Uncountable free abelian groups via κ-computability

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Outline

1. **Complexity of being free and computing a basis in countable groups**
2. κ-computability
3. Complexity of freeness in uncountable groups
4. Cone avoiding and coding in bases for uncountable groups
5. Singular cardinals

In this talk, all groups are abelian, and $V = L$.

Countable free abelian groups

A group is *free* if it has a subset, called a *basis*, such that every element of the group can be uniquely specified as a linear combination of basis elements with integer coefficients.

Results of Downey and Melnikov on countable groups, coded as subsets of ω.

Naively, determining whether a coded group G is free is Σ^1_1. But they show:

- If a group G is free, it has an G'-computable basis.
- The set of free groups on ω is Π^0_3-complete.

Definition. A set $A \subseteq G$ is P-independent if it is linearly independent, and for each $g \in G$ and nonzero $n \in \mathbb{Z}$, if ng is in the span of A, then g is in the span of A.

They build a basis by maintaining P-indepence as an invariant.

How to extend to uncountable groups?

Countable free abelian groups

Example.
Let $G = \mathbb{Z}^{\omega+1}$, the free group on generators $e_0, e_1, \ldots$ and e_ω.
Let $A = \{p_i e_i + e_\omega : i < \omega\}$, where p_i is the ith prime.
Although A is P-independent, it cannot be extended to a basis for G.

When trying to build an uncountable basis by transfinite initial segments, this becomes a real obstacle!

Theorem (Greenberg-Turetsky-W). If κ is a successor cardinal, then $\{G : G \text{ is free}\}$ is $\Sigma^1_1(L_\kappa)$-complete.

Informally, there is no invariant (similar to P-independence or otherwise) that will allow construction of bases by transfinite initial segments.

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κ-computability

Use α-recursion theory restricted to *regular* cardinals κ.

The usual recursion theory:

- A computation is a process running for ω steps. At each step, one can perform arbitrary manipulations involving a bounded finite set of numbers.
- A computable set is any set whose membership relation can be decided by such a process.
- Equivalent definition: a computable subset of ω is is any set definable by a pair of complementary Σ^0_1 formulas.

κ-computability

κ-recursion theory:

Definition: A computable subset of κ is defined by a pair of complementary $\Sigma^0_1(L_\kappa)$ formulas.

Intuition:

- A κ-finite set is an element of L_α for some $\alpha < \kappa$. (Think of bounded sets of ordinals of cardinality strictly less than κ.)
- A computation is a process, running for κ many steps. At each step, one can perform arbitrary manipulations on κ-finite sets.

κ-computability

Definition. A subset of κ is $\Sigma^1_1(L_\kappa)$ if

$$\alpha \in X \iff \exists Y \subseteq \kappa (L_\kappa \models \varphi(Y, \alpha))$$

where φ is a formula in the language of set theory. Such a set is $\Sigma^0_1(L_{\kappa^+})$.

Theorem (Fokina, Friedman, Knight & Miller). The set

$$\{A \subseteq \kappa : A \text{ contains a club (closed unbounded set)}\}$$

is $\Sigma^1_1(L_\kappa)$-complete.

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Complexity of being a free group

- Filtration, freeness and clubs
- Twisting and bad starts for a basis
- Computability of κ-finite free groups

Filtration, freeness and clubs

If $G \subseteq \kappa$, then G can be decomposed as

$$G = \cup_{\alpha<\kappa} G_\alpha$$

where $G_\alpha = G \cap \alpha$. This decomposition is called a *filtration*.

- If $H \subseteq G$ are groups, we say H *divides* G if

  $$G = H \oplus K$$

  for another group $K \subseteq G$.
- Every subgroup of a free group is free.
- If G is free and $G = H \oplus K$, then to find a basis for G, it suffices to take a union of a basis for H and a basis for K.

Filtration, freeness and clubs

- If G is free, then let B be a basis. There is a club $C \subseteq \kappa$ such that for each $\alpha \in C$, G_α is the free group generated by some subset of B (of cardinality less than κ).
- Suppose there is a club $C \subseteq \kappa$ such that for each $\alpha, \beta \in C$ with $\alpha < \beta$, we have $G_\alpha | G_\beta$. Then we can build a basis for G as an increasing union of cohereing bases for G_α for $\alpha \in C$.

Definition. For any group $G \subseteq \kappa$, let

$$\mathrm{Div}(G) = \{\alpha < \kappa : (\forall \beta > \alpha)[G_\alpha | G_\beta]\}.$$

A group G is free if and only if $\mathrm{Div}(G)$ contains a club. So determining freeness is difficult if sets of the form $\mathrm{Div}(G)$ are general enough that it is difficult to determing whether they contain clubs.

Twisting and bad starts for a basis

Theorem. If κ is a successor cardinal, then $\{G : G \text{ is free}\}$ is $\Sigma^1_1(L_\kappa)$-complete.

Idea. Given a set A, which may or may not contain a club, build G by initial segments G_α.

If $\alpha \in A$, make $G_{\alpha+1}$ by just adding a new basis element to what you had.

If $\alpha \notin A$, make $G_{\alpha+1}$ by adding new elements around what you already had (in the style of the $\mathbb{Z}^{\omega+1}$ example) to make sure that $\alpha \notin \mathrm{Div}(G)$. (We say G_α is "twisted".)

Proof: uses a variant of the principle $\square$, which holds in L.

In fact, the above theorem holds as long as κ is not weakly compact.

On the other hand:
Theorem. If κ is weakly compact, then $\{G : G \text{ is free}\}$ is $\Pi^0_2(L_\kappa)$.

Computability of κ-finite free groups

Let us call κ *difficult* if κ is (regular and) not weakly compact.

We have seen: If κ is difficult, then $\{G \subseteq \kappa : G \text{ is free}\}$ is $\Sigma^1_1(L_\kappa)$-complete.

Consider $C_\kappa = \{G \in L_\kappa : G \text{ is free}\}$.

Theorem.

- If κ is the successor of a cardinal that is not difficult, then C_κ is κ-computable.
- If κ is the successor of a difficult cardinal, then C_κ is $\Sigma^0_1(L_\kappa)$-complete.
- If κ is a limit, then C_κ computes $\emptyset'(L_\kappa)$, but is not complete.

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Computing, coding, and cone avoiding in bases

- Bases are not easily computed.
- But not much can be coded into them (cone avoidance).
- Characterization of what can be coded.

Observe: Suppose κ is a successor. For every $X \in \Delta^1_1(L_\kappa)$ (for any reasonable definition of $\Delta^1_1(L_\kappa)$) there is a computable $G \subseteq \kappa$ such that X does not compute a basis for G.

If there were an X that computed every basis, $\{G \subseteq \kappa : G \text{ is free}\}$ would be Δ^1_1, not Σ^1_1-complete.

Cone Avoidance

Recall:

$$\mathrm{Div}(G) = \{\alpha < \kappa : (\forall \beta > \alpha)[G_\alpha | G_\beta]\}.$$

Theorem. Suppose that G is κ-computable and $X \not\leq_T \mathrm{Div}(G)$. Then there is a basis of G that does not compute X.

Consequence. If $X \not\leq_T \emptyset''(L_\kappa)$, then there is a basis B of G such that $X \not\leq_T B$.

Theorem. If κ is not the successor of a difficult cardinal, then for all κ-computable G, $\mathrm{Div}(G)$ is $\emptyset'(L_\kappa)$ computable.

Consequence. If κ is not the successor of a difficult cardinal, and $X \not\leq_T \emptyset'(L_\kappa)$, then there is a basis B of G such that $X \not\leq_T B$.

Coding in bases

Proposition. For every κ, there is a κ-computable G such that every basis of G computes $\emptyset'$.

Start building a free group on generators $b_0, b_1, \ldots, b_\alpha$. If you see α enter $\emptyset'(L_\kappa)$, add an element equal to $b_\alpha/2$.

Theorem. If κ is the successor of a difficult cardinal, then there is a κ-computable G, all bases for which compute $\emptyset''(L_\kappa)$.

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Singular cardinals - many questions

A cardinal is *singular* if it is not regular ($\mathrm{cof}(\kappa) < \kappa$).

κ-computability for singular cardinals:

- Computation as a pair of $\Sigma^0_1(L_\kappa)$ formulas is still well-defined.
- Computation length only $\mathrm{cof}(\kappa)$ breaks many results.

Theorem. If $\mathrm{cof}(\kappa) = \aleph_0$,

- The index set of the κ-computable free groups is $\Pi^0_2(L_\kappa)$-complete.
- If X computes a cofinal ω-sequence in κ, then every κ-computable free group has an $X \oplus \emptyset'$-computable basis.

Thank you!