Distributing Entanglement over Separable Quantum Networks

Karthik Mohan, Sung Won Yun, Joonwoo Bae

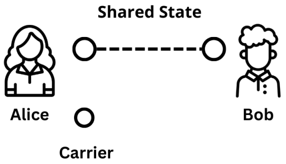
January 9, 2024

KAIST

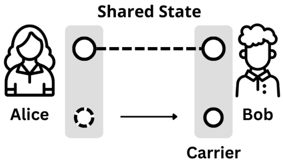
Preliminaries

Overview

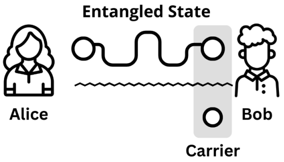
Entanglement Distribution Protocol



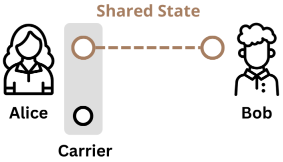
Alice and Bob share a separable state (dotted line) + a carrier state



Carrier transmission and Local Operations (gray box) on each end



Separability in $C|AB$ (Jagged line).Entanglement in $A|BC$(curved line)



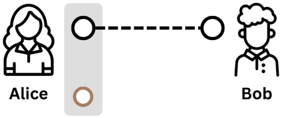
Shared state

$$\alpha_{AB} = \lambda_{\phi^+}|\phi^+\rangle\langle\phi^+|_{AB} + \lambda_{\phi^-}|\phi^-\rangle\langle\phi^-|_{AB} + \lambda_{\psi^+}|\psi^+\rangle\langle\psi^+|_{AB} + \lambda_{\psi^-}|\psi^-\rangle\langle\psi^-|_{AB}$$

Without loss of generality

$$\boxed{\frac{1}{2} \geq \lambda_{\phi^+} \geq \lambda_{\phi^-} \geq \lambda_{\psi^+} \geq \lambda_{\psi^-}}$$

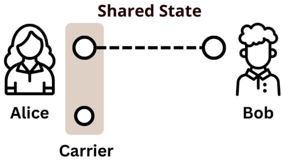
Resources



Carrier State

$$\alpha_C = \frac{1}{2}\left(\mathcal{I} + s.\sigma_z\right)$$

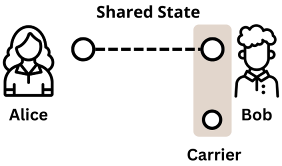
$$s \in (-1, 1)$$



Local Transformation

$$\mathcal{U}_{AC} = |0\rangle\langle 0|_A \otimes \mathcal{I}_C + |1\rangle\langle 1|_A \otimes \sigma_{x,C}$$

Resources

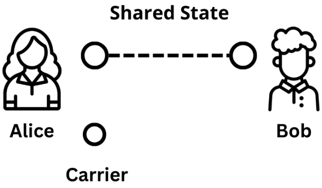


Local Transformation

$$\mathcal{U}_{BC} = |0\rangle\langle 0|_B \otimes \mathcal{I}_C + |1\rangle\langle 1|_B \otimes \sigma_{x,C}$$

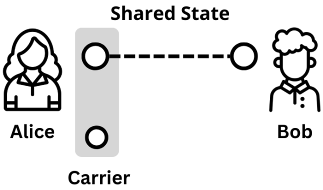
What are the carrier states that we can use to distribute entanglement?

Two-party Scenario



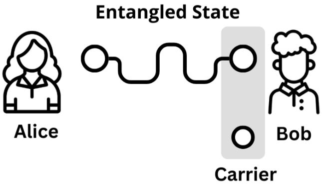
(I) $\alpha_{ABC} = \alpha_{AB} \otimes \alpha_C$

Two-party Scenario



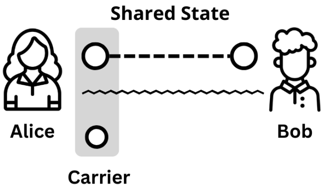
$$\text{(I)} \quad \alpha_{ABC} = \alpha_{AB} \otimes \alpha_C$$

$$\text{(II)} \quad \beta_{ABC} = \mathcal{U}_{AC} \left(\alpha_{ABC}\right) \mathcal{U}_{AC}^{\dagger}$$



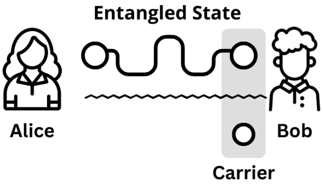
$$\text{(I)}\quad \alpha_{ABC} = \alpha_{AB} \otimes \alpha_C$$
$$\text{(II)}\quad \beta_{ABC} = \mathcal{U}_{AC}\left(\alpha_{ABC}\right)\mathcal{U}_{AC}^{\dagger}$$
$$\text{(III)}\quad \gamma_{ABC} = \mathcal{U}_{BC}\left(\beta_{ABC}\right)\mathcal{U}_{BC}^{\dagger}$$

Two-party Scenario - Conditions



$$\forall\, i \quad \boxed{\Upsilon_i\left(\beta_{ABC}\right)^{T_C} \geq 0}$$

Separability in $AB|C$ partition



$$\exists\, j \text{ s.t } \boxed{\Upsilon_j\left(\gamma_{ABC}\right)^{T_A} < 0}$$

Distillable entanglement in $A|BC$ partition

Analysis-1

$$\forall\, i \; \boxed{0 \leq \Upsilon_i \left(\beta_{ABC}\right)^{T_C}} \qquad \exists\, j \text{ s.t } \boxed{\Upsilon_j \left(\gamma_{ABC}\right)^{T_A} < 0}$$

Example Eigenvalue:

$$\frac{1}{2}\left(\lambda_{\phi^-} - s\lambda_{\phi^+}\right)$$

Since $0 \leq \Upsilon_i \left(\beta_{ABC}\right)^{T_C}$

$$0 \leq \frac{1}{2}\left(\lambda_{\phi^-} - s\lambda_{\phi^+}\right)$$

$$\therefore s \leq \frac{\lambda_{\phi-}}{\lambda_{\phi+}}$$

Example Eigenvalue:

$$\frac{1}{4}\left(1 - 2\lambda_{\phi^+} - s\left(1 - 2\lambda_{\phi^-}\right)\right)$$

Since $\Upsilon_j \left(\gamma_{ABC}\right)^{T_A} < 0$

$$\frac{1}{4}\left(1 - 2\lambda_{\phi^+} - s\left(1 - 2\lambda_{\phi^-}\right)\right) < 0$$

$$\therefore \frac{1-2\lambda_{\phi+}}{1-2\lambda_{\phi-}} < s$$

We can use similar analysis on eigenvalues of $\left(\beta_{ABC}\right)^{T_C}$ and $\left(\gamma_{ABC}\right)^{T_A}$ resp.

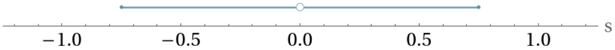
Carrier bounds

Main Result

$$\min\left[\frac{\Lambda_{\phi^+}}{\Lambda_{\phi^-}}, \frac{\Lambda_{\psi^+}}{\Lambda_{\psi^-}}\right] < |s| \leq \min\left[\frac{\lambda_{\phi^-}}{\lambda_{\phi^+}}, \frac{\lambda_{\psi^-}}{\lambda_{\psi^+}}\right]$$

where $\Lambda_i = 1 - 2\lambda_i$

Example: $\lambda_{\phi^+} = \frac{1}{2}, \lambda_{\phi^-} = \frac{3}{8}, \lambda_{\psi^+} = \lambda_{\psi^-} = \frac{1}{16}$

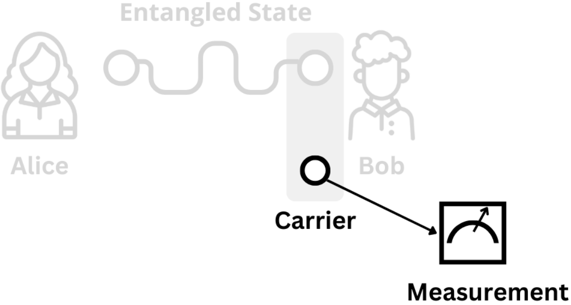


$\mathrm{s} \in \{-\frac{3}{4}, \frac{3}{4}\}, \ \mathrm{s} \neq 0$

Probabilistic Entanglement Distribution

Entangled state is obtained probabilistically after measuring the carrier

$$\gamma_{ABC} = c\gamma^1_{AB} \otimes |0\rangle\langle 0|_C + (1-c)\gamma^2_{AB} \otimes |1\rangle\langle 1|_C$$



$$s = \frac{1}{2} \quad c = \frac{11}{16}$$

$$\gamma^1_{AB} = \left\{\frac{6}{11}\ ;\ \frac{9}{22}\ ;\ \frac{1}{44}\ ;\ \frac{1}{44}\right\}$$

$$\gamma^2_{AB} = \left\{\frac{2}{5}\ ;\ \frac{3}{10}\ ;\ \frac{3}{20}\ ;\ \frac{3}{20}\right\}$$

γ^1_{AB} contains entanglement

Question

Can Bob apply transformations to his qubits after the final CNOT operation to make the protocol deterministic?

Say $\Gamma_{ABC} = \sum_j \left(\mathcal{I}_A \otimes \mathcal{K}_j\right) \gamma_{ABC} \left(\mathcal{I}_A \otimes \mathcal{K}_j^\dagger\right)$ and $\Gamma_{AB} = Tr_C\left(\Gamma_{ABC}\right)$

Can we find the Kraus operators that give us distillable entanglement?

$$\exists\, i \text{ s.t } \Upsilon_i\left(\Gamma_{AB}\right)^{T_A} < 0?$$

Transformations

If s > 0

$$\mathcal{K}_1 = \mathcal{I}_B \otimes |0\rangle\langle 0|_C$$

$$\mathcal{K}_2 = |0\rangle\langle 0|_B \otimes |1\rangle\langle 1|_C$$

$$\mathcal{K}_3 = |0\rangle\langle 1|_B \otimes |1\rangle\langle 1|_C$$

$$\mathcal{I} = \sum_i \mathcal{K}_i^\dagger \mathcal{K}_i$$

If s < 0

$$\mathcal{K}_1 = \mathcal{I}_B \otimes |1\rangle\langle 1|_C$$

$$\mathcal{K}_2 = |0\rangle\langle 0|_B \otimes |0\rangle\langle 0|_C$$

$$\mathcal{K}_3 = |0\rangle\langle 1|_B \otimes |0\rangle\langle 0|_C$$

$$\mathcal{I} = \sum_i \mathcal{K}_i^\dagger \mathcal{K}_i$$

When is $\Upsilon_i \left(\Gamma_{AB}\right)^{T_A} < 0$

Deterministic Entangelement Distribution - Simplification

When is $\Upsilon_i\left(\Gamma_{AB}\right)^{T_A} < 0$

Simplifying Assumption

$$\lambda_{\psi^+} = \lambda_{\psi^-} \text{ and } \lambda_{\phi^+} = \frac{1}{2}$$

Probabilistic Bounds

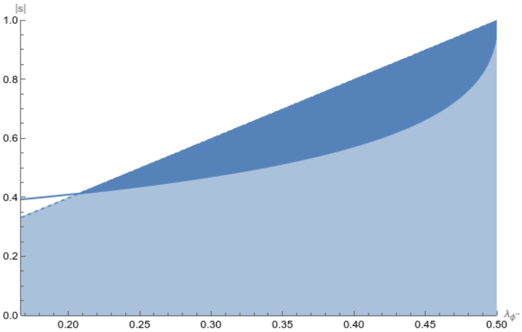
$$0 < |s| \leq 2\lambda_{\phi^-}$$

Deterministic Bounds

$$\frac{3-2\lambda_{\phi^-}-\sqrt{4\lambda_{\phi^-}^2-20\lambda_{\phi^-}+9}}{4\lambda_{\phi^-}} < |s| \leq 2\lambda_{\phi^-}$$

Deterministic Entangelement Distribution - Examples

Possible Region of $|s|$ for Entanglement Distribution

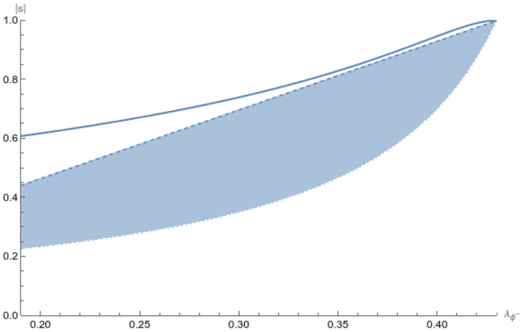


Deterministic Probabilistic

There is deterministic entanglement distribution when $\lambda_{\phi^-} \in \{\frac{1}{2}(\sqrt{2}-1), \frac{1}{2}\}$

Deterministic Entangelement Distribution - Examples

Possible Region of $|s|$ for Entanglement Distribution

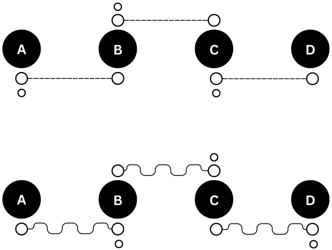


■ Deterministic ■ Probabilistic

There is no deterministic entanglement distribution

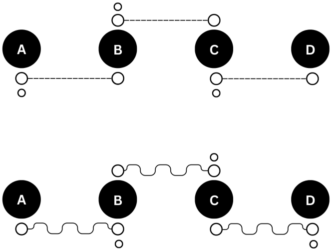
Applications

Linear Network

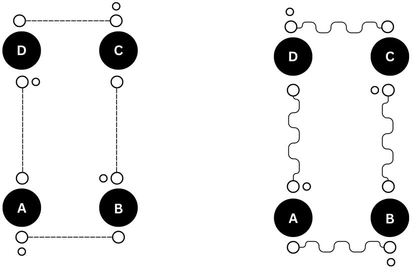


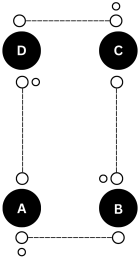
Network Entanglement Distribution

Linear Network



Circular Network



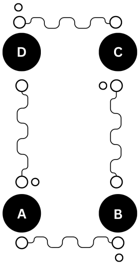


Initial State

$$\text{Shared State: } \alpha_4^{\text{Circ}} = \alpha_{a_1 b_1} \otimes \alpha_{b_2 c_1} \otimes \alpha_{c_2 d_1} \otimes \alpha_{d_2 a_2}$$

$$\text{Carrier State: } \eta_4^{\text{Circ}} = \eta_a \otimes \eta_b \otimes \eta_c \otimes \eta_d$$

Network Entanglement Distribution - Circular



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$$\text{Carrier State: } \eta_4^{\text{Circ}} = \eta_a \otimes \eta_b \otimes \eta_c \otimes \eta_d$$

Final State

$$\Delta^{\text{Final}} = \gamma_{a_1 b_1 \eta_a} \otimes \gamma_{b_2 c_1 \eta_b} \otimes \gamma_{c_2 d_1 \eta_c} \otimes \gamma_{d_2 a_2 \eta_d}$$

Summary

1. We have characterized carrier states in terms of shared Bell state coefficients that can be used to distribute entanglement between two parties
2. We have shown that deterministic entanglement distribution is possible using an uncorrelated carrier state which remains separable
3. We have adapted the entanglement distribution protocol to networks giving us networks with entangled states between nearest neighbors

References

(1) T. S. Cubitt, F. Verstraete, W. Dür, and J. I. Cirac, "Separable States Can Be Used To Distribute Entanglement," vol. 91, no. 3, p. 037902.

(2) D. Deutsch, A.Ekert, R. Jozsa, C. Macchiavello, S. Popescu, and A. Sanpera, "Quantum Privacy Amplification and the Security of Quantum Cryptography over Noisy Channels," vol.77, no. 13, pp. 2818–2821

(3) J. Bae, T. Cubitt, and A. Acín, "Nonsecret correlations can be used to distribute secrecy," vol. 79, no. 3, p. 032304.

(4) A. Kay, "Using Separable Bell-Diagonal States to Distribute Entanglement," vol. 109, no. 8, p. 080503