Basis Theorems and Models of WKL_0

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Definition

$\mathcal{L}_2$: The language of 2nd order arithmetic.
$M = (\mathbb{N}^M, S)$: $\mathcal{L}_2$-structure. $\mathbb{N}^M$: the 1st order part of M,
S: the 2nd order part of M.

Note: We use ω to denote the standard natural numbers, and $\mathbb{N}$ to denote the 1st part of a $\mathcal{L}_2$-structure. If $\mathbb{N}^M \simeq \omega$ then M is called ω-model.

Subsystems of 2nd order arithmetic

$\mathrm{RCA}_0 = \mathrm{I}\Sigma^0_1 +$ recursive sets exist.
$\mathrm{WKL}_0 = \mathrm{RCA}_0 +$ any infinite binary tree has a path.
$\mathrm{ACA}_0 = \mathrm{RCA}_0 +$ arithmetically definable sets exist.

General Form of Basis Theorems

Basis theorems have the following form in general:
Let T be a computable infinite binary tree. Then T has a path s.t. (some conditions).

Examples of Basis Theorems:

Low Basis Theorem(LBT)

T has a low path.

Hyperimmune-free Basis Thorem(HFBT)

T has a hyperimmune-free path.

Fact

There is a computable tree T s.t. each path of T is regarded as a countable ω-model of WKL_0.

Combining this fact and basis theorems, we can make a model of WKL_0 with some properties:

Fact

There is an ω-model of WKL_0 which includes only low/hyperimmune-free sets.

In this talk,
(1) we formalize this argument in 2nd order arithmetic,
(2) decide the strength of low/hyperimmune-free basis theorem.

Definition(low set)

For $A \subseteq \omega$, A' is one of Σ^0_1-complete set.
We say that a set A is low if $A' \leq_T \emptyset'$.

The jump operator is formalized in 2nd order arithmetic as follows: Let $\Phi(e, m, A)$ be a Σ^0_1-universal formula. For $A \subseteq \mathbb{N}$, $A' = \{(e, m) : \Phi(e, m, A)\}$.

Definition(hyperimmune-free set)

Let $X \subseteq \omega$. X is hyperimmune-free if
$\forall f \leq_T X \exists g \leq_T \varnothing \ (f < g)$.

From now on, we assume T to be a binary tree.

Relativized Low Basis Thoerem

ACA_0 proves

$$\forall X \forall T \leq_T X (|T| = \infty \rightarrow \exists Y \in [T]\ (Y \bigoplus X)' \leq_T X').$$

Relativized Hyperimmune-free Basis Theorem

ACA_0 proves

$$\forall X \forall T \leq_T X (|T| = \infty \rightarrow \exists Y \in [T](\forall f \leq_T Y \exists g \leq_T X (f < g))).$$

So, they are strong versions of weak König's lemma.

Lemma 1

In WKL_0, $\exists X \Pi^0_1$ is also Π^0_1.

Intuitively, a Π^0_1 formula $\varphi(X)$ corresponds to a effectively closed set of a Cantor space. Thus we can find a binary tree T s.t. $[T] = \{X : \varphi(X)\}$. Moreover this argument can be formalized in RCA_0.
[Proof] Let $\varphi(X)$ be a Π^0_1 formula and T be a binary tree s.t. $\forall X(X \in [T] \leftrightarrow \varphi(X))$. Then

$$\mathrm{WKL}_0 \vdash \exists X \varphi(X) \leftrightarrow |T| = \infty.$$

Clearly $|T| = \infty$ is Π^0_1. $\square$

Lemma 2

WKL_0 proves the compactness of the Cantor space $2^{\mathbb{N}}$. That is, for any Π^0_1 formula $\varphi(X, n)$,

$$\mathrm{WKL}_0 \vdash (\forall n \exists X \forall i < n\ \varphi(X, i)) \rightarrow \exists X \forall n\ \varphi(X, n).$$

Intuitively, each n defines a closed subset $\mathcal{C}_n$ of $2^{\mathbb{N}}$ s.t. $\mathcal{C}_n = \{X : \varphi(X, n)\}$. Hence, this theorem says that if $\{\mathcal{C}_n\}_{n\in\mathbb{N}}$ has finite intersection property, then $\bigcap_n \mathcal{C}_n \neq \varnothing$.

This can be proved by similar way of Lemma 1.

Low basis theorem (ACA_0)

$\forall X \forall T \leq_T X(|T| = \infty \to \exists Y \in [T]\ (Y \bigoplus X)' \leq_T X')$.

By universal Σ^0_1 formula, we can take an enumeration $\{\varphi_e(m, A)\}_{e \in \mathbb{N}}$ of all Π^0_1 formulas. Then,
$RCA_0 \vdash \forall A\ (A' \equiv_T \{(e, m) : \varphi_e(m, A)\})$.
We will show that
$\exists Y \in [T](\{(e, m) : \varphi_e(m, Y \bigoplus X)\} \leq_T X')$.

Claim: $\exists Y \in [T](\{(e,m) : \varphi_e(m, Y \bigoplus X)\} \leq_T X')$.
[Proof] By using arithmetical comprehension, we can define a maximal subseaquence $\{(e_i, m_i)\}_i$ of $\mathbb{N}^2$ s.t.

$$\exists i \forall Z(\varphi_{e_i}(n_i, Z \bigoplus X) \leftrightarrow Z \in [T]),$$
$$\forall k \exists Z \forall i < k \ (\varphi_{e_i}(m_i, Z \bigoplus X)).$$

By lemma 1, $\{(e_i, m_i)\}_{i \in \mathbb{N}}$ is computable from X'.
Now there exists Y s.t.

$$\forall i \, \varphi_{e_i}(m_i, Y \bigoplus X),$$
$$\forall e, m \, (\varphi_e(m, Y \bigoplus X) \leftrightarrow (e,m) \in \{(e_i, m_i)\}_i).$$

Thus $\{(e,m) : \varphi_e(m, Y \bigoplus X)\} \leq_T \{(e_i, m_i)\}_i \leq_T X'$. $\square$

We can make $\mathcal{L}_2$-structures in a $\mathcal{L}_2$-structure. These structures are called coded structure.

Definition(Coded Structure)

A coded structure is a pair $(N, \{W_n\}_n)$ s.t.
(1) N has $0^N, +^N$ and so on,
(2) each W_n is a subset of N.
We say a coded structure $(N, \{W_n\})$ is an ω-structure if $N = \mathbb{N}, 0^N = 0, +^N = +$ and so on.

Fact

There exists a Π^0_1 formula $\psi(X, M)$ s.t.
(1) $\mathrm{WKL}_0 \vdash \forall X \exists M \psi(X, M)$,
(2) $\mathrm{ACA}_0 \vdash \psi(X, M) \to M$ is a coded ω-model of WKL_0 including X.

Theorem (ACA_0)

For any $X \subseteq \mathbb{N}$, there exists a coded ω-model M of WKL_0 s.t. $X \in M$ and $\forall Y \in M (Y \bigoplus X)' \leq_T X'$.

[Proof] Let $X \subseteq \mathbb{N}$ and $\psi(X, M)$ be a Π^0_1 formula as in the previous fact. Then we can define an X-computable tree T s.t. $M \in [T] \leftrightarrow \psi(X, M)$. Since $\exists M \psi(X, M)$, $[T] \neq \varnothing$. By applying low basis theorem to T, we can get $M \in [T]$ s.t. $(M \bigoplus X)' \leq_T X'$. Now, if $Y \in M$ then $Y \leq_T M$ and hence $(Y \bigoplus X)' \leq X'$. $\square$

By similar argument, we can show the followings:

Theorem

ACA_0 proves realativized hyperimmune-free basis theorem:
$\forall X \forall T \leq_T X(|T| = \infty \rightarrow \exists Y \in [T]$
$(Y$ is X-hyperimmune-free$))$.

Theorem (ACA_0)

For any $X \subseteq \mathbb{N}$, there exists a coded ω-model M of WKL_0 s.t. $X \in M$ and $\forall Y \in M(Y$ is X-hyperimmune-free$)$.

Fact

Every noncomputable low set is not hyperimmune-free.

Corollary

Let M, M' be countable ω-models of WKL_0 s.t.
$X \in M \Rightarrow X$ is low , $X \in M' \Rightarrow X$ is hyperimmune-free.
Then $M \cap M' = \mathrm{REC}$.

Corollary

In RCA_0 the following relations hold.
$\mathrm{WKL}_0 < \mathrm{LBT} < \mathrm{ACA}_0$,
$\mathrm{WKL}_0 < \mathrm{HFBT} < \mathrm{ACA}_0$.

Note: We have already shown that $\leq$ holds.
[Proof] M, M' : ω-models of WKL_0 s.t.
M includes only low sets,
M' includes only hyperimmune-free sets.
(1st inequality) $M \models \mathrm{WKL}_0 + \neg\mathrm{HFBT}$,
$M' \models \mathrm{WKL}_0 + \neg\mathrm{LBT}$.
(2nd inequality) Since $\emptyset' \notin M, M'$, we have
$M \models \mathrm{LBT} + \neg\Sigma^0_1\text{-CA}$,
$M' \models \mathrm{HFBT} + \neg\Sigma^0_1\text{-CA}$. $\square$

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