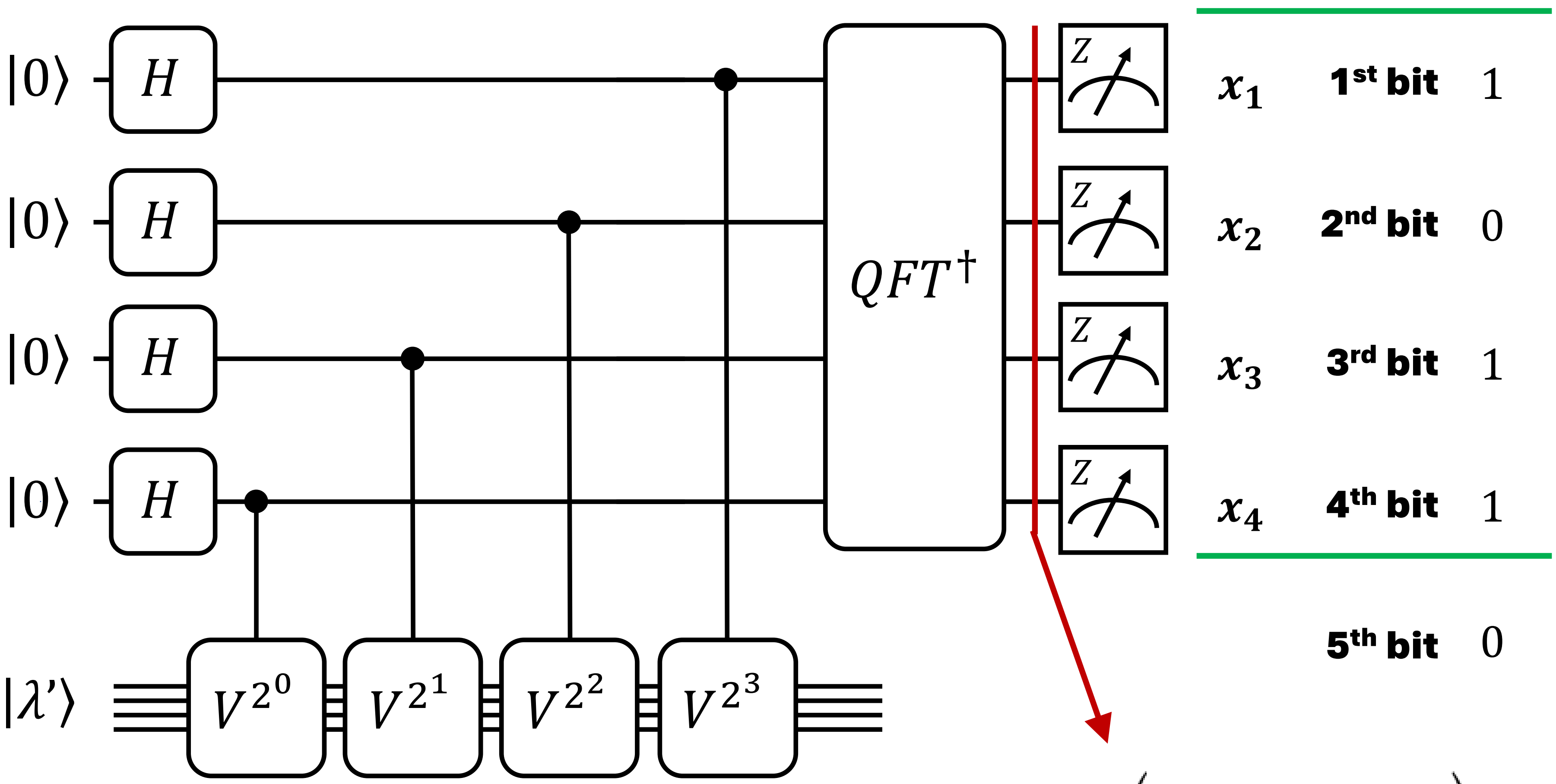
3. Modified phase estimation algorithm

Left shift:
Bit operation



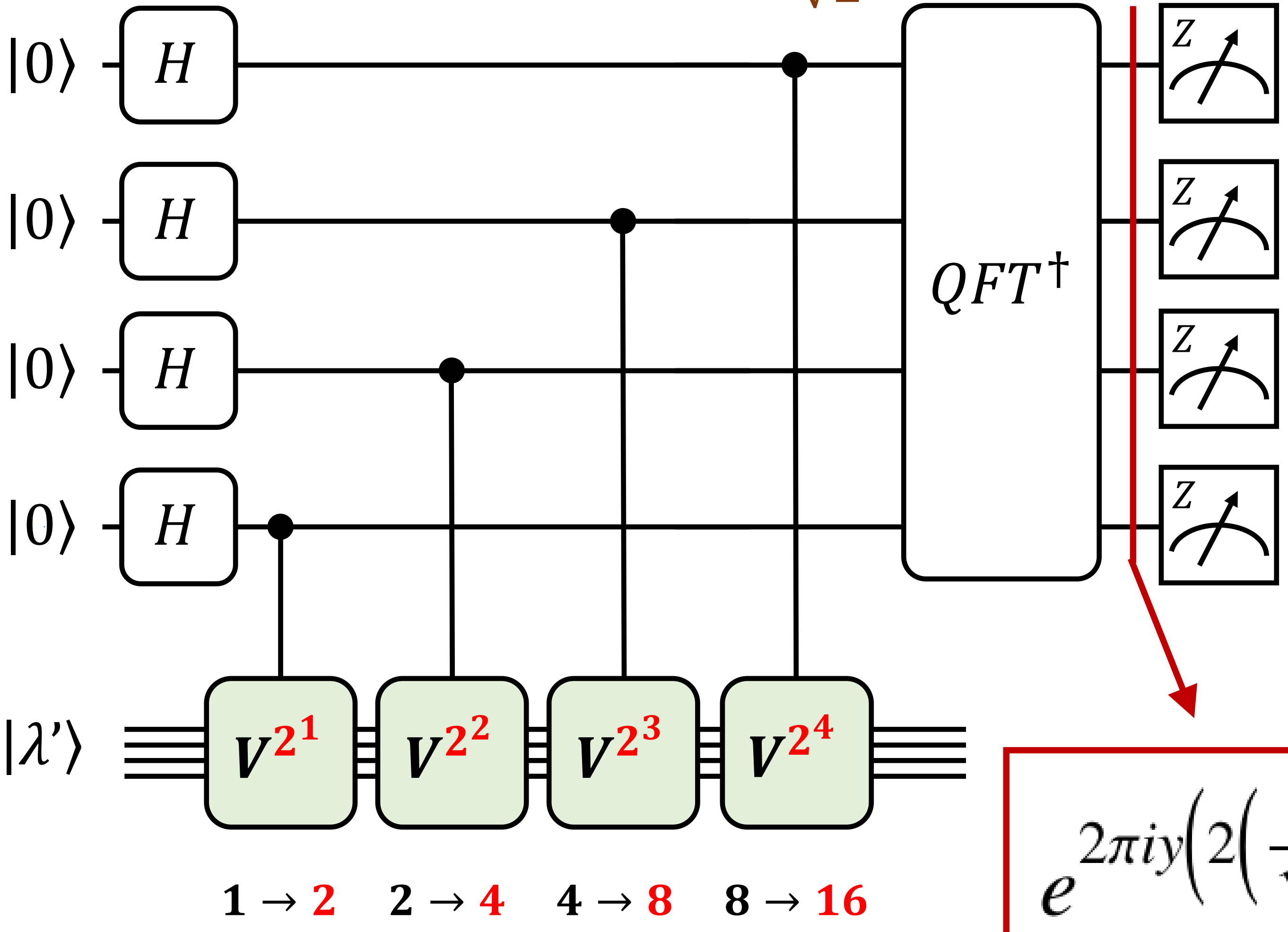
Observation: Phase $\frac{1}{\sqrt{2}}$=0.10110⋯

$$V|\lambda'\rangle = e^{2\pi i\left(\frac{1}{\sqrt{2}}\right)}|\lambda'\rangle,$$

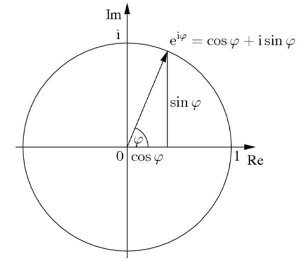


$$e^{2\pi i y\left(\frac{1}{\sqrt{2}} - \frac{x_1 x_2 x_3 x_4}{16}\right)}|x_1\rangle|x_2\rangle|x_3\rangle|x_4\rangle|\lambda'\rangle$$

Observation: Phase $\frac{1}{\sqrt{2}}=0.10110\cdots$



$$e^{2\pi i y((0.011010\cdots)-\frac{x_1x_2x_3x_4}{16})}$$



$$e^{2\pi i y((1.011010\cdots)-\frac{x_1x_2x_3x_4}{16})}$$

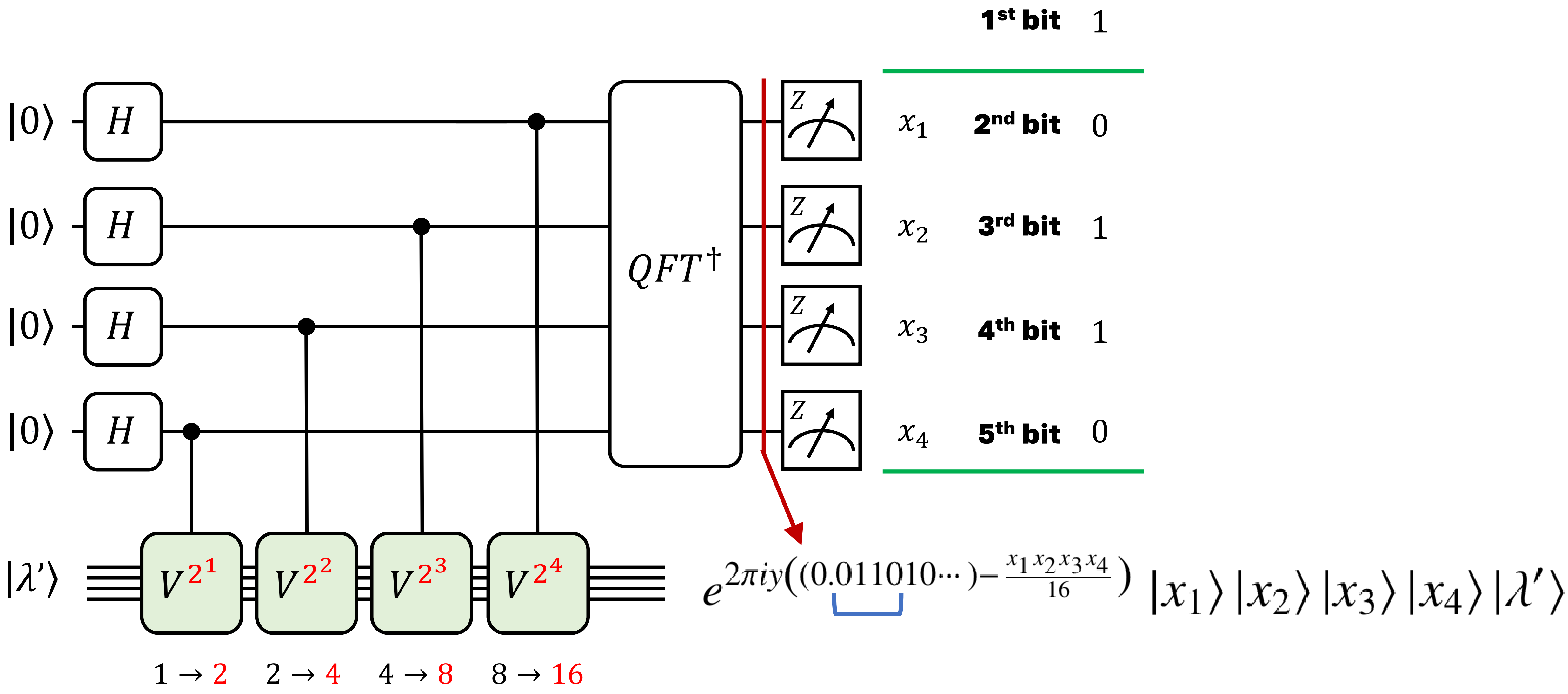
Left shift

$$e^{2\pi i y(2(0.1011010\cdots)-\frac{x_1x_2x_3x_4}{16})}$$

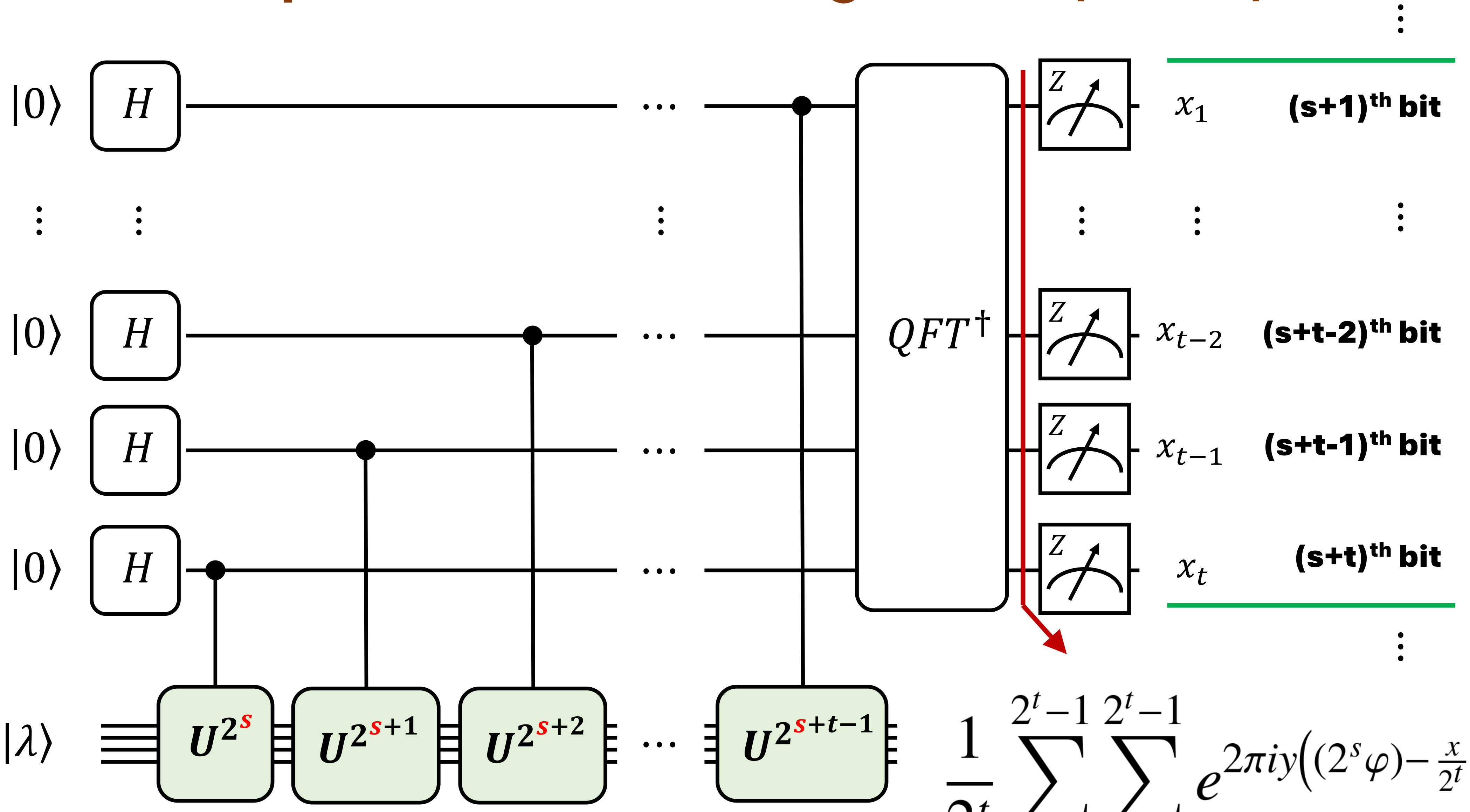
$$e^{2\pi i y\left(2\left(\frac{1}{\sqrt{2}}\right)-\frac{x_1x_2x_3x_4}{16}\right)}|x_1\rangle|x_2\rangle|x_3\rangle|x_4\rangle|\lambda'\rangle$$

$$e^{2\pi i y\left(\frac{1}{\sqrt{2}}-\frac{x_1x_2x_3x_4}{16}\right)}|x_1\rangle|x_2\rangle|x_3\rangle|x_4\rangle|\lambda'\rangle$$

Observation: Phase $\frac{1}{\sqrt{2}}$=0.10110⋯



Modified phase estimation algorithm (MPEA)



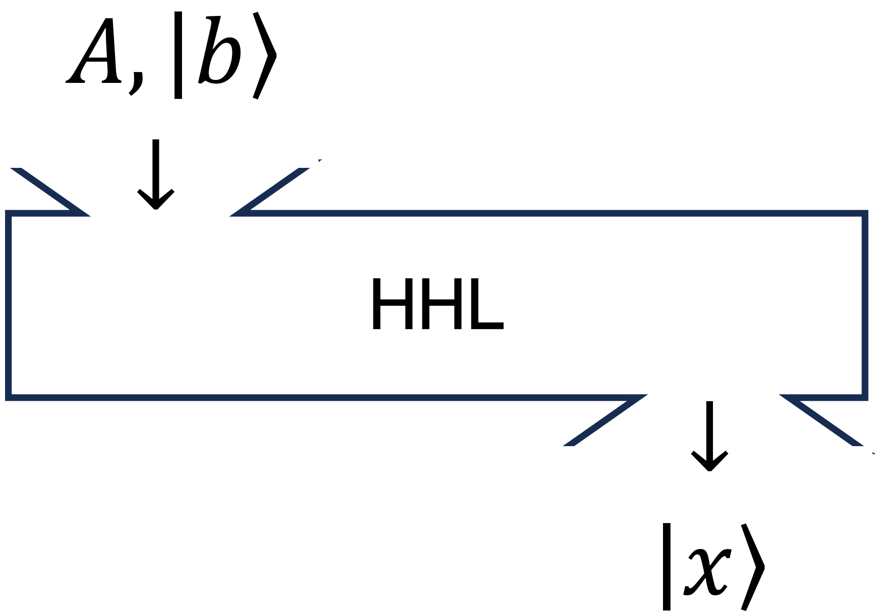
$$\frac{1}{2^t}\sum_{x=0}^{2^t-1}\sum_{y=0}^{2^t-1} e^{2\pi i y\left((2^s\varphi)-\frac{x}{2^t}\right)}|x\rangle\otimes|\lambda\rangle$$

4. Application ?

HHL algorithm

$$A\vec{x} = \vec{b}$$

HHL algorithm is a quantum algorithm for numerically solving a system of linear equations.



[Harrow, Hassidim, and Lloyd, PRL **103**, 150502 (2009)]

[Scott Aaronson, Nature Physics **11**, 291–293 (2015)]

Solving linear equation

A: Hermitian

$$A|x\rangle = |b\rangle$$

$$\boldsymbol{A} = \sum_{i} \lambda_i |\lambda_i\rangle\langle\lambda_i|$$

$$|\boldsymbol{b}\rangle = \sum_{i} \alpha_i |\lambda_i\rangle$$

When A is invertible,

$$A|x\rangle = |b\rangle$$

$$\Rightarrow \quad A^{-1}A|x\rangle = A^{-1}|b\rangle$$

$$\Rightarrow \quad |x\rangle = A^{-1}|b\rangle = \sum_{i} \frac{\alpha_i}{\lambda_i} |\lambda_i\rangle$$

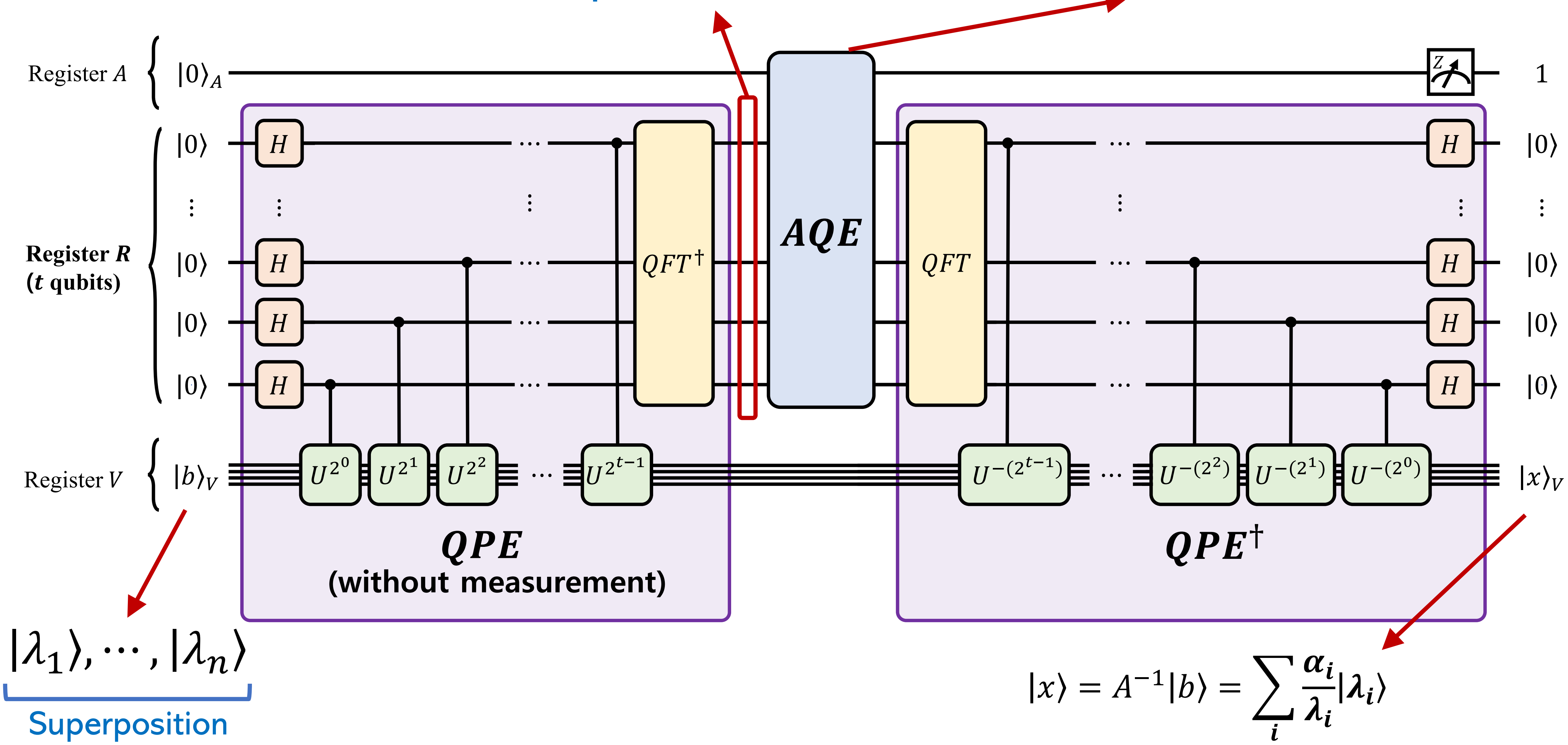
In practice, we don't know eigenvalues and eigenvectors!

HHL also doesn't know them!

Circuit for HHL

Register $\boldsymbol{R}$ contains estimation information of each phase

$$|0\rangle_A \otimes |\lambda_i\rangle_R \mapsto \left(\sqrt{1 - \frac{c^2}{{\lambda_i}^2}}\, |0\rangle_A + \frac{c}{\lambda_i} |1\rangle_A \right) \otimes |\lambda_i\rangle$$



Condition for exponential speedup

HHL provides an exponential speedup over classical counterparts.

When? $\Rightarrow$ **Small condition number**

Condition number

Among the eigenvalues λ_i of a normal matrix A

$$\kappa(A) = \frac{|\lambda_{\max}(A)|}{|\lambda_{\min}(A)|}$$

Small condition number: Observation

$$|\lambda_{\max} - \lambda_{\min}| \leq \frac{1}{2^n} \quad \Rightarrow \quad |\lambda_i - \lambda_j| \leq \frac{1}{2^n}$$

$$\lambda_{\max} = 0.\,\boxed{b_1 b_2 \cdots b_n}\,\boxed{c_{n+1} c_{n+2} \cdots}$$

$$\lambda_{\min} = 0.\,\boxed{b_1 b_2 \cdots b_n}\,\boxed{d_{n+1} d_{n+2} \cdots}$$

fixed | different

$\Rightarrow$

$$\lambda_1 = 0.\,b_1 b_2 \cdots b_n \cdots$$

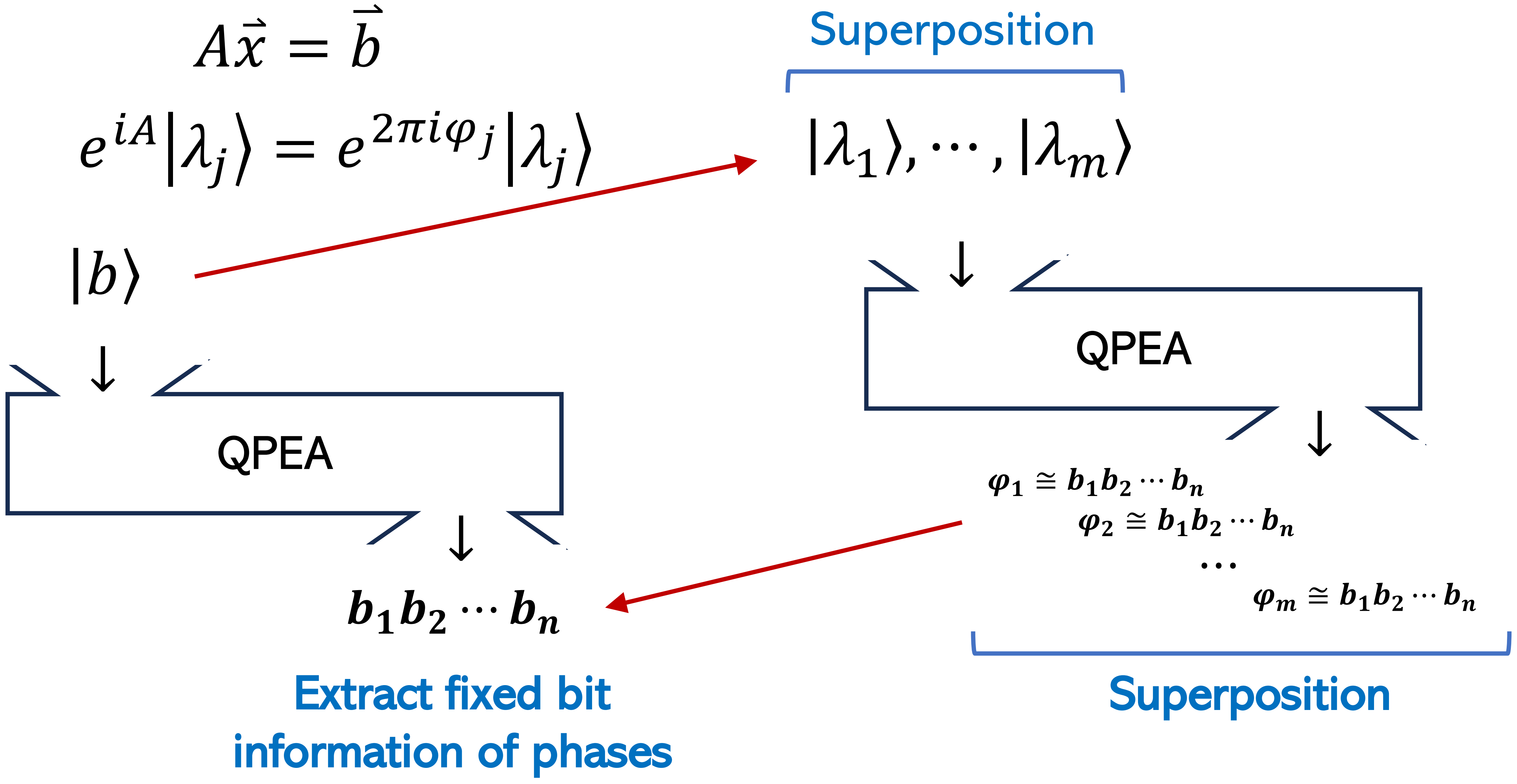
$$\lambda_2 = 0.\,b_1 b_2 \cdots b_n \cdots$$

$$\vdots$$

$$\lambda_m = 0.\,b_1 b_2 \cdots b_n \cdots$$

fixed | different

Small condition number: **Back to QPEA**



Small condition number: QPEA → **MPEA**

From the prior QPEA…

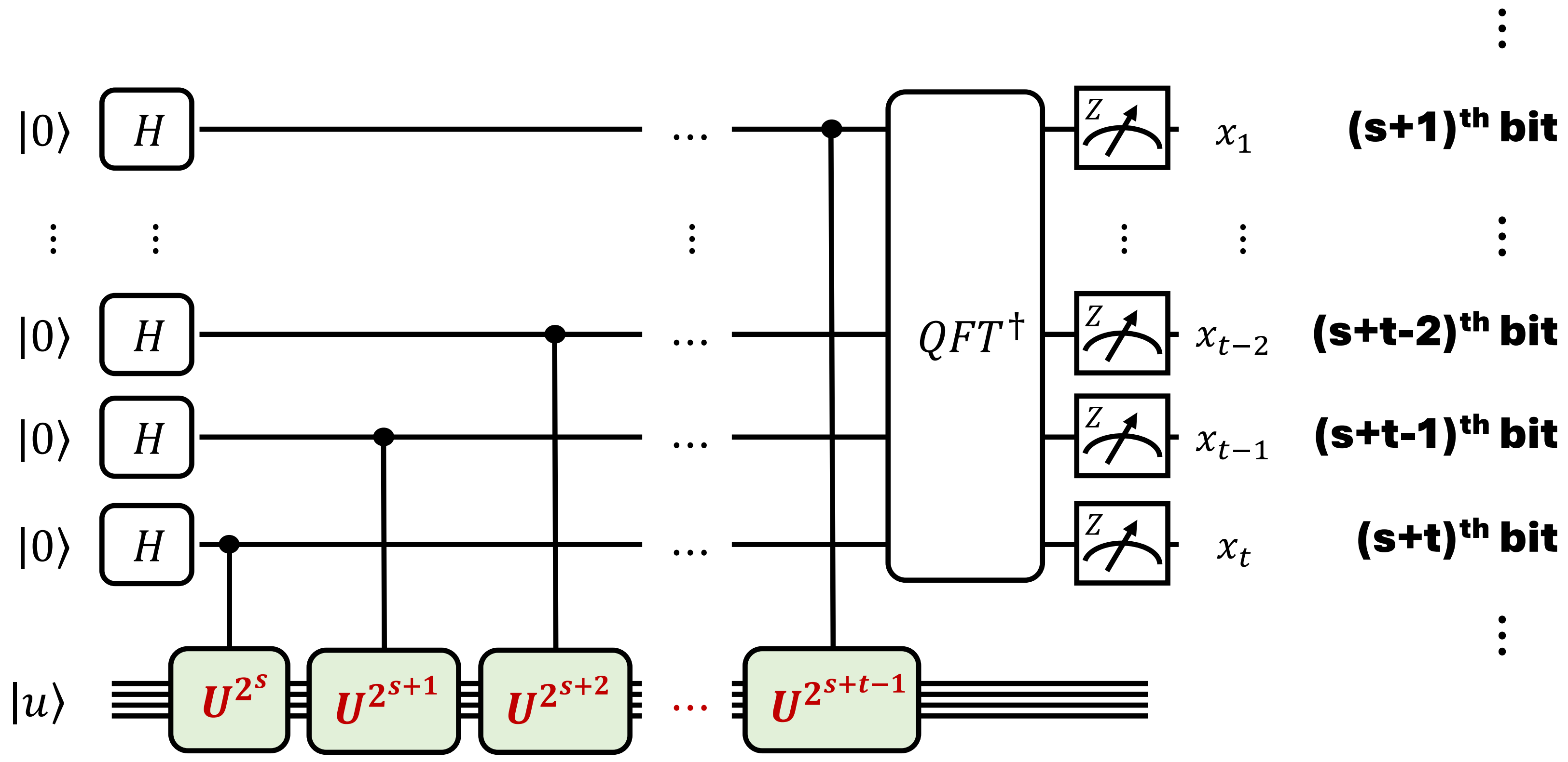
$$\lambda_1 = 0.b_1 b_2 \cdots b_s \cdots$$

$$\lambda_2 = 0.b_1 b_2 \cdots b_s \cdots$$

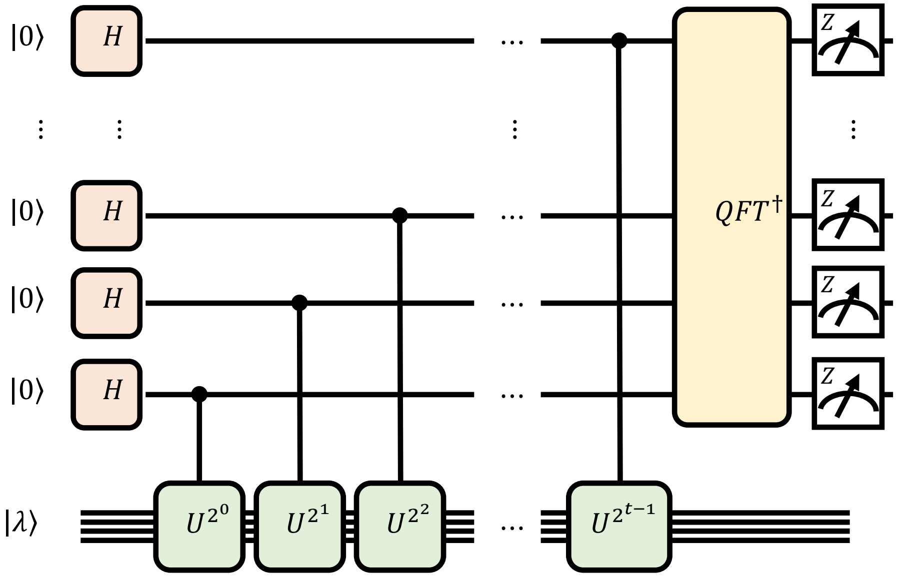
$$\vdots$$

$$\lambda_m = 0.b_1 b_2 \cdots b_s \cdots$$

Already know! No idea…



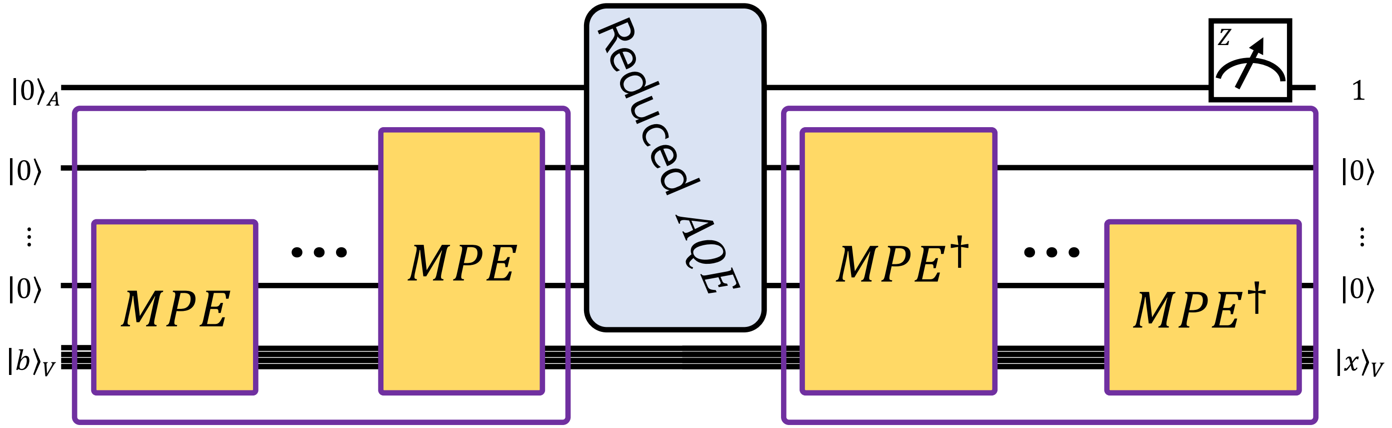
Our circuit implementation method



#1. Perform QPEA

#2. Analyze measurement outcomes

#3. Perform simplified HHL



Reduced qubits and reduced gates!

→ Improved performance under NISQ!

Example: 10 qubits + **small condition number**

$$A|x\rangle = |b\rangle$$

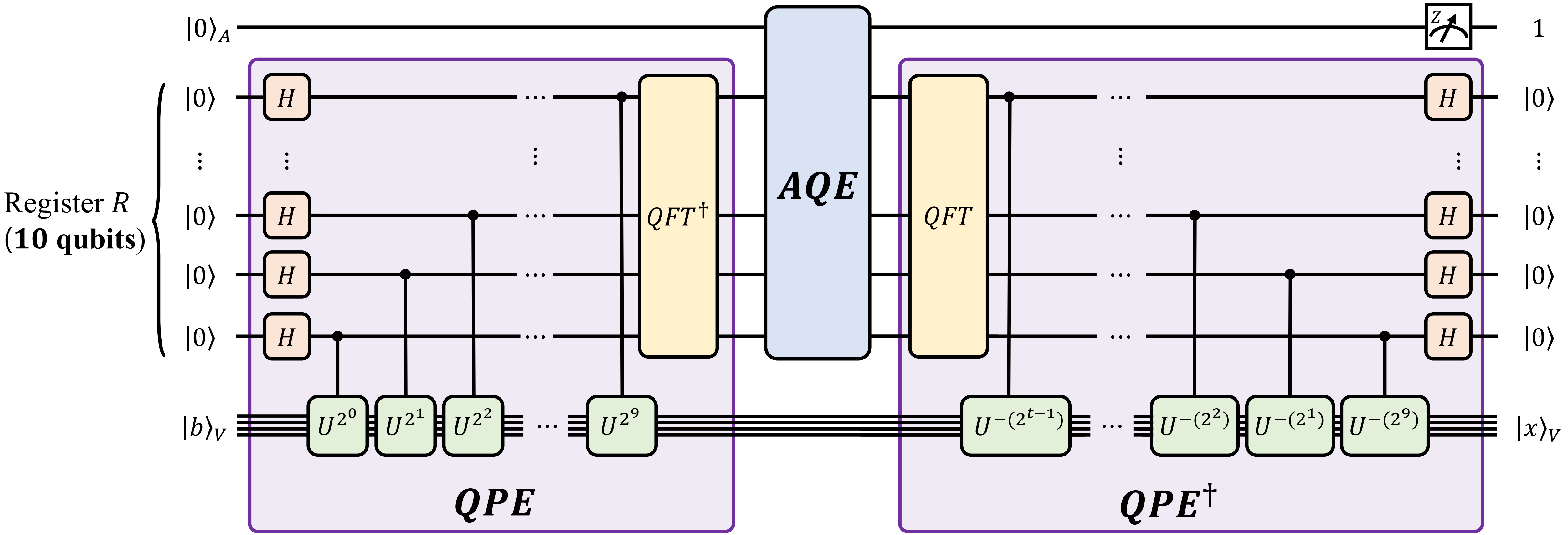
Assume that **all eigenvalues of A have the following binary representation**

$$\boldsymbol{\lambda_i = 0.0\,1\,\blacksquare\,\blacksquare\,0\,1\,\blacksquare\,\blacksquare\,1\,1}$$

Fixed! Fixed! Fixed!

Assume that we **use10 bits as the second register R**

Example: 10 qubits, **original HHL** for $A|x\rangle = |b\rangle$



H : 10 U : 1,023 $QFT^\dagger$: 60

Example: 10 qubits, **our method** for $A|x\rangle = |b\rangle$

#1. Perform QPEA

$$m_1 \quad m_2 \quad m_3 \quad m_4 \quad m_5 \quad m_6 \quad m_7 \quad m_8 \quad m_9 \quad m_{10}$$

$$m_1 \quad m_2 \quad m_3 \quad m_4 \quad m_5 \quad m_6 \quad m_7 \quad m_8 \quad m_9 \quad m_{10}$$

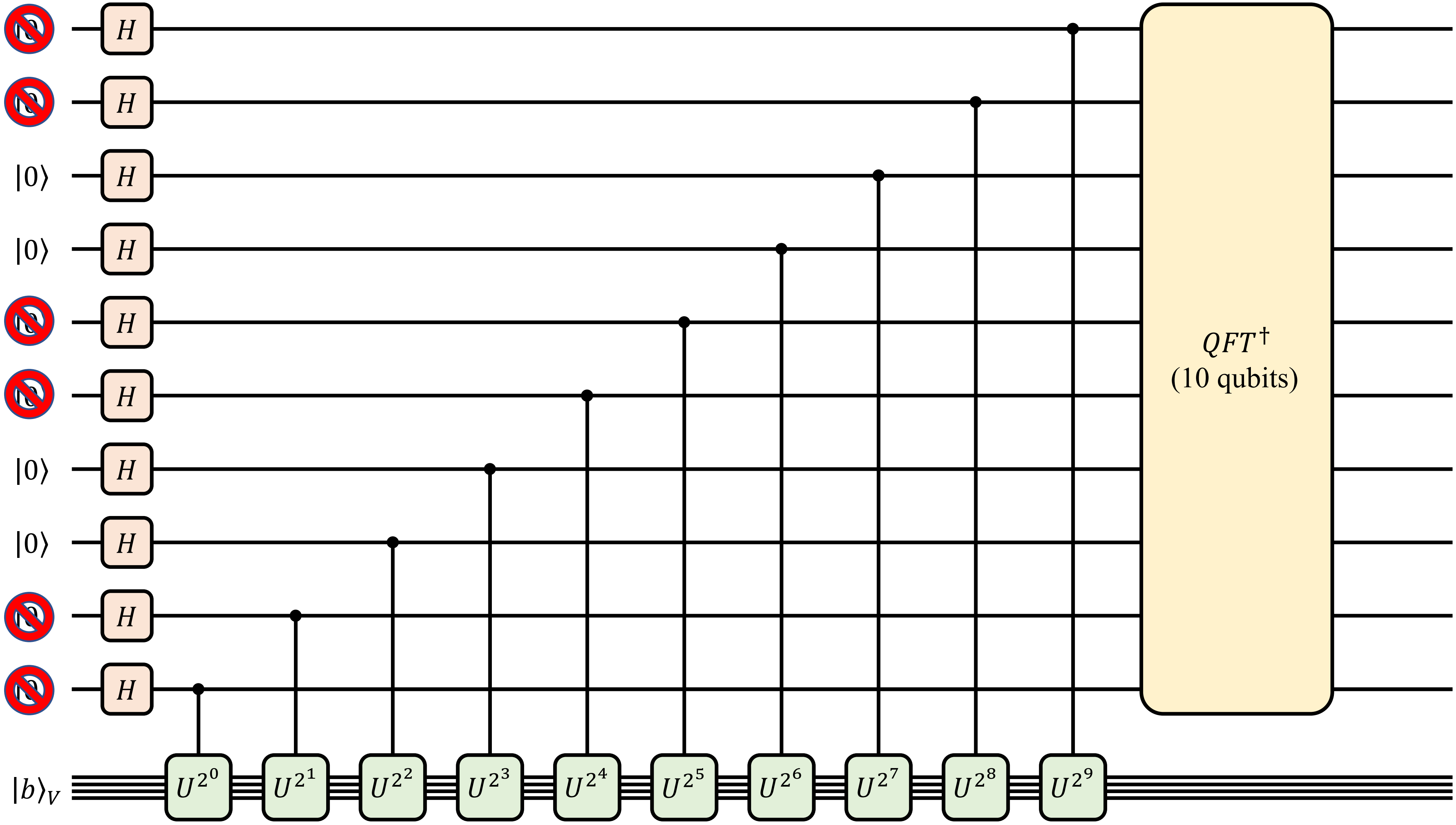
$$m_1 \quad m_2 \quad m_3 \quad m_4 \quad m_5 \quad m_6 \quad m_7 \quad m_8 \quad m_9 \quad m_{10}$$

#2. Analyze (compare) measurement outcomes

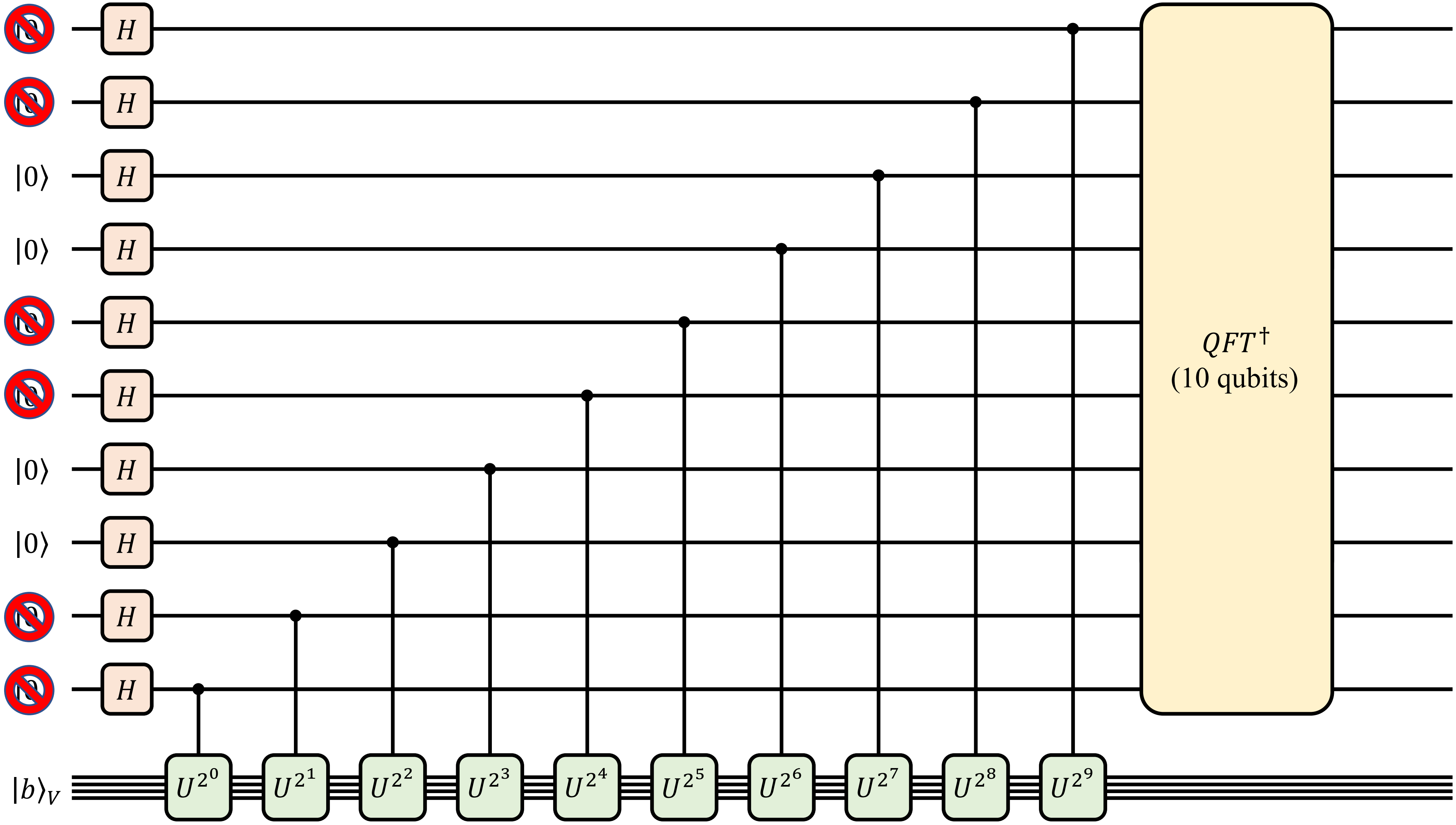
$$\lambda_i = \boxed{0 \quad 1} \quad b_3 \quad b_4 \quad \boxed{0 \quad 1} \quad b_7 \quad b_8 \quad \boxed{1 \quad 1}$$

Fixed! Fixed! Fixed!

Example: 10 qubits, 1st **part** of original HHL

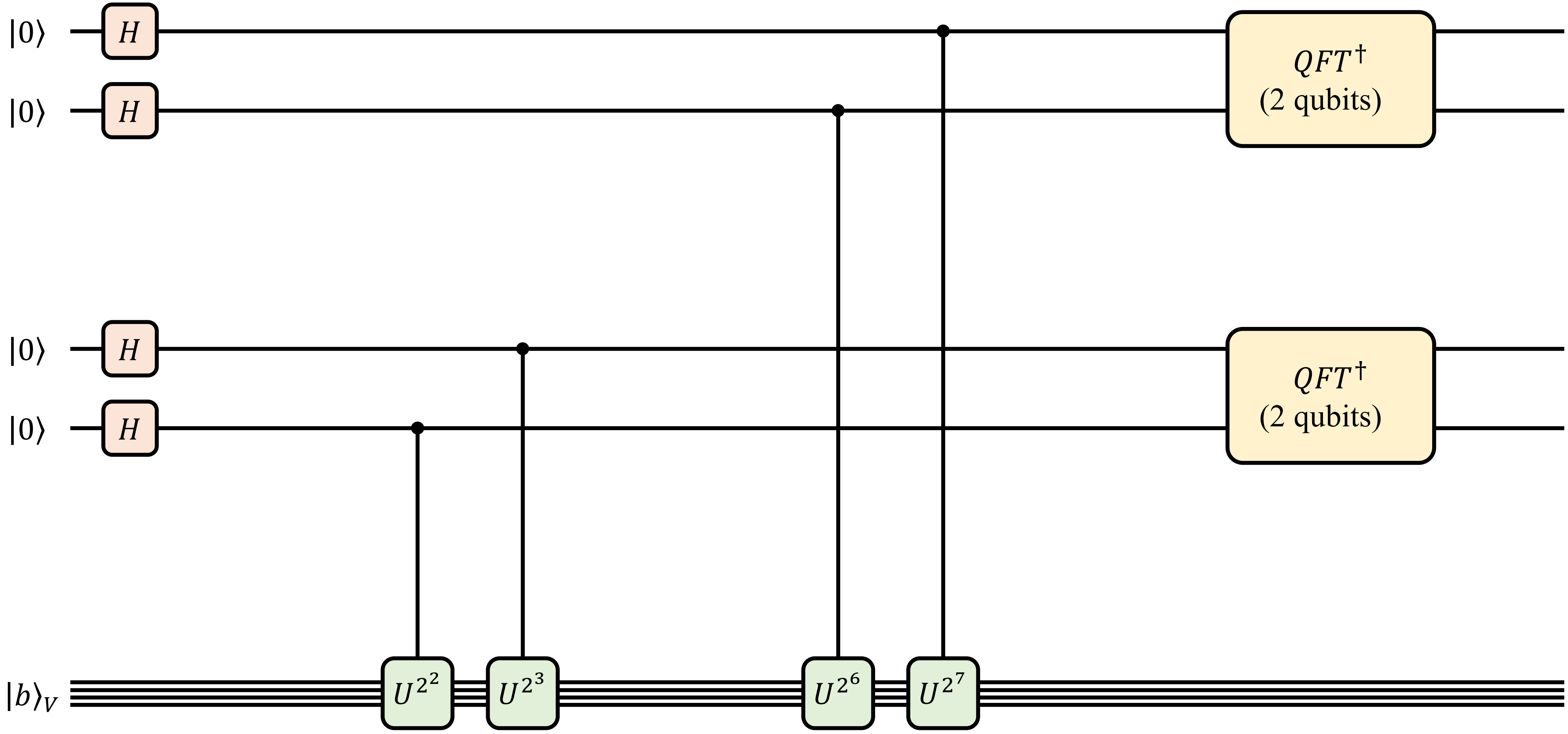


How does the implementation change?



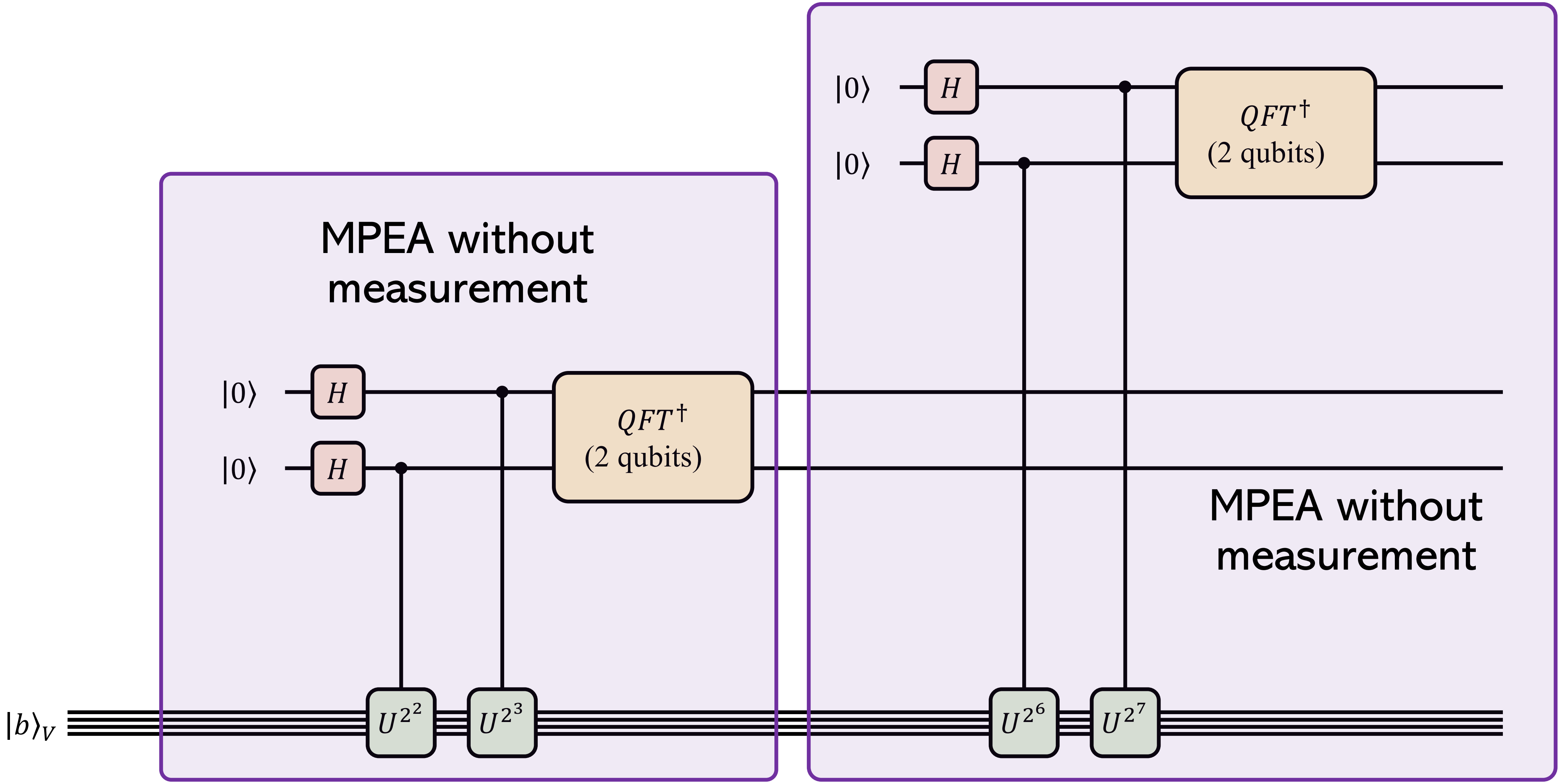
How does the implementation change?

How does it work?

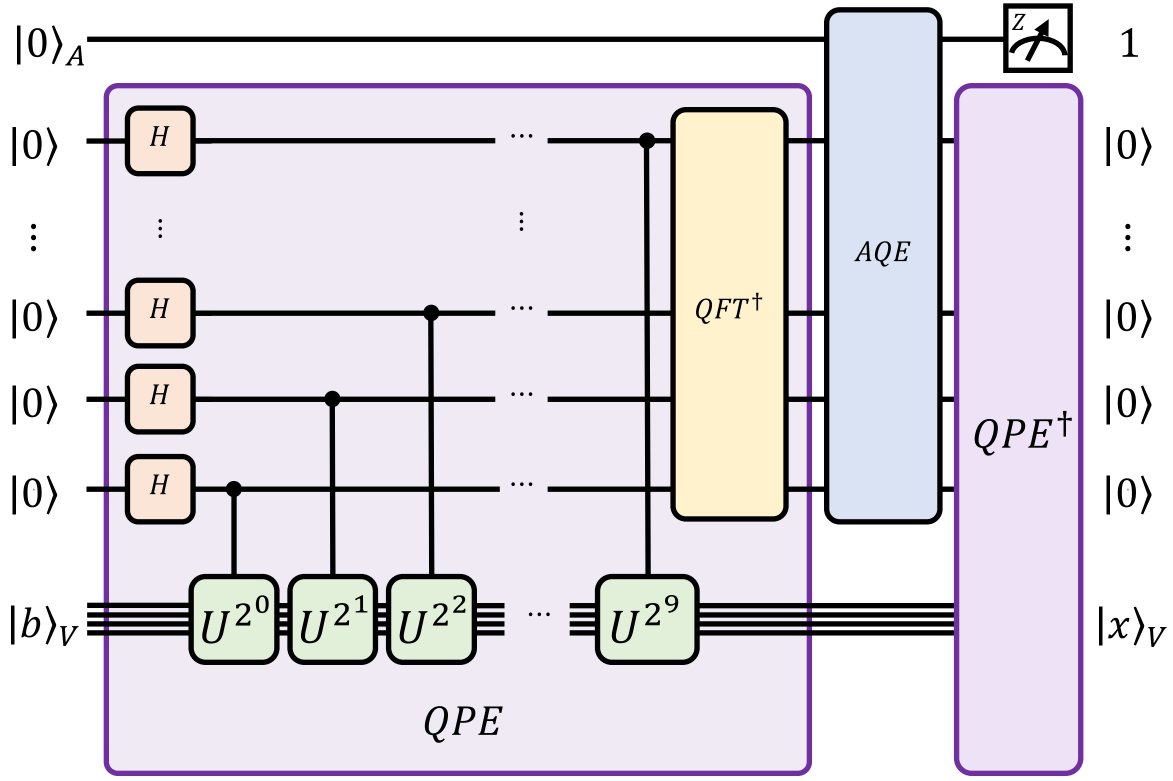


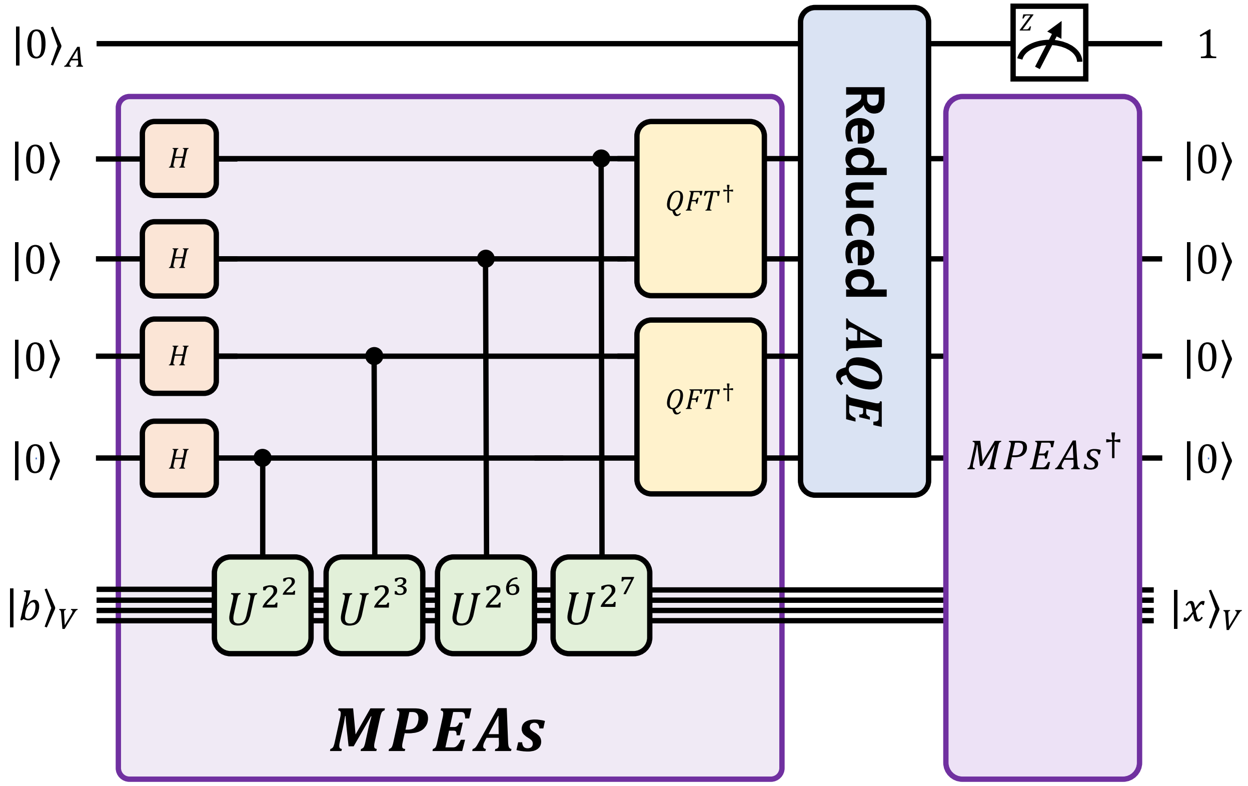
How does the implementation change?

How does it work?



Example: 10 qubits, **original HHL vs. ours**





H : 10 U : 1,023 $QFT^\dagger$: 60

H : 4 U : 204 $QFT^\dagger$: 10

Conclusion

Conclusion

- Bit Shift + QPEA → MPEA

- Simplifying implementation of HHL

- Experimental verification using classical and quantum computers

- Is our implementation method still faster than the conjugate gradient method?

Thank you!

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Performance of QPEA

Suppose we wish to approximate φ to an accuracy 2^{-n}, that is, we choose $e = 2^{t-n} - 1$. By making use of $t = n + p$ qubits in the phase estimation algorithm we see from (5.34) that the probability of obtaining an approximation correct to this accuracy is at least $1 - 1/2(2^p - 2)$. Thus to successfully obtain φ accurate to n bits with probability of success at least $1 - \epsilon$ we choose

$$t = n + \left\lceil \log \left(2 + \frac{1}{2\epsilon} \right) \right\rceil . \tag{5.35}$$

Previous result

Hybrid quantum linear equation algorithm and its experimental test on IBM Quantum Experience

Yonghae Lee[1], Jaewoo Joo[2,3] & Soojoon Lee[1,2,4]

- MPEA was *not* devised
- Small condition number was *not* mentioned
- # of qubits were *not* reduced

[Lee, Joo, and Lee, SR **9**, 4778 (2019)]