On Hilbert's tenth problem over subrings of $\mathbb{Q}$

Takao Yuyama

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Tokyo Institute of Technology

Hilbert's tenth problem

A Diophantine equation is an equation of the form $f(\vec{x}) = 0$ for some (possibly multivariate) polynomial $f \in \mathbb{Z}[x_1, x_2, \ldots]$.

Hilbert's tenth problem; $\mathrm{HTP}(\mathbb{Z})$ (Hilbert, 1900)

Is there an algorithm to decide whether there exists an integer solution of a given Diophantine equation?

MRDP theorem (Matiyasevich, 1970)

No such algorithm exist. That is, $\mathrm{HTP}(\mathbb{Z})$ is undecidable.

More precisely, every computably enumerable set (c.e. set) in $\mathbb{Z}^n$ is a Diophantine set in $\mathbb{Z}$.

Hilbert's tenth problem over the rational numbers

It is natural to generalize the original Hilbert's tenth problem to arbitrary ring R.

Hilbert's tenth problem over a ring R; $\mathrm{HTP}(R)$

Is there an algorithm to decide whether there exists a solution in R of a given Diophantine equation from $R[x_1, x_2, \ldots]$?

The undecidability status for $R = \mathbb{Q}$ is still open.

Open problem ($\mathrm{HTP}(\mathbb{Q})$)

Is $\mathrm{HTP}(\mathbb{Q})$ undecidable?

Known difficulty

The only known method to prove the undecidability of $\mathrm{HTP}(R)$ for a ring R is the following proposition.

Proposition

If $\mathbb{Z}$ admits a Diophantine model in a ring R, then $\mathrm{HTP}(R)$ is undecidable. In particular, if $\mathbb{Z}$ is a Diophantine set over a ring R, then $\mathrm{HTP}(R)$ is undecidable.

However, the above proposition cannot be appliable to $R = \mathbb{Q}$ when we assuming some plausible number-theoretic condition.

Theorem (Cornelissen-Zahidi, 2000)

The integers $\mathbb{Z}$ does not admit a Diophantine model in $\mathbb{Q}$ under the Mazur conjecture.

Activating computability theory

For $W \subseteq \mathbb{P} := \{\text{prime numbers}\}$, define $R_W := \mathbb{Z}[W^{-1}] \subseteq \mathbb{Q}$ and

$$\mathrm{HTP}(R_W) := \{ f \in \mathbb{Z}[x_1, x_2, \ldots] \mid f \text{ has a solution in } R_W \}.$$

Eisenträger-Miller-Park-Shlapentokh (2017) observed:

The set of subrings of $\mathbb{Q}$ is isomorphic to the Cantor space $2^{\mathbb{P}}$

There is a bijection

$$\begin{array}{ccc} 2^{\mathbb{P}} & \stackrel{\sim}{\longleftrightarrow} & \{\text{subring } R \subseteq \mathbb{Q}\} \\ W & \longmapsto & R_W = \mathbb{Z}[W^{-1}] \\ \{ p \mid 1/p \in R \} & \longleftarrow & R. \end{array}$$

Basic facts

For any $W \in 2^{\mathbb{P}}$, we have $R_W \equiv_{\mathrm{T}} W$ and

- $\mathrm{HTP}(R_W)$ is c.e. in W. In particular, $W \leq_{\mathrm{T}} \mathrm{HTP}(R_W) \leq_{\mathrm{T}} W'$,
- $\mathrm{HTP}(\mathbb{Z}) \equiv \emptyset'$ and $\mathrm{HTP}(\mathbb{Q}) \leq_{\mathrm{T}} \mathrm{HTP}(R_W)$.

Open set associating to a polynomial

Definition (Miller, 2016)

For a polynomial $f \in \mathbb{Z}[x_1, x_2, \dots]$,

- $\mathcal{A}(f) := \{\, W \in 2^{\mathbb{P}} \mid f$ has a solution in $R_W \,\}$: open set in $2^{\mathbb{P}}$,
- $\mathcal{C}(f) := \operatorname{int}(\overline{\mathcal{A}}) = \left\{ W \in 2^{\mathbb{P}} \;\middle|\; \exists V \in 2^{\mathbb{P}} \begin{bmatrix} W \subseteq V,\ V \text{ is cofinite, } f \text{ does} \\ \text{not have a solution in } R_V \end{bmatrix} \right\}$,
- $\mathcal{B}(f) := \partial\mathcal{A}(f) = \left\{ W \in 2^{\mathbb{P}} \;\middle|\; \begin{array}{l} f \text{ does not have a solution in } R_W, \\ \forall V \in 2^{\mathbb{P}} \begin{bmatrix} W \subseteq V,\ V \text{ is cofinite } \Longrightarrow \\ f \text{ has a solution in } R_V \end{bmatrix} \end{array} \right\}$,
- $\mathcal{B} := \bigcup_{f \in \mathbb{Z}[x_1, x_2, \dots]} \mathcal{B}(f)$: meager set in $2^{\mathbb{P}}$.

HTP-genericity

Definition (Miller, 2016)

A set $W \in 2^{\mathbb{P}}$ is HTP-generic if $W \notin \mathcal{B}$.

Since $\mathcal{B}$ is meager, there are comeager many HTP-generic sets.

Proposition (Eisenträger-Miller-Park-Shlapentokh, 2017)

For any finite set $A \subseteq \mathbb{P}$, $\mathrm{HTP}(R_{\mathbb{P}-A}) \leq_{\mathrm{T}} \mathrm{HTP}(\mathbb{Q})$.

This proposition yields the following one.

Proposition (Miller, 2016)

If $W \in 2^{\mathbb{P}}$ is an HTP-generic set, then $\mathrm{HTP}(R_W) \leq_{\mathrm{T}} W \oplus \mathrm{HTP}(\mathbb{Q})$.

We can construct co-infinie HTP-generic set $W \leq_{\mathrm{T}} \mathrm{HTP}(\mathbb{Q})$, which satisfies $\mathrm{HTP}(R_W) \equiv_{\mathrm{T}} \mathrm{HTP}(\mathbb{Q})$.

HTP-completeness versus HTP-nontriviality

Definition (Miller, 2019^+)

A set $W \in 2^{\mathbb{P}}$ is HTP-complete if $W' \leq_1 \mathrm{HTP}(R_W)$ ($\Rightarrow W' \equiv_{\mathrm{T}} \mathrm{HTP}(R_W)$).

Proposition

If there exists $W \in 2^{\mathbb{P}}$ such that it is HTP-complete and HTP-generic, then $\mathrm{HTP}(\mathbb{Q}) >_{\mathrm{T}} \emptyset$.

However:

Theorem (Miller, 2019^+)

The set of HTP-complete sets is meager and null in $2^{\mathbb{P}}$.

So we introduce more suitable notion for undecidability proof.

Definition (Y.)

A set $W \in 2^{\mathbb{P}}$ is HTP-nontrivial if $W <_{\mathrm{T}} \mathrm{HTP}(R_W)$ (i.e., $\mathrm{HTP}(R_W) \not\leq_{\mathrm{T}} W$).

Main Theorem 1

We characterize the undecidability of $\mathrm{HTP}(\mathbb{Q})$ in terms of HTP-nontriviality.
Define $\mathcal{N} := \{\, W \in 2^{\mathbb{P}} \mid W$ is HTP-nontrivial $\}$.

Theorem (Y.)

The following conditions are equivalent.

1. $\mathrm{HTP}(\mathbb{Q}) >_{\mathrm{T}} \emptyset$,
2. $\mathcal{N}$ *is comeager in* $2^{\mathbb{P}}$,
3. $\mathcal{N}$ *is not meager in* $2^{\mathbb{P}}$,
4. $\mathcal{N} \cap \overline{\mathcal{B}} \neq \emptyset$.

Proof sketch.

(1) ⇒ (2). If $\mathrm{HTP}(\mathbb{Q}) >_{\mathrm{T}} \emptyset$, then there are comeager many sets incomparable with $\mathrm{HTP}(\mathbb{Q})$. Then we have $W \not\geq_{\mathrm{T}} \mathrm{HTP}(\mathbb{Q}) \leq_{\mathrm{T}} \mathrm{HTP}(R_W)$, i.e., $W <_{\mathrm{T}} \mathrm{HTP}(R_W)$ for such W.

(2) ⇒ (3) ⇒ (4). easy.

(4) ⇒ (1). For $W \in \mathcal{N} \cap \overline{\mathcal{B}}$, we have

$$W <_{\mathrm{T}} \mathrm{HTP}(R_W) \leq_{\mathrm{T}} W \oplus \mathrm{HTP}(\mathbb{Q}). \qquad \square$$

Note that undecidability proof along this direction work even if $\mathrm{HTP}(\mathbb{Q}) <_{\mathrm{T}} \emptyset'$!

Comparing with Miller's result

Miller has showed the following result.

Theorem (Miller, 2016)

For $C \in 2^\omega$, the following conditions are equivalent.

1. $C \leq_{\mathrm{T}} \mathrm{HTP}(\mathbb{Q})$,
2. $\{\, W \in 2^{\mathbb{P}} \mid C \leq_{\mathrm{T}} \mathrm{HTP}(R_W) \,\} = 2^{\mathbb{P}}$,
3. $\{\, W \in 2^{\mathbb{P}} \mid C \leq_{\mathrm{T}} \mathrm{HTP}(R_W) \,\}$ *is not meager.*

However, undecidability proofs in this direction need to construct some fixed set C.

Banach-Mazur game

Definition

For $\mathcal{A} \subseteq 2^\omega$, Banach-Mazur game for $\mathcal{A}$ (denoted by $\mathrm{BM}(\mathcal{A})$) is an infinite game played by Player I and II. They choose increasing strings $\sigma_s \in 2^{<\omega}$ in turns, and Player I wins if and only if $f = \bigcup_{s\in\omega} \sigma_s \in \mathcal{A}$.

$$\begin{array}{llllllll} \mathrm{I}: & \sigma_0 & & & \sigma_2 & & & \cdots \\ & & \searrow\!\prec & \nearrow\!\prec & & \searrow\!\prec & \nearrow & \\ \mathrm{II}: & & & \sigma_1 & & & \sigma_3 & \end{array}$$

$$\begin{aligned} \mathrm{I} \uparrow \mathrm{BM}(\mathcal{A}) &:\Longleftrightarrow \text{Player I has a winning strategy,} \\ \mathrm{II} \uparrow \mathrm{BM}(\mathcal{A}) &:\Longleftrightarrow \text{Player II has a winning strategy.} \end{aligned}$$

Proposition

$$\begin{aligned} \mathrm{I} \uparrow \mathrm{BM}(\mathcal{A}) &\Longleftarrow \mathcal{A} \text{ is comeager,} \\ \mathrm{II} \uparrow \mathrm{BM}(\mathcal{A}) &\Longleftrightarrow \mathcal{A} \text{ is meager.} \end{aligned}$$

Main Theorem 2

Theorem (Y.)

The following conditions are equivalent.

1. $\mathrm{HTP}(\mathbb{Q}) >_{\mathrm{T}} \emptyset$,
2. $\mathrm{I} \uparrow \mathrm{BM}(\mathcal{N})$,
3. $\mathrm{II} \not\uparrow \mathrm{BM}(\mathcal{N})$.

In particular, $\mathrm{BM}(\mathcal{N})$ *is determined.*

Proof.

(1) $\Rightarrow$ (2). If $\mathrm{HTP}(\mathbb{Q}) >_{\mathrm{T}} \emptyset$, then $\mathcal{N}$ is comeager and $\mathrm{I} \uparrow \mathrm{BM}(\mathcal{N})$.

(2) $\Rightarrow$ (3). clear.

(3) $\Rightarrow$ (1). If $\mathrm{II} \not\uparrow \mathrm{BM}(\mathcal{N})$, then $\mathcal{N}$ is not meager and $\mathrm{HTP}(\mathbb{Q}) >_{\mathrm{T}} \emptyset$. □

Partial result

Theorem (Y.)

The set of m-nontrivial rings $\mathcal{N}_{\mathrm{m}} = \{\, W \in 2^{\mathbb{P}} \mid W <_{\mathrm{m}} \mathrm{HTP}(R_W) \,\}$ *is comearger in* $2^{\mathbb{P}}$.

Proof sketch.

For each computable function $h\colon \omega \to \omega$,

$$\{\, W \in 2^{\mathbb{P}} \mid W \leq_{\mathrm{m}} \mathrm{HTP}(R_W) \text{ via } h \,\} - \mathcal{B}$$

is closed and nowhere dense in $2^{\mathbb{P}} - \mathcal{B}$. □

Question

How about tt-nontrivial rings $\mathcal{N}_{\mathrm{tt}} := \{\, W \in 2^{\mathbb{P}} \mid W <_{\mathrm{tt}} \mathrm{HTP}(R_W) \,\}$?

References

- G. Cornelissen, K. Zahidi, Topology of Diophantine Sets: Remarks on Mazur's Conjectures, *Hilbert's Tenth Problem: Relations with Arithmetic and Algebraic Geometry (Ghent, 1999)*, 253–260, Contemp. Math. Vol. 270, *Amer. Math. Soc., Providence, RI*, 2000.
- K. Eisenträger et al., As easy as $\mathbb{Q}$: Hilbert's tenth problem for subrings of the rationals and number fields, Trans. Amer. Math. Soc, **369**(11): 8291–8315, 2017.
- R. G. Miller, Baire Category Theory and Hilbert's Tenth Problem Inside $\mathbb{Q}$, in CiE 2016: Pursuit of the Universal, LNCS Vol. 9709, Springer, 2016, 343–352.
- R. G. Miller, HTP-complete rings of rational numbers, arXiv Preprint, 2019, `http://arxiv.org/abs/1907.03147`.