Maximum confidence measurement for qubit states and its certification

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Maximum-confidence measurement for qubit states

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Contextual Advantages and Certification for Maximum-Confidence Discrimination

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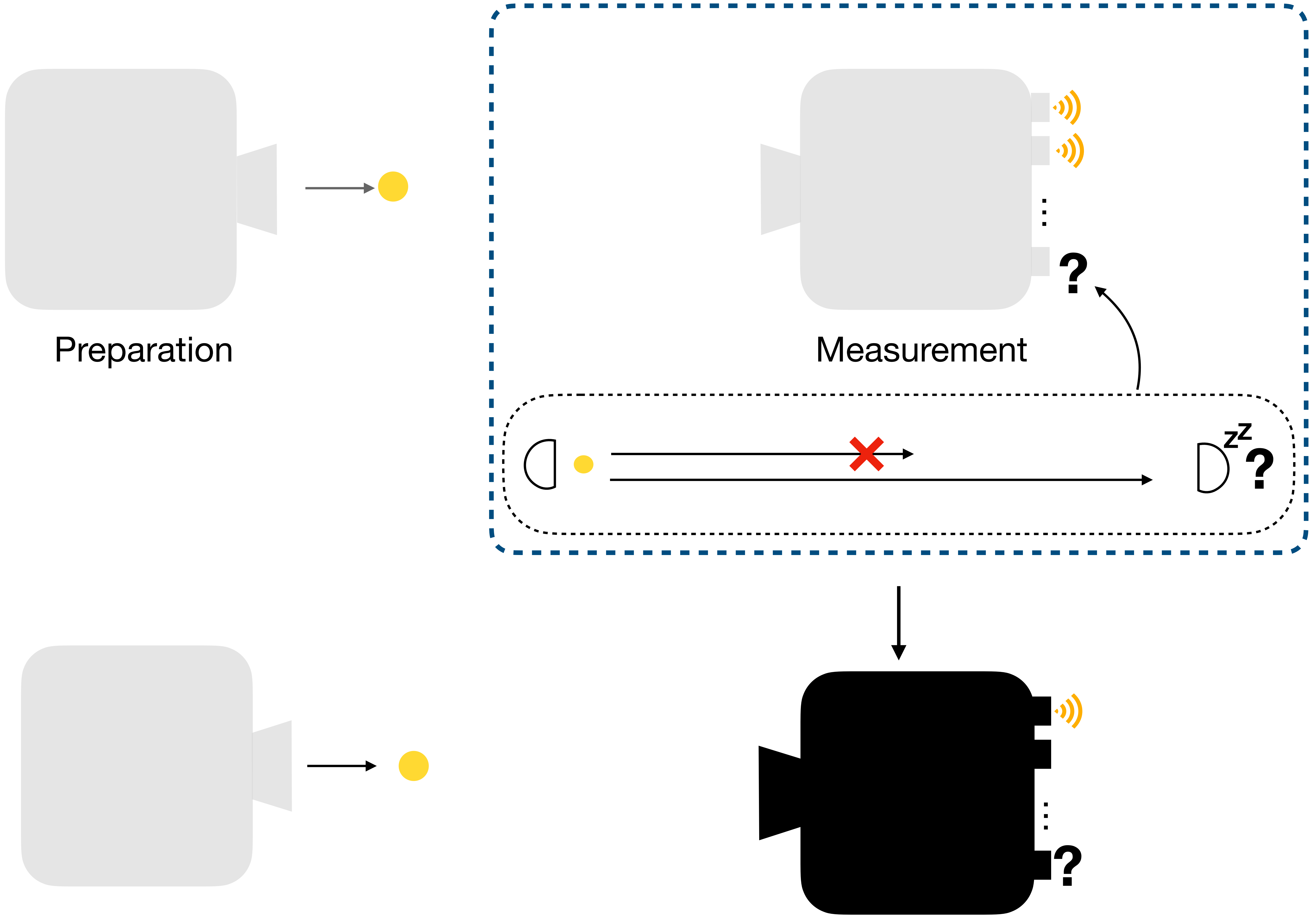
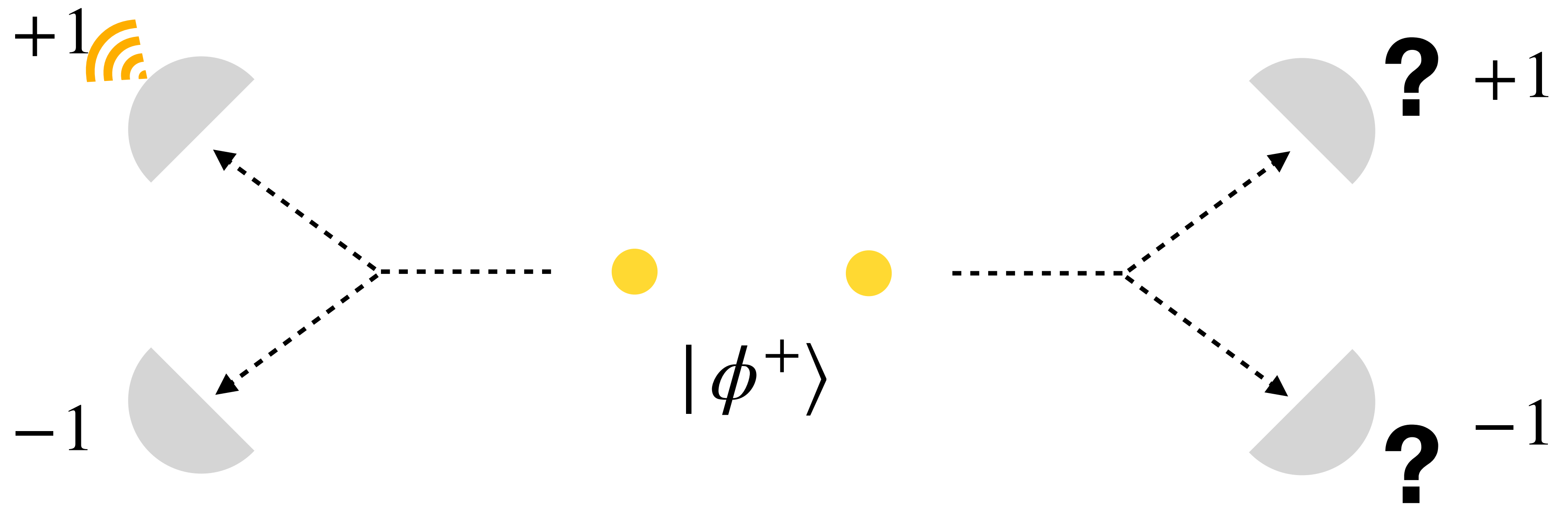
Motivation: Quantum information processing with imperfect devices





A challenge for observing violation of bell inequality

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I. Quantum state discrimination: Maximum confidence measurement (MCM)

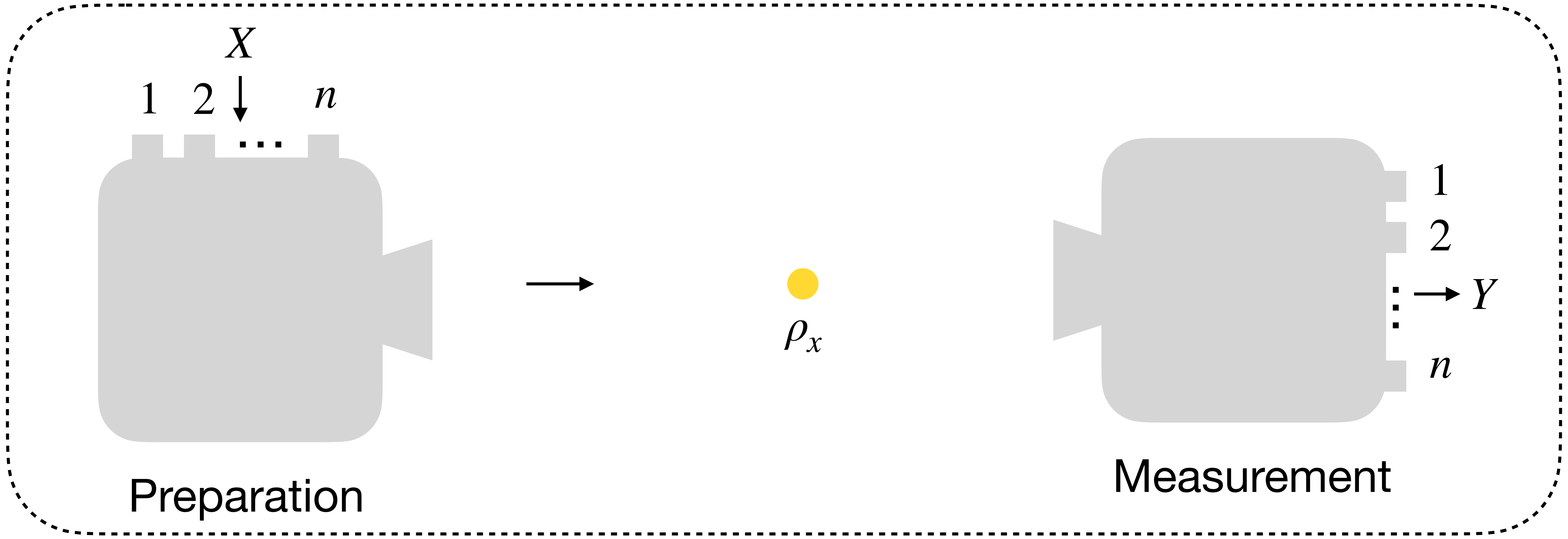
II. A semi-definite programming (SDP) approach to an MCM

III. Construction of MCMs for qubit states

IV. Certification of MCMs with an uncharacterized measurement device

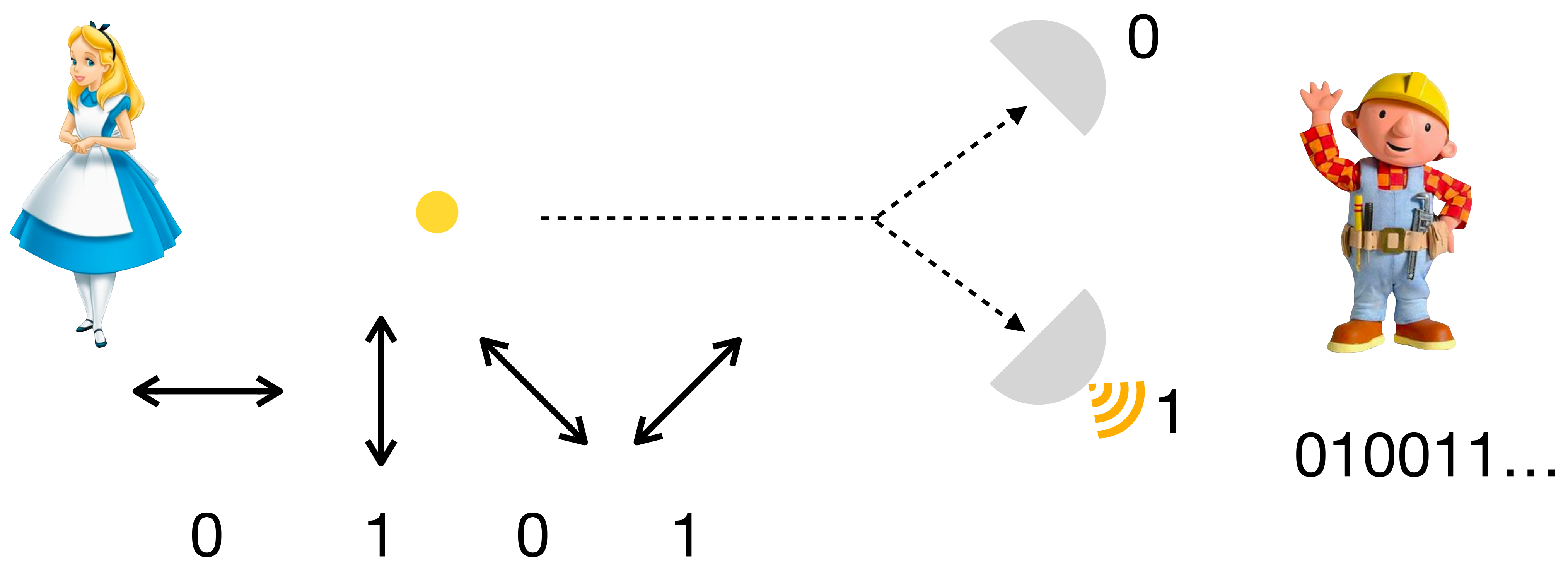
The problem of quantum state discrimination

A prepare-and-measure experiment

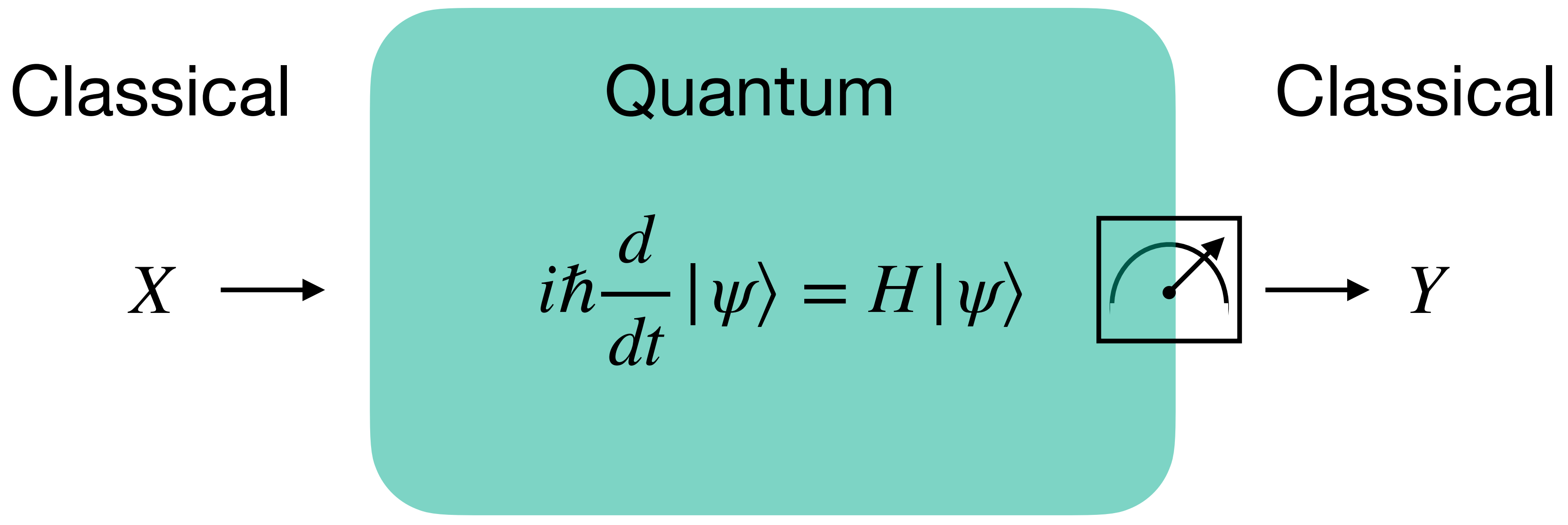


$Y = x \longrightarrow X = x$

Guessing about the state preparation



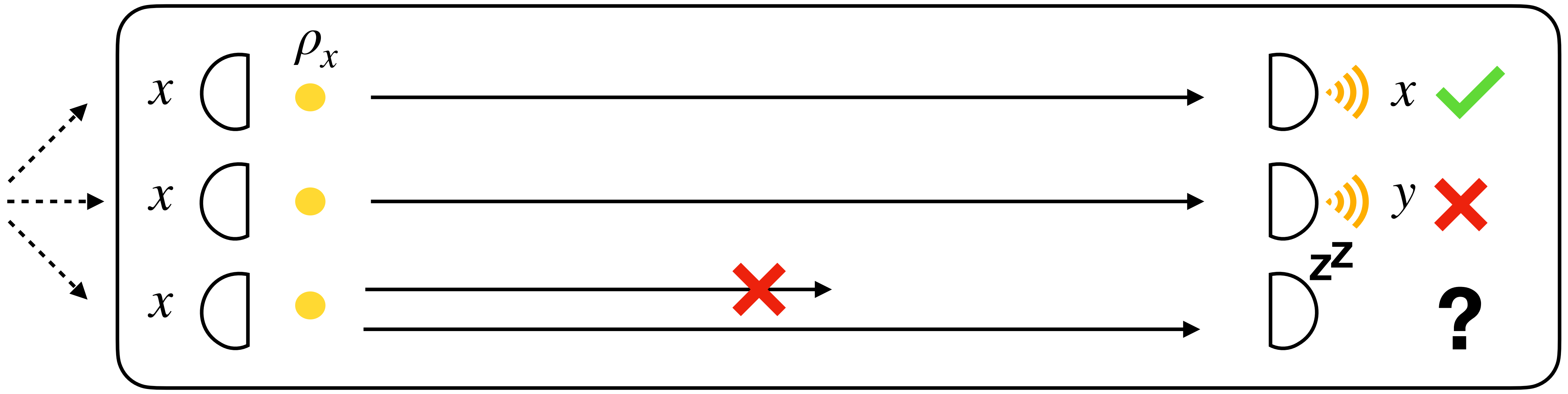
Quantum communication



Quantum Information processing

Quantum state discrimination in a realistic scenario

$q_x = P_{\mathrm{P}}(x)$



Correct guess

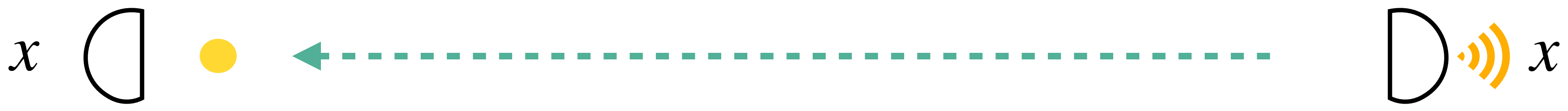
Incorrect guess

No conclusion

ensemble

$\{q_x, \rho_x\}_{x=1}^{n}, \rho = \sum_x q_x \rho_x$

Confidence: $C(x) = P_{\mathrm{P|M}}(x\,|\,x) = \dfrac{P_{\mathrm{P}}(x)P_{\mathrm{M|P}}(x\,|\,x)}{P_{\mathrm{M}}(x)} = \dfrac{q_x \mathrm{tr}[\rho_x M_x]}{\mathrm{tr}[\rho M_x]}$



Confidence is determined only from a detected event

Comparisons of different state discrimination strategies

Figure of merit in state discrimination

$$\max f(C(x))$$

$$\max\langle C(x)\rangle = \max \sum_x \mathrm{tr}[\rho M_x]C(x)$$

Minimum-error discrimination

Condition: $\sum_{x=1}^{n} M_x = I$

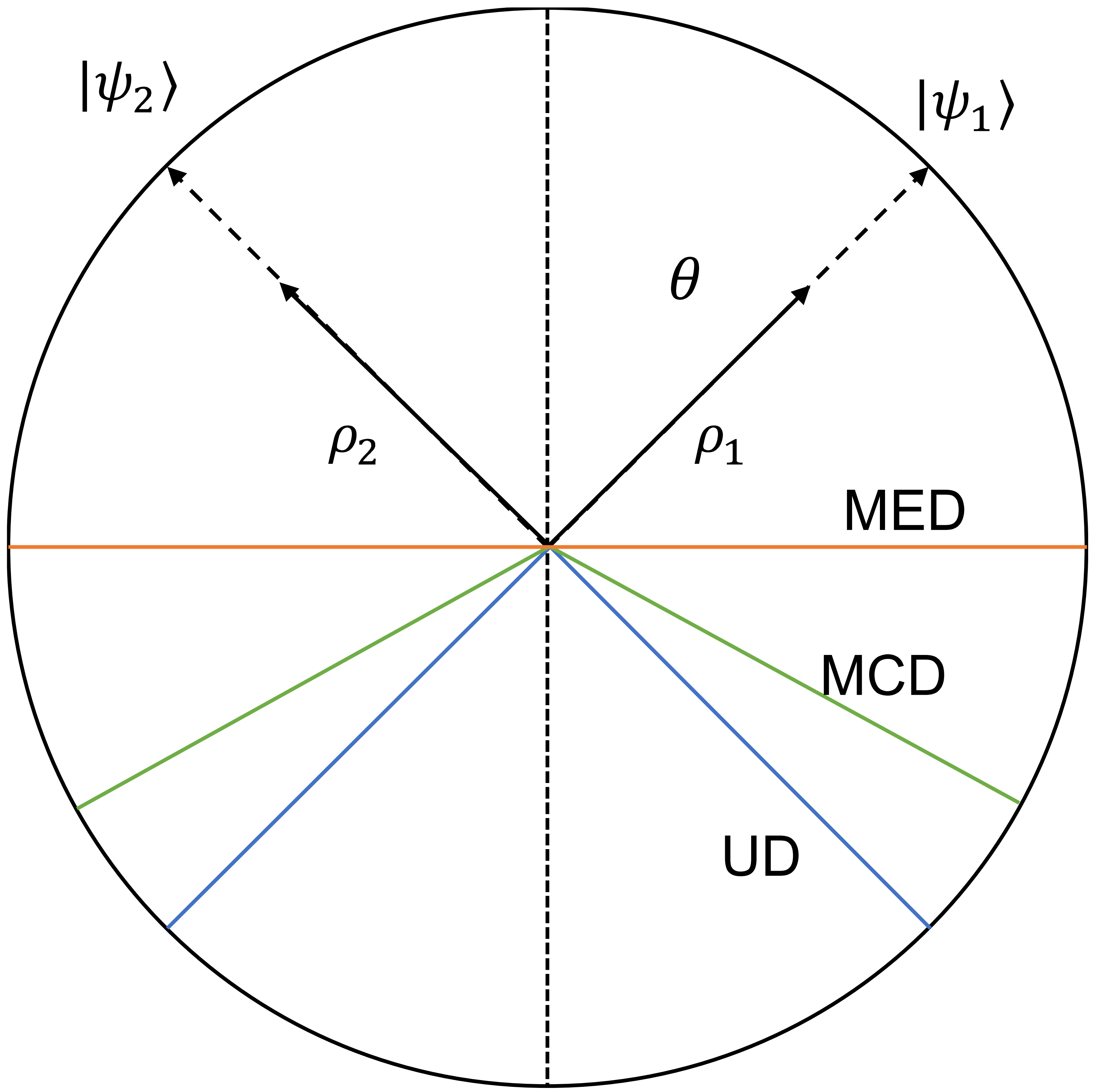
$$\max C(x)$$

Maximum confidence measurement

Unambiguous discrimination

$$\max C(x) = 1 \,\forall x$$

Condition: states must be linearly independent



$$\rho_x = (1-p)|\psi_x\rangle\langle\psi_x| + p\frac{I}{2}$$

MCMs do not have the weaknesses of MED and UD

Maximum confidence measurement (MCM)

Maximum confidence : $\max C(x) = \max_{M_x \geq 0} \frac{q_x \mathrm{tr}[\rho_x M_x]}{\mathrm{tr}[\rho M_x]}$

$$= \max_{Q_x \geq 0, \mathrm{tr}[Q_x]=1} \mathrm{tr}[\tilde{\rho}_x Q_x]$$

$$= || \sqrt{\rho}^{-1} q_x \rho_x \sqrt{\rho}^{-1} ||_\infty$$

$\{\tilde{\rho}_x = \sqrt{\rho}^{-1} q_x \rho_x \sqrt{\rho}^{-1}, Q_x = \frac{\sqrt{\rho} M_x \sqrt{\rho}}{\mathrm{tr}[\rho M_x]}\}$

$Q_x^* = \arg\max \mathrm{tr}[\tilde{\rho}_x Q_x]$
is the eigen projector of $\tilde{\rho}_x$ with the largest eigenvalue.

Maximum confidence measurement : $M_x^* = \arg\max_{M_x \geq 0} \frac{q_x \mathrm{tr}[\rho_x M_x]}{\mathrm{tr}[\rho M_x]}$

$= a_x \Pi_x$ where $\Pi_x = \frac{\sqrt{\rho}^{-1} Q_x^* \sqrt{\rho}^{-1}}{\mathrm{tr}[\sqrt{\rho}^{-1} Q_x^* \sqrt{\rho}^{-1}]}$

In general, $\sum_{x=1}^{n} M_x^* \leq I$ so additional measurement outcome is necessary

$$M_0 = I - \sum_{x=1}^{n} M_x^*$$

A POVM is $\{M_0, M_1^*, \ldots, M_n^*\}$

S. Croke. et al. "Maximum confidence quantum measurements." *Physical review letters* 96.7 (2006): 070401.

A semi-definite programming (SDP) approach to an MCM

Semi-definite programming (SDP) is one form of convex optimization.

The problem of MCM is SDP.

$$\max C(x) = \max_{Q_x \geq 0, \mathrm{tr}[Q_x]=1} \mathrm{tr}[\tilde{\rho}_x Q_x], \text{ where } \tilde{\rho}_x = \sqrt{\rho}^{-1} q_x \rho_x \sqrt{\rho}^{-1}$$

A technique in convex optimization can be applied

The optimality conditions

Lagrangian stability : $\rho = \mu_x \rho_x + (1 - \mu_x)\sigma_x$

where $0 \leq \mu_x \leq 1$ and σ_x is a non-full-rank quantum state

$$\max C(x) = \frac{q_x}{\mu_x}$$

Complementary slackness : $\mathrm{tr}[M_x \sigma_x] = 0$

Geometry of a qubit MCM

Original problem

$$\max C(x) = \max_{M_x \geq 0} \frac{q_x \mathrm{tr}[\rho_x M_x]}{\mathrm{tr}[\rho M_x]}$$

$\longrightarrow$

New problem

$$\rho = \mu_x \rho_x + (1-\mu_x)\sigma_x$$

$$\mathrm{tr}[M_x \sigma_x] = 0$$

$\mathrm{tr}[\sigma_x^2] = 1$

For a pure state ρ_x,

$$\mu_x = \frac{1-\mathrm{tr}[\rho^2]}{2(1-\mathrm{tr}[\rho\rho_x])}$$

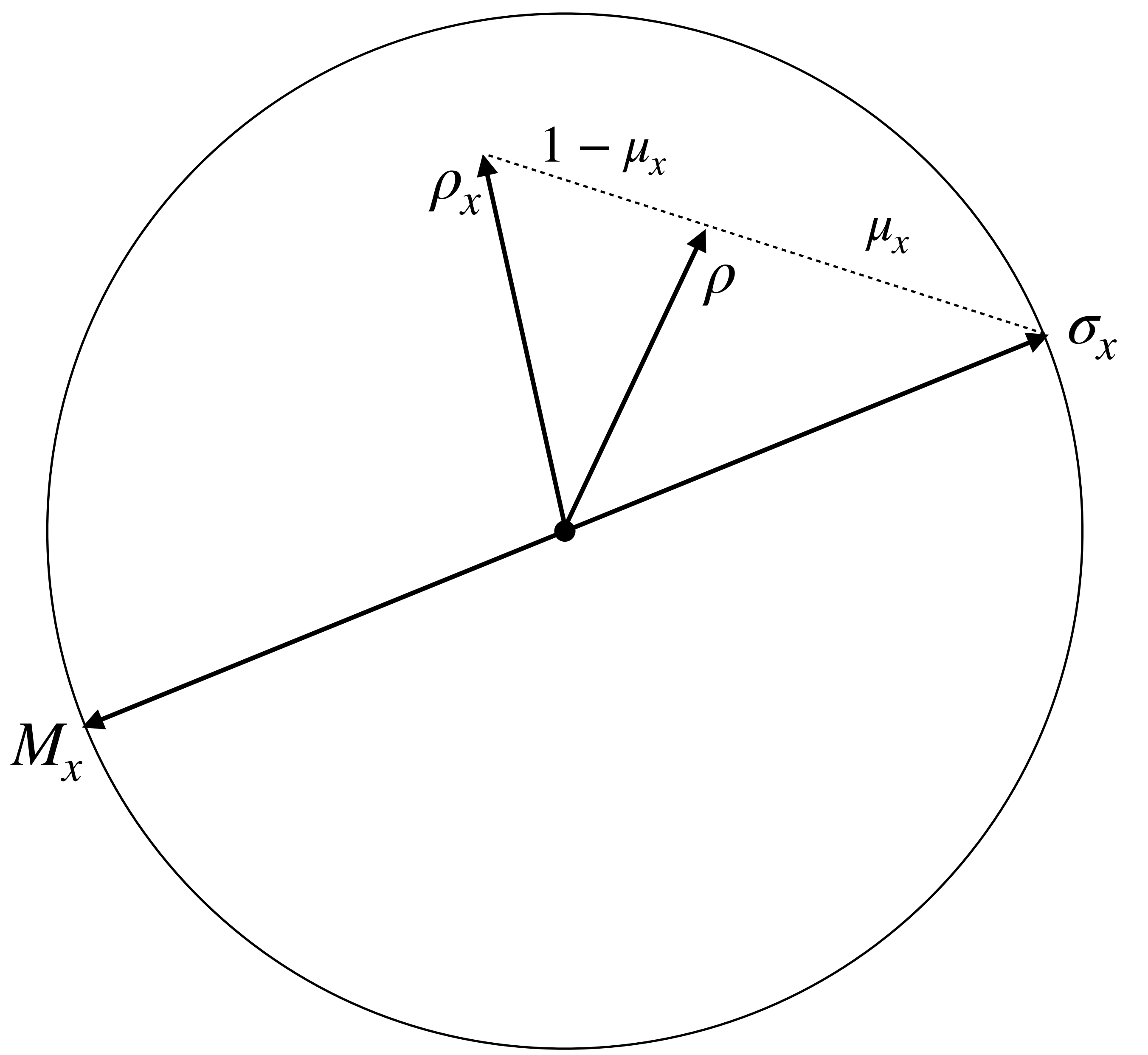
For a mixed state ρ_x,

$$\mu_x = \frac{(1-\mathrm{tr}[\rho\rho_x]) - Det(\rho,\rho_x)}{1-\mathrm{tr}[\rho_x^2]}$$

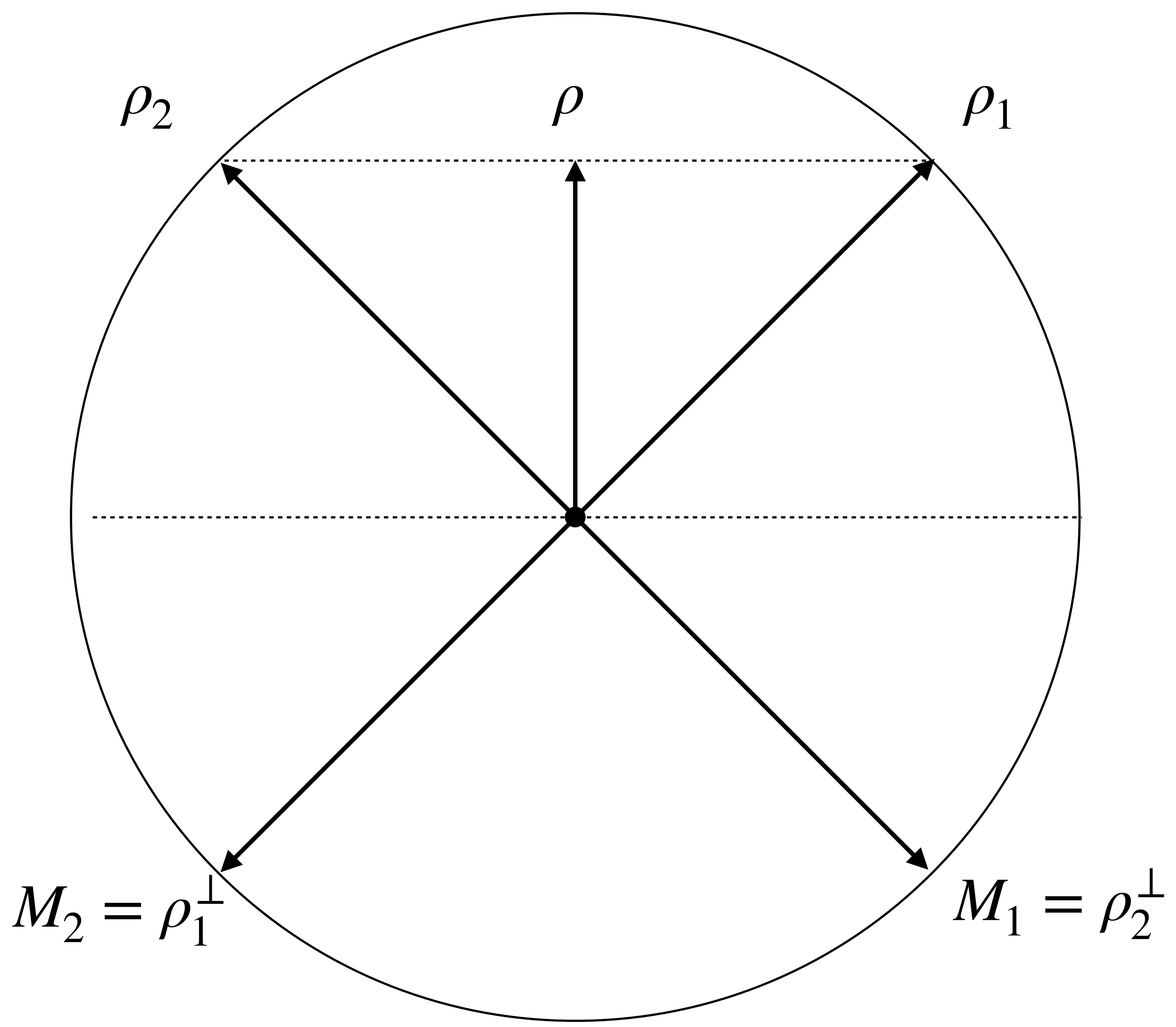
where

$$Det(\rho,\rho_x) = [(1-\mathrm{tr}[\rho\rho_x])^2 - (1-\mathrm{tr}[\rho^2])(1-\mathrm{tr}[\rho_x^2])]^{\frac{1}{2}}$$

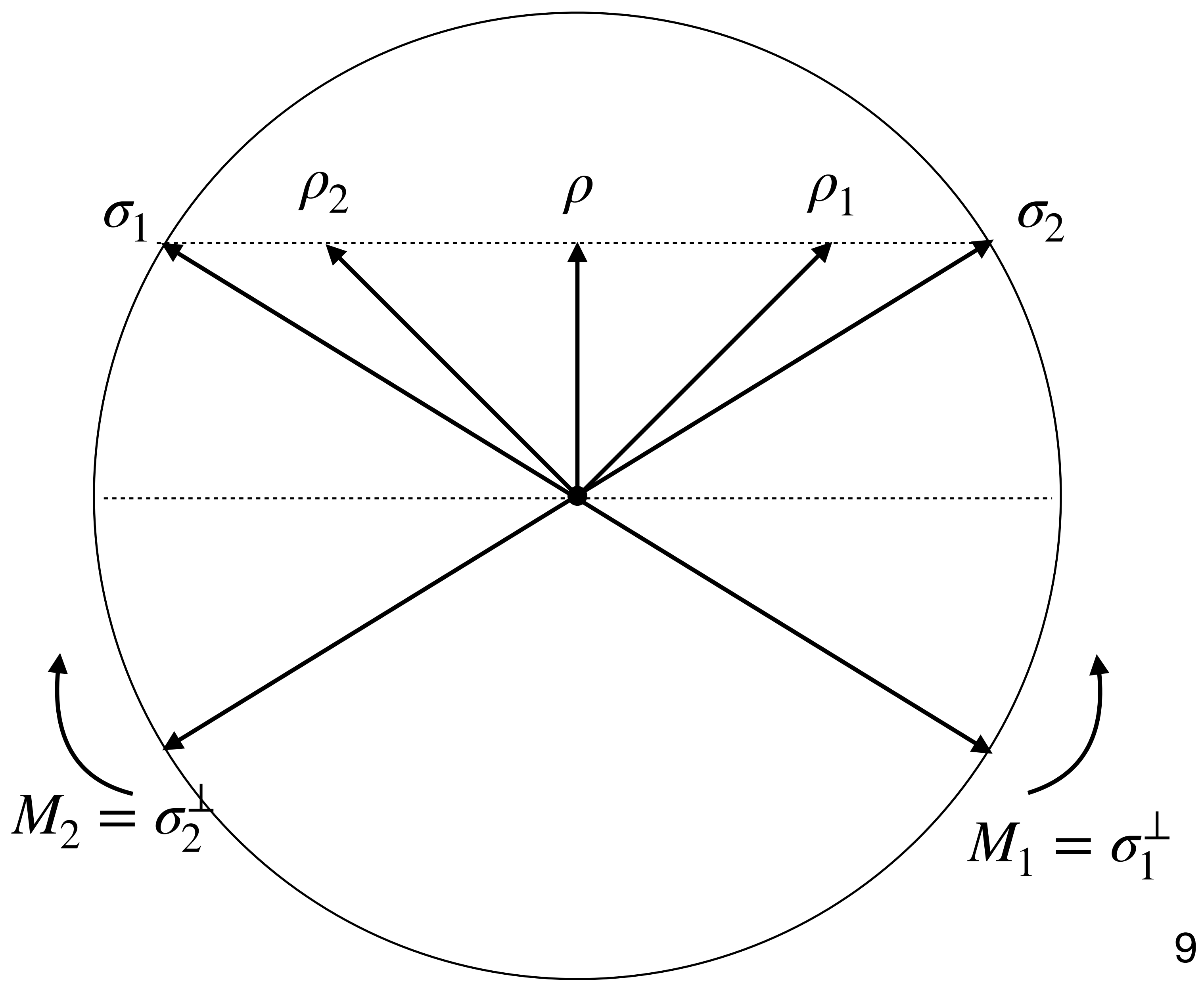
The optimality conditions for a qubit MCM



Two pure states

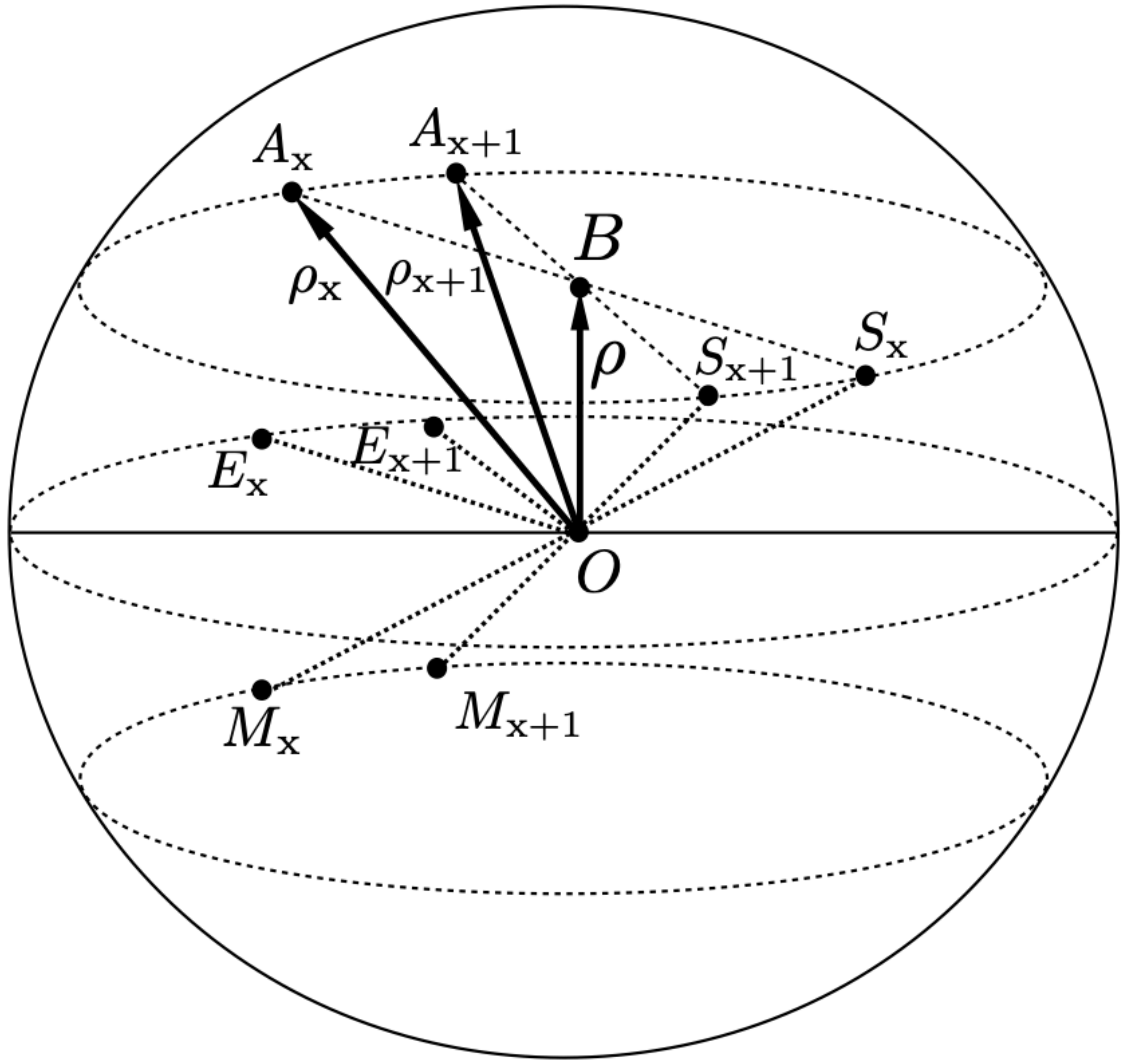


Two mixed states



Various qubit ensembles

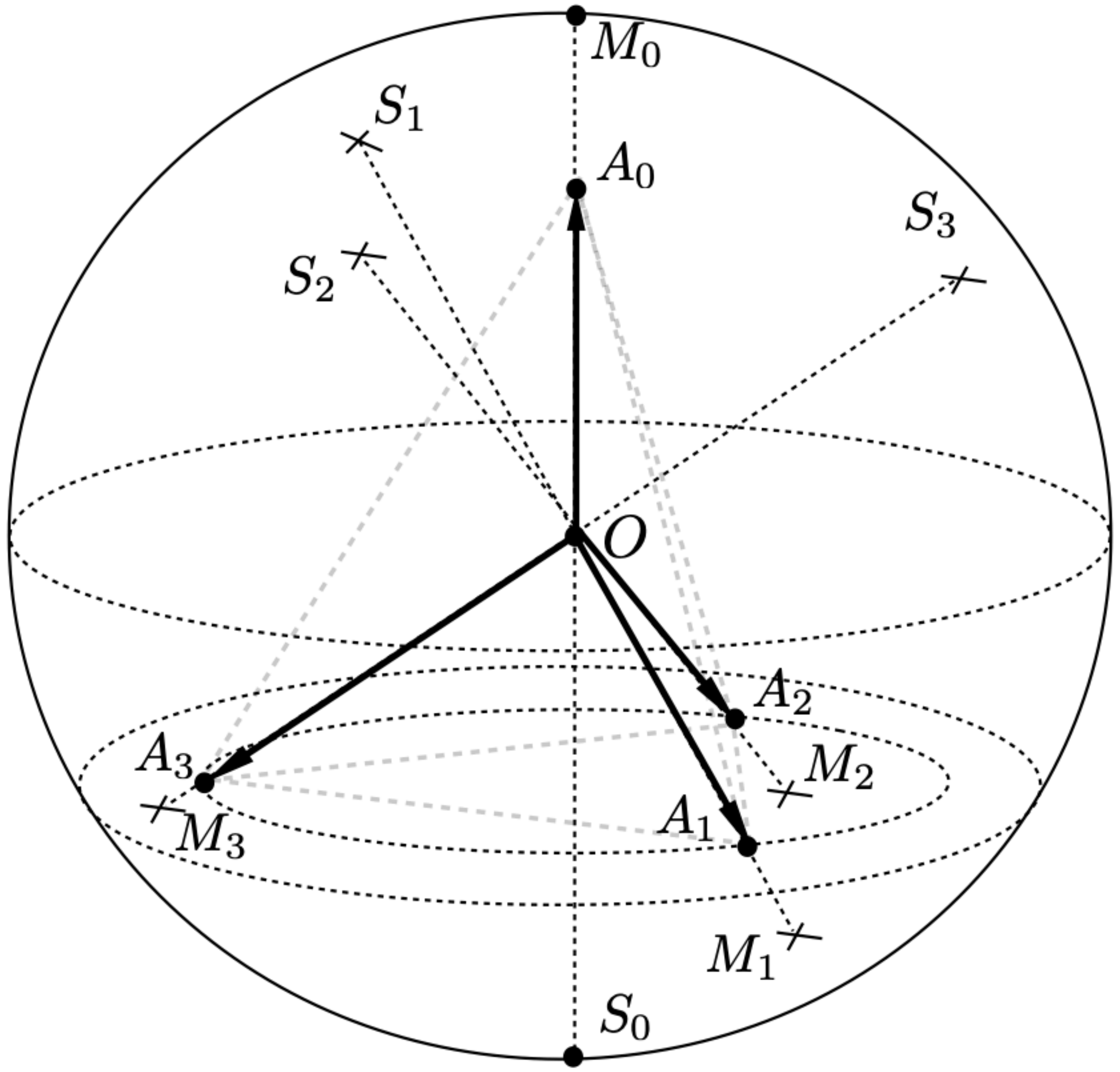
Ex 1. Geometrically uniform states



$$|\psi_x\rangle = \cos\frac{\theta}{2}|0\rangle + e^{\frac{2\pi i x}{n}}\sin\frac{\theta}{2}|1\rangle, x = 1,\ldots,n$$

$$\max C(x) = \frac{2}{n}, \forall x$$

Ex 2. Noisy symmetric, informationally complete (SIC) states



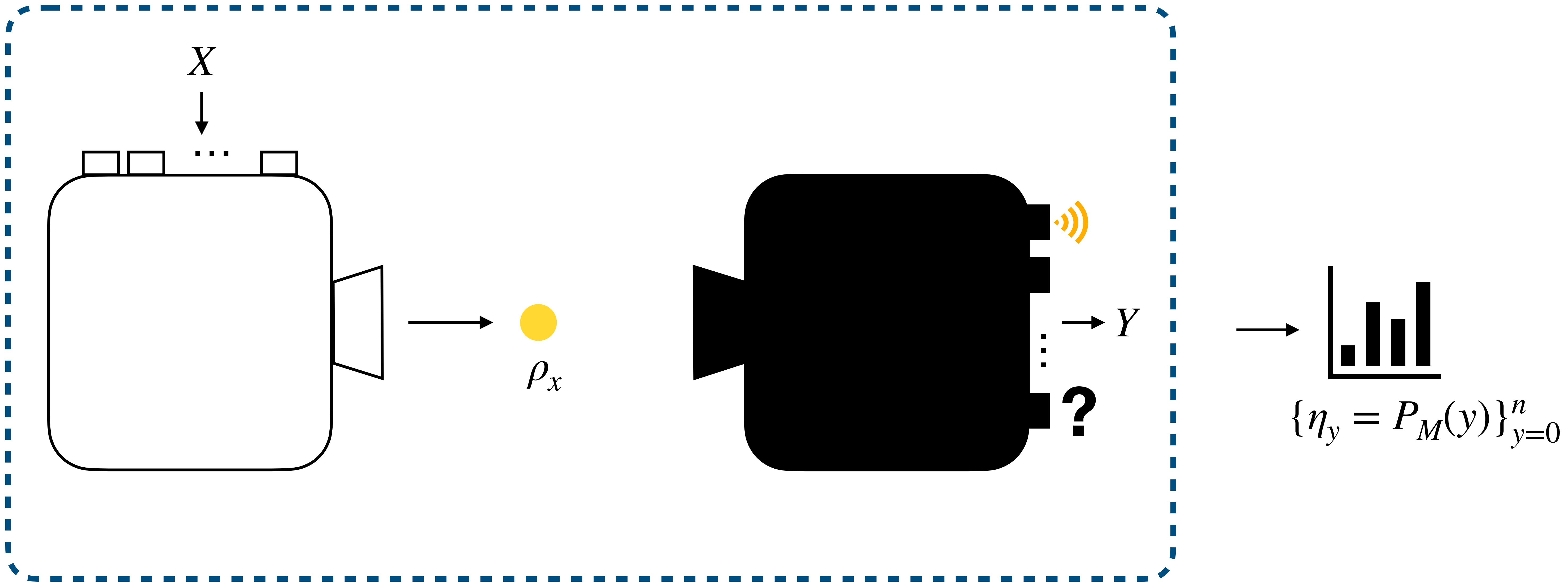
$$|\psi_1\rangle = |0\rangle,\ |\psi_x\rangle = \sqrt{\frac{1}{3}}|0\rangle + e^{\frac{2\pi i(x-1)}{3}}\sqrt{\frac{2}{3}}|1\rangle, x = 2,3,4$$

$$\rho_x = (1-p)|\psi_x\rangle\langle\psi_x| + p\frac{I}{2}$$

$$\max C(x) = \frac{2-p}{4}, \forall x$$

Measurement device as a black box

A semi-device-independent scenario



How much can we trust the measurement device that performs a state discrimination task?

SDP formulation for certifying MCMs

Given the outcome statistics $\{\eta_y\}_{y=0}^{n}$, one can certify

maximize $\langle C(y)\rangle_\alpha = \sum_{y=1}^{n} \alpha_y C(y) = \sum_y \frac{\alpha_y q_y}{\eta_y} \mathrm{tr}[\rho_y M_y]$

subject to $M_y \geq 0, \sum_{y=0}^{n} M_y = I$

$\mathrm{tr}[\rho M_y] = \eta_y, y = 0,1,\ldots,n$

The optimality conditions

Lagrangian stability : $\forall y = 1,\ldots,n$

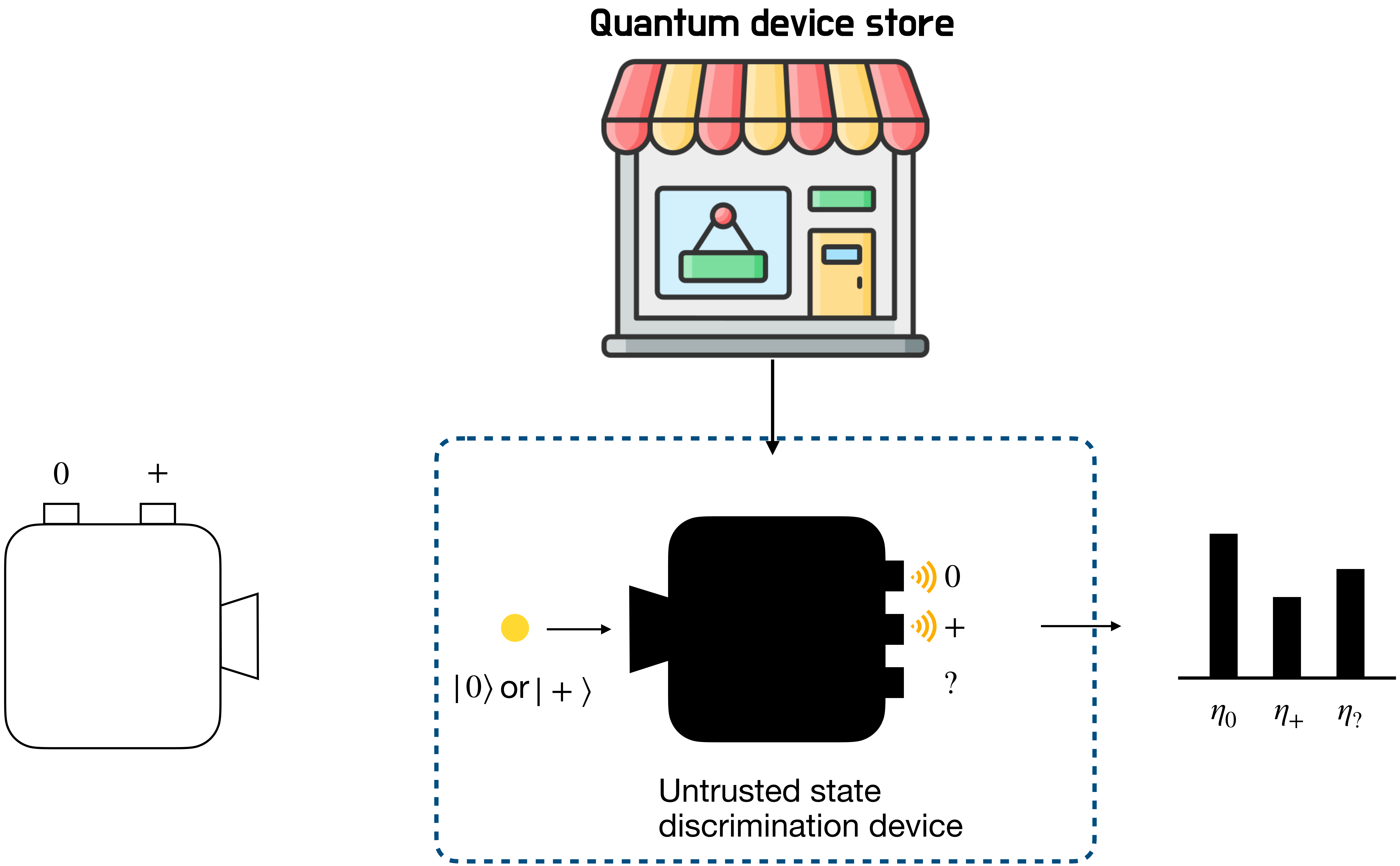
$K = \alpha_y \frac{q_y}{\eta_y} \rho_y + r_y \sigma_y - s_y \rho,$

and $K = r_0 \sigma_0$

Complementary slackness : $\forall y = 0,\ldots,n$

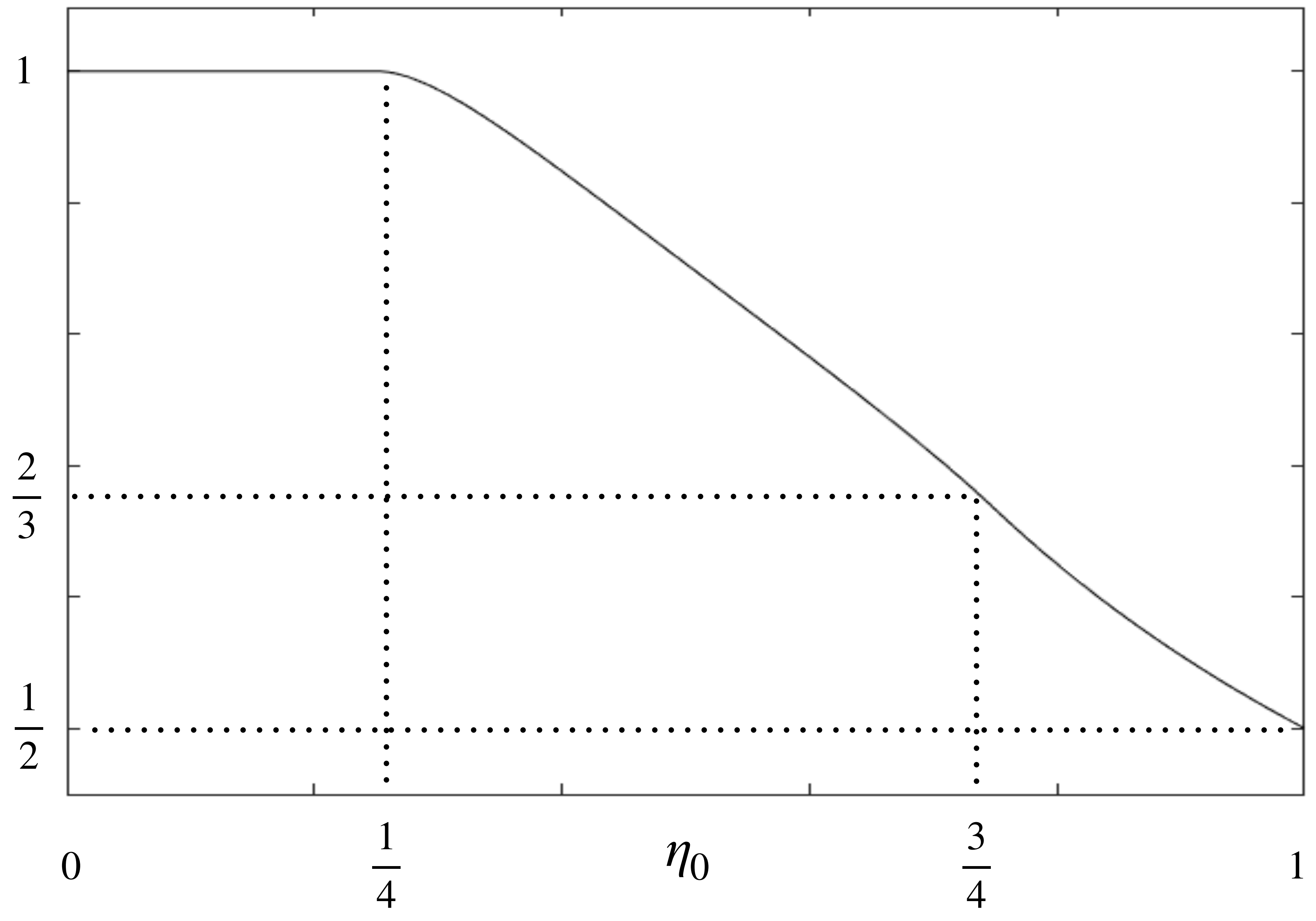
$r_y \mathrm{tr}[M_y \sigma_y] = 0$

Certification of a two-state-discrimination device



Certification of MCMs in a two-state discrimination task

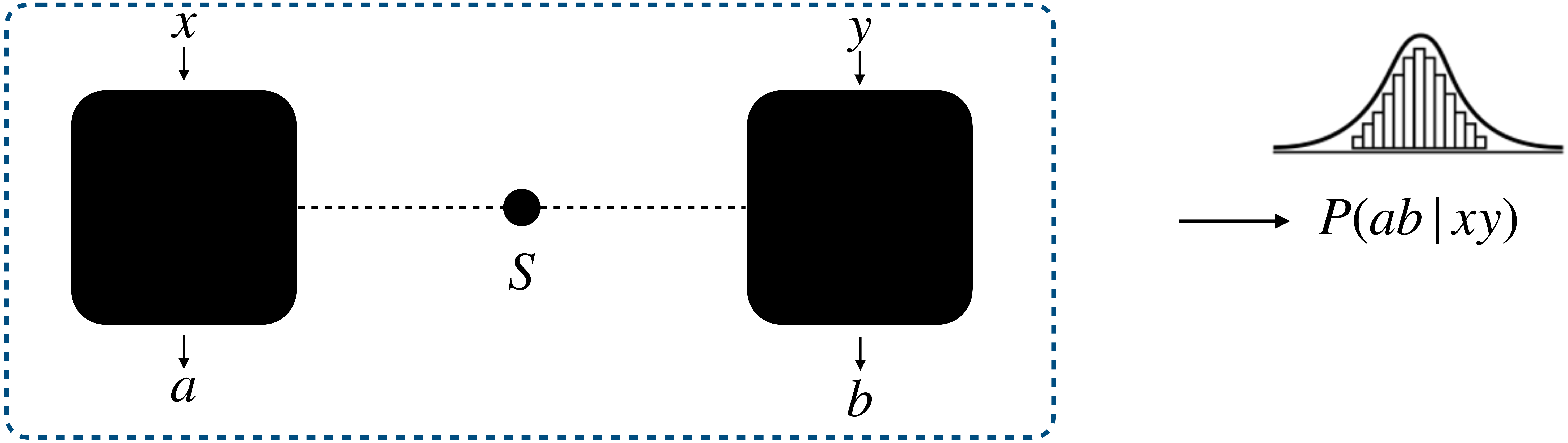
Maximum confidence : $\max C(0)$



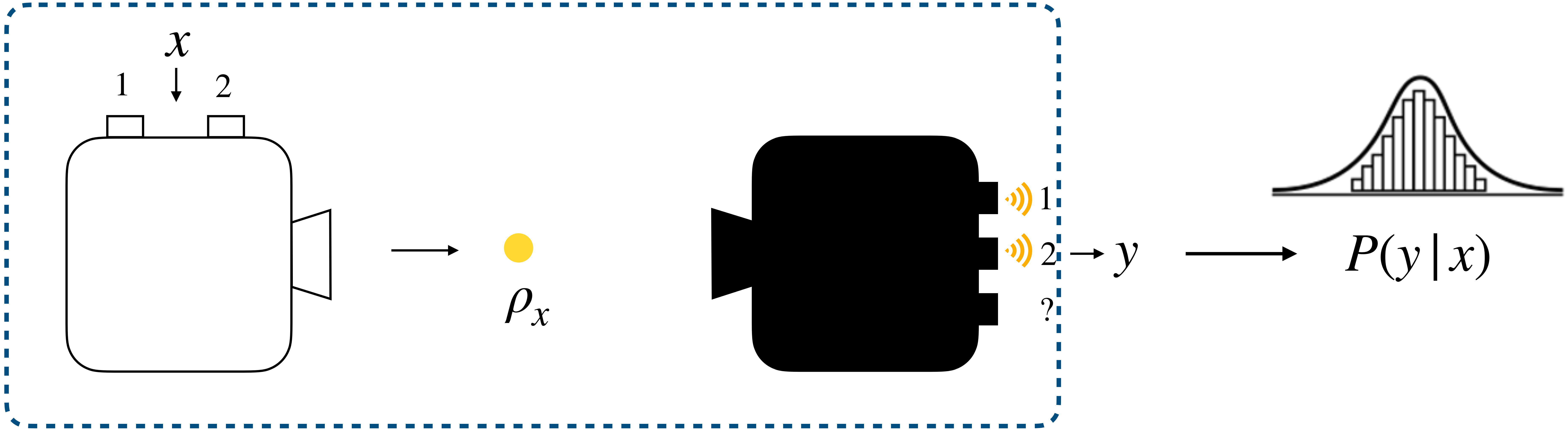
$$\max C(0) = \begin{cases} 1, & \text{for } 0 \le \eta_0 < \frac{1}{4} \\ \frac{1}{2} + \frac{1}{4\eta_0}\sqrt{4\eta_0 - 4\eta_0^2 - \frac{1}{2}}, & \text{for } \frac{1}{4} \le \eta_0 < \frac{3}{4} \\ \frac{1}{2\eta_0}, & \text{for } \frac{3}{4} \le \eta_0 \le 1 \end{cases}$$

1. When $\eta_0 > 0.25$, UD is not possible.
2. There is a trade-off between the maximum confidence and the outcome rate.
3. A linear combition of confidence, such as the average guessing probablity $\max \eta_0 C(0) + \eta_+ C(+)$, can also be certified.

Certification of a quantum measurement

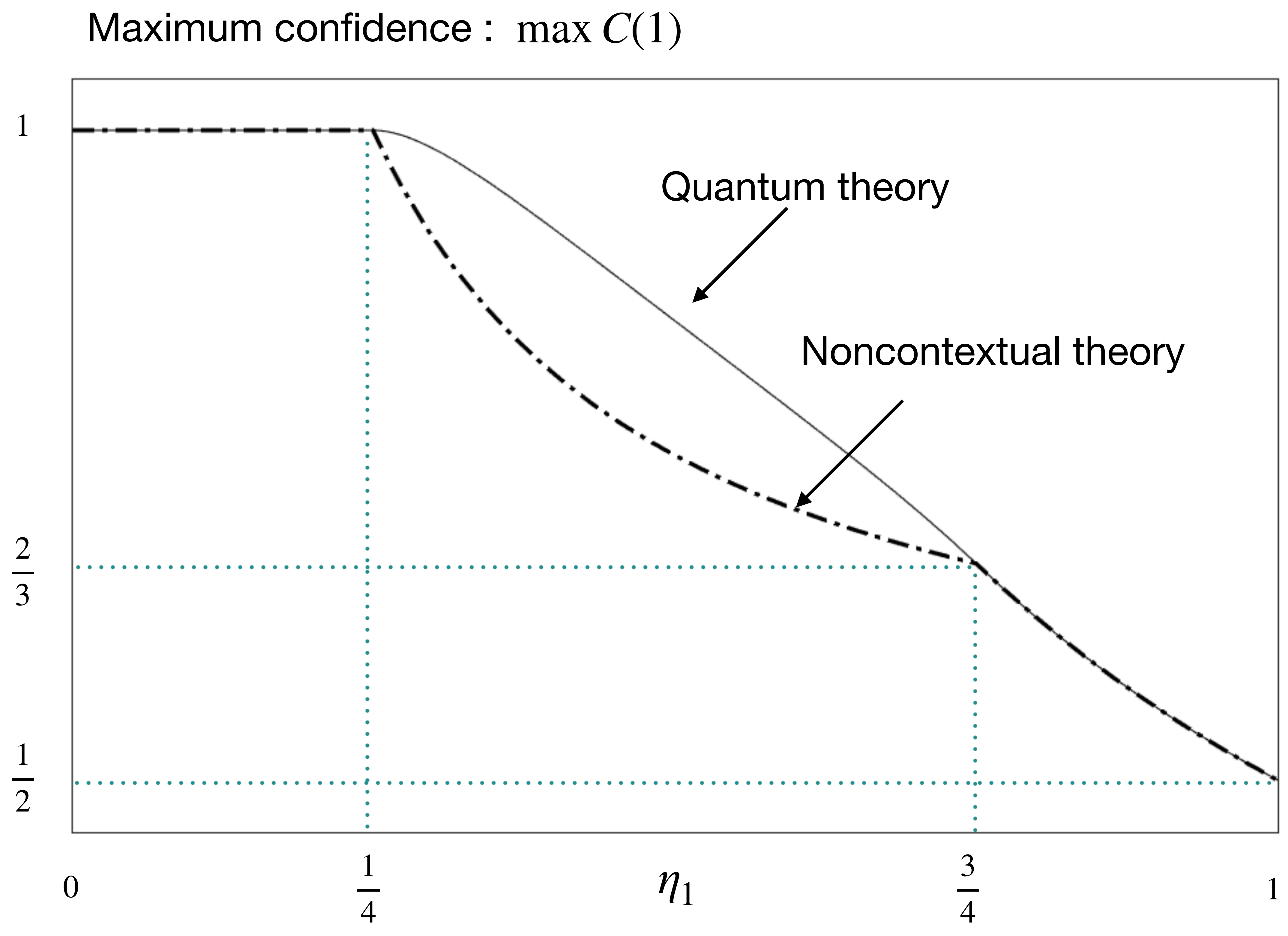


The violation of Bell inquality certifies a quantum device



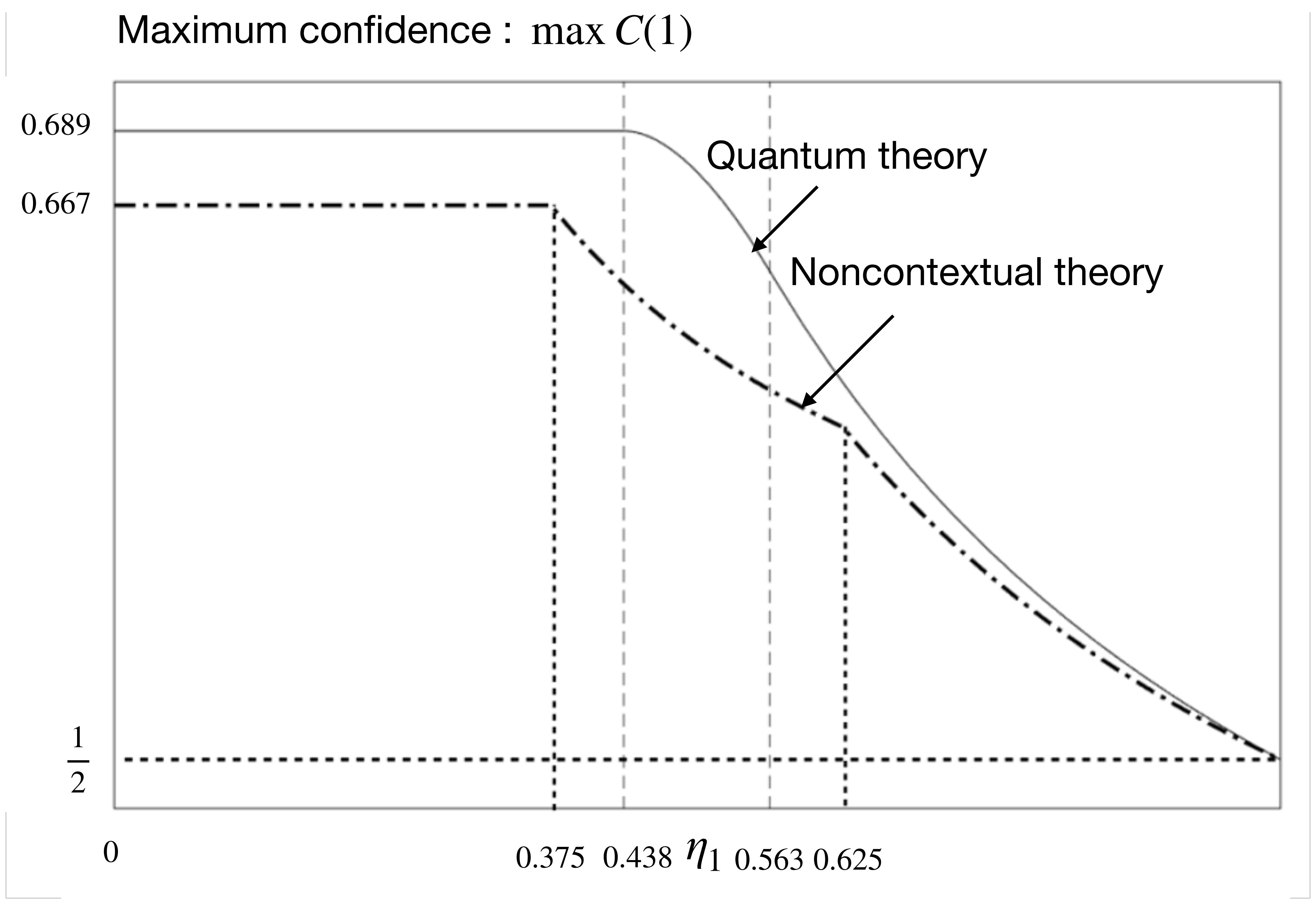
Can we certify a quantum device by observing the violation of noncontextual inequality?

Certifiable maximum confidence in quantum and non contextual theories



Noiseless preparation

$|\psi_1\rangle = |0\rangle, |\psi_2\rangle = |+\rangle$



Noisy preparation

$\rho_x = (1-p)|\psi_x\rangle\langle\psi_x| + p\frac{I}{2}, x = 1,2$

Summary

I. MCM is a state discrimination strategy that can be implemented in the presence of undetected events.

II. The problem of MCM is SDP, so the optimality conditions can be obtained.

III. MCM for qubit states is obtained from the geometry of the ensemble.

IV. MCMs can be certified from the outcome statistics.

Thank you !

Collaborators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Prof. Jonatan Brask