Contextual advantages in maximum confidence measurements

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Contextual advantages and certification for maximum confidence discrimination Kieran Flatt, Hanwool Lee, Carles Roch i Carceller, Jonatan Bohr Brask, Joonwoo Bae

PRX Quantum 3, 030337

- 1. What is contextuality? What constrains possible hidden variables models of quantum theory?
- 2. Generalised contextuality
- 3. Quantum advantages in state discrimination
- 4. Practical applications semi-device-independent experiments, randomness generation

Question: can quantum theory be understood as a hidden variable model?

J. von Neumann, "Mathematical Foundations of Quantum Mechanics"

First axiomatic formulation of quantum physics

As an argument against <u>all</u> hidden variable theories, he makes this assumption:

 $\langle a\hat{A} + b\hat{B} \rangle = a\langle \hat{A} \rangle + b\langle \hat{B} \rangle$

However, explicit hidden variables models can be constructed (e.g., Bohmian mechanics).

His assumption later became known as 'noncontextuality'.



Gleason-Busch theorem

Consider a function $P(A): B(\mathcal{H}) \to [0,1]$ satisfying linearity: P(aA + bB) = aP(A) + bP(B)

Then it must take the form of the Born rule. Linearity over observables extends to linearity over probabilities.

 $\text{Consider a function} \quad P(A): B(\mathcal{H}) \to [0,1] \quad \text{satisfying linearity:} \quad P(aA+bB) = aP(A) + bP(B)$

Then it must take the form of the Born rule. *Linearity over observables extends to linearity over probabilities*.

This can be used to rule out the possibility of a deterministic hidden variable model. Deterministic here means all probabilities are 0 or 1. Consider the observables $\{Q, \bar{Q}\}$ and $\{R, \bar{R}\}$.

W.l.o.g. can let P(Q) = P(R) = 0 for some system. Then consider observable qQ + rR

P(qQ + rR) = qP(Q) + rP(R) = 0

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$$P(qQ + rR) = qP(Q) + rP(R) = 0$$

But we can also decompose this into its eigenbasis: $\, qQ + rR = s_1S + s_2ar{S}$. Let $\, P(S) = 0$. Then:

$$P(qQ + rR) = P(s_1S + s_2\bar{S}) = s_1P(S) + s_2P(\bar{S}) = s_2 \neq 0$$

Thus we cannot assign deterministic probabilities to all observables.

P. Busch, Phys. Rev. Lett. 91, 120403

NONCONTEXTUALITY

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Older notion of contextuality

1. Cannot construct **deterministic** hidden variable model for quantum states



2. Different linear decompositions of the same observables or states in quantum theory cannot have the same representation in a hidden variable model.



Generalised Contextuality

1. Can have probabilistic dependence on hidden variables



2. Want to take into account other equivalent processes

Generalised Contextuality

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Generalised Contextuality

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Ontological models





Contextuality / Noncontextuality

Preparation Noncontextuality. A theory is preparation noncontextual if it represents operationally equivalent preparations with the same epistemic state.



Noncontextual operational models

Operational models are representations of experimental procedures. We represent quantum states as probability distributions (epistemic states) and measurements by response functions.

Contextuality concerns the possibility of representing quantum theory as a hidden variable model. A theory is noncontextual if it represents equivalent experimental processes by the same mathematical objects.

Analogous constraint would be nonlocality of some hidden variable models.

We will now look at consequences of this property for information processing....

Minimum error state discrimination



Minimum error approach maximises figure of merit $P_g = \frac{1}{2}P(\pi_0|\psi_0) + \frac{1}{2}P(\pi_1|\psi_1)$

Maximum value is the Helstrom bound $P_g^{max} = \frac{1}{2} \left(1 + \sqrt{1 - |\langle \psi_0 | \psi_1 \rangle|^2} \right)$

Bayesian confidence less than one, e.g., $f\phi\phi$, $|+\rangle$ we have $P(\psi_0|\pi_0) = 0.85$

Contextual advantages for minimum error state discrimination



Does it apply to more general forms of state discrimination?

- Unambiguous state discrimination
- Maximum confidence measurements

Does it have applications in practical scenarios?

- Semi-device-independence
- Randomness generation

Intuitive explanation of minimum error advantage

From the maximally mixed state, we have: $\frac{1}{2}\mu_1(\lambda) + \frac{1}{2}\overline{\mu}_1(\lambda) = \frac{1}{2}\mu_2(\lambda) + \frac{1}{2}\overline{\mu}_2(\lambda)$

Now consider $\lambda \in \operatorname{supp}[\mu_1(\lambda)] \cap \operatorname{supp}[\mu_2(\lambda)]$

This implies, on that region: $\mu_1(\lambda) = \mu_2(\lambda)$

So, we have

Probability of success



Contextual advantages for minimum error state discrimination



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David Schmid and Robert W. Spekkens, Phys. Rev. X 8, 011015

Maximum confidence measurements



Confidence: $C(i) = P(\psi_i | \pi_i) = \frac{P(\psi_i)P(\pi_i | \psi_i)}{P(\pi_i)}$ Guessing probability $P_g = \sum_i C(i)$

Maximum confidence measurements are those which maximise this value for all states in the desired ensemble.

Maximum confidence measurements

$$p_{0} = p_{1} = 1/2$$

$$|\psi_{0}\rangle |\psi_{1}\rangle$$

$$\int \int \int \int \int \int \frac{\pi_{0}}{\pi_{1}} \pi_{2}$$

Confidence: $C(i) = P(\psi_i | \pi_i) = \frac{P(\psi_i)P(\pi_i | \psi_i)}{P(\pi_i)}$ Guessing probability $P_g = \sum_i C(i)$ Includes an inconclusive outcome, for which we cannot draw any conclusions about the prepared state.

Can be considered a generalisation of unambiguous state discrimination:

$$C(i) = 1 \quad \forall \ i$$

Latter only possible for linearly-independent ensembles

Most natural scenario for practical experimental state discrimination.

Maximum confidence measurements are those which maximise this value for all states in the desired ensemble.

Sarah Croke et al Phys. Rev. Lett. 96, 070401

Contextual advantages for noisy maximum confidence measurements

 $\max C(1)$



Ensemble is:

$$\rho_1 = p \frac{I}{2} + (1-p) |0\rangle \langle 0|$$

$$\rho_2 = p \frac{I}{2} + (1-p) |+\rangle \langle +$$

Cannot be discriminated unambiguously.

Note: there is also an advantage in terms of the inconclusive outcome rate.

This is the most natural scenario for demonstrating a quantum advantage taking into account non-detections and noise.

Advantages for different figures of merit



Certified maximum confidence scenario



This is the semi-device-independent scenario

It's the most realistic scenario for experiments and viable quantum communications

Want to know (a) how much you can trust a device (b) whether your device is doing something quantum

Contextual advantages for certified maximum confidence measurements

For a fixed value of $\operatorname{P}(\pi_1)$:

Quantum

Semi-device-independence linearises the confidence

This allows us to solve the problem with an SDP

Noncontextual theory

Noncontextuality constrains the space of possible response functions

By hand, we can construct the maximally confident response functions



Contextual advantages for certified maximum confidence measurements

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Certified randomness generation from state discrimination



Eve's guessing probability

Randomness generated

 $p_g = \sum_x p_x \sum_\lambda q_\lambda \max_b p(b|x,\lambda)$

 $\mathbf{H}_{min} = -\log_2(p_g)$

Quantum vs. noncontextual semi-device-independent randomness certification Carles Roch i Carceller, Kieran Flatt, Hanwool Lee, Joonwoo Bae, Jonatan Bohr Brask

Certified randomness generation from state discrimination



Summary

Noncontextual theories are those in which the representation of individual states does not depend upon the properties of the ensemble in which they are created.

This property fails for quantum theory, justifying its use as a notion of nonclassicality.

We showed that this has consequences for information processing: in particular, state discrimination. Maximum confidence discrimination brings us closer to experimental realisation of loophole-free contextual advantages.

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Thank you for listening!