Measurement-Based Estimation of Observables

Detecting Entanglement by State Preparation and Fixed Measurements

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Contents

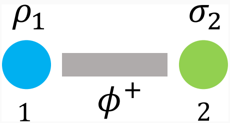
Introduction

- Estimating observables requires various measurement settings in general, e.g.,

$$W = \frac{1}{2}\mathbb{1}\otimes\mathbb{1} - |\psi^-\rangle\langle\psi^-| = \frac{1}{2}(\mathbb{1}\otimes\mathbb{1} + X\otimes X + Y\otimes Y + Z\otimes Z)$$

- A fixed measurement setting is advantageous for some systems, e.g., distributed sensing
- This work: Measurement-based estimation of observables
  - *Duplex state* preparation + fixed local measurements $\rightarrow$ Observable estimation
  - Focus on entanglement witnesses (EWs)
  - Application of the quantum teleporation scheme
- cf. Measurement-based quantum computation
  - Graph state preparation + various local measurements $\rightarrow$ Quantum dynamics (unitary transformation)

Positive or negative semi-definite operators



- $\mathcal{O}$: an observable that has only positive or negative eigenvalues

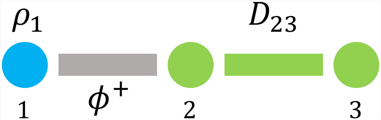
$$\mathcal{O}^T = k\sigma, \text{ where } \sigma \geq 0, k = \mathrm{tr}[\mathcal{O}], \mathrm{tr}[\sigma] = 1$$

$$|\phi_d^+\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} |kk\rangle \,, \; d : \text{dimension}$$

Then

$$\begin{aligned} \mathrm{tr}[\rho\mathcal{O}] &= \mathrm{tr}[d\, |\phi_d^+\rangle\langle\phi_d^+|_{12}\, \rho_1 \otimes \mathcal{O}_2^T] \\ &= d\, k\, \mathrm{tr}[\, |\phi_d^+\rangle\langle\phi_d^+|_{12}\, \rho_1 \otimes \sigma_2] \end{aligned}$$

Single-qubit observables



- $\mathcal{O}$: an single-qubit observable that has positive and negative eigenvalues

$$\mathcal{O}^T = \lambda_+ |\lambda_+\rangle\langle\lambda_+| - \lambda_- |\lambda_-\rangle\langle\lambda_-|, \quad \lambda_\pm \geq 0$$

$$c_+ = \frac{\lambda_+}{\lambda_+ + \lambda_-}, c_- = \frac{\lambda_-}{\lambda_+ + \lambda_-}$$

- D_{23}: duplex state

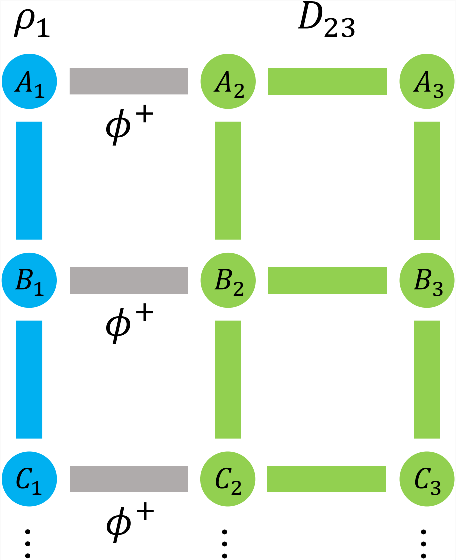
$$D_{23} = c_+ |\lambda_+\rangle\langle\lambda_+|_2 \otimes |1\rangle\langle 1|_3 + c_- |\lambda_-\rangle\langle\lambda_-|_2 \otimes |0\rangle\langle 0|_3, \text{ or,}$$

$$D_{23} = |D_{23}\rangle\langle D_{23}|, |D_{23}\rangle = \sqrt{c_+} |\lambda_+\rangle_2 |1\rangle_3 + \sqrt{c_-} |\lambda_-\rangle_2 |0\rangle_3$$

- Expected value of the observable can be calculated as

$$\mathrm{tr}[\rho_1 \otimes D_{23} |\phi^+\rangle\langle\phi^+|_{12} \otimes (|1\rangle\langle 1| - |0\rangle\langle 0|)_3] = \frac{\mathrm{tr}[\rho\mathcal{O}]}{2(\alpha + \beta)}$$

General observables



General observables

- $\mathcal{O}$: an observable on an n-partite d-dimensional system

$$\mathcal{O}^T = \sum_{k=0}^{d^n-1} \lambda_k \left|\lambda_k\right\rangle\!\left\langle\lambda_k\right| , \quad \left\langle\lambda_i\middle|\lambda_j\right\rangle = \delta_{ij}$$

$$c_k = \frac{|\lambda_k|}{\sum_k |\lambda_k|}$$

- D_{23}: duplex state

$$D_{23} = \sum_k c_k \left|\lambda_k\right\rangle\!\left\langle\lambda_k\right|^{(2)} \otimes \left|k\right\rangle\!\left\langle k\right|^{(3)} , \text{ or}$$

$$D_{23} = \left|D_{23}\right\rangle\!\left\langle D_{23}\right| , \left|D_{23}\right\rangle = \sum_k \sqrt{c_k} \left|\lambda_k\right\rangle^{(2)} \otimes \left|k\right\rangle^{(3)}$$

- Expected value of $\mathcal{O}$

$$\mathrm{tr}[\,\left|\phi_d^+\right\rangle\!\left\langle\phi_d^+\right|^{\otimes n} \rho_1 \otimes D_{23}(\sum_{k\in\mathbb{P}} \left|k\right\rangle\!\left\langle k\right| - \sum_{k\in\mathbb{N}} \left|k\right\rangle\!\left\langle k\right|)_3] = \frac{\mathrm{tr}[\rho\mathcal{O}]}{d^n(\sum_k |\lambda_k|)}$$

where $\mathbb{P} = \{k \in [d^n] : \lambda_k > 0\}, \ \mathbb{N} = \{k \in [d^n] : \lambda_k < 0\}$ and $[x] = \{0, \ldots, x-1\}$.

The entanglement witness (EW)

- A bipartite quantum state ρ is separable if it can be written as

$$\rho = \sum_i p_i \psi_i^{(A)} \otimes \phi_i^{(B)}.$$

  Otherwise, ρ is entangled.

- An entanglement witness (EW) W is an observable s.t.

$$\boxed{\begin{aligned} \forall \sigma_{sep} \in \text{SEP} :& \text{tr}[\sigma_{sep} W] \geq 0, \\ \exists \rho_{ent} \in \text{ENT} :& \text{tr}[\rho_{ent} W] < 0, \end{aligned}}$$

  where SEP and ENT denote the set of separable and entangled states.

- The EW W detects the entangled state ρ_{ent}.
- A quantum state is entangled iff there exists an EW that detects the state.

Notations

- Projections onto a d-dimensional Bell states

$$P_{st} = |\phi_{st}\rangle\langle\phi_{st}|\,,\ \text{where}\ |\phi_{st}\rangle = \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1} e^{2\pi i jt/d}\,|j, j+s\rangle\,,$$

  for $s, t \in \{0, \dots, d-1\}$
- Separable Bell-diagoanl projectors

$$\Pi_s = \sum_{t=0}^{d-1} P_{st} = \sum_{j=0}^{d-1} |j, j+s\rangle\langle j, j+s| \quad \text{for}\ s \in \{0, \dots, d-1\}.$$

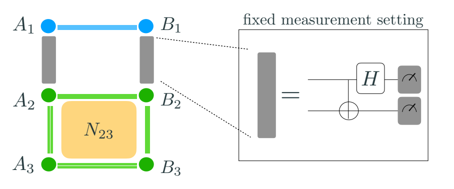
- The flip (permutation) operator

$$\mathbb{F} = d\, P_{00}^{T_A} = \sum_{i,j=0}^{d-1} |i, j\rangle\langle j, i|$$

- Projections onto symmetric and anti-symmetric subspaces

$$\mathcal{S} = \frac{\mathbb{1} + \mathbb{F}}{2},\ \mathcal{A} = \frac{\mathbb{1} - \mathbb{F}}{2}.$$

The framework



1. $\rho_1 = \rho^{(A_1B_1)}$: the state of interest
2. $N_{23} = N^{(A_2B_2A_3B_3)}$: the duplex state realizing EW W
3. Bell measurements onto $P_{00}^{(A_1A_2)} \otimes P_{00}^{(B_1B_2)}$ leave the final state

$$\Lambda^{(1\to3)}(\rho) = R^{(A_3B_3)} = \frac{\mathrm{tr}_{12}[\rho_1 \otimes N_{23}\, P_{00}^{(A_1A_2)} \otimes P_{00}^{(B_1B_2)}]}{\mathrm{tr}[\rho_1 \otimes N_{23}\, P_{00}^{(A_1A_2)} \otimes P_{00}^{(B_1B_2)}]}$$

4. Check the final singlet fraction $f(\rho, N) \equiv \mathrm{tr}[R^{(3)} P_{00}^{(3)}] = \langle\phi_{00}|R|\phi_{00}\rangle$
   - $f(\rho, N) \overset{?}{>} \eta \implies \rho$ is entangled.
   - $\eta = \mathbf{sup}_{\rho_{sep}} f(\rho_{sep}, N) \in [\frac{1}{d}, 1)$ for separable states $\rho_{sep}^{(1)}$.

Constructing the duplex state from the given EW

- Suppose that an EW $W = \sum_j c_j W_j$ is given, where $W_j \geq 0,\ c_j \in \mathbb{R}$.
- For some pure/mixed Bell projections $P_j \geq 0$ and probability p_j, define

$$N_{23} = \sum_j p_j W_j^{(2)} \otimes P_j^{(3)},$$

such that
$$W^{T^{(2)}} \propto \mathrm{tr}_3[N_{23}(\eta \mathbb{1} - P_{00})^{(3)}].$$

$$\begin{aligned}\mathrm{tr}[\rho_1 W^{(1)}] &= \mathrm{tr}[\rho_1 \otimes W^{T^{(2)}}\ d^2 P_{00}^{(A_1A_2)} \otimes P_{00}^{(B_1B_2)}] \\ &\propto \mathrm{tr}[\rho_1 \otimes N_{23}\ P_{00}^{(A_1A_2)} \otimes P_{00}^{(B_1B_2)} \otimes (\eta \mathbb{1} - P_{00})^{(3)}].\end{aligned}$$

$$R^{(3)} = \frac{\mathrm{tr}_{12}[\rho_1 \otimes N_{23}\ P_{00}^{(A_1A_2)} \otimes P_{00}^{(B_1B_2)}]}{\mathrm{tr}[\rho_1 \otimes N_{23}\ P_{00}^{(A_1A_2)} \otimes P_{00}^{(B_1B_2)}]}$$

- Finally, $$f(\rho, N) = \mathrm{tr}[R^{(3)} P_{00}^{(3)}] > \eta \iff \mathrm{tr}[\rho W] < 0$$
- The choice of the duplex state is not unique.

Duplex states can detect all bipartite entangled states

- Define

$$E_d(N) \equiv \sup_{A,B} \frac{\mathrm{tr}[(A \otimes B)N(A^\dagger \otimes B^\dagger)P_{00}]}{\mathrm{tr}[(A \otimes B)N(A^\dagger \otimes B^\dagger)]}$$

  where A, B are matrices such that $A : \mathcal{H}_A \to \mathbb{C}^d$ and $B : \mathcal{H}_B \to \mathbb{C}^d$.
- A bipartite state ρ is entangled iff for all $d \geq 2$ and $\eta \in [\frac{1}{d}, 1)$, there exists a bipartite state N such that $E_d(N) \leq \eta$ and $E_d(\rho \otimes N) > \eta$.
- If $E_d(N) \leq \eta$, then $W = \mathrm{tr}_2[N^{(12)}(\eta\mathbb{1} - P_{00})^{(2)}]$ can be proved to be an entanglement witness.
- The set of such EWs detects all bipartite entangled states.
- The problem of finding N given ρ is at least as hard as finding an entanglement witness that detects ρ.

$$E_d(N) \stackrel{?}{=} \sup_{\rho_{sep}} f(\rho_{sep}, N)$$

Masanes, L. (2006). All bipartite entangled states are useful for information processing. Physical Review Letters, 96(15).

The flip witness that detects entangled Werner state

- Werner state $\omega^{(1)}$

$$\omega = p\frac{\mathcal{A}}{\mathrm{tr}[\mathcal{A}]} + (1-p)\frac{\mathcal{S}}{\mathrm{tr}[\mathcal{S}]} \quad \text{for } p \in [0,1].$$

  ω is non-PPT and entangled iff $p > \frac{1}{2}$.
- An EW $W_F = \mathbb{F} = \mathcal{S} - \mathcal{A}$ detects entangled Werner state.
- A duplex state

$$N_F^{(23)} = \frac{1}{d+2}\left(\frac{\mathbb{1}-\mathbb{F}}{d^2-d}\right)^{(A_2B_2)} \otimes P_{00}^{(A_3B_3)} + \frac{d+1}{d+2}\left(\frac{\mathbb{1}+\mathbb{F}}{d^2+d}\right)^{(A_2B_2)} \otimes \left(\frac{\mathbb{1}-P_{00}}{d^2-1}\right)^{(A_3B_3)}$$

  is PPT in $A_2A_3|B_2B_3$ and undistillable.*
- The final state $R^{(3)}$ is 1-distillable* iff ω is entangled.

$$f(\omega, N_F) > \frac{1}{d} \iff \mathrm{tr}[\omega W_F] < 0$$

Vollbrecht, K. G., & Wolf, M. M. (2002). Activating distillation with an infinitesimal amount of bound entanglement. Physical Review Letters, 88(24). ,

1-distillable state: a single copy of the state can be converted into a entangled qubit pair shared between Alice and Bob by LOCC.

Bell diagonal EWs

- Let $\vec{\lambda} = (\lambda_0, \ldots, \lambda_{d-1})$ where $\lambda_s \geq 0, \sum_{s=0}^{d-1} \lambda_s = 1$.
- Bell diagonal EWs (e.g. Reductioin EW, Choi EW)

$$W_{BD}(\vec{\lambda}) = \sum_{s=0}^{d-1} \lambda_s \Pi_s - P_{00} \quad \text{for some } \vec{\lambda}$$

- $W_{BD}(\vec{\lambda})$ corresponds to the mixed/pure duplex states

$$N_{BD}^{(23)}(\vec{\lambda}) = \sum_{s=0}^{d-1} \lambda_s \frac{1}{d} \sum_{t=0}^{d-1} P_{st}^{(2)} \otimes P_{st}^{(3)} \text{ or}$$

$$N_{BD}^{(23)}(\vec{\lambda}) = |N'_{BD}\rangle\langle N'_{BD}|, \quad |N'_{BD}\rangle^{(23)} = \sum_{s,t=0}^{d-1} \sqrt{\frac{\lambda_s}{d}} |\phi_{st}\rangle^{(2)} |\phi_{st}\rangle^{(3)}$$

with the threshold value $\eta = \lambda_0$.

$$f(\rho, N_{BD}(\vec{\lambda})) > \lambda_0 \iff \mathrm{tr}[\rho W_{BD}(\vec{\lambda})] < 0$$

$P_{st} = |\phi_{st}\rangle\langle\phi_{st}|, \quad |\phi_{st}\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} e^{2\pi i jt/d} |j, j+s\rangle, \quad \Pi_s = \sum_{t=0}^{d-1} P_{st} = \sum_{j=0}^{d-1} |j, j+s\rangle\langle j, j+s|$

Reduction witness

- W_{red} is decomposable (cannot detect PPT entangled state).
- Take $\vec{\lambda} = (\frac{1}{d}, \ldots, \frac{1}{d})$, then

$$W_{red} = \sum_{s=0}^{d-1} \frac{1}{d} \Pi_s - P_{00} = \frac{1}{d} \mathbb{1} - P_{00}$$

$$N_{red}^{(23)} = \frac{1}{d^2} \sum_{s=0}^{d-1} \sum_{t=0}^{d-1} P_{st}^{(2)} \otimes P_{st}^{(3)}, \text{ or}$$

$$|N'_{red}\rangle^{(23)} = \sum_{s,t=0}^{d-1} |\phi_{st}\rangle^{(2)} \, |\phi_{st}\rangle^{(3)}$$

with the threshold value $\eta = \frac{1}{d}$.

$$f(\rho, N_{red}) > \frac{1}{d} \iff \mathrm{tr}[\rho W_{red}] < 0$$

Duplex states of Reduction witness

- $N_{red}^{(23)}$ is a direct generalization of Smolin state into d-dimension.
  - For $d = 2$, Smolin state N_{Smolin} is PPT and undistillable.*

$$N_{Smolin} = \frac{1}{4}(\phi_{AB}^{+} \otimes \phi_{CD}^{+} + \phi_{AB}^{-} \otimes \phi_{CD}^{-} + \psi_{AB}^{+} \otimes \psi_{CD}^{+} + \psi_{AB}^{-} \otimes \psi_{CD}^{-})$$

  - For $d \geq 3$, $N_{red}^{(23)}$ can be non-PPT, and the distillablity is unknown.

- $|N'_{red}\rangle^{(23)}$ is two copies of $|\phi^{+}\rangle$ when $d = 2$

$$|N'_{red}\rangle^{(23)} = |\phi^{+}\rangle^{(A_2 A_3)} \otimes |\phi^{+}\rangle^{(B_2 B_3)} \text{ if } d = 2$$

  - For $d = 2$, the scheme is merely the quantum teleporation in two parties (Alice and Bob) followed by a singlet fraction measurement.
  - For $d \geq 3$, $|N'_{red}\rangle^{(23)}$ cannot be simplified to copies of an identical state.

Smolin, J. A. (2001). Four-party unlockable bound entangled state. Physical Review A, 63(3).

Choi Witness

- W_{Choi} is nondecomposable (can detect PPT entangled state).
- Take $\vec{\lambda} = (\frac{2}{3}, \frac{1}{3}, 0)$ where $d = 3$, then

$$W_{Choi} = \frac{2}{3}\Pi_0 + \frac{1}{3}\Pi_1 - P_{00}$$

$$N^{(23)}_{Choi} = \frac{2}{9}\sum_{t=0}^{2} P^{(2)}_{0t} \otimes P^{(3)}_{0t} + \frac{1}{9}\sum_{t=0}^{2} P^{(2)}_{1t} \otimes P^{(3)}_{1t}$$

$$|N'_{PBD}\rangle^{(23)} = \frac{\sqrt{2}}{3}\sum_{t=0}^{2} |\phi_{0t}\rangle^{(2)} \, |\phi_{0t}\rangle^{(3)} + \frac{1}{3}\sum_{t=0}^{2} |\phi_{1t}\rangle^{(2)} \, |\phi_{1t}\rangle^{(3)}$$

$$f(\rho, N_{Choi}) > \frac{2}{3} \iff \mathrm{tr}[\rho W_{Choi}] < 0$$

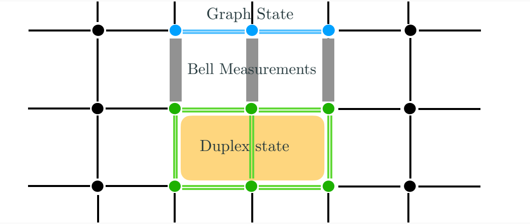
- The 3-dimensional Bell diagonal state*

$$\rho_{BD} = \frac{2}{7}P_{00} + \frac{1}{7}\frac{\Pi_1}{3} + \frac{4}{7}\frac{\Pi_2}{3}$$

is PPT entangled, and detected by W_{Choi} and $N^{(23)}_{Choi}$.

Horodecki, P., Horodecki, M., & Horodecki, R. (1999). Bound entanglement can be activated. Physical Review Letters, 82(5), 1056–1059.

Extension to multipartite systems



- Consider Greenberger–Horne–Zeilinger (GHZ) states $\psi_{abc} = |\psi_{abc}\rangle\langle\psi_{abc}|$ for $a, b, c \in \{0, 1\}$,

$$|\psi_{abc}\rangle = Z^a \otimes X^b \otimes X^c \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle).$$

- GHZ state $\psi_{000}^{(A_1B_1C_1)}$ is detected by the EW

$$W_{GHZ} = \frac{1}{2}\mathbb{1}^{(A_1B_1C_1)} - \psi_{000}^{(A_1B_1C_1)}$$

Extension to the multipartite systems

- The corresponding duplex states are

$$N_{GHZ}^{(23)} = \frac{1}{8} \sum_{a,b,c=0}^{1} \psi_{abc}^{(A_2 B_2 C_2)} \otimes \psi_{abc}^{(A_3 B_3 C_3)}$$

$$|N'_{GHZ}\rangle^{(23)} = \frac{1}{\sqrt{8}} \sum_{a,b,c=0}^{1} |\psi_{abc}\rangle^{(A_2 B_2 C_2)} \, |\psi_{abc}\rangle^{(A_3 B_3 C_3)}$$

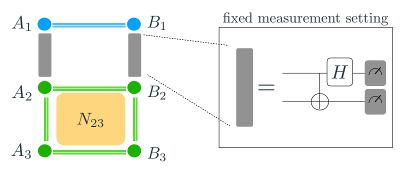
  with the threshold value $\eta = \frac{1}{2}$.
- If the final state has a higher overlap with GHZ state than $\frac{1}{2}$, then the state of interest is entangled.

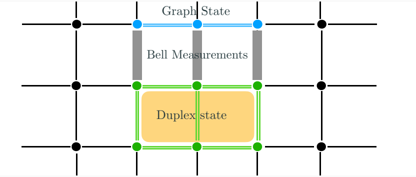
$$f(\rho, N_{GHZ}) = \langle\psi_{000}| \frac{\mathrm{tr}_{12}[\rho_1 \otimes N_{GHZ}^{(23)} \phi_{A_1 A_2}^{+} \otimes \phi_{B_1 B_2}^{+} \otimes \phi_{C_1 C_2}^{+}]}{\mathrm{tr}[\rho_1 \otimes N_{GHZ}^{(23)} \phi_{A_1 A_2}^{+} \otimes \phi_{B_1 B_2}^{+} \otimes \phi_{C_1 C_2}^{+}]} |\psi_{000}\rangle$$

$$f(\rho, N_{GHZ}) > \frac{1}{2} \iff \mathrm{tr}[\rho W_{GHZ}] < 0.$$

- The framework can be extended to the general graph states.

Summary





- General observables can be estimated by state preparation + fixed measurements.
- The duplex states and Bell measurements can detect all bipartite entangled states.
- The final singlet fraction above a threshold value confirms that the state of interest is entangled.
- The framework can detect multipartite graph state as well.

THANK YOU

A PPT duplex state cannot detect a PPT entangled state

- Let $N_{PPT}^{(23)}$ be PPT and $\rho_{PPT}^{(1)}$ be PPT entangled state.
  - $\rho_{PPT}^{(1)} \otimes N_{PPT}^{(23)}$ is PPT and undistillable.
  - $f(\rho_{PPT}, N_{PPT})$ is upper bounded by $\frac{1}{d}$.
- A PPT duplex state $N_{PPT}^{(23)}$ cannot detect a PPT entangled state, i.e., $N_{PPT}^{(23)}$ can only produce decomposable EW.
- If W is nondecomposable, then $N^{(23)}$ must be non-PPT.
- The converse does not hold
  - An EW constructed from a non-PPT duplex state is not always nondecomposable.
  - The 3-dimensional Smolin state $N_{red}^{(23)}$ is non-PPT, but the reduction witness W_{red} is decomposable.

Optimal decomposable EWs

- Decomposable EWs cannot detect PPT entangled state.
- An optimal decomposable EW $W_{dec} = Q^{T_A}$ where $Q \geq 0$.
- Let $\mathrm{tr}Q = 1$ and $\lambda = \max_i |\lambda_i|$ where $\{\lambda_i\}$ are eigenvalues of Q^{T_A}.
- The corresponding duplex state is

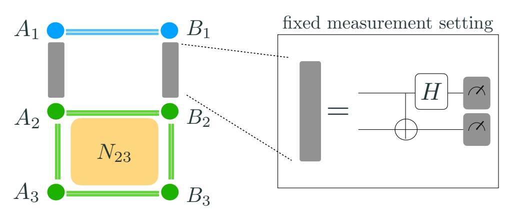
$$N_{dec}^{(23)} = \frac{\lambda d^2 - 1}{\lambda d^3 + d - 2} \left(\frac{\lambda \mathbb{1} - Q^{T_A}}{\lambda d^2 - 1} \right)^{(A_2 B_2)} \otimes P_{00}^{(A_3 B_3)} + \frac{\lambda d^3 - \lambda d^2 + d - 1}{\lambda d^3 + d - 2} \left(\frac{\lambda \mathbb{1} + Q^{T_A}}{\lambda d^2 + 1} \right)^{(A_2 B_2)} \otimes \left(\frac{\mathbb{1} - P_{00}}{d^2 - 1} \right)^{(A_3 B_3)}$$

with the threshold value $\eta = \frac{1}{d}$.

$$f(\rho, N_{dec}) > \frac{1}{d} \iff \mathrm{tr}[\rho W_{dec}] < 0$$

- The flip witness $W_F = \frac{1}{d}\mathbb{F}$ is a special case of W_{dec} with $Q = P_{00}$.

Bell diagonal EWs



- Bell diagonal EWs detect entangled Bell diagonal states.
  - $\rho_{BD} = \lambda_0 P_{00} + \sum_{s=1}^{d-1} \lambda_s \frac{\Pi_s}{d}$ is entangled iff $\exists s : \lambda_0 > \lambda_s$.*
- If $\rho_{sep}^{(1)}$ is separable, then $f(\rho_{sep}, N_{BD}(\vec{\lambda}))$ is upper bounded by λ_0.
  - A separable state $\sigma = \frac{1}{d} P_{00} + \sum_{s=1}^{d-1} \frac{1}{d} \frac{\Pi_s}{d} \longrightarrow f(\sigma, N_{BD}(\vec{\lambda})) = \lambda_0$.
- $f(\rho, N_{BD}(\vec{\lambda})) > \lambda_0$ implies that ρ_1 is entangled.
- Bell diagonal EWs contain
  - Reduction witness W_{red} (decomposable)
  - Choi witness W_{Choi} (3-dimension, nondecomposable)
  - generalized Choi witnesses (d-diemension, nondecomposable)

Chruściński, D. & Kossakowski, A. (2010). Bell diagonal states with maximal abelian symmetry. Physical Review A, 82(6).

Breuer-Hall witness

- Let U be a skew-symmetric unirary operator: $UU^\dagger = \mathbb{1}, U^T = -U$.
- Let $\mathbb{F}' = (\mathbb{1} \otimes U)\mathbb{F}(\mathbb{1} \otimes U^\dagger)$.
- The Breuer-Hall witness and the duplex state

$$\begin{aligned} W_{BH} =& \frac{1}{d}(\mathbb{1} - \mathbb{F}') - P_{00} \\ N_{BH}^{(23)} =& c_0 \frac{1}{d^2} \sum_{s=0}^{d-1} \sum_{t=0}^{d-1} P_{st}^{(2)} \otimes P_{st}^{(3)} \\ &+ c_1 \left(\frac{\mathbb{1} + \mathbb{F}'}{d^2 + d} \right)^{(2)} \otimes P_{00}^{(3)} \\ &+ c_2 \left(\frac{\mathbb{1} - \mathbb{F}'}{d^2 - d} \right)^{(2)} \otimes \left(\frac{\mathbb{1} - P_{00}}{d^2 - 1} \right)^{(3)} \end{aligned}$$

where $c_0 = \frac{2d^2-2d}{3d^2-3d+2}, c_1 = \frac{d+1}{3d^2-3d+2}, c_2 = \frac{d^2-d+1}{3d^2-3d+2}$.

- The threshold value $\eta = \frac{1}{d}$.

$$f(\rho, N_{BH}) > \frac{1}{d} \iff \mathrm{tr}[\rho W_{BH}] < 0$$