Training parameterized quantum circuits for optimal measurement

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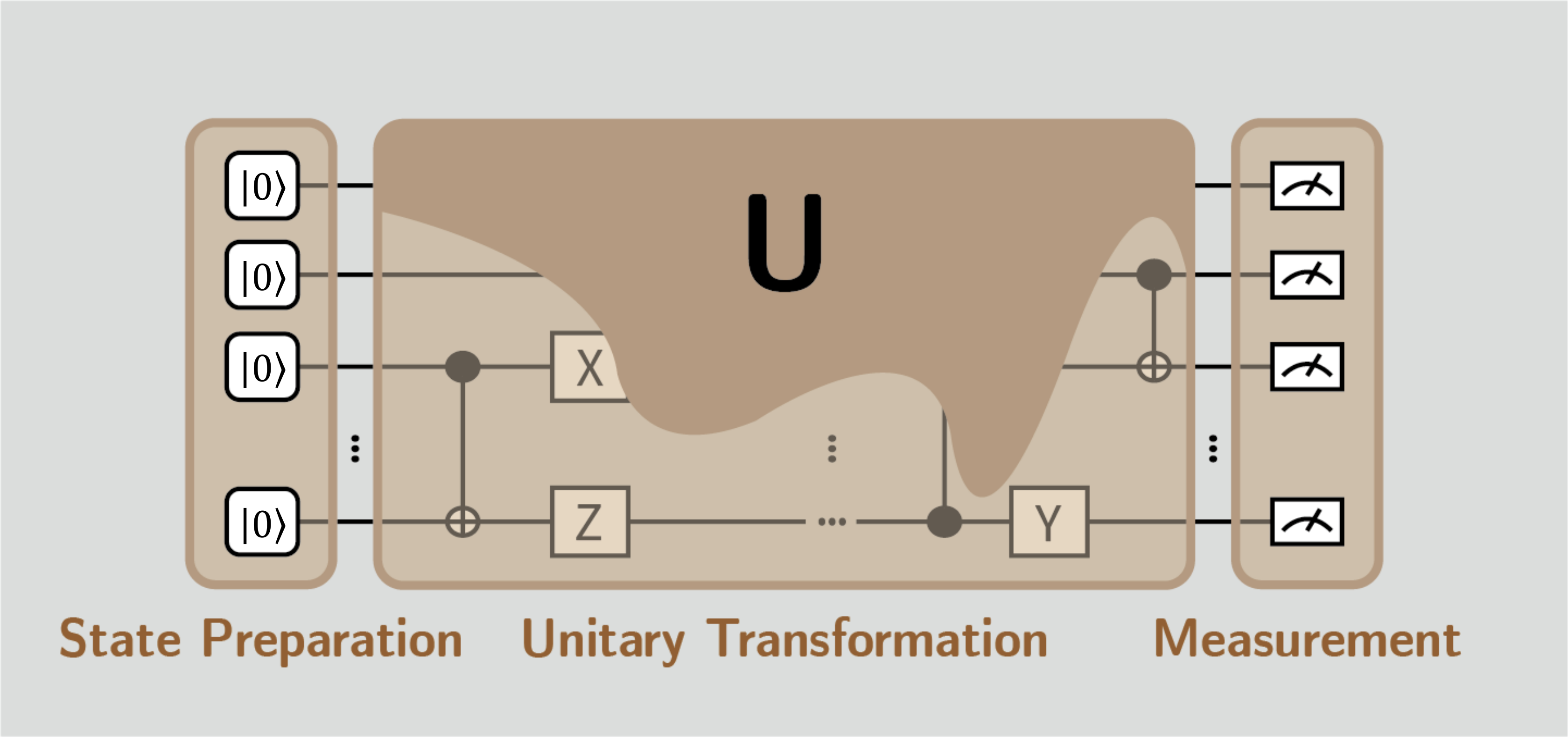
Background: circuit model



- In the standard circuit model, qubit preparations are initialized to $|0\rangle^{\otimes n}$ and measurements are made in the $\{|0\rangle, |1\rangle\}^{\otimes n}$ basis

- The main task in the circuit model of quantum computation is ***manipulation of unitray transformations.***

Background: POVM

- In Popescu et al.'s 1995 Study

1. Projective measurement on Werner state (by Werner)

    - Result: Local
      (Explained by local hidden variable)

2. Successive measurement including ***POVM***

    - Result: ***Nonlocality***
      (Due to inequality Violated)

- POVM in circuit model

- ***Naimark's theorem*** states that POVM can be realized with the aid of ancillary qubits.

Sandu Popescu, Bell's inequalities and density matrices: Revealing hidden nonlocality, Phys. Rev. Lett. 74 (1995) 2619–2622.

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   - Result: Local
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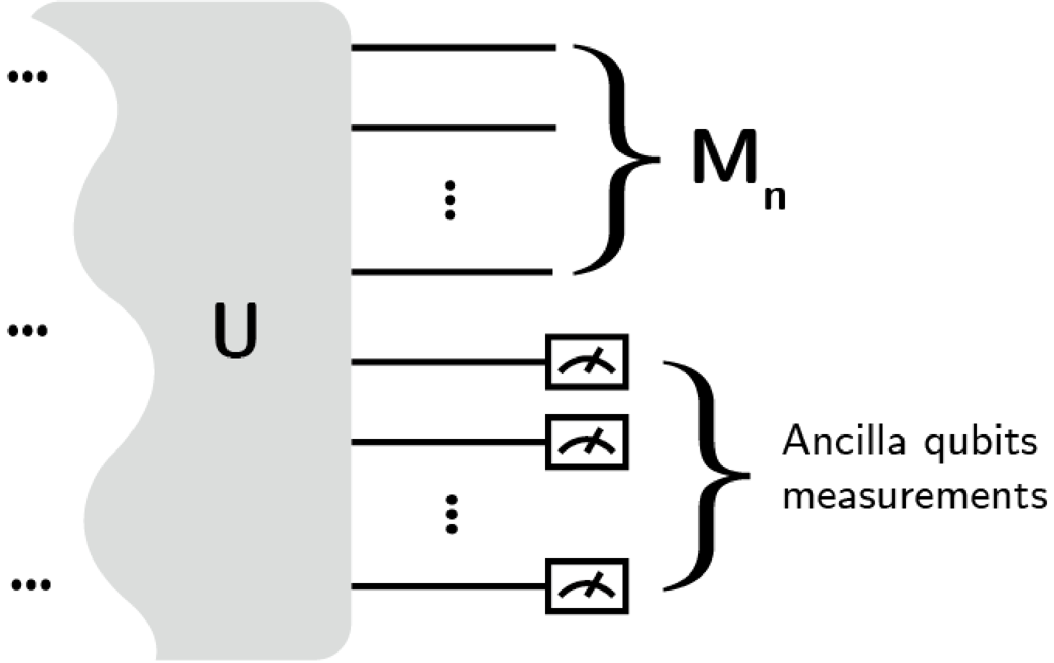
- POVM through a circuit model



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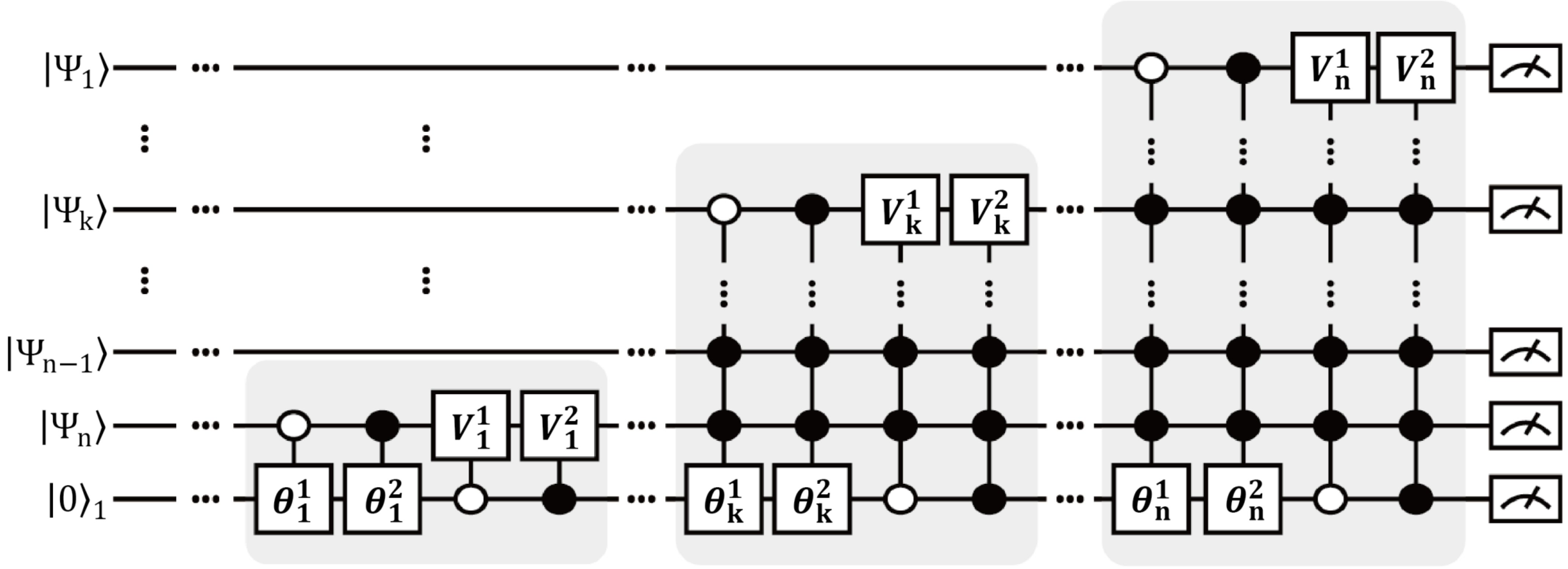
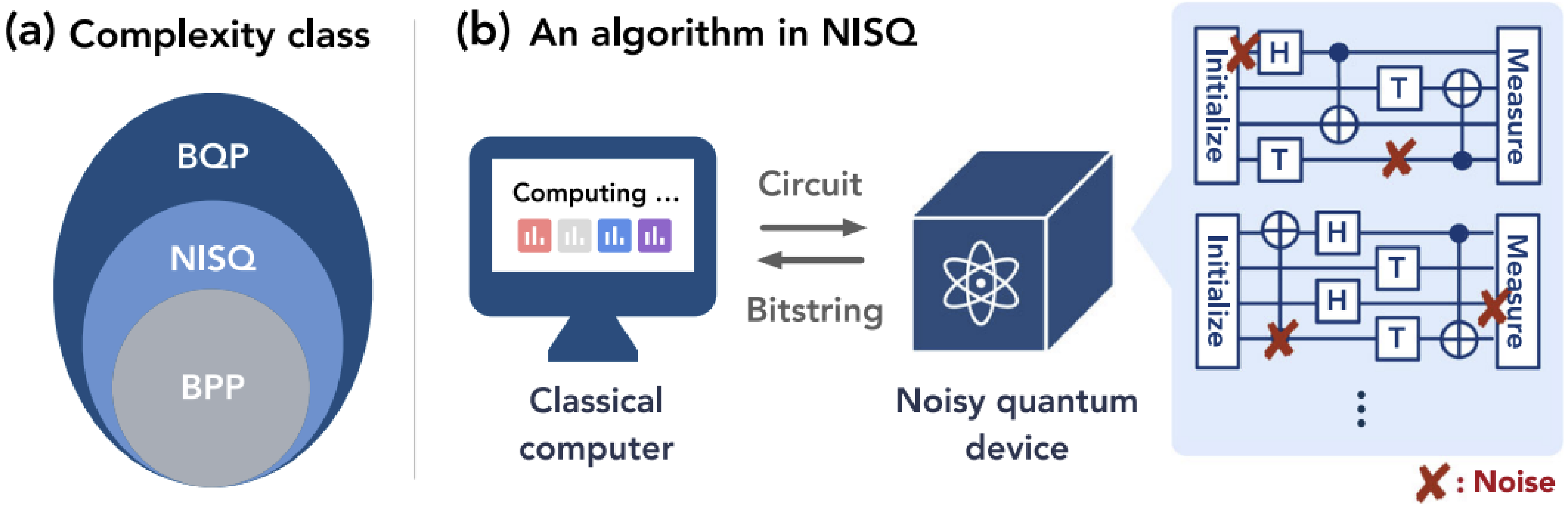
Can we efficiently prepare POVM with NISQ technologies?

- For general POVM approach to measurement operator universally, Yordan S. et al. (2019) provides insights
- However, the circuitry proposed by Yordan S. has a depth that ***might not be suitable for NISQ devices***.



Yordan S. Yordanov and Crispin H. W. Barnes. Implementation of a general single-qubit positive operator-valued measure on a circuit-based quantum computer. Physical Review A, 100(6), dec 2019.

Noisy intermediate scale quantum (NISQ)



- Quantum circuit must have shallow depth on NISQ
- The *[Variational Quantum Eigensolver (VQE)]* is a preferred method for NISQ devices.

Chen, S., Cotler, J., Huang, H. Y., & Li, J. (2023, September 26). The complexity of NISQ. Nature Communications, 14(1). https://doi.org/10.1038/s41467-023-41217-6

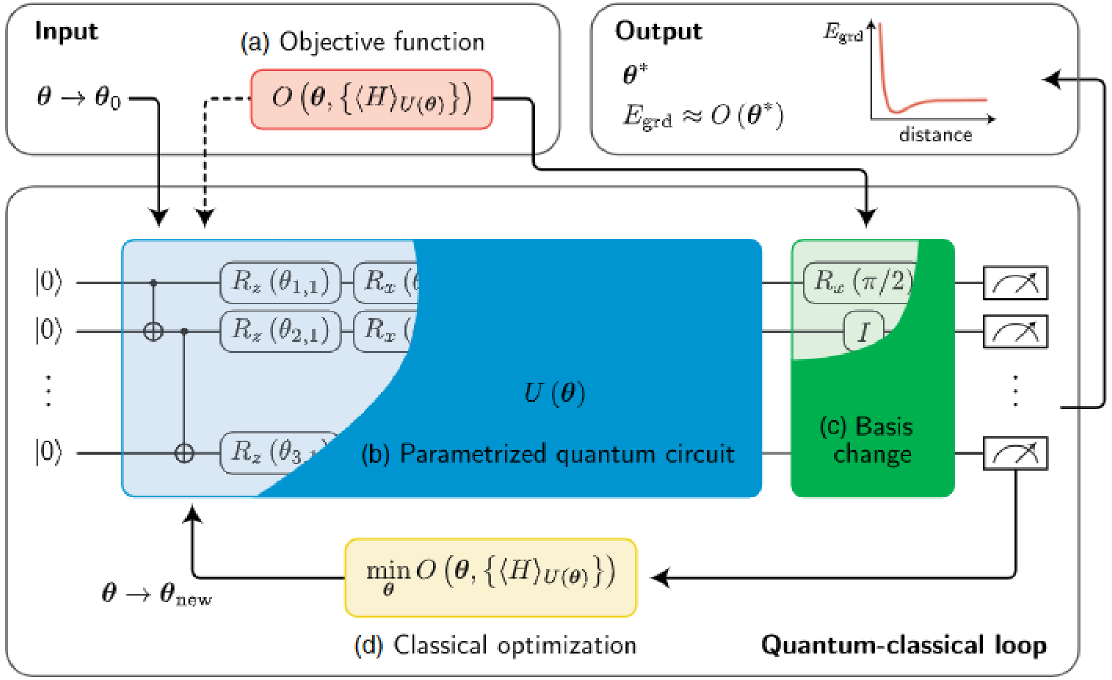
Universal quantum circuit for NISQ: variational quantum eigensolver(VQE)

- NISQ circuits are not universal comprehensive. However, the parameterized quantum circuit (PQC) closely mirrors a universal quantum framework.

- Comparison; VQE serves as NISQ's alternative to universal quantum circuits

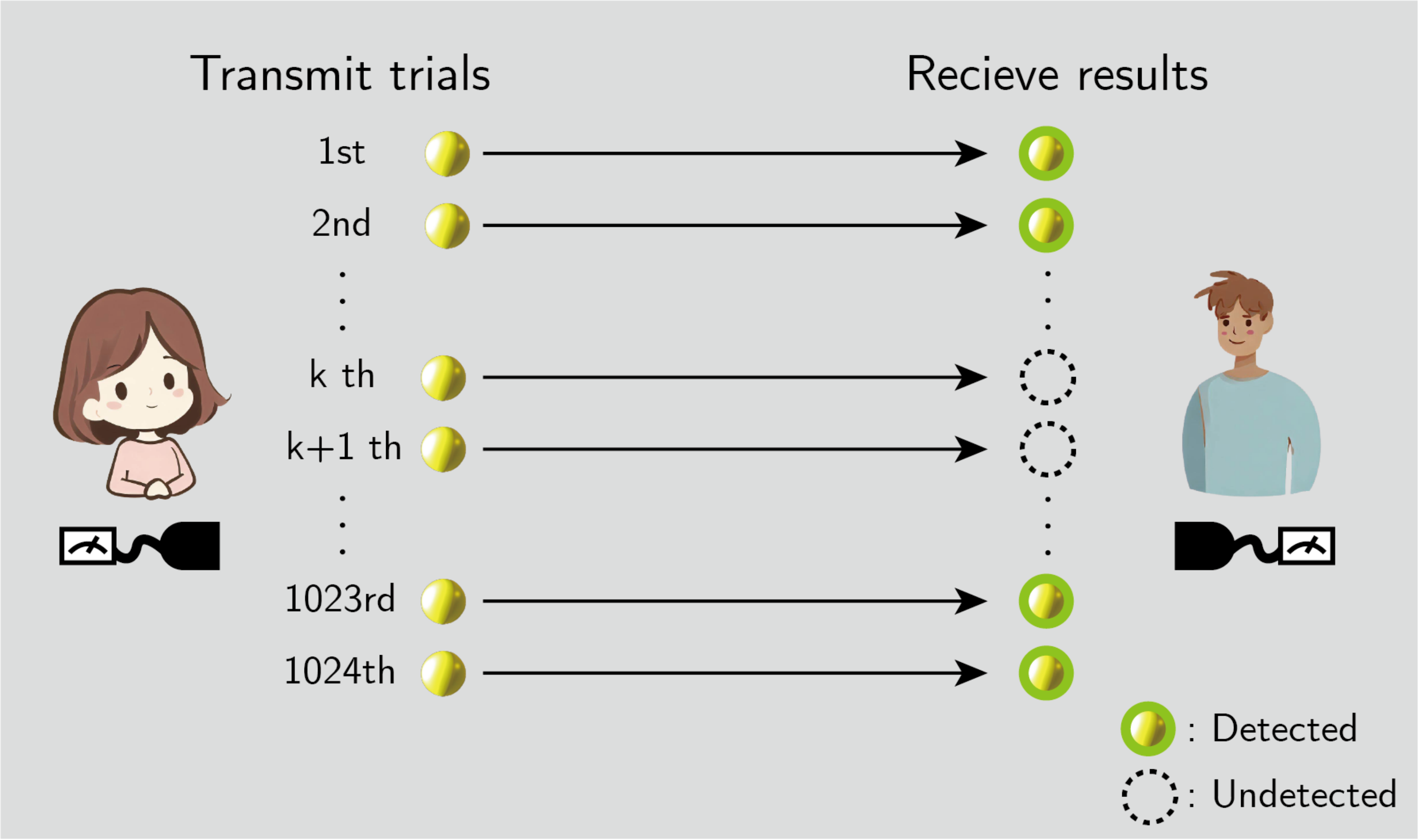
	Universal quantum circuit	NISQ circuit
Error	Fully controlled by QECC	No QECC or Mitigation
Depth	Arbitrary depth	Shallow depth
Construction	CNOT + single qubit gates	VQE(PQC + Machine Learning)
Universality	O	X

- VQE(PQC + Machine Learning)



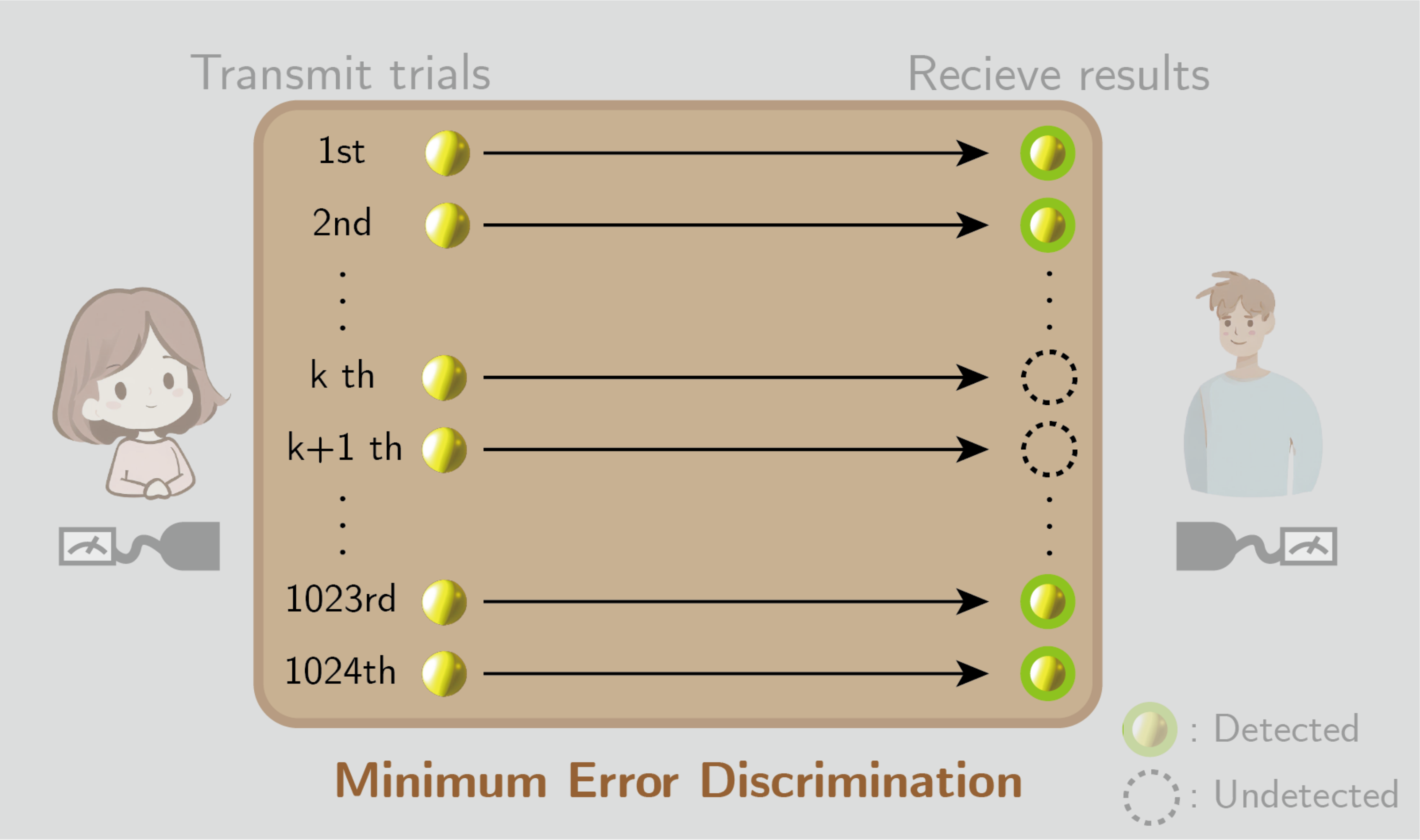
Kishor Bharti, Alba Cervera-Lierta, Thi Ha Kyaw, Tobias Haug, Sumner Alperin-Lea, Abhinav Anand, Matthias Degroote, Hermanni Heimonen, Jakob S. Kottmann, Tim Menke, Wai-Keong Mok, Sukin Sim, Leong-Chuan Kwek,and Alán Aspuru-Guzik. Noisy intermediate-scale quantum algorithms. Rev. Mod. Phys., 94:015004, Feb 2022.

Undetected Measurement Outcome Problem



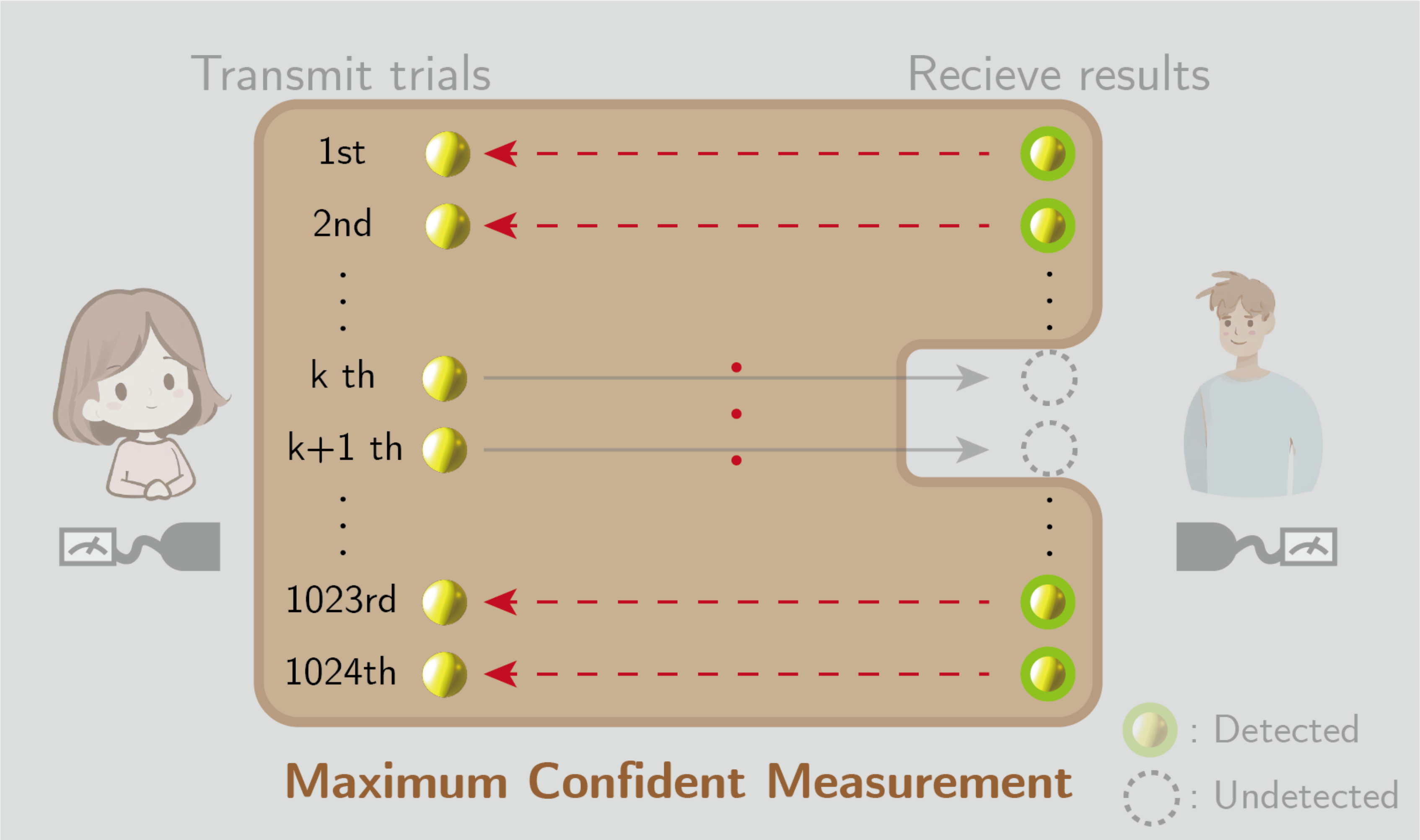
- Challenge: sometimes photons go undetected, which leads to unobserved measurement results.

Undetected Measurement Outcome Problem



- The technique of minimum error discrimination includes all results, even if some outcomes go undetected

Undetected Measurement Outcome Problem



- Making realistic devices fitted with designed quantum information tasks

Maximum confident measurement (MCM)

- Quantum measurement

when a ρ_j is prepared from ensemble $S = \{q_i, \rho_i\}_{i=1}^{n}$ and the POVM E_i is utilized,

$$P_{M|P}(i|j) = tr[\rho_j E_i]$$

- Utilizing the Bayes' rule, retrodiction of state preparation with measurement outcomes gives:

$$P_{P|M}(i|i) = \frac{p_P(i)P_{M|P}(i|i)}{P_M(i)}$$

- Maximum Confidence, C_i

MCM seeks a POVM element that maximizes the

$$C(i) = \max_{E_i} \frac{q_i\, tr[\rho_i E_i]}{tr[\rho E_i]}$$

q_i or $p_P(i)$: Priori Probability
$E_i \geq 0$ and $\Sigma_i E_i = I$
$P_M(i)$: probability of an outcome i
M: Measurement
P: Preparation

Sarah Croke, Erika Andersson, Stephen M. Barnett, Claire R. Gilson, and John Jeffers. Maximum confidence quantum measurements. Physical Review Letters, 96(7), feb 2006.

Maximum confident measurement (MCM)

- Divide and multiply by $\sqrt{\rho}$

$$C(i) = \max_{E_i} \frac{q_i \, tr[\rho_i E_i]}{tr[\rho E_i]} = \max_{E_i} \frac{q_i \, tr[\sqrt{\rho}^{-1} \rho_i \sqrt{\rho}^{-1} \, \sqrt{\rho} E_i \sqrt{\rho}]}{tr[\rho E_i]}$$

- Define $\tilde{\rho}_i = \sqrt{\rho}^{-1} q_i \rho_i \sqrt{\rho}^{-1}$ and $Q_i = \frac{\sqrt{\rho} E_i \sqrt{\rho}}{tr[\rho E_i]}$ ($Tr[Q_i] = 1$), subsequently, it can be expressed as a linear optimization problem:

$$C(i) = \max_{Q_i \geq 0, tr[Q_i]=1} tr[\tilde{\rho}_i Q_i]$$

$q_i \; or \; p_P(i)$: Priori Probability
$E_i \geq 0$ and $\Sigma_i E_i = I$
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Maximum confident measurement (MCM)

- Given that $tr[Q_i] = 1$, we can define $Q_i = |v_i\rangle\langle v_i|$, where

$$|v_i\rangle = U(\theta)|i-1\rangle \;\; (0 \leq i \leq \lceil \log_2 n \rceil)$$

  - n is number of states in ensemble.
  - $U(\theta)$ denotes parameterized unitary transformation by the vector θ

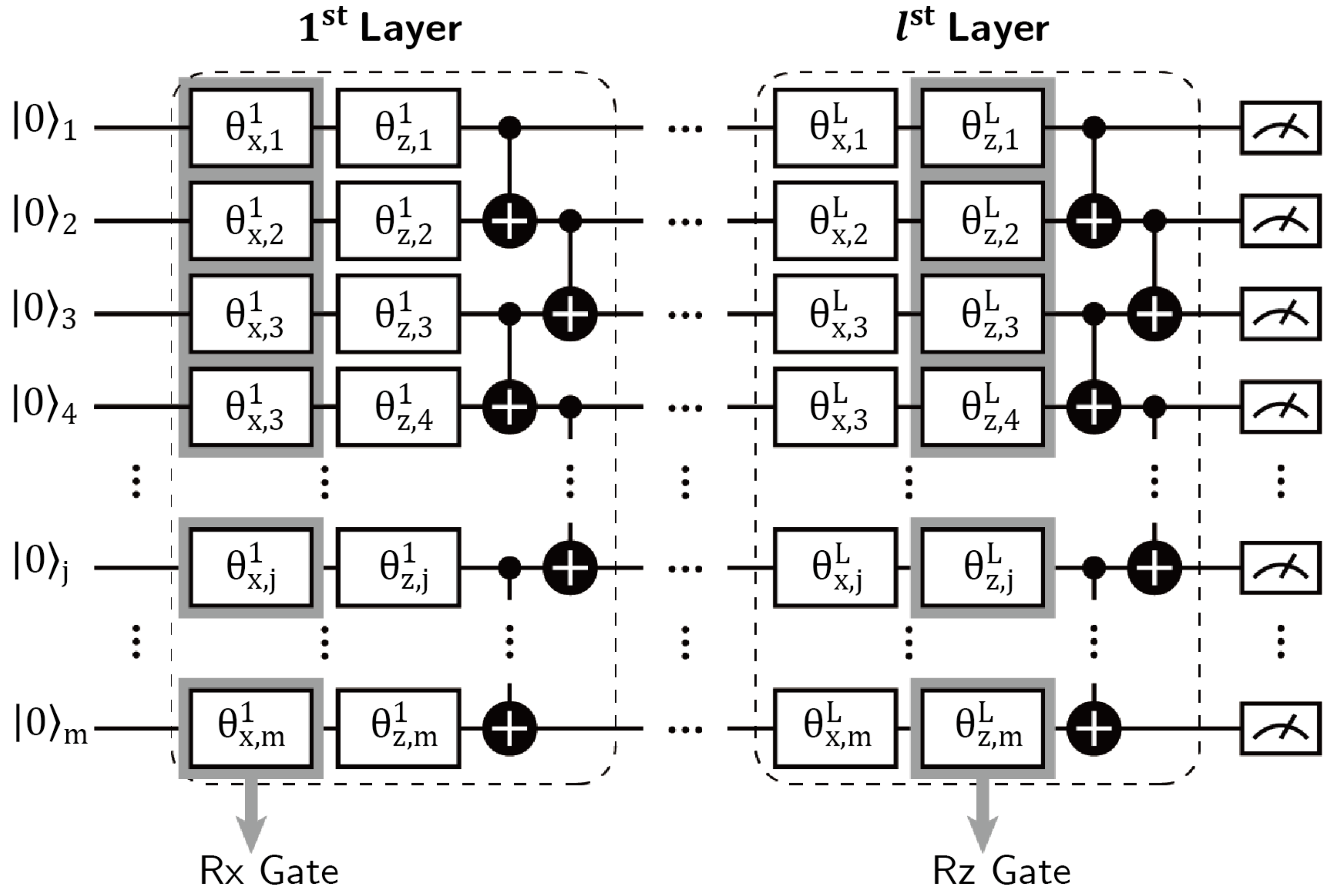
- Substituting this into the MCM formulation, we obtain:

$$C(i) = \max Tr[U^\dagger \sqrt{\rho}^{-1} q_i \rho_i \sqrt{\rho}^{-1} U |i-1\rangle\langle i-1|]$$

The ***key challenge*** here is to ***identify the unitary transformation*** defined in $\lceil \log_2 n \rceil$ qubits that optimizes C_i.

PQC: NISQ friendly circuit model

- ***<u>To address NISQ limitations, We utilized PQC as an ansatz in VQE for optimal MCM</u>***
- PQC, Hardware - efficient ansatz



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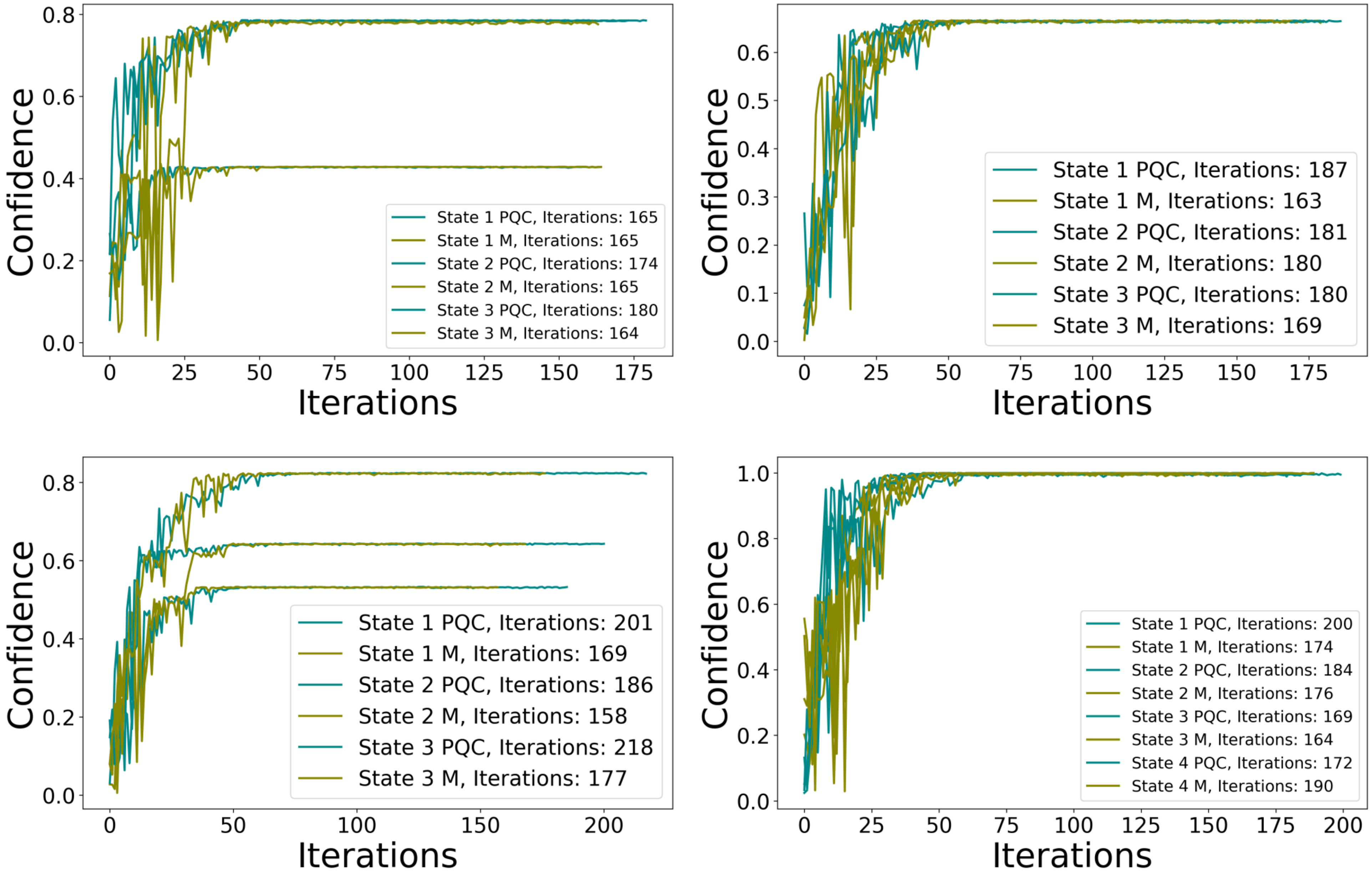
Comparison: PQC vs Exact measurement circuit

- Results



- States

- Left up

$|0\rangle, \sqrt{\frac{2}{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle,$
$\sqrt{\frac{2}{3}}|0\rangle - \sqrt{\frac{2}{3}}|1\rangle$

- Right up

$\cos\theta\,|0\rangle + \sin\theta\,|1\rangle,$
$\cos\theta\,|0\rangle + e^{2\pi i/3}\sin\theta\,|1\rangle,$
$\cos\theta\,|0\rangle + e^{-2\pi i/3}\sin\theta\,|1\rangle$

- Left down

$\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle,$
$\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle, \frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle,$

- Right down

$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),$
$\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle), |00\rangle, |++\rangle$

- Num. of parameters

PQC: 12 (3 layers, depth 12)
Exact: 11 (depth 5)

Approach 2. ensemble measurement

- Measurement can be effectively addressed by maximum confidence measurement (MCM) as it does not consider undetected results.

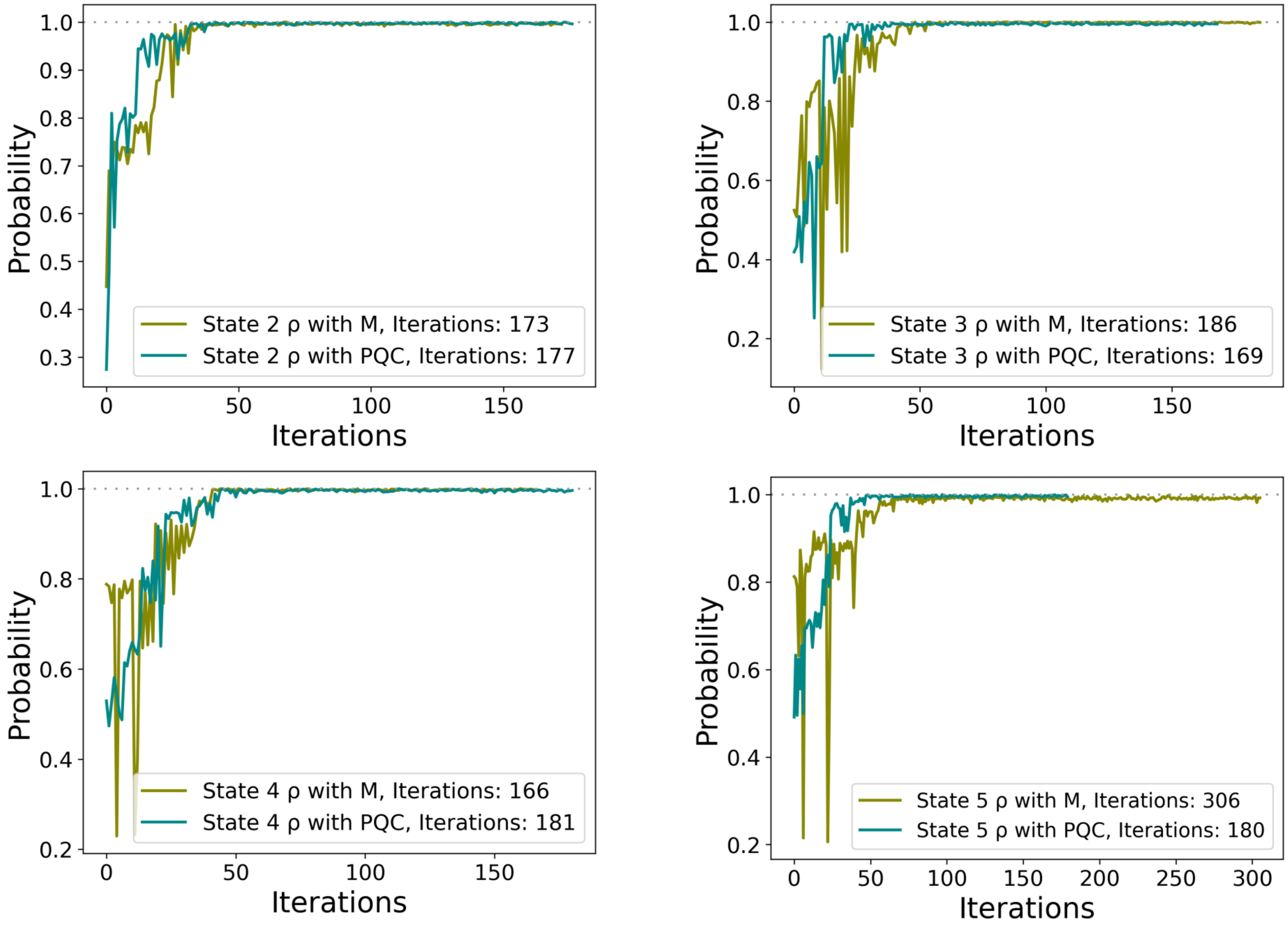
- Furthermore, We can consider ***increasing the effectiveness of detected outcomes***. This can be done by Measurements for the ensemble ρ itself

- The main challenge to overcome is identifying the measurement operator M that minimizes the error rate.

$$P_{error} = 1 - \sum_{i=1}^{n} Tr[\rho_i M_i] = 1 - Tr[\rho M_1]$$

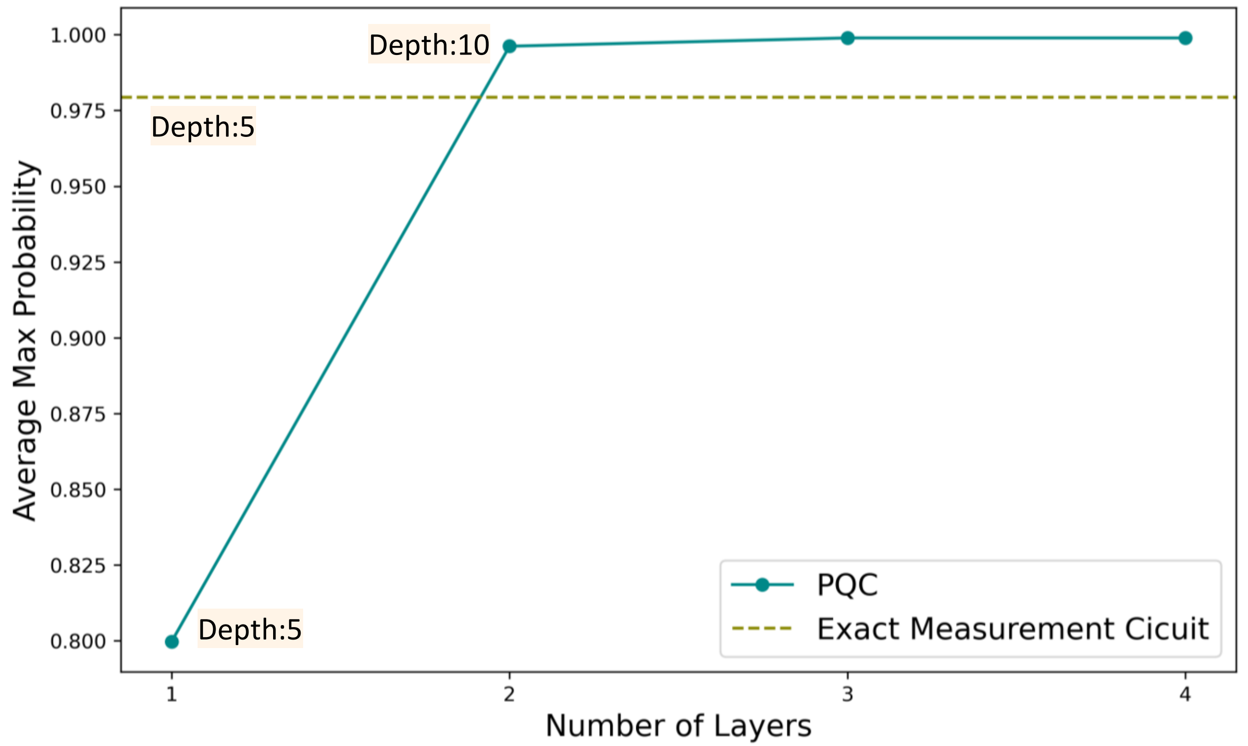
$$\rho = \sum_{i=1}^{n} \rho_i , \qquad \sum_{i=1}^{2} M_i = \mathbb{I}$$

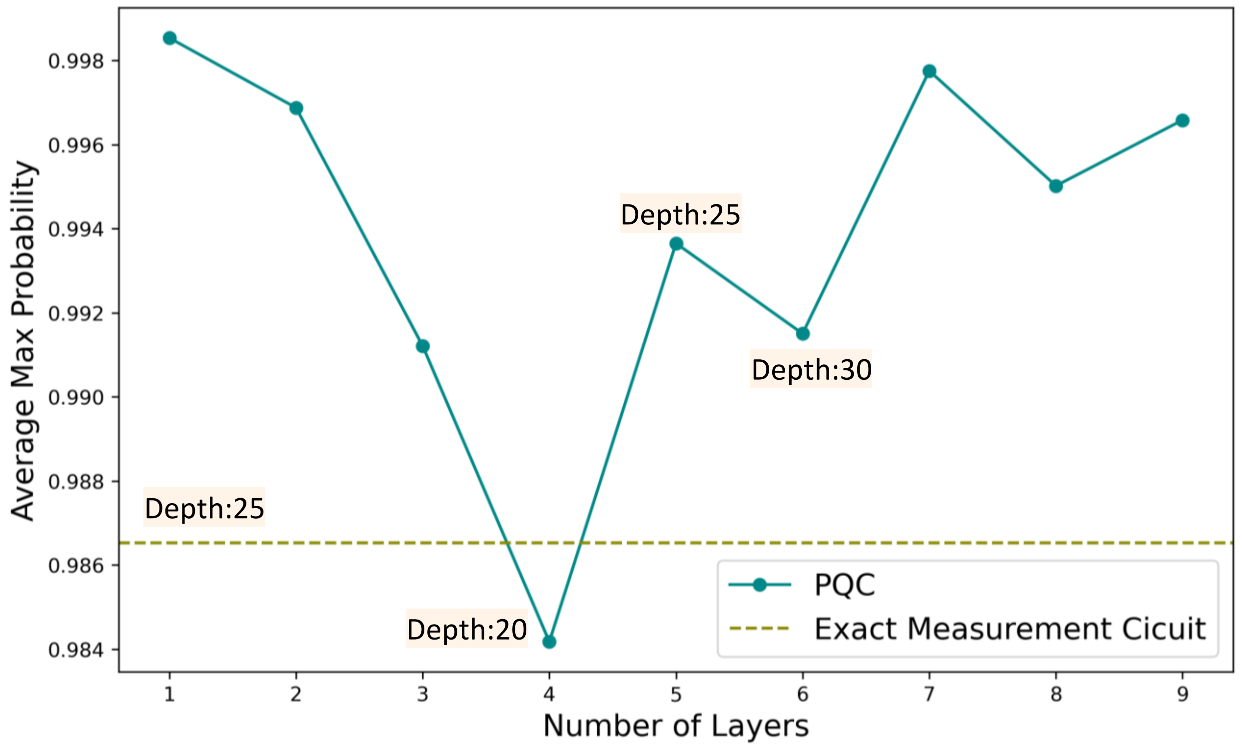
Results: ensemble measurement



Results: ensemble measurement

- Averaged max probabilities, *10 times*





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$|0\rangle, \sqrt{\frac{2}{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle, \sqrt{\frac{2}{3}}|0\rangle - \sqrt{\frac{2}{3}}|1\rangle$

- States

$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle),$
$|00\rangle, |++\rangle$

Approach 3. noise tolerance (depolarization noise)

- Suppose the ensemble include depolarizing noise. Let $\mathcal{N}[\cdot]$ is a depolarizing noise channel and p is the parameter that is related to depolarizing noise. $\mathbb{I}$ is identity matrix.

$$\sigma_i \coloneqq \mathcal{N}[\rho_i] = (1-p)\rho_i + p\frac{\mathbb{I}}{2}$$

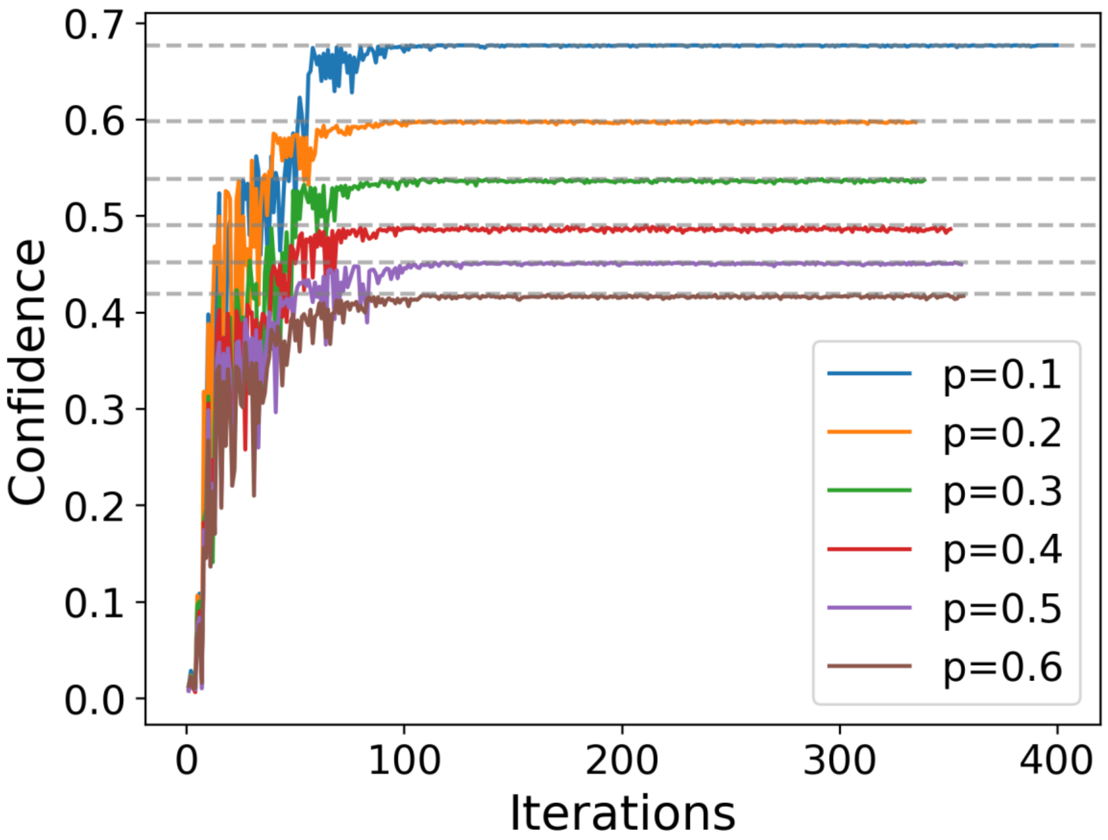
- For evaluating noise resilience, we may examine two scenarios:

  - Access to the precise depolarizing noise ***is unavailable***; We identify ρ_i from a noise-affected ensemble $\sigma = \mathcal{N}[\rho]$

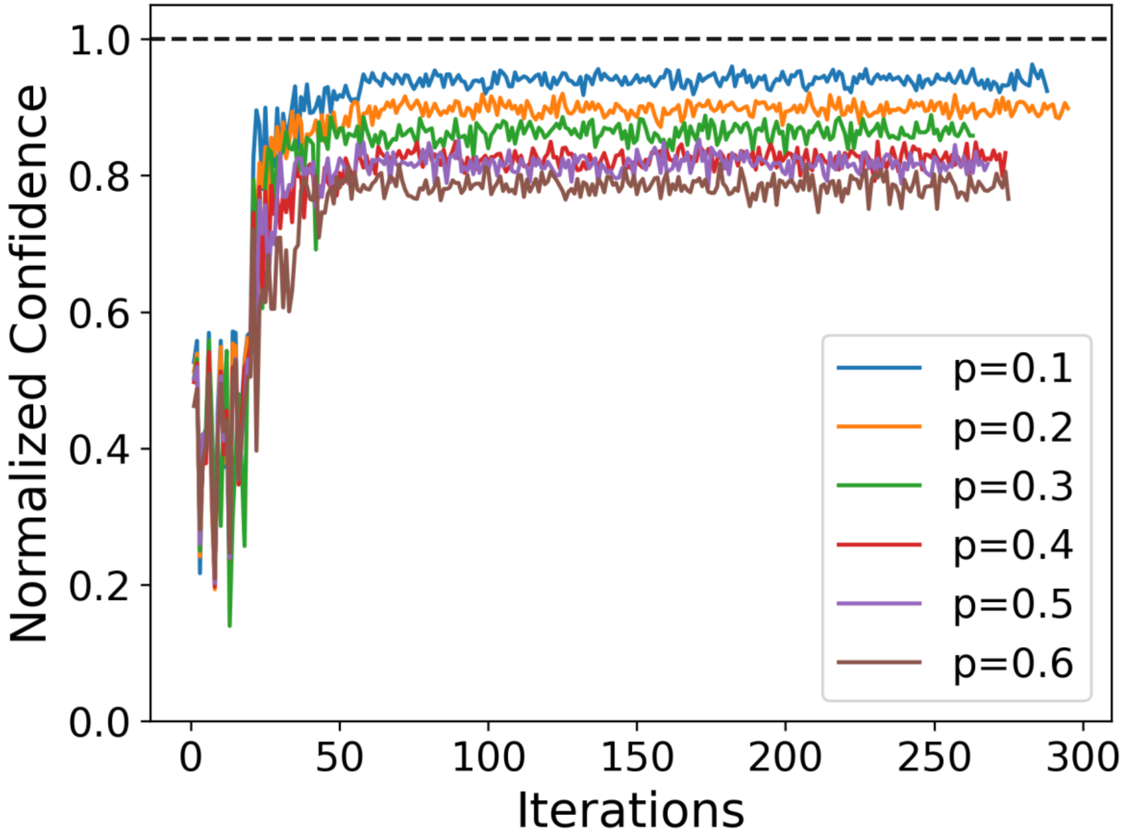
  - ***With access to the depolarizing noise***, we determine σ_i from a noise-affected ensemble $\sigma = \mathcal{N}[\rho]$

Results: noise tolerance

- Measurement ρ_2 from σ



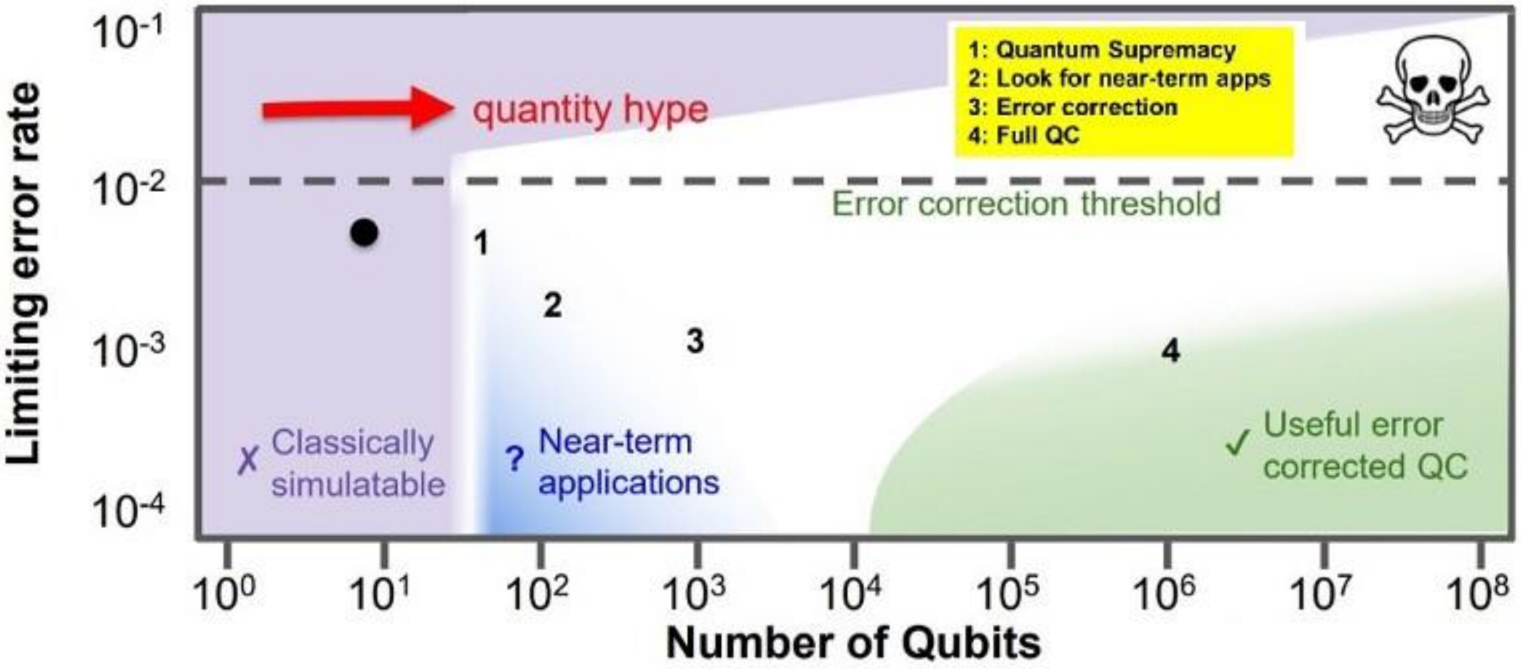
- Measurement σ_2 from σ



Summary and Conclusion

- We presented an implementation of MCMs in a quantum circuit model with realistic quantum devices.
- MCMs are useful in the NISQ era: they take undetected events into account to conclude about which state has been prepared.
- We demonstrated a construction of MCM with PQCs, which are NISQ friendly, in a hybrid quantum-classical manner with VQE.
- We demonstrated a circuit that boosts detected results efficiency.
- The usefulness of PQCs
    - PQCs vs Exact Circuit demonstrated for an MCM: PQCs are both cost-effective and precise in the NISQ regime.
    - We displayed PQC's resilience to depolarizing noise.

Qubit quality vs Quantity relationship



- As the number of qubits increases, the acceptable or tolerable error rate decreases

- **Near-term Applications (NISQ Era)**: These quantum systems are beyond classical simulation capabilities but are still prone to errors.

- **Error correction threshold:** Past this threshold, error correction becomes ineffective.

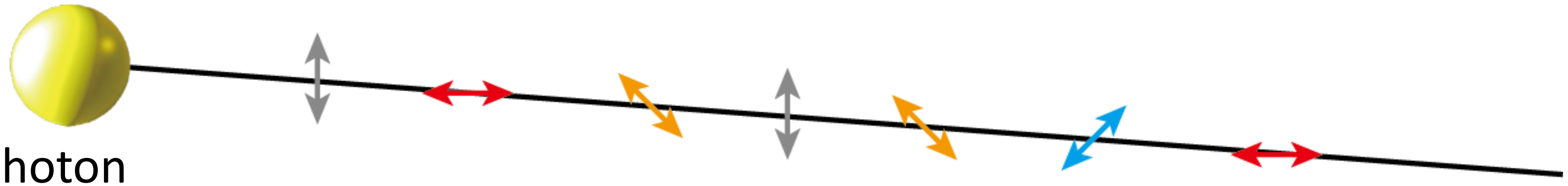
What is the quantum communication?

- BB84 protocol, Charles Bennet et al. (1984)

Alice

Random bit	0	1	1	0	1	0	1
Random sending basis	+	+	X	+	X	X	+
Photon polarization	↑	→	↘	↑	↘	↗	→
random measuring basis	+	X	X	X	+	X	+
Photon polarization measures	↑	↗	↘	↗	→	↗	→
Public discussion of basis							
Shared secret key	0		1			0	1

Bob



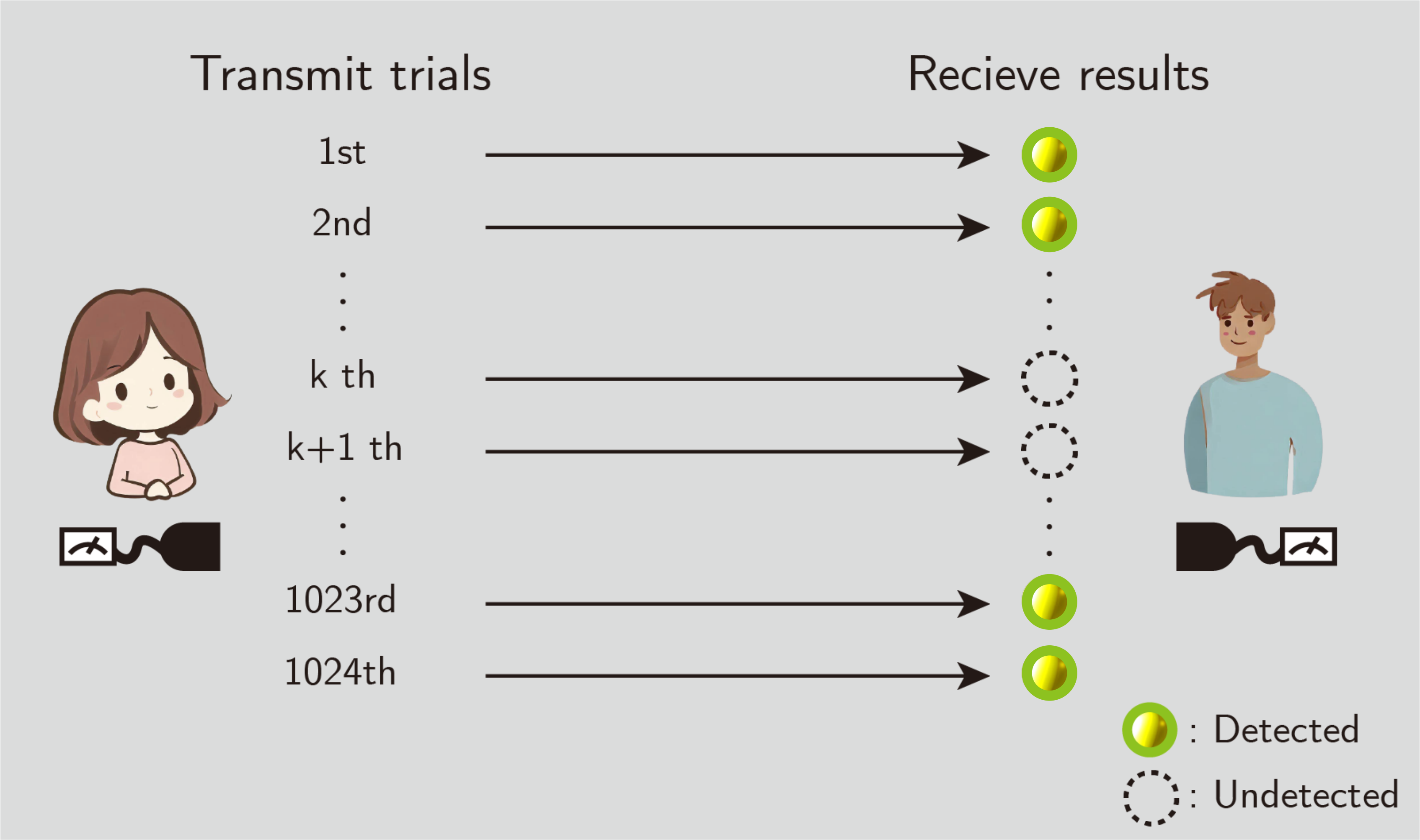
Photon

C. H. Bennett and G. Brassard, in Proceedings of IEEE International Conference on Computer, Systems, and Signal Processing _IEEE, New York, 1984_, p. 175.

Maximum Confident Measurement (MCM)

- Challenge: sometimes photons go undetected, which leads to unobserved measurement results.

Maximum Confident Measurement (MCM)



- Challenge: sometimes photons go undetected, which leads to unobserved measurement results.

Minimum Error Discrimination (MED) and Unambiguous State Discrimination (USD)

- Suppose state discrimination between $|\psi_0\rangle, |\psi_1\rangle$. State $|\psi_x\rangle$ is prepared with probability q_x $x \in \{0,1\}$ and measurement devices output $y \in \{0,1,2\}$. When $y = 2$, we denote it inconclusive outcome

  - $\eta_{err} = q_0 \Pr(y = 1|x = 0) + q_1 \Pr(y = 0|x = 1)$
  - $\eta_{inc} = q_0 \Pr(y = 2|x = 0) + q_1 \Pr(y = 2|x = 1)$

- MED minimize the η_{err} under $\eta_{inc} = 0$

- USD minimize the η_{inc} under $\eta_{err} = 0$

Approach 1. Realization of MCM through VQA and PQC

- Sets 2

$\psi_1 = |0\rangle$

$\psi_2 = \sqrt{\frac{2}{3}}|0\rangle + \sqrt{\frac{1}{3}}|1\rangle$

$\psi_3 = \sqrt{\frac{2}{3}}|0\rangle - \sqrt{\frac{1}{3}}|1\rangle$

$\rho = \begin{pmatrix} \frac{7}{9} & 0 \\ 0 & \frac{2}{9} \end{pmatrix}$

- Sets 3

$\psi_1 = \cos\theta\,|0\rangle + \sin\theta\,|1\rangle$
$\psi_2 = \cos\theta\,|0\rangle + e^{2\pi i/3}\sin\theta\,|1\rangle$
$\psi_3 = \cos\theta\,|0\rangle + e^{-2\pi i/3}\sin\theta\,|1\rangle$

$\rho = \begin{pmatrix} \cos^2\theta & 0 \\ 0 & \sin^2\theta \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$

- Sets 4

$\psi_1 = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle$

$\psi_2 = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$

$\psi_3 = \frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle$

$\rho = \begin{pmatrix} 0.45118 & 0.1178 \\ 0.117851 & 0.548816 \end{pmatrix}$

- Sets 5

$\psi_1 = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$\psi_2 = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$

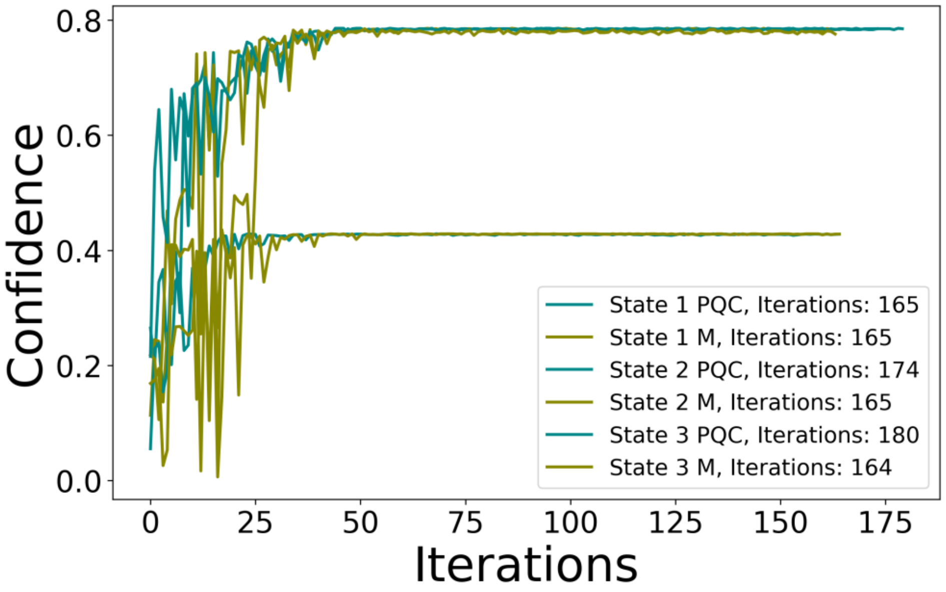
$\psi_3 = |00\rangle$

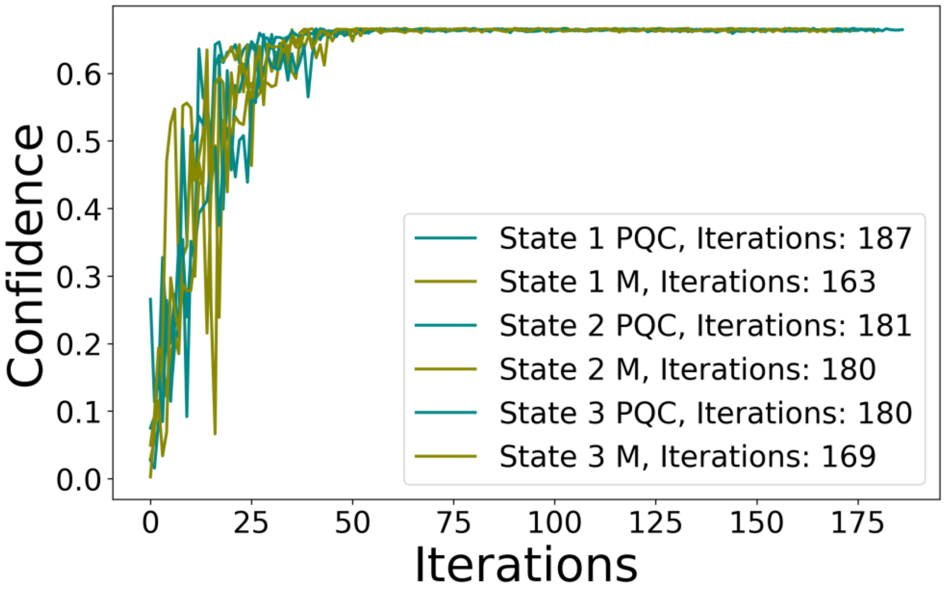
$\psi_4 = |+\,+\rangle$

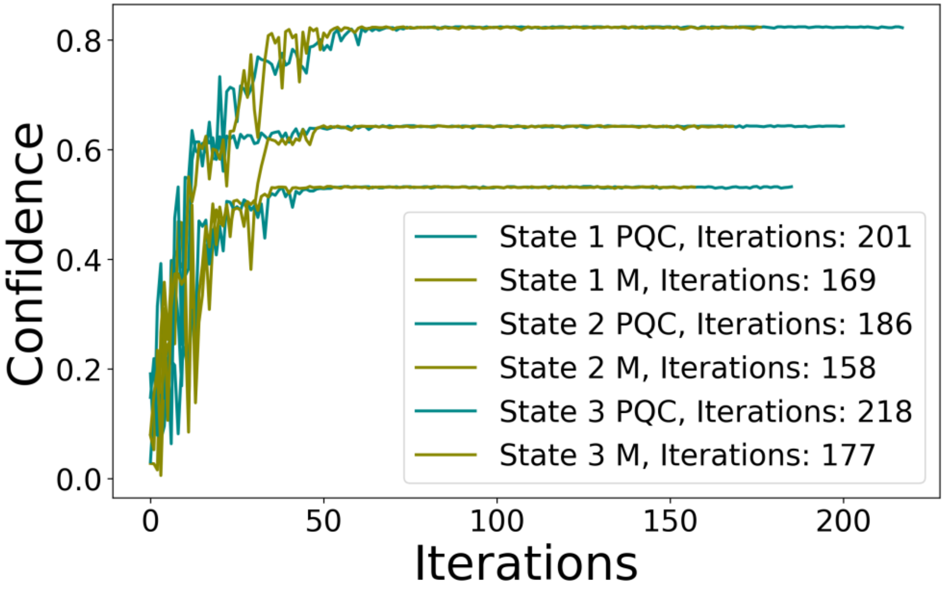
$\rho = \begin{pmatrix} \frac{7}{16} & \frac{1}{16} & \frac{1}{16} & \frac{3}{16} \\ \frac{1}{16} & \frac{3}{16} & -\frac{1}{16} & \frac{1}{16} \\ \frac{1}{16} & -\frac{1}{16} & \frac{3}{16} & \frac{1}{16} \\ \frac{3}{16} & \frac{1}{16} & \frac{1}{16} & \frac{3}{16} \end{pmatrix}$

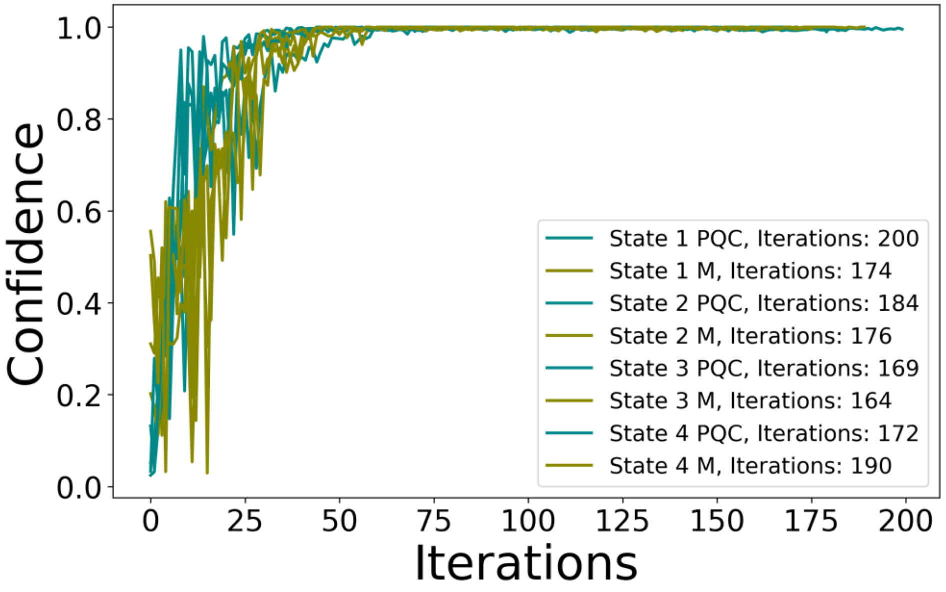
Comparison: PQC vs Exact measurement circuit

- Results









- States

  - Left up
    $|0\rangle, \sqrt{\frac{2}{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle,$
    $\sqrt{\frac{2}{3}}|0\rangle - \sqrt{\frac{2}{3}}|1\rangle$
  - Right up
    $\cos\theta\,|0\rangle + \sin\theta\,|1\rangle,$
    $\cos\theta\,|0\rangle + e^{2\pi i/3}\sin\theta\,|1\rangle,$
    $\cos\theta\,|0\rangle + e^{-2\pi i/3}\sin\theta\,|1\rangle$
  - Left down
    $\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle,$
    $\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle, \frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle,$
  - Right down
    $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),$
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  - Num. of parameters
    PQC: 12 (3 layers, depth 12)
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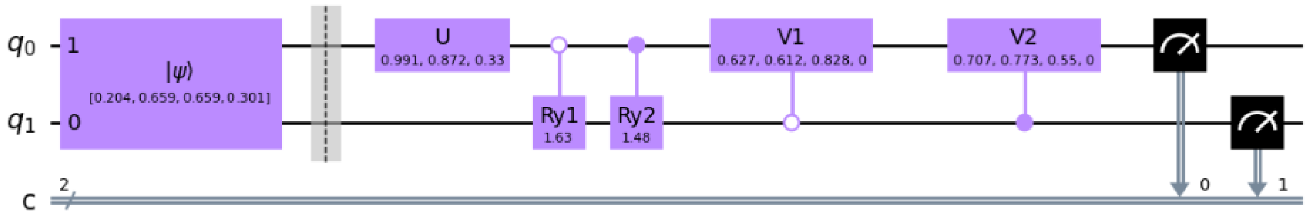
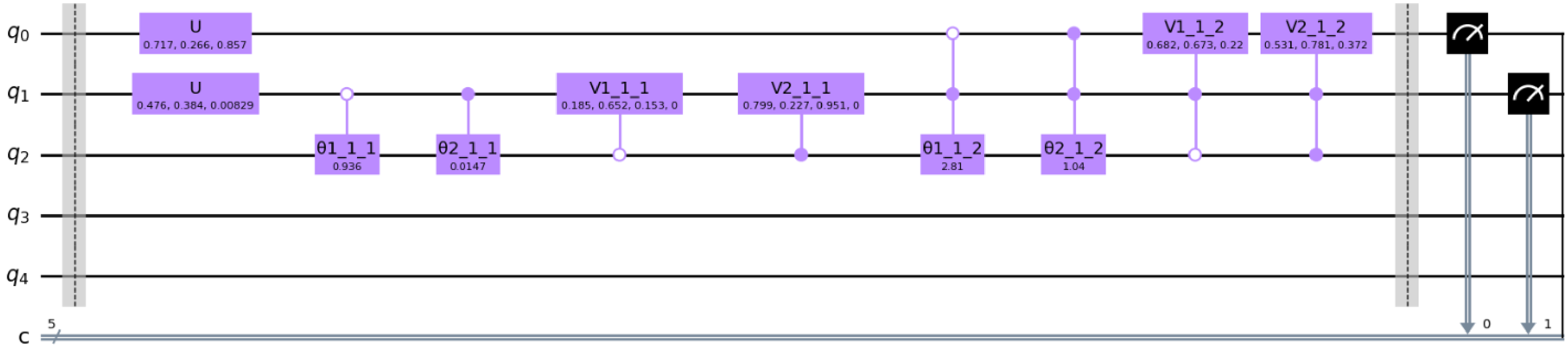
Approach 2. ensemble measurement

- 1 qubit ensemble state

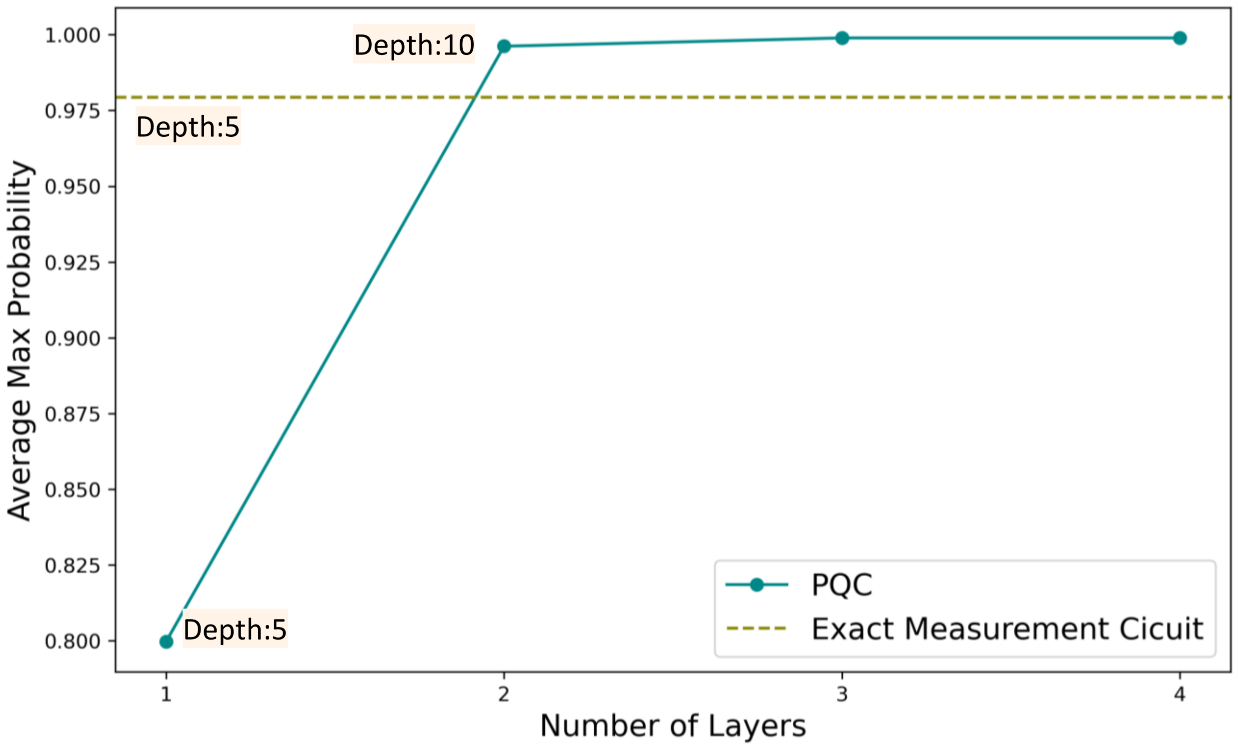


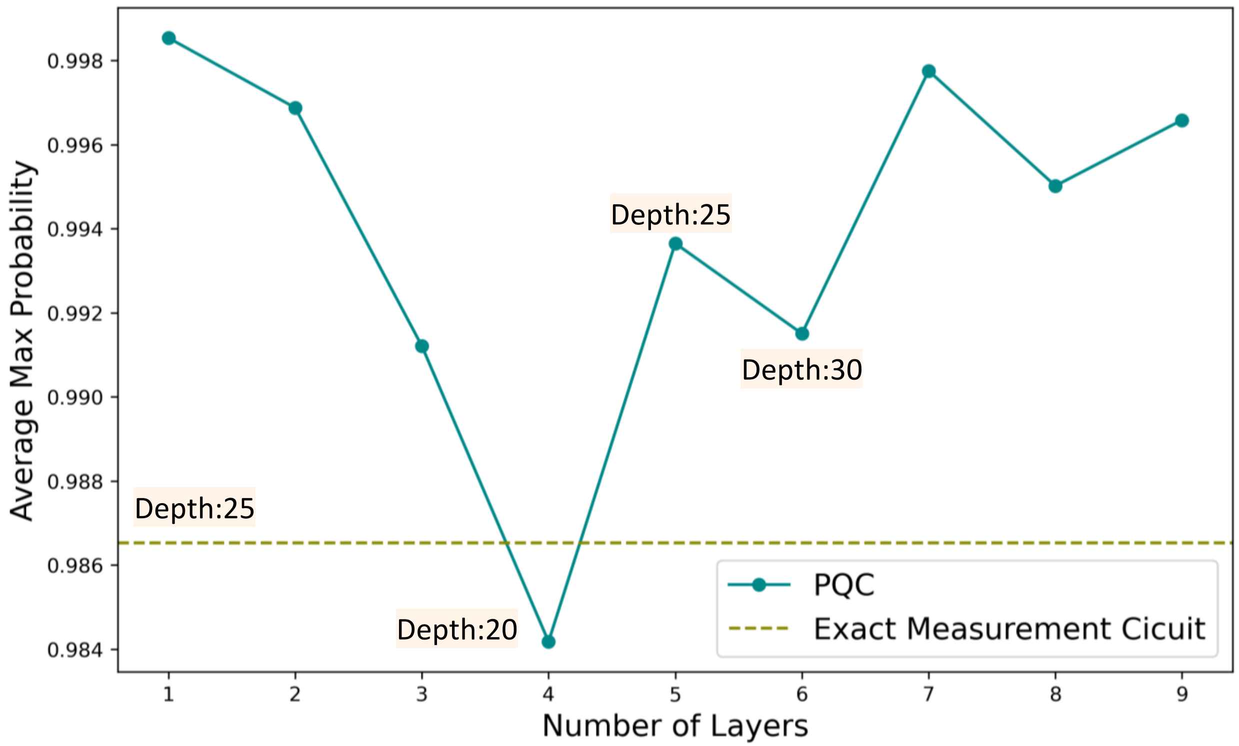
- 2 qubit ensemble state



Results: ensemble measurement

- Averaged max probabilities, <u>*10 times*</u>





- States

$|0\rangle, \sqrt{\frac{2}{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle, \sqrt{\frac{2}{3}}|0\rangle - \sqrt{\frac{2}{3}}|1\rangle$

- States

$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle),$
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Results: ensemble measurement

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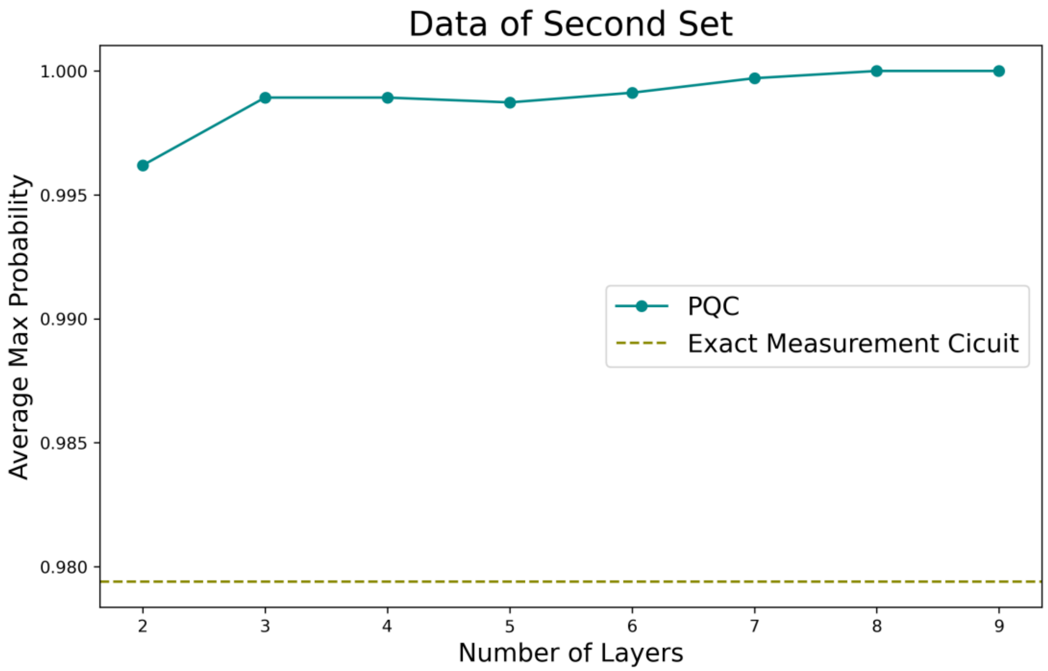
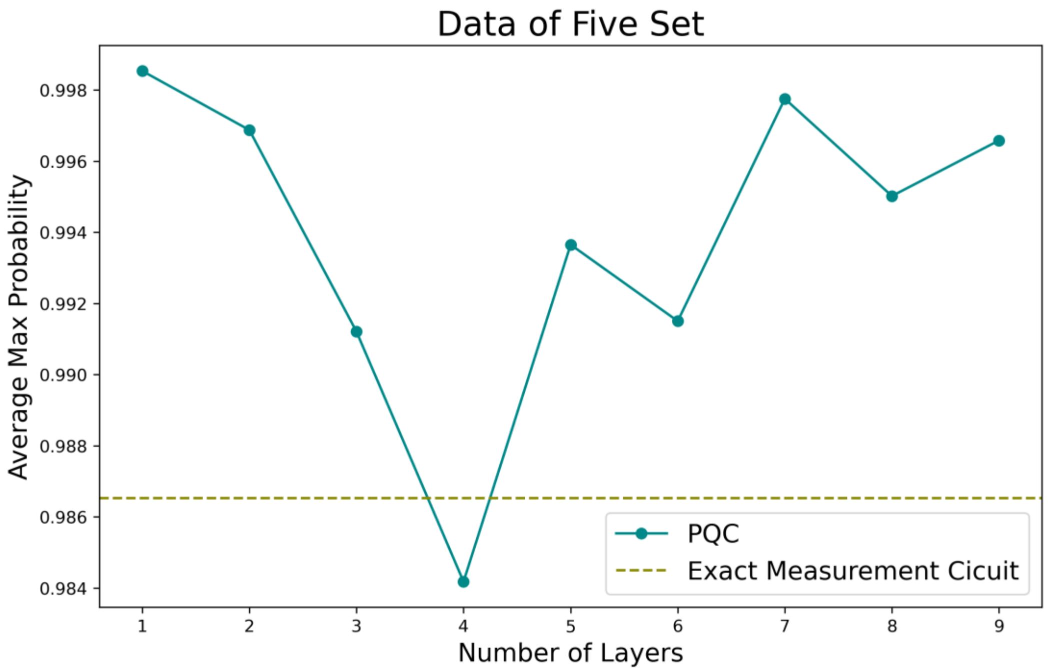


$|0\rangle, \sqrt{\frac{2}{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle,$
$\sqrt{\frac{2}{3}}|0\rangle - \sqrt{\frac{2}{3}}|1\rangle$



$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),$
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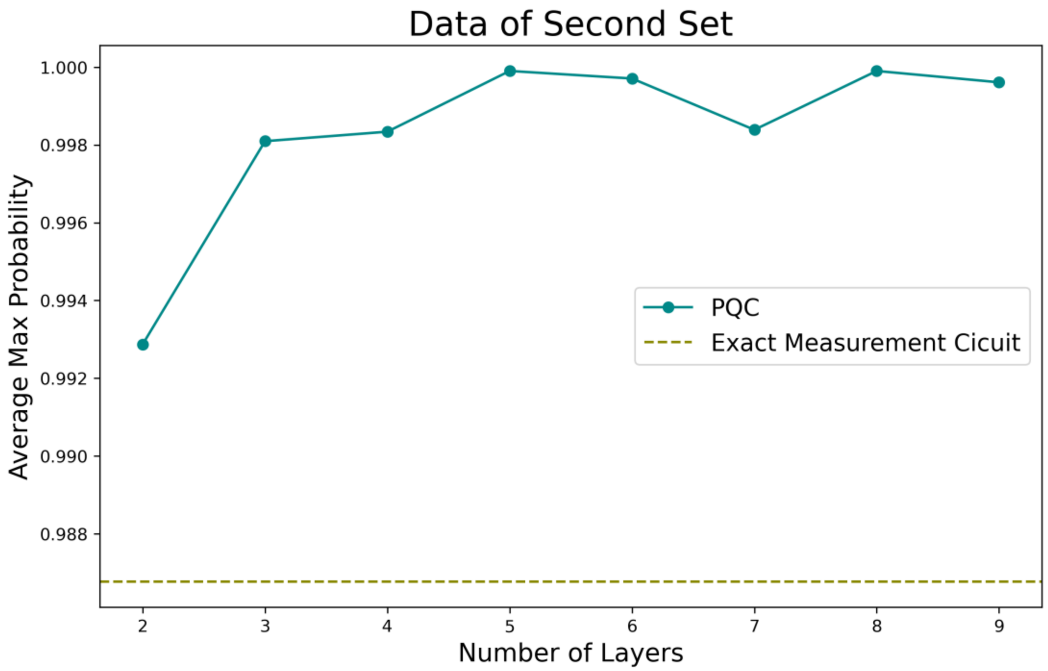
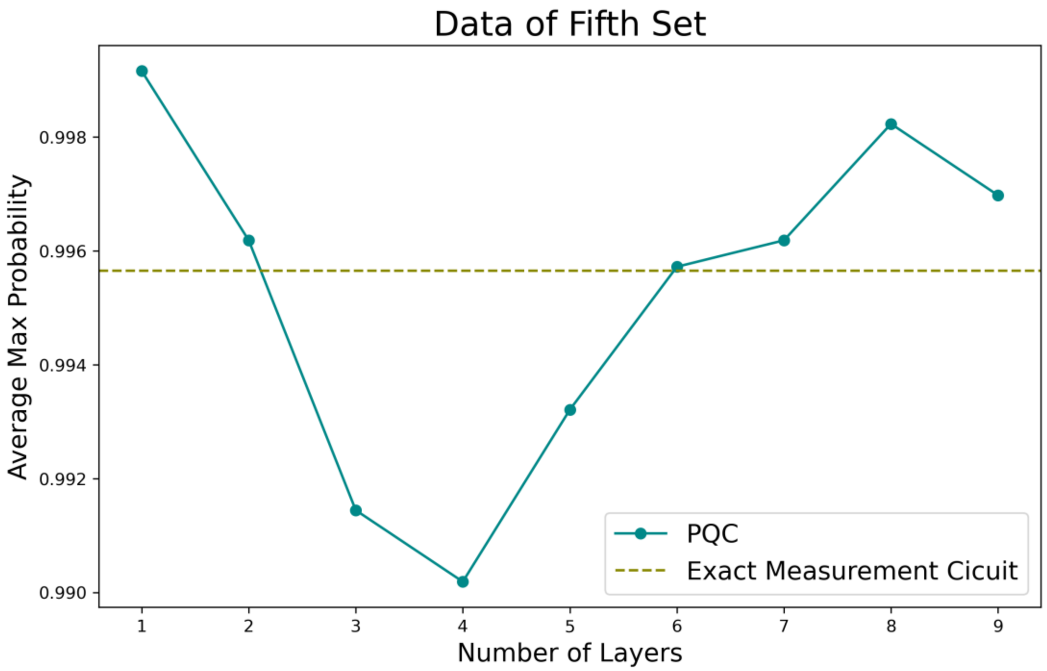
$|00\rangle, |++\rangle$

Results: ensemble measurement

- Averaged max probabilities, <u>*20 times*</u>



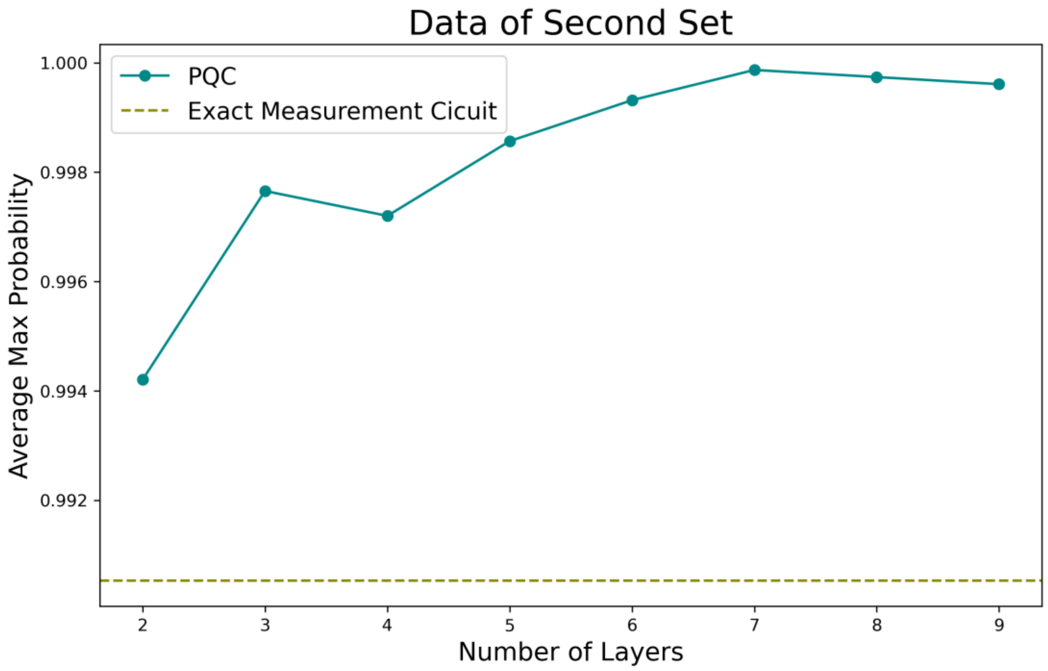
$$|0\rangle, \sqrt{\frac{2}{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle, \sqrt{\frac{2}{3}}|0\rangle - \sqrt{\frac{2}{3}}|1\rangle$$

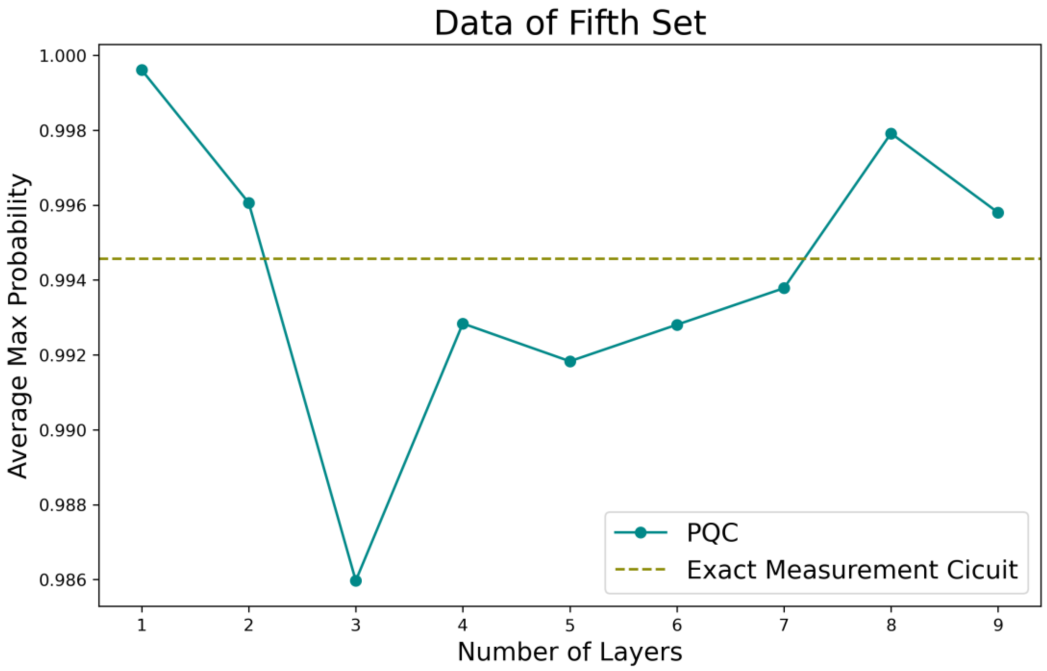


$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle), |00\rangle, |++\rangle$$

Results: ensemble measurement

- Averaged max probabilities, <u>*30 times*</u>



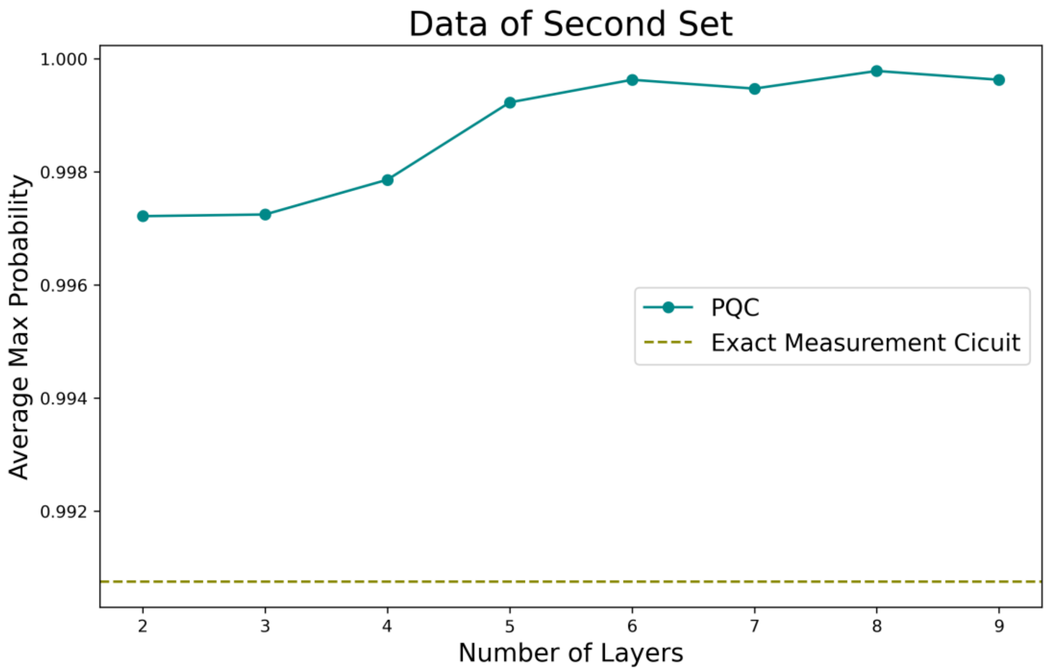


$|0\rangle, \sqrt{\frac{2}{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle,$
$\sqrt{\frac{2}{3}}|0\rangle - \sqrt{\frac{2}{3}}|1\rangle$

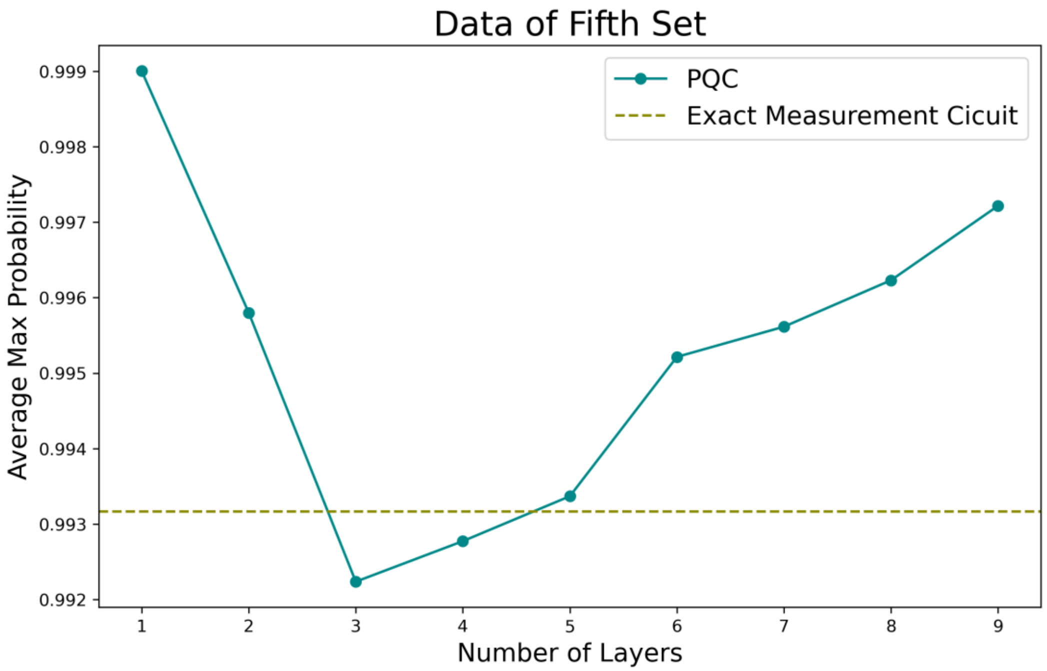
$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),$
$\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle),$
$|00\rangle, |++\rangle$

Results: ensemble measurement

- Averaged max probabilities, <u>*100 times*</u>



$|0\rangle, \sqrt{\frac{2}{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle,$
$\sqrt{\frac{2}{3}}|0\rangle - \sqrt{\frac{2}{3}}|1\rangle$



$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),$
$\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle),$
$|00\rangle, |++\rangle$

Approach 3. noise tolerance (depolarization noise)

- Measurement ρ_i from σ

$$C(i) = \max_{E_i} \frac{q_i\ tr[\rho_i E_i]}{tr[\sigma E_i]} = \max_{E_i} \frac{q_i\ tr[\sqrt{\sigma}^{-1} \rho_i \sqrt{\sigma}^{-1}\ \sqrt{\sigma} E_i \sqrt{\sigma}]}{tr[\sigma E_i]} = \max_{Q_i \geq 0, tr[Q_i]=1} tr[\widetilde{\rho}_i Q_i]$$

$$\tilde{\rho}_i = \sqrt{\sigma}^{-1} q_i \rho_i \sqrt{\sigma}^{-1} \text{ and } Q_i = \frac{\sqrt{\sigma} E_i \sqrt{\sigma}}{tr[\sigma E_i]}$$

- Measurement σ_i from σ

$$C(i) = \max_{E_i} \frac{q_i\ tr[\sigma_i E_i]}{tr[\sigma E_i]} = \max_{E_i} \frac{q_i\ tr[\sqrt{\sigma}^{-1} \sigma_i \sqrt{\sigma}^{-1}\ \sqrt{\sigma} E_i \sqrt{\sigma}]}{tr[\sigma E_i]} = \max_{Q_i \geq 0, tr[Q_i]=1} tr[\widetilde{\sigma}_i Q_i]$$

$$\tilde{\sigma}_i = \sqrt{\sigma}^{-1} q_i \sigma_i \sqrt{\sigma}^{-1} \text{ and} Q_i = \frac{\sqrt{\sigma} E_i \sqrt{\sigma}}{tr[\sigma E_i]}$$

Approach 3. noise tolerance (depolarization noise)

$$\psi_1 = |0\rangle$$
$$\psi_2 = \sqrt{\tfrac{2}{3}}|0\rangle + \sqrt{\tfrac{1}{3}}|1\rangle$$
$$\psi_3 = \sqrt{\tfrac{2}{3}}|0\rangle - \sqrt{\tfrac{1}{3}}|1\rangle$$
$$\rho = \begin{pmatrix} \frac{7}{9} & 0 \\ 0 & \frac{2}{9} \end{pmatrix}$$

$$\sigma_i \coloneqq \mathcal{N}[\rho_i] = (1-p)\rho_i + p\frac{\mathbb{I}}{2}$$
$$\sigma \coloneqq \mathcal{N}[\rho] = (1-p)\rho + p\frac{\mathbb{I}}{2}$$

$$\tilde{\rho}_2 = \sqrt{\sigma}^{-1} q_2 \rho_2 \sqrt{\sigma}^{-1}$$
$$= \begin{pmatrix} \frac{4}{(14+9p)\left(\frac{2}{4+9p}+\frac{4}{14+9p}\right)} & \frac{2\sqrt{2}}{\sqrt{4+9p}\sqrt{14+9p}\left(\frac{2}{4+9p}+\frac{4}{14+9p}\right)} \\ \frac{2\sqrt{2}}{\sqrt{4+9p}\sqrt{14+9p}\left(\frac{2}{4+9p}+\frac{4}{14+9p}\right)} & \frac{2}{(4+9p)\left(\frac{2}{4+9p}+\frac{4}{14+9p}\right)} \end{pmatrix}$$

$$\sigma_i \coloneqq \mathcal{N}[\rho_i] = (1-p)\rho_i + p\frac{\mathbb{I}}{2}$$
$$\tilde{\sigma}_2 = \sqrt{\sigma}^{-1} q_2 \sigma_2 \sqrt{\sigma}^{-1}$$
$$= \begin{pmatrix} \frac{(4+3p)(4+9p)}{44+108p+54p^2} & \frac{\sqrt{2}\sqrt{(4+9p)(14+9p)}}{22+54p+27p^2} \\ \frac{\sqrt{2}\sqrt{(4+9p)(14+9p)}}{22+54p+27p^2} & \frac{(2+3p)(14+9p)}{44+108p+54p^2} \end{pmatrix}$$