Review on the strong measure zero σ-ideal and Yorioka's σ-ideals

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Young Logicians Gathering in Japan
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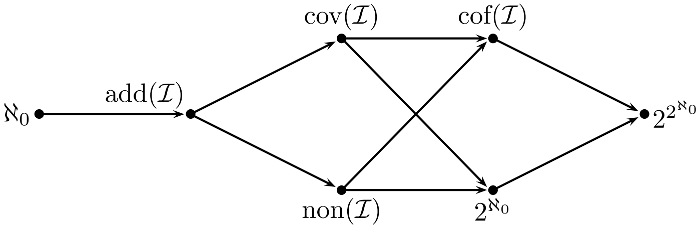
Provable inequalities



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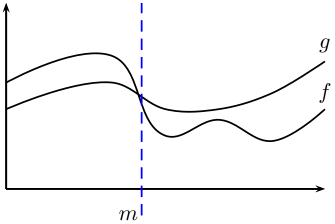

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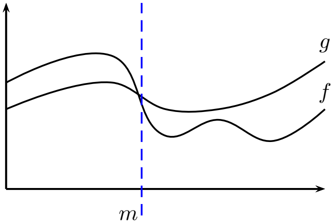

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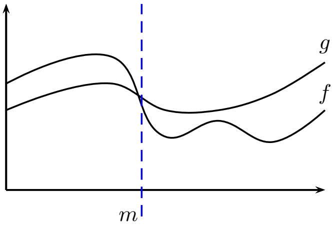

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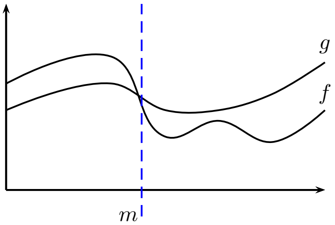

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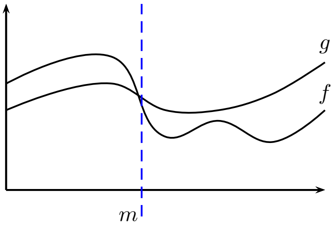

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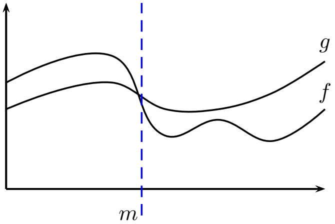

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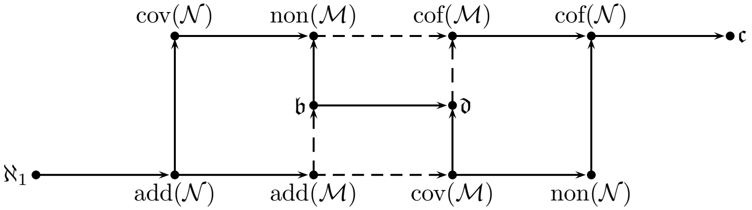
Cichoń's diagram



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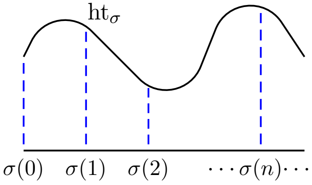

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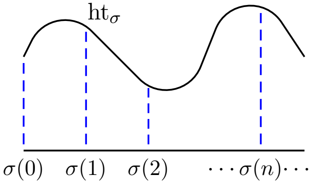

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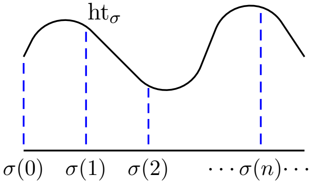

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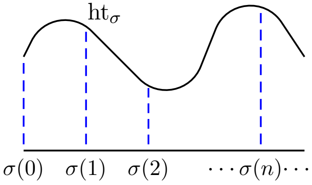

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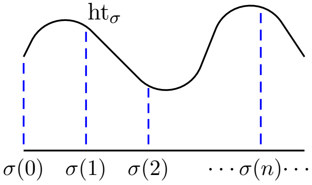

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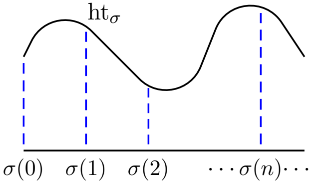

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$$\begin{aligned}[\sigma]_\infty :&= \{x \in 2^\omega : \forall n < \omega \exists m \geq n (\sigma(m) \subseteq x)\} \\ &= \bigcap_{n<\omega} \bigcup_{m \geqslant n} [\sigma(m)]\end{aligned}$$

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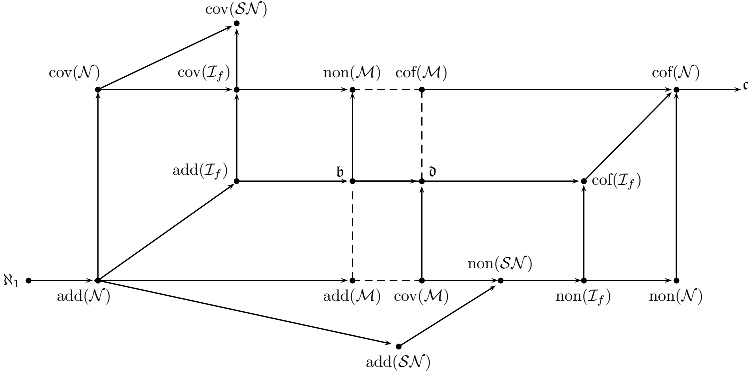
(2) $\mathcal{SN} = \bigcap\{\mathcal{I}_f : f \text{ increasing}\}$.

Extended Cichoń's diagram



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Playground

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Are there $\aleph_1$-many null sets whose union is not null, while we need $\aleph_2$-many null sets to cover 2^ω? i.e., $\bigcup_{\alpha<\omega_1} N_\alpha \notin \mathcal{N}$ for some $N_\alpha \in \mathcal{N}$ $(\alpha < \omega_1)$, but $\bigcup_{\xi<\omega_2} N'_\xi = 2^\omega$ for some $N'_\xi \in \mathcal{N}$ $(\xi < \omega_2)$, i.e.,

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Are there $\aleph_1$-many null sets whose union is not null, while we need $\aleph_2$-many null sets to cover 2^ω? i.e., $\bigcup_{\alpha<\omega_1} N_\alpha \notin \mathcal{N}$ for some $N_\alpha \in \mathcal{N}$ $(\alpha < \omega_1)$, but $\bigcup_{\xi<\omega_2} N'_\xi = 2^\omega$ for some $N'_\xi \in \mathcal{N}$ $(\xi < \omega_2)$, i.e., $\mathrm{add}(\mathcal{N}) = \aleph_1$ and $\mathrm{cov}(\mathcal{N}) = \aleph_2$?

Defintion

A forcing notion $\mathbb{P}$ is a pair $\langle \mathbb{P}, \leq \rangle$ where $\mathbb{P} \neq \emptyset$ and $\leq$ is a relation on $\mathbb{P}$ that satisfies reflexivity and transitivity.

Review forcing

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The forcing notion is choosen in such away that its elements represents *potencial aproximation* of some *special object*, which is called *generic object* we would like to create, but that typically does not exists in the initial universe called ground model. Intuitively, a forcing notion is used to construct *special objects*. Forcing allows us to extend a transitive model V of ZFC to other transitive model $V[G]$ of ZFC through a generic object G. This generic object is, in practice, a new subset of $\mathbb{P}$ in V.

Examples

In Cohen's model,

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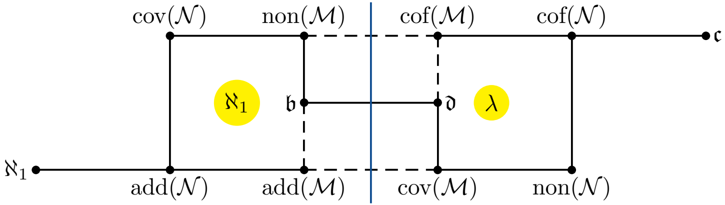

Define Cohen forcing (denoted $\mathbb{C}_\lambda$) as
$\mathbb{C}_\lambda := \{[[s]] : [s] \in \mathrm{BAIRE}(2^{\omega\times\lambda})/\mathcal{M}(2^{\omega\times\lambda})\}$ ordered by $\supseteq$:
$[[s]] \leq [[t]]$ if $[s] \setminus [t] \in \mathcal{M}$.

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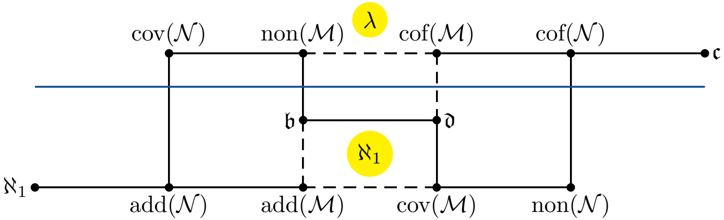

Define random forcing (denoted $\mathbb{B}_\lambda$) as
$\mathbb{B}_\lambda := \{[[s]] : [s] \in \mathrm{BAIRE}(2^{\omega\times\lambda})/\mathcal{N}(2^{\omega\times\lambda})\}$ ordered by $\supseteq$:
$[[s]] \leq [[t]]$ if $[s] \setminus [t] \in \mathcal{N}$.

Examples

In Hechler's model,

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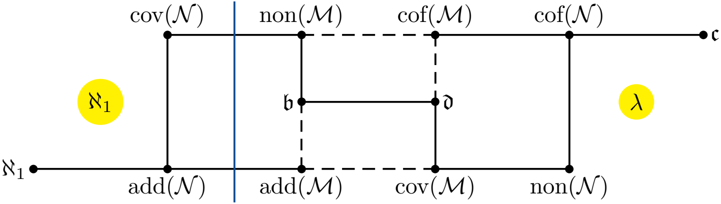
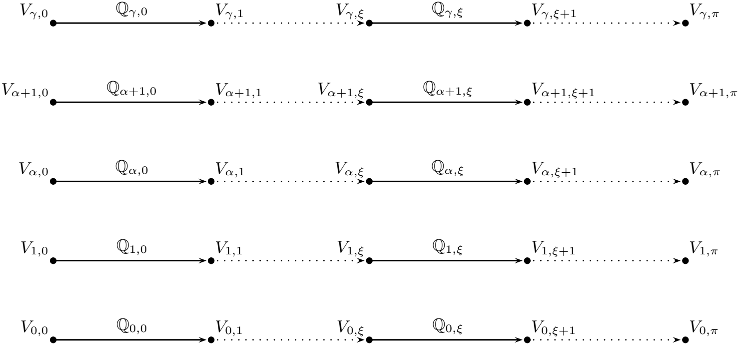


Define Hechler forcing (denoted $\mathbb{D}$) as
$\mathbb{D} := \{(s, f) : s \in \omega^{<\omega}, f \in \omega^\omega \text{ and } s \subseteq f\}$ ordered by

$$(t, g) \leq (s, f) \text{ iff } s \subseteq t \text{ and } f \leq g.$$

Matrix iteration



Examples

In a Mejía's model (2013),

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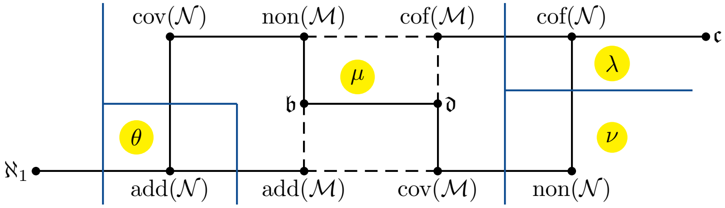

It is consistent with ZFC that

$$\mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N}).$$

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In a Brendle, C. and Mejía model (2018),

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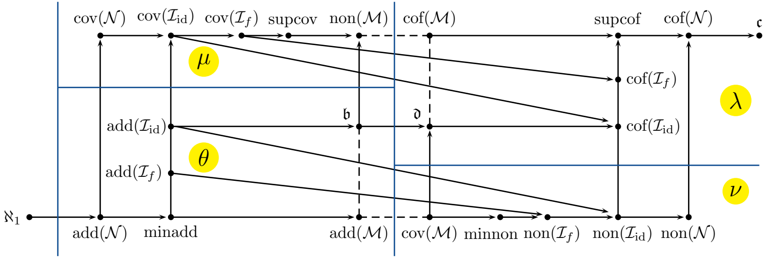

It is consistent with ZFC:

(i) $\mathrm{add}(\mathcal{I}_f) < \mathrm{cov}(\mathcal{I}_f) < \mathrm{non}(\mathcal{I}_f) < \mathrm{cof}(\mathcal{I}_f)$ for any $f \in \omega^{\omega}$",

(ii) $\mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N})$, and

(iii) $\mathrm{add}(\mathcal{M}) < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) < \mathrm{cof}(\mathcal{M})$.

Examples

Questions

Is it consistent with ZFC that

(a) $\mathrm{add}(\mathcal{SN}) < \mathrm{cov}(\mathcal{SN}) < \mathrm{non}(\mathcal{SN}) < \mathrm{cof}(\mathcal{SN})$?

Examples

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Is it consistent with ZFC that

(a) $\mathrm{add}(\mathcal{SN}) < \mathrm{cov}(\mathcal{SN}) < \mathrm{non}(\mathcal{SN}) < \mathrm{cof}(\mathcal{SN})$?

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Questions

Is it consistent with ZFC that

(a) $\mathrm{add}(\mathcal{SN}) < \mathrm{cov}(\mathcal{SN}) < \mathrm{non}(\mathcal{SN}) < \mathrm{cof}(\mathcal{SN})$?

(b) $\mathrm{add}(\mathcal{SN}) < \mathrm{non}(\mathcal{SN}) < \mathrm{cov}(\mathcal{SN}) < \mathrm{cof}(\mathcal{SN})$?

Theorem (C., Mejía and Rivera-Madrid 2019)

It is consistent with ZFC that

$$\mathrm{add}(\mathcal{SN}) = \mathrm{non}(\mathcal{SN}) = \aleph_1 < \mathrm{cov}(\mathcal{SN}) = \aleph_2 = \mathfrak{c} < \mathrm{cof}(\mathcal{SN}).$$

Examples

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Is it consistent with ZFC that

(a) $\mathrm{add}(\mathcal{SN}) < \mathrm{cov}(\mathcal{SN}) < \mathrm{non}(\mathcal{SN}) < \mathrm{cof}(\mathcal{SN})$?

(b) $\mathrm{add}(\mathcal{SN}) < \mathrm{non}(\mathcal{SN}) < \mathrm{cov}(\mathcal{SN}) < \mathrm{cof}(\mathcal{SN})$?

Theorem (C., Mejía and Rivera-Madrid 2019)

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$$\mathrm{add}(\mathcal{SN}) = \mathrm{non}(\mathcal{SN}) = \aleph_1 < \mathrm{cov}(\mathcal{SN}) = \aleph_2 = \mathfrak{c} < \mathrm{cof}(\mathcal{SN}).$$

Theorem (C.)

It is consistent with ZFC that

$$\mathrm{add}(\mathcal{SN}) = \mathrm{cov}(\mathcal{SN}) < \mathrm{non}(\mathcal{SN}) < \mathrm{cof}(\mathcal{SN}).$$

Open Problems

Questions

It is consistent with ZFC that

(I) $\mathrm{add}(\mathcal{I}_f) < \mathrm{non}(\mathcal{I}_f) < \mathrm{cov}(\mathcal{I}_f) < \mathrm{cof}(\mathcal{I}_f)$ for all increasing function $f \in \omega^\omega$?

Open Problems

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It is consistent with ZFC that

(I) $\mathrm{add}(\mathcal{I}_f) < \mathrm{non}(\mathcal{I}_f) < \mathrm{cov}(\mathcal{I}_f) < \mathrm{cof}(\mathcal{I}_f)$ for all increasing function $f \in \omega^\omega$?

(II) $\mathrm{add}(\mathcal{SN}) < \mathrm{cov}(\mathcal{SN}) < \mathrm{non}(\mathcal{SN}) < \mathrm{cof}(\mathcal{SN})$?

Open Problems

Questions

It is consistent with ZFC that

(I) $\mathrm{add}(\mathcal{I}_f) < \mathrm{non}(\mathcal{I}_f) < \mathrm{cov}(\mathcal{I}_f) < \mathrm{cof}(\mathcal{I}_f)$ for all increasing function $f \in \omega^\omega$?

(II) $\mathrm{add}(\mathcal{SN}) < \mathrm{cov}(\mathcal{SN}) < \mathrm{non}(\mathcal{SN}) < \mathrm{cof}(\mathcal{SN})$?

(III) $\mathrm{add}(\mathcal{SN}) < \mathrm{non}(\mathcal{SN}) < \mathrm{cov}(\mathcal{SN}) < \mathrm{cof}(\mathcal{SN})$?.

Moreover,

Question IV

Is it consistent with ZFC that

$$\mathrm{add}(\mathcal{I}_f) < \mathrm{cov}(\mathcal{I}_f) < \mathrm{non}(\mathcal{I}_f) < \mathrm{cof}(\mathcal{I}_f) \text{ for all increasing } f \in \omega^\omega$$

and

$$\mathrm{add}(\mathcal{SN}) < \mathrm{cov}(\mathcal{SN}) < \mathrm{non}(\mathcal{SN}) < \mathrm{cof}(\mathcal{SN}) \text{ simultaneously?}$$

Thank you for your attention!