Nonlocal and quantum advantages in network coding for multiple access channels

Ashutosh Rai

(Korea Advanced Institute of Science and Technology)

Nagoya-KAIST GEnKO Workshop
(Nagoya University, Dec. 7th-8th, 2023)

Network coding for multiple access channels:

In collaboration with:

Jiyoung Yun

Seung-Hyun Nam

Hyun-Young Park

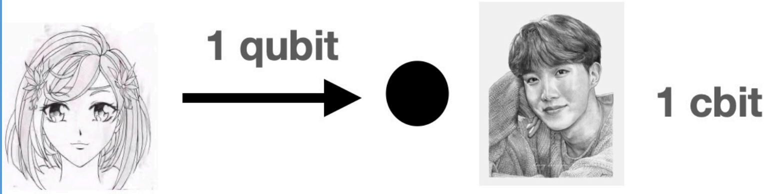
Prof. Si-Hyeon Lee

Prof. Joonwoo Bae

Point-to-Point Communication

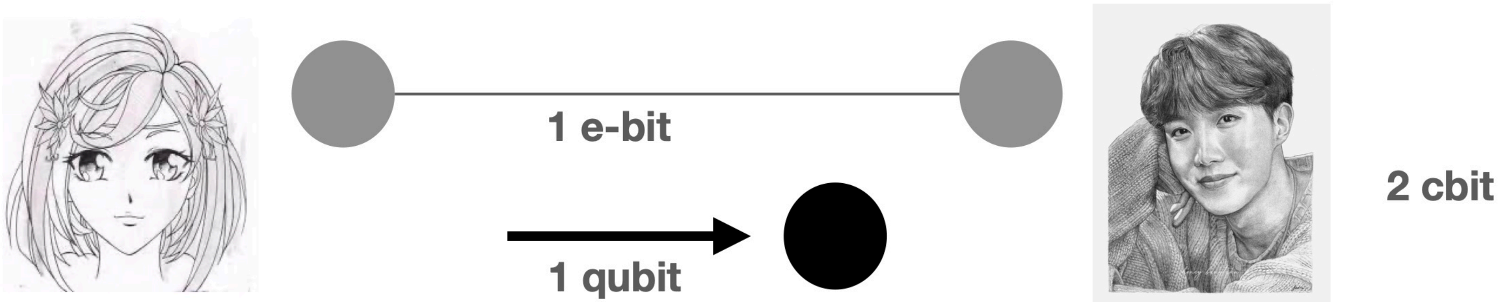
Entanglement Assisted Classical Communication over Quantum Channels

Holevo’s bound



QM respects no-signaling

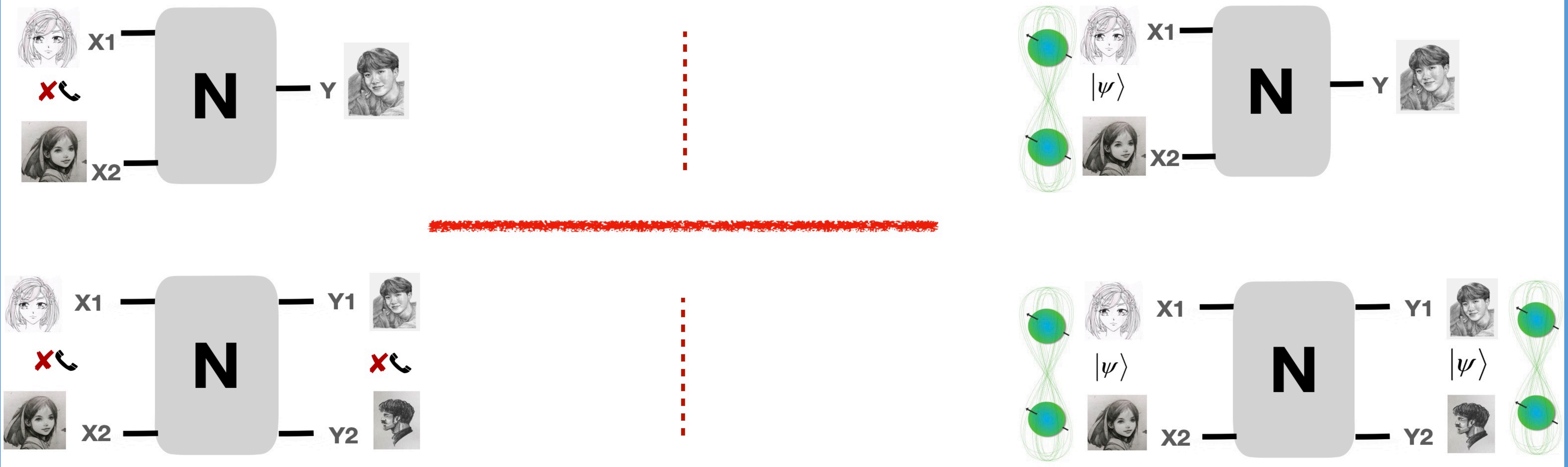
Super dense coding (1 + 0 = 2)



Network Communication

Entanglement Assisted Classical Communication over Classical Channels

Multiple Access Channel: P (Y | X1, X2)



Interference Channel: P (Y1, Y2 | X1, X2)

Some recent works

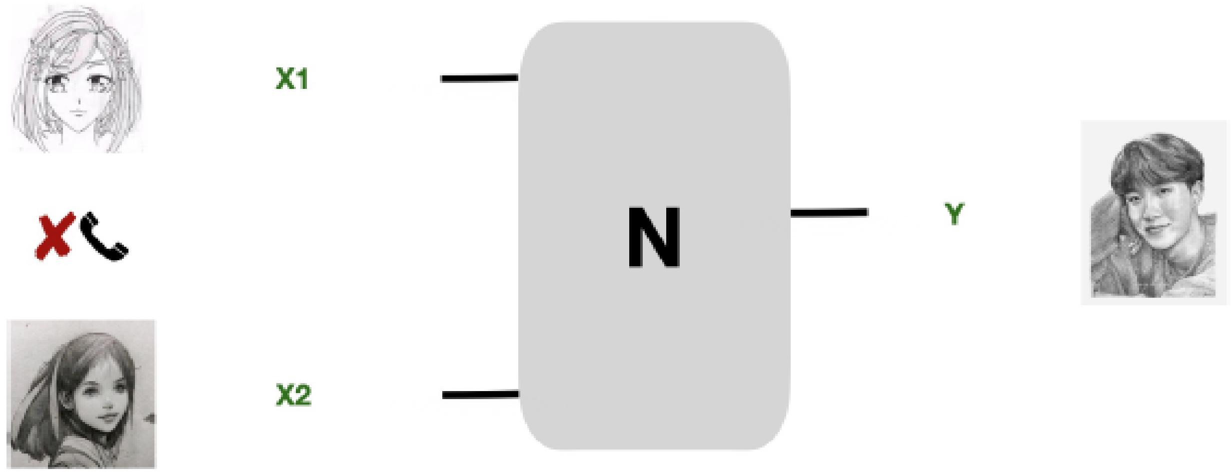
Quantum and superquantum enhancements to two-sender, two-receiver channels, Y. Quek and P. W. Shor, Phys. Rev. A 95, 052329 (2017).

Entanglement-Enabled Communication, J. Nötzel, IEEE Journal on Selected Areas in Information Theory (Volume: 1, Issue: 2, August 2020).

Playing games with multiple access channels, F. Leditzky, M. A. Alhejji, J. Levin and G. Smith, Nat. Comm. 11, 1497 (2020).

Nonlocal network coding in interference channels, J. Yun, A. Rai, and J. Bae, Phys. Rev. Lett. 125, 150502 (2020).

Discrete Memoryless Two-Senders One-Receiver Multiple Access Channels

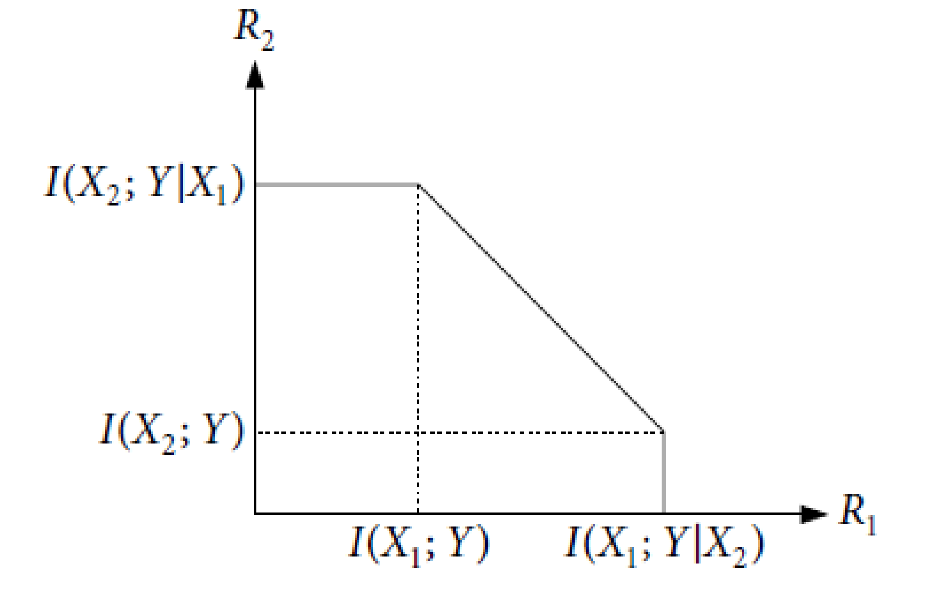


Single letter charecterization of the capacity region

$$R_1 \leq I(X1; Y|X2)$$
$$R_2 \leq I(X2; Y|X1)$$
$$R_1 + R_2 \leq I(X1, X2; Y)$$

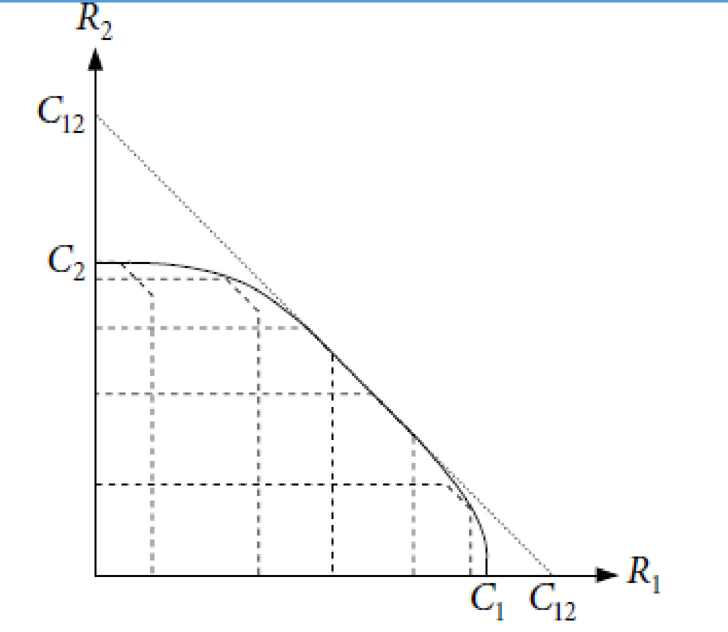
Capacity Region: $\text{Closure}\ \left(\cup_{\pi(x_1)\pi(x_2)}(R_1, R_2)\right)$

Sum Capacity: $C_{sum} = \max_{\pi(x_1)\pi(x_2)}(R_1 + R_2) = \max_{\pi(x_1)\pi(x_2)} I(X1, X2; Y)$

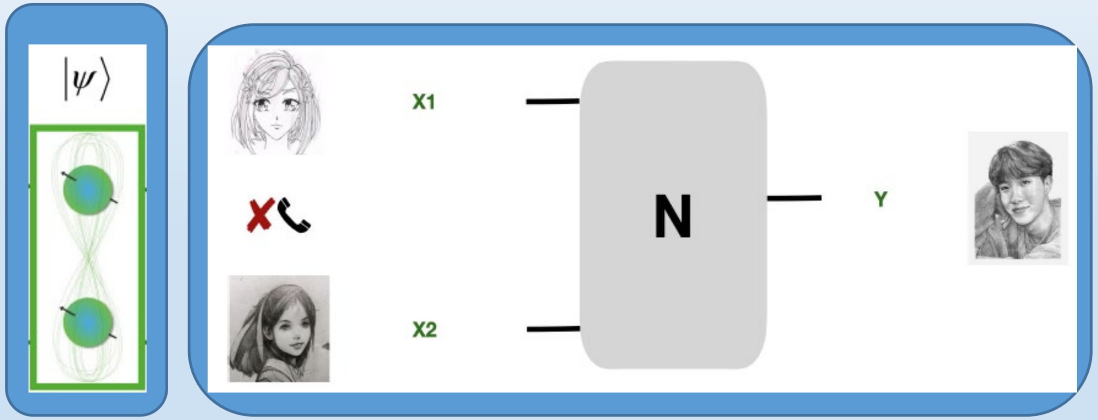


Achievable rate region for some fixed product distribution of inputs.

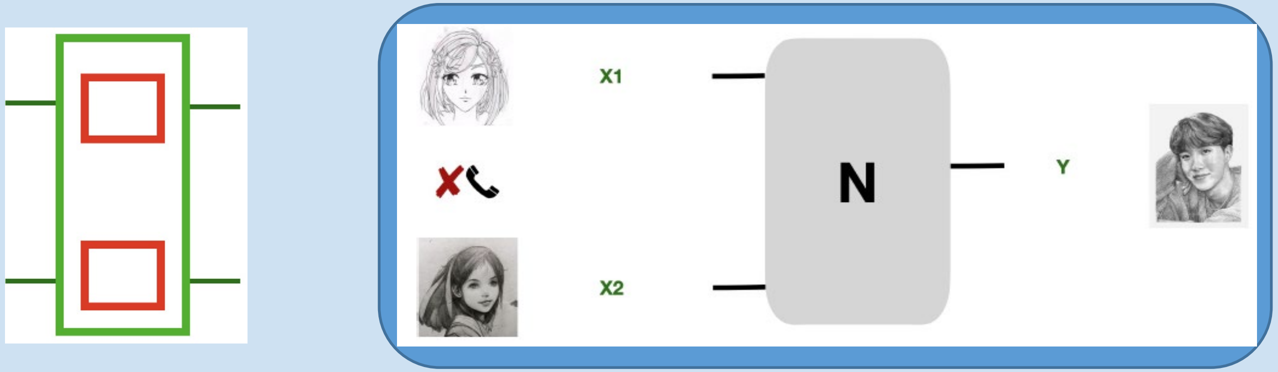
Capacity region is the closure of union of all achievable rate regions.



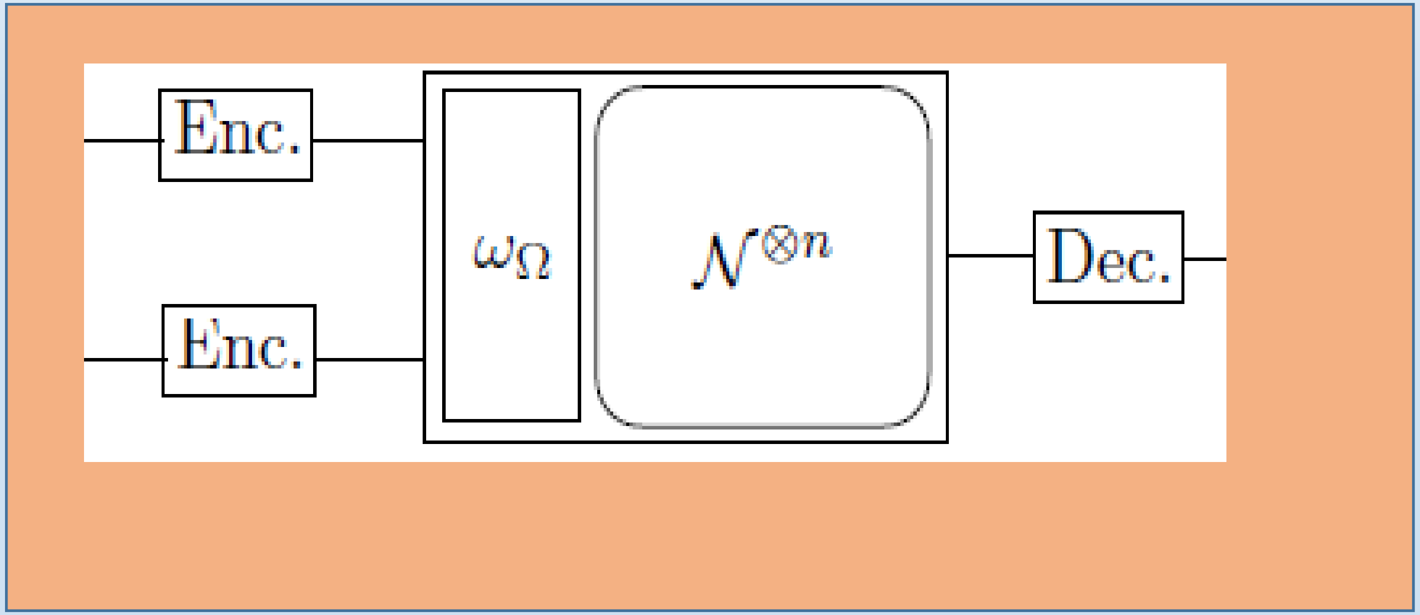
Entanglement assisted multiple access channel. Capacity region may expand !



Non-signaling cooperation assisted multiple access channel.

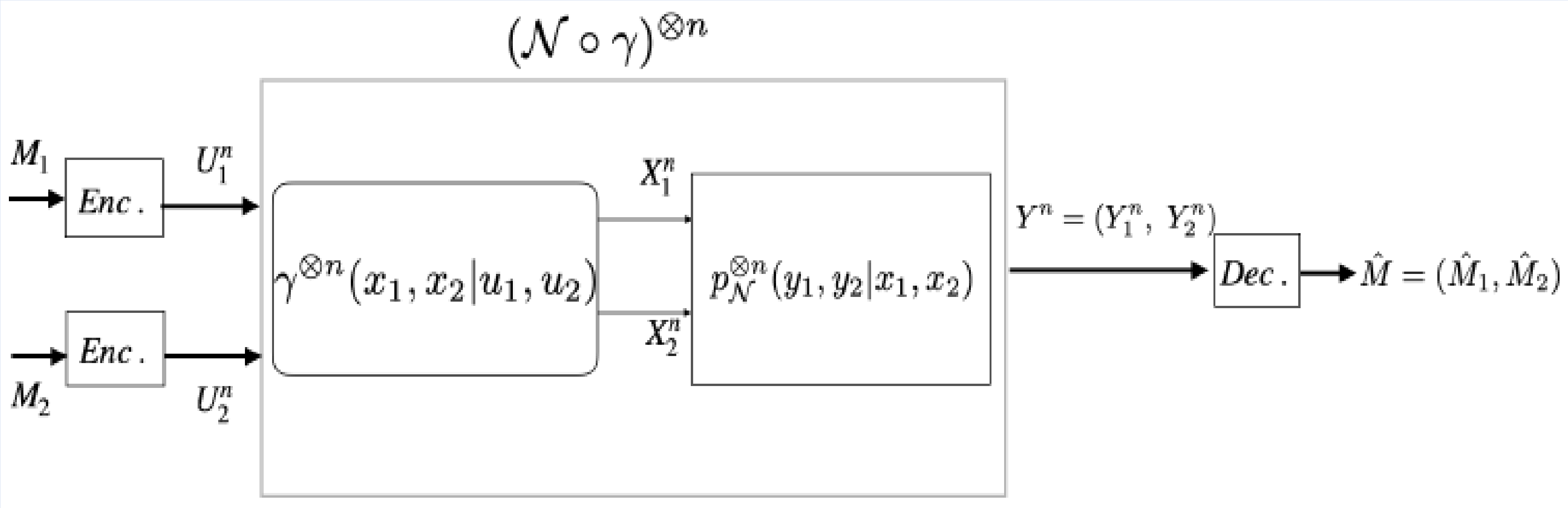


NOT KNOWN: Single letter formula for capacity region !



$$\omega \in \Omega_n^r \text{ for } r \in \{L, Q, NS\}$$

Restricted model for Non-signaling cooperation to n-copies of the channel $\omega = \gamma^{\otimes n}$



We get a single letter formula for inner bound to capacity region of assisted channel

$$\omega \in \Omega_n^r \text{ for } r \in \{L, Q, NS\}$$

Non-signaling cooperation between senders

Local correlations

$$\gamma^{L}(x_1, x_2|u_1, u_2) = \sum_{\lambda} p(\lambda)\, p(x_1|u_1, \lambda)\, p(x_2|u_2, \lambda)$$

Quantum correlations

$$\gamma^{Q}(x_1, x_2|u_1, u_2) = Tr\left[\, \rho_{A_1A_2}\, \Pi_{x_1}^{u_1} \otimes \Pi_{x_2}^{u_2} \,\right]$$

No-signaling correlations

$$\sum_{x_2} \gamma^{NS}(x_1, x_2|u_1, u_2) = \gamma^{NS}(x_1\,|u_1\,)$$

$$\sum_{x_1} \gamma^{NS}(x_1, x_2|u_1, u_2) = \gamma^{NS}(x_2\,|u_2\,)$$

Single letter formula for achievable rate region (inner bounds on capacity region)

$$R^r \neq C^r$$

For any $r \in \{L, Q, NS\}$,

$$\mathcal{R}^r = \mathrm{cl}\left(\left\{(R_1, R_2) : \begin{array}{c} R_1 \leq I(U_1; Y|U_2), \\ R_2 \leq I(U_2; Y|U_1), \\ R_1 + R_2 \leq I(U_1, U_2; Y), \end{array} \text{ for some } p(u_1)p(u_2) \text{ and } \gamma \in \Gamma^r \right\}\right),$$

$$\mathcal{R}^r_s = \sup_{\gamma \in \Gamma^r} \sup_{p(u_1)p(u_2)} I(U_1, U_2; Y).$$

Goal: Capacity separation results with assistance of non-signaling cooperation.

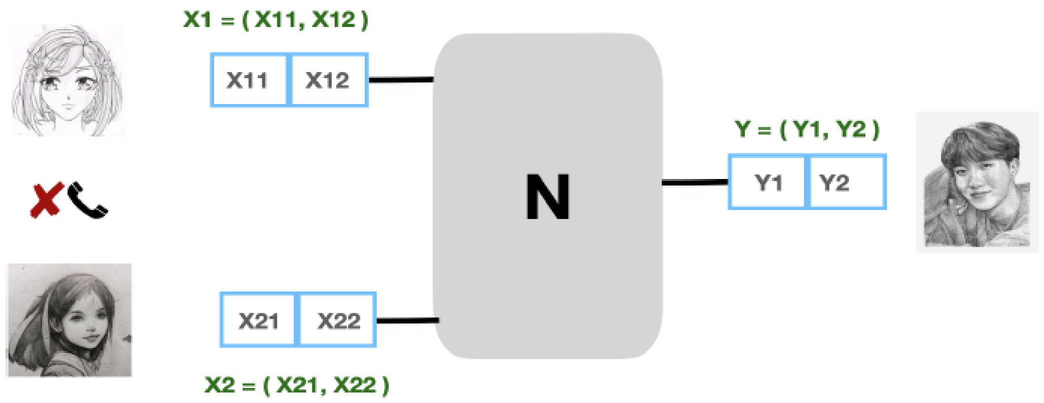
$$\mathcal{C}_s \leq \mathcal{R}^L_s \leq \mathcal{R}^Q_s \leq \mathcal{R}^{NS}_s$$

Noisy Channel Models

Proof of concept

Example

Example: Two-Senders One-Receiver Multiple Access Channels

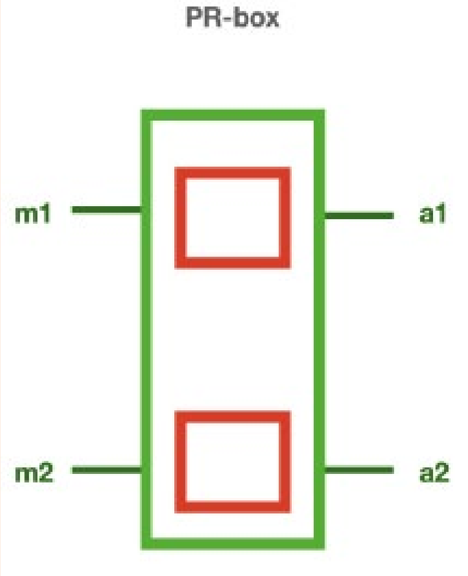


All variables are bits: $\{0,1\}$

$$\mathrm{P}_{\mathrm{N}}(y_1, y_2 | x_{11}, x_{12}; x_{21}, x_{22}) = \begin{cases} \delta(y_1, x_{11})\ \delta(y_2, x_{21}), & \text{if } x_{12} \oplus x_{22} = x_{11} x_{21} \\ 1/4 & \text{otherwise.} \end{cases}$$

Resource: Popescu-Rohrlich nonlocal box

$m_1, m_2 \in \{0,1\}$



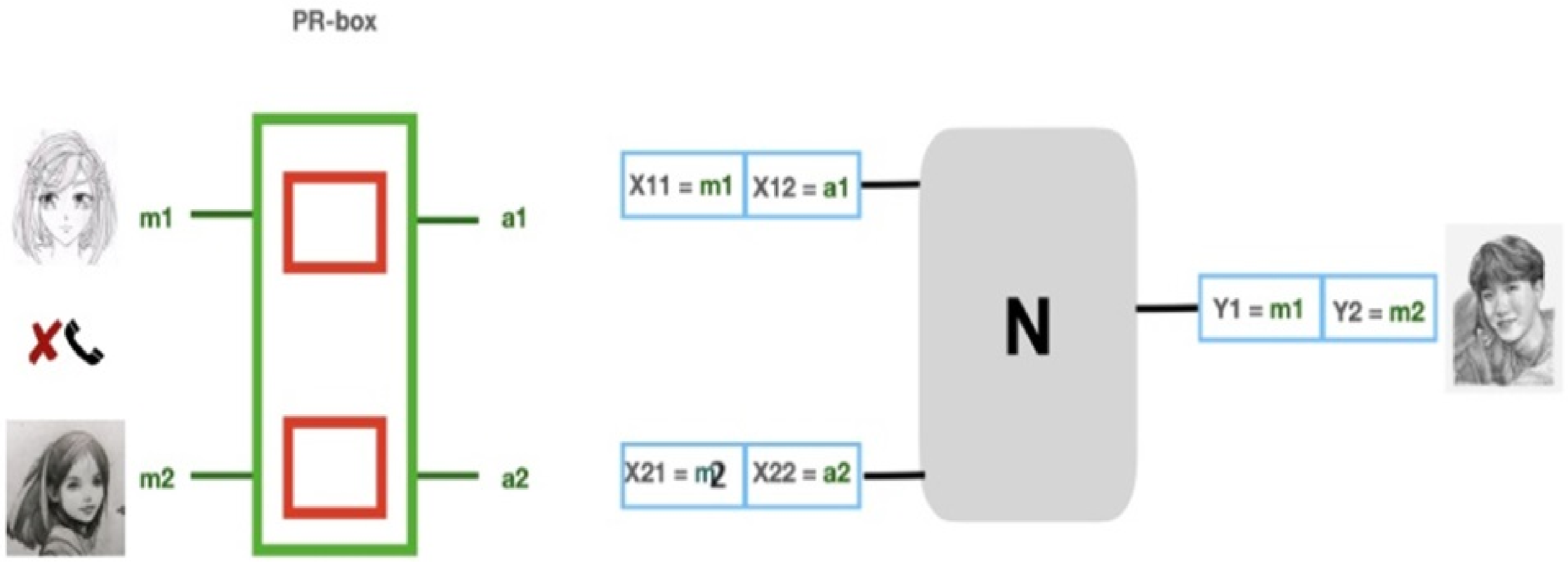
$a_1, a_2 \in \{0,1\}$

$$P(a_1, a_2 | m_1, m_2) = \frac{1}{2}\,\delta(\,m_1 m_2, a_1 \oplus a_2\,)$$

PR-box correlation

$a_1a_2 \vert m_1m_2$	00	01	10	11
00	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
01	0	0	0	$\frac{1}{2}$
10	0	0	0	$\frac{1}{2}$
11	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0

Sum-Capacity Enhancement



$$R_1^{PR} = I(M1; Y1, Y2|M2) = 1$$

$$R_2^{PR} = I(M2; Y1, Y2|M1) = 1$$

$$R_1^{PR} + R_2^{PR} = I(M1, M2; Y1, Y2) = 2$$

$$\text{Sum Capacity} = R_1^{PR} + R_2^{PR} = 2$$

Key features

- Consider some multiple access channel $\mathcal{N} : X \equiv (X1, X2) \to Y \ \equiv \mathrm{P}_{\mathcal{N}}(\ y \mid x \equiv (x_1, x_2)\ )$.
- Suppose the channel is asymmetric on a bi-partition $X = \widetilde{X} + \widetilde{\widetilde{X}}$

$$\mathcal{N} = \begin{cases} \widetilde{\mathcal{N}} & \text{if } x = (x_1, x_2) \in \widetilde{X} \\ \widetilde{\widetilde{\mathcal{N}}} & \text{if } x = (x_1, x_2) \in \widetilde{\widetilde{X}} \end{cases}$$

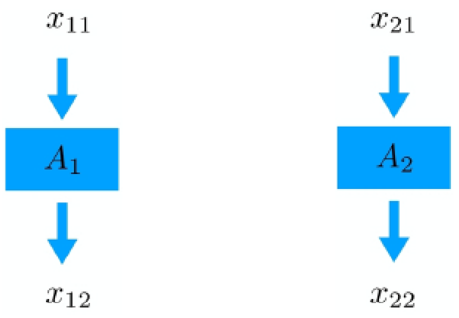
- Then if senders share nonlocally-correlating encoder $\mathrm{E}(\mathrm{x}_1, \mathrm{x}_2 | \mathrm{m}_1, \mathrm{m}_2)$ such that $\mathrm{P}_{\mathrm{E}}(\mathrm{x} \in \widetilde{\mathrm{X}}) > \mathrm{P}_{\mathrm{SR}}(\mathrm{x} \in \widetilde{\mathrm{X}})$, one may get quantum and nonlocal advantage.
- Nonlocal correlations shared between senders and receivers has potential to increase capacity region and sum-capacity of the entanglement assisted network channels.
- Motivates channels N defined with reference to some Nonlocal Games.

Nonlocal Games

2-party Nonlocal-Game G

- Question Set: $\mathcal{X}_{11} = \mathcal{X}_{21} = \{0, ..., d-1\}$

  Answer Set: $\mathcal{X}_{12} = \mathcal{X}_{22} = \{0, ..., D-1\}$



- Winning condition:

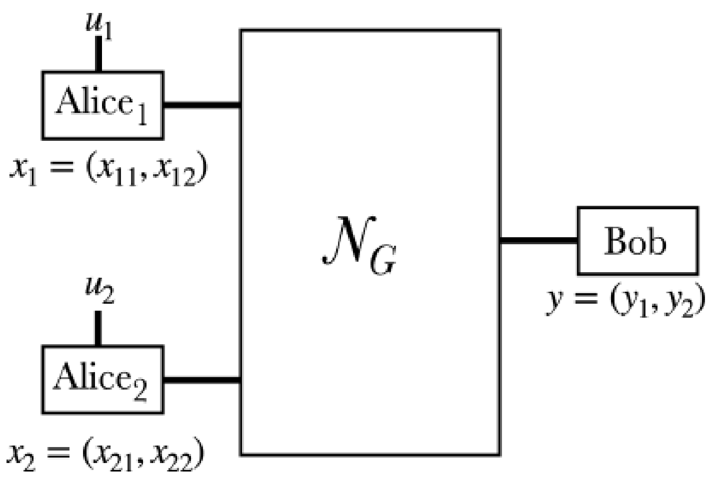
$$x = (x_{11}, x_{12}; x_{21}, x_{22}) \in \mathcal{W}_G \subset \mathcal{X}_{11} \times \mathcal{X}_{12} \times \mathcal{X}_{21} \times \mathcal{X}_{22}$$

- Winning probability: $\omega = \sum_{x \in \mathcal{W}_G} p(x)$
- Nonlocal Game: $\omega_L < \omega_{NL}$

Channel Model

2-sender MAC with reference to a Nonlocal Game G

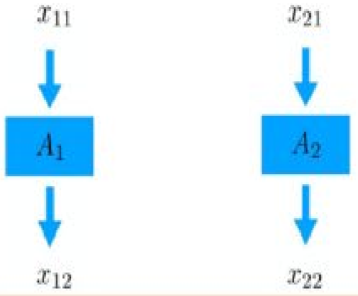
$$\mathcal{N}_{\mathsf{G}} = \begin{cases} \mathcal{N}_{\mathsf{G_W}} & \text{if inputs} \in \mathcal{W}_G, \\ \mathcal{N}_{\mathsf{G_L}} & \text{otherwise.} \end{cases}$$



Example: Magic square game (Quantum pseudotelepathy game)

Question Set: $\mathcal{X}_{11} = \mathcal{X}_{21} = \{0,\ 1,\ 2\}$

Answer Set: $\mathcal{X}_{12} = \mathcal{X}_{22} = \{000, 001, 010, 011, 100, 101, 110, 111\}$



$$x_{12} = (a_1^0, a_1^1, a_1^2)$$

$$x_{22} = (a_2^0, a_2^1, a_2^2)$$

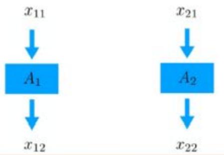
A2 / A1	0	1	2
0			
1			
2			

$$W_{MS} = \left\{ (x_{11}, x_{12};\ x_{21}, x_{22}) :\ a_1^0 \oplus a_1^1 \oplus a_1^2 = 0,\ a_2^0 \oplus a_2^1 \oplus a_2^2 = 1,\ a_1^{x_{11}} = a_2^{x_{21}} \right\}.$$

$\omega_{cl} = 8/9$ $\omega_Q = 1$

Question Set: $\mathcal{X}_{11} = \mathcal{X}_{21} = \{0,\ 1,\ 2\}$

Answer Set: $\mathcal{X}_{12} = \mathcal{X}_{22} = \{000, 001, 010, 011, 100, 101, 110, 111\}$



$$x_{12} = (a_1^0, a_1^1, a_1^2)$$

$$x_{22} = (a_2^0, a_2^1, a_2^2)$$

A2 / A1	0	1	2
0			
1			
2			

$$\mathcal{W}_{MS} = \Big\{(x_{11}, x_{12};\ x_{21}, x_{22}) :\ a_1^0 \oplus a_1^1 \oplus a_1^2 = 0,\ a_2^0 \oplus a_2^1 \oplus a_2^2 = 1,\ a_1^{x_{11}} = a_2^{x_{21}}\Big\}.$$

$$|\Psi\rangle_{A_1A_2} = \frac{1}{2}\big(|00\rangle_{A_1}|11\rangle_{A_2} - |01\rangle_{A_1}|10\rangle_{A_2} - |10\rangle_{A_1}|01\rangle_{A_2} + |11\rangle_{A_1}|00\rangle_{A_2}\big).$$

$$\mathcal{U}_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} i & 0 & 0 & 1 \\ 0 & -i & 1 & 0 \\ 0 & i & 1 & 0 \\ 1 & 0 & 0 & i \end{pmatrix}, \quad \mathcal{U}_1 = \frac{1}{2}\begin{pmatrix} i & 1 & 1 & i \\ -i & 1 & -1 & i \\ i & 1 & -1 & -i \\ -i & 1 & 1 & -i \end{pmatrix}, \quad \mathcal{U}_2 = \frac{1}{2}\begin{pmatrix} -1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \end{pmatrix};$$

$$\mathcal{V}_0 = \frac{1}{2}\begin{pmatrix} i & -i & 1 & 1 \\ -i & -i & 1 & -1 \\ 1 & 1 & -i & i \\ -i & i & 1 & 1 \end{pmatrix}, \quad \mathcal{V}_1 = \frac{1}{2}\begin{pmatrix} -1 & i & 1 & i \\ 1 & i & 1 & -i \\ 1 & -i & 1 & i \\ -1 & -i & 1 & -i \end{pmatrix}, \quad \mathcal{V}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}.$$

Optimal Quantum Protocol

I. Players A_1 and A_2 share a pair of singlet state.

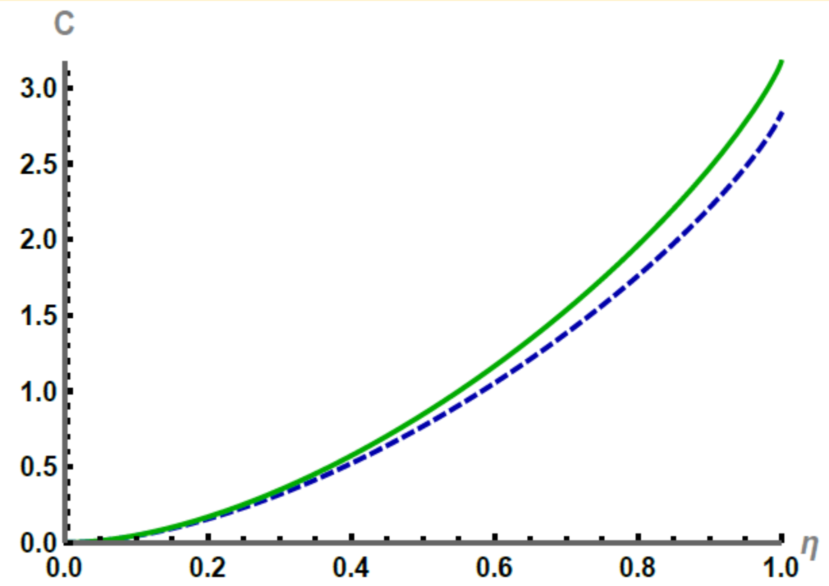
II. On questions x_{11} and x_{21} they apply respective local unitaries $U_{x_{11}}, U_{x_{21}}$ to their local two qubits followed by measurement in standard basis.

III. Player A_1 answer the first two bits with measurement outcome and third one such that the parity is even.

IV. Player A_2 answer the first two bits with measurement outcome and third one such that parity is odd.

Depolarizing MAC channel based on the Magic square game

$$\mathcal{N}_{MS}^{\eta}=\begin{cases}\eta\left(\prod_{k=1}^{2}\delta_{(x_{k1},y_k)}\right)+\frac{1-\eta}{9}, & \text{if } (x_{11},x_{12},x_{21},x_{22})\in\mathcal{W}_{MS}\\ \frac{1}{9} & \text{otherwise.}\end{cases}$$



$$C_s \leq \boldsymbol{R_s^L} < \boldsymbol{R_s^Q} \leq C_s^Q$$

Summary

1. We derived bounds on sum-capacities of 2-sender one receiver multiple access channels based on nonlocal games for the resource set L, Q, NS
2. We showed capacity enhancement results when resource set varies from L, Q, NS.
3. We showed that for certain class of quantum pseudotelepathy games the quantum upper bound is achieved.
4. From more practical side, we like to extend our findings to the converse problem: given asymmetric-network channels with multiple senders and receivers, when can entanglement assistance lead to expansion of the capacity region.

Thanks for your attention.