How to compute with an infinite time Turing machine?

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Motivations

- Ordinals as time for computation.
- Peculiar ordinal properties.
- Proof of mathematical properties from an algorithmic point of view.

Ordinals

Definition (Ordinal)

Transitive well-ordered set for the membership relation.

$$0 := \emptyset$$
$$1 := \{0\} = \{\emptyset\}$$
...
$$\omega := \{0, 1, 2, 3, \cdots\}$$
$$\omega + 1 := \{0, 1, 2, 3, \cdots, \omega\}$$
...
$$\omega.2 := \{0, 1, 2, \cdots, \omega, \omega + 1, \omega + 2 \ldots\}$$

- If α is an ordinal, then $\alpha \cup \{\alpha\}$, denoted $\alpha + 1$ is called successor of α and is an ordinal;
- let A be a set of ordinal numbers, then $\alpha = \bigcup_{\beta \in A} \beta$ is a limit ordinal.

Encoding countable ordinals

Countable ordinal = well order on $\mathbb{N}$.

Encoding (Encoding countable ordinals by reals)

Let $<$ be an order on the natural numbers.
*The real r is a code for the order-type of $<$ if, for $i = \langle x, y \rangle$, the i-th bit of r is **1** if and only if $x < y$.*

Example: $\omega.2 = \omega + \omega \rightsquigarrow$ even integers lower than odd integers.

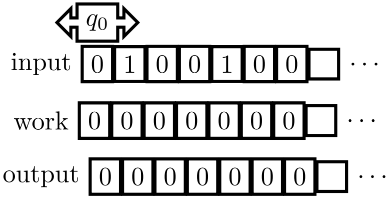
$$0 = \langle 0, 0 \rangle \; 1 = \langle 0, 1 \rangle \cdots \; r = 0_0 1_1 0_2 0_3 0_4 1_5 0_6 1_7 1_8 1_9 1_{10} \cdots$$

Structure of infinite time Turing machines (ITTM)

- 3 right-infinite tapes
- a single head
- binary alphabet $\{0, 1\}$

- additional special limit state $\lim$
- computation steps are indexed by ordinals

Configuration

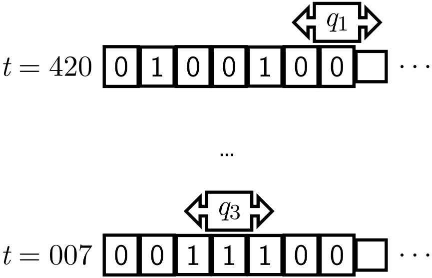


Operating an ITTM

Configuration at $\alpha + 1$.

$\rightsquigarrow$

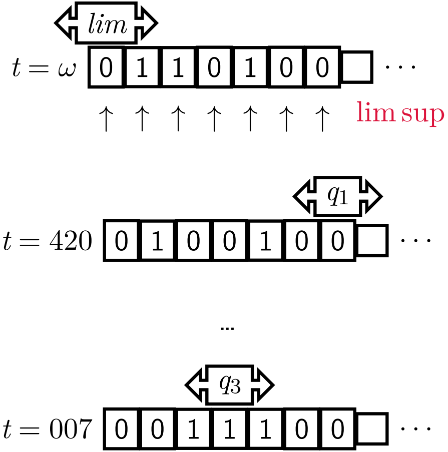
Configuration at α.



Operating an ITTM

Configuration limit:

- head: initial position;
- state: lim;
- each cell: *lim sup* of cell values before.



Computational power

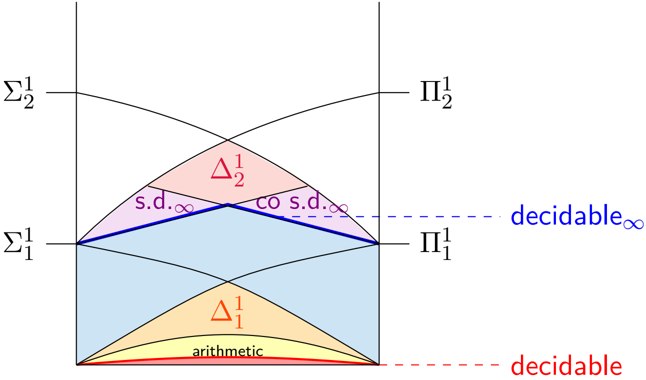


Figure: Projective hierarchy

Halting

- Machines halt when they reach the halting state.
- We consider the strong stabilisation of cells at 0.

Theorem (Hamkins, Lewis [HL00])

Either an ITTM halts in a countable numer of steps, or it begins looping in a ***countable number of steps****.*

- We focus on the halting problem on 0.

Clockable and writable ordinals

Two natural notions:

Definition (Clockable ordinal)

*α clockable: there exists an ITTM that **halts** on input $000\ldots$ in exactly α steps of computation.*

Definition (Writable ordinal)

*α writable: there exists an ITTM that **writes a code** for α on input $000\ldots$ and halts.*

Supremum

Theorem (Welch [Wel09])

The supremum of the clockable ordinals is equal to the supremum of the writable ordinals. It is called λ.

λ is a rather large countable ordinal...

Let's count!

Count with a clockable ordinal $\rightsquigarrow$ Clock.

Like an hourglass, execute operations while clocking the desired ordinal.

Speed-up lemma (Hamkins, Lewis [HL00])

If p halts on 0 in $\alpha + n$ steps, then there exists p' which halts on 0 in α steps (and computes the same). $\rightsquigarrow$ ***limit*** *ordinals*

Count with a writable ordinal $\rightsquigarrow$ Empty an order.

It is about counting through the encoding of an ordinal.

...

What about the particularities of these ordinals?

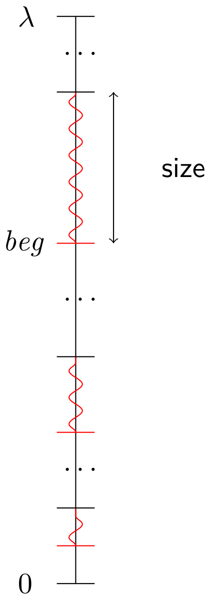
Gap

There exist writable ordinals that are not clockable such that:

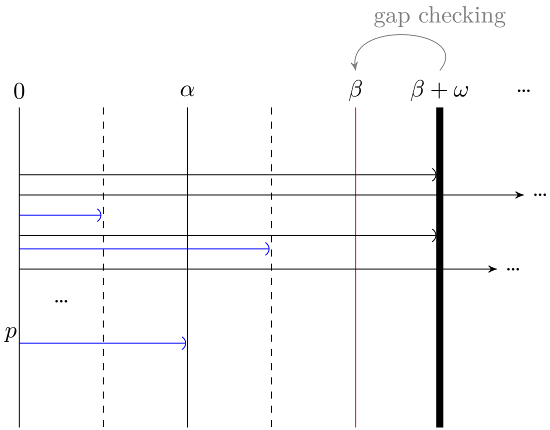
- they form intervals;
- these intervals have limit sizes.

Definition (Gap)

Intervals of not clockable ordinals.


λ
beg
size
0

Proof of gap existence



Simulation of all programs on input 0.

In blue: halting programs. In red: limit step, begins a gap?

Proof of gap existence

But ...does the algorithm halt?

Halting of the algorithm, proof by contradiction:

- Above λ, by definition, there are no clockable ordinals.
- If no gaps before λ, thus beginning of gap detected **at** λ.
- **Contradiction**.

...

What does the literature say about gaps?

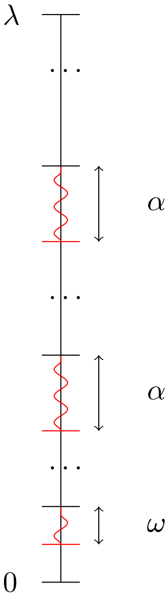
Sizes of gaps

Theorem (Hamkins, Lewis [HL00])

Above any clockable ordinal, the first gap has size ω.

Theorem (Hamkins, Seabold [HS01])

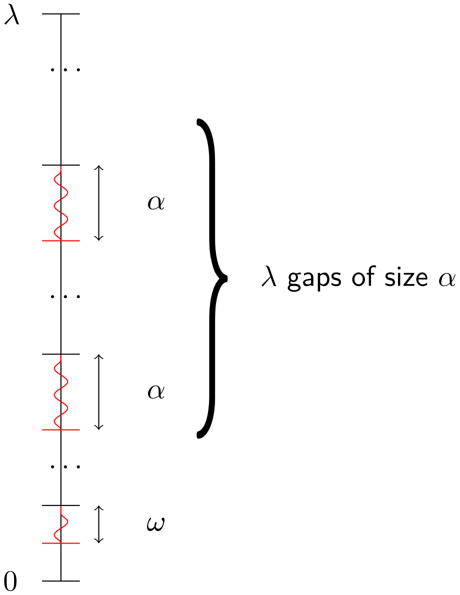
For all writable limit ordinals α, there exists a gap having size exactly α.

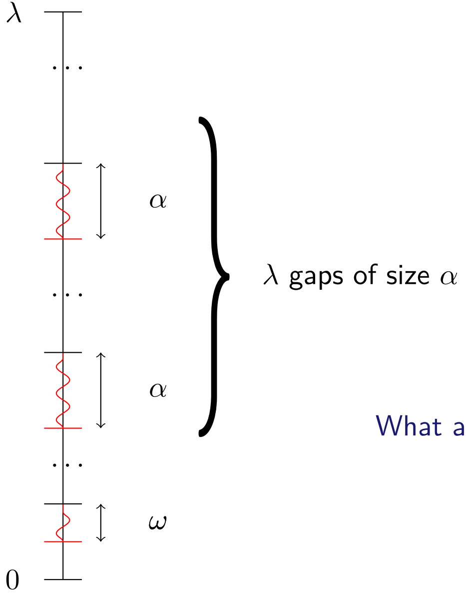

λ
...
α
...
α
...
ω
0

Gaps cofinal in λ

Theorem (Hamkins, Lewis [HL00])

If α is a writable ordinal, the order-type of gaps having size at least α is λ.


λ
α
λ gaps of size α
α
ω
0



What about the ordinals in gaps ?

Admissible ordinals

Property

*A limit ordinal α is admissible if and only if there **doesn't exist** a function f from $\gamma < \alpha$ to α such that:*

- *f is unbounded (no greatest element in α) and*
- *f is Σ_1-definable in L_α.*

Constructible hierarchy

Definition (Constructible hierarchy L)

- *$L_0 = \emptyset$;*
- *$L_{\alpha+1} = def(L_\alpha)$;*
- *if α is a limit ordinal, $L_\alpha = \bigcup_{\beta<\alpha} L_\beta$;*

Application: reals of L_λ are the writable reals.

Definability

Let M be a set and F be the set of the formulas of the language $\{\in\}$.

Definition (Definability)

X is definable on a model $(M, \in)$ if:

- *there exists a formula $\varphi \in F$,*
- *there exists $a_1, \ldots, a_n \in M$*

such that $X = \{x \in M : \varphi(x, a_1, \ldots, a_n)$ is true in $(M, \in)\}$.

$def(M) = \{X \subset M : X$ is definable on $(M, \in)\}$.

Admissible ordinals

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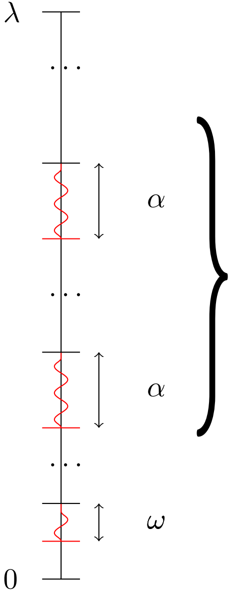
Beginning of gaps and logic

Theorem (Welch [Wel09])

Gaps begin at admissible ordinals.

...

What do we say about gaps?



λ gaps of size α, beginning at admissibles

How is the size distributed?

Existence of a very big gap

Theorem (PhD)

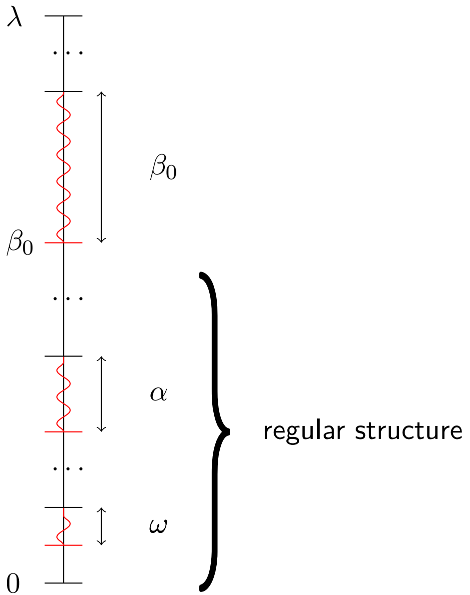
There exists a gap g such that $beginning(g) = size(g)$.

Structure of gaps before β_0

Let β_0 be the beginning and the size of the first gap g such that $beginning(g) = size(g)$.

Theorem (PhD)

*Before β_0, the function that maps α to the beginning of the first gap of **size** α is **increasing**.*


λ
...
β_0
β_0
...
α
...
ω
0
regular structure

Beginning of gaps and logic

Theorem (PhD)

The ordinal β_0 begins the β_0-th gap.
This is also the β_0-th admissible ordinal.

infinite time Turing machines

=

model for algorithms proving logical properties

Conclusion

Questions:

- Characterization of admissible ordinals by gaps?
- Gaps in other transfinite models of computation?
- ITTMs and other fields of Mathematics/CS?

Another result

Theorem (ITTM are equivalent to)

Any Infinite Time Turing Machine can be simulated by some computable (hence continuous) ordinary differential equation and vice-versa.

Consequences

infinite time Turing machines
$=$
model for algorithms proving logical properties

Continuous ordinary differential equations $\equiv$ Infinite time Turing machines.

Consequences and questions

- Applying transfinite techniques to Analysis.
- Transposing Analysis questions to transfinite computations.
- 2 dual views for the same computability questions.
- discrete transfinite time $=$ continuous time.

Other transdisciplinary aspects, an example

Other applications of ITTM using cheap non-standard analysis:

- asymptotic limit of a sequence for results about computability
- extension to an index set different from $\mathbb{N}$
- expression of ITTM computations

Thank you for your attention.