# HYP with Finite Mind-Changes: On Kechris-Martin Theorem and a Solution to Fournier's Question

Takayuki Kihara[1]

Nagoya University, Japan

Oberwolfach Workshop: Computability Theory 2021
April 27, 2021

[1] partially supported by JSPS KAKENHI Grant 19K03602, and the JSPS-RFBR Bilateral Joint Research Project JPJSBP120204809

## Theorem (Kechris-Martin 197x?)

Under the axiom of determinacy (AD), the Wadge rank of the $\omega$-th level of the decreasing difference hierarchy over $\underset{\sim}{\mathbf{\Pi}}^1_1$ is $\omega_2$.

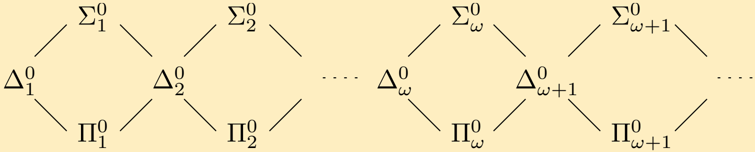
Kechris and Martin have located the pointclass $\Gamma$ such that $o(\Delta) = \omega_2$ with the aid of Theorem 1.2. Namely, let $\Gamma$ be the class of $\omega$-$\underset{\sim}{\mathbf{\Pi}}^1_1$ sets, that is, sets of the form

$$A = \bigcup_{n<\omega} A_{2n} - A_{2n+1}$$

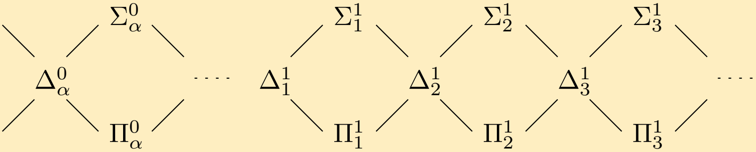
where $\langle A_n : n < \omega \rangle$ is a decreasing sequence of $\underset{\sim}{\mathbf{\Pi}}^1_1$ sets. Then $\Gamma$ is nonselfdual, and both $\Gamma$ and $\check{\Gamma}$ are closed under intersections with $\underset{\sim}{\mathbf{\Pi}}^1_1$ sets. By Theorem 1.2 we have $o(\Delta) \geq \omega_2$. By analyzing the ordinal games associated to Wadge games involving sets in $\Delta$, Martin showed $o(\Delta) \leq \omega_2$. Thus $o(\Delta) = \omega_2$.

— J. Steel, Closure properties of pointclasses, In *Wadge Degrees and Projective Ordinals: The Cabal Seminar, Volume II*.

### Borel hierarchy / arithmetical hierarchy / hyperarithmetical hierarchy

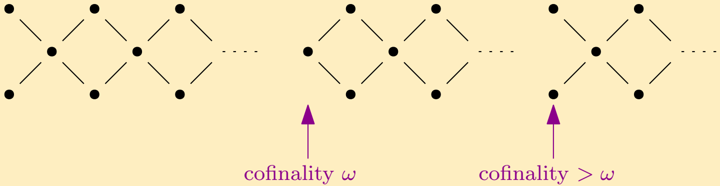



### Projective hierarchy / analytical hierarchy




$\boldsymbol{\Delta}^0_1$ = clopen; $\boldsymbol{\Sigma}^0_1$ = open; $\boldsymbol{\Pi}^0_1$ = closed; $\boldsymbol{\Sigma}^0_2 = F_\sigma$; $\boldsymbol{\Pi}^0_2 = G_\delta$;
$\boldsymbol{\Delta}^1_1$ = Borel; $\boldsymbol{\Sigma}^1_1$ = analytic; $\boldsymbol{\Pi}^1_1$ = coanalytic

## Wadge hierarchy (1970s)

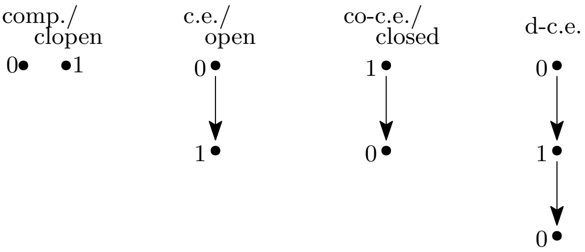



- Wadge degree: Ultimate measure for topological complexity
  Let $X$ and $Y$ be topological spaces, $A \subseteq X$ and $B \subseteq Y$,

$$A \leq_W B \iff \exists \text{ continuous } \theta\colon X \to Y \quad \forall x \in X \quad [x \in A \longleftrightarrow \theta(x) \in B]$$

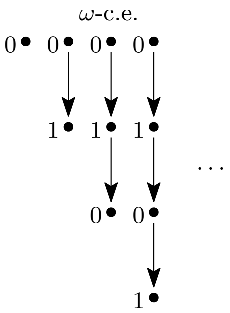
- $A <_W B \iff B$ is topologically more complicated than $A$.
- (AD) The subsets of $\omega^\omega$ are semi-well-ordered by $\leq_W$.
- This assigns an ordinal rank to each subset of $\omega^\omega$.

Tree/Forest-representation of various $\Delta^0_2$ sets:




- (computable/clopen) Given an input $x$, effectively decide $x \notin A$ (indicated by $\mathbf{0}$) or $x \in A$ (indicated by $\mathbf{1}$).
- (c.e./open) Given an input $x$, begin with $x \notin A$ (indicated by $\mathbf{0}$) and later $x$ can be enumerated into $A$ (indicated by $\mathbf{1}$).
- (co-c.e./closed) Given an input $x$, begin with $x \in A$ (indicated by $\mathbf{1}$) and later $x$ can be removed from $A$ (indicated by $\mathbf{0}$).
- (d-c.e.) Begin with $x \notin A$ (indicated by $\mathbf{0}$), later $x$ can be enumerated into $A$ (indicated by $\mathbf{1}$), and $x$ can be removed from $A$ again (indicated by $\mathbf{0}$).

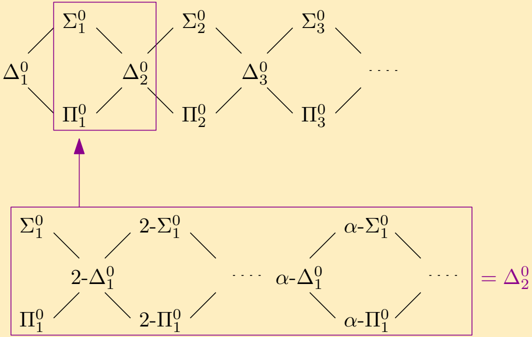
Forest-representation of a complete $\omega$-c.e. set:

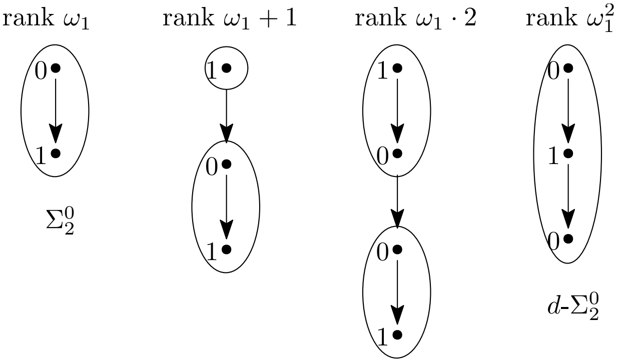



($\omega$-c.e.) The representation of "$\omega$-c.e." is a forest consists of linear orders of finite length (a linear order of length $n + 1$ represents "$n$-c.e.").

- Given an input $x$, effectively choose a number $n \in \omega$ giving a bound of the number of times of mind-changes until deciding $x \in A$.

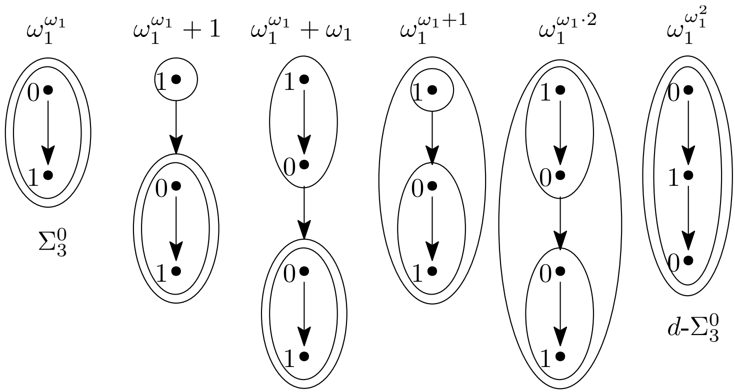
- The term $\mathbf{0}$, $\mathbf{1} = \boldsymbol{\emptyset}$, $\omega^\omega$, resp. = rank $\mathbf{0}$
- The term $\mathbf{0}^{\rightarrow}\mathbf{1}$ = Open; $\mathbf{1}^{\rightarrow}\mathbf{0}$ = Closed. (rank $\mathbf{1}$)
- The term $\mathbf{0}^{\rightarrow}\mathbf{1}^{\rightarrow}\mathbf{0}$: Difference of two open sets (rank $\mathbf{2}$)
- The term $\mathbf{0}^{\rightarrow}\mathbf{1}^{\rightarrow}\mathbf{0}^{\rightarrow}\mathbf{1}$: Difference of three open sets (rank $\mathbf{3}$)
- Boolean combination of finitely many open sets (rank finite)
- The $\alpha$-th level of the difference hierarchy (rank $\alpha$)






Tree/Forest-representation of $\underset{\sim}{\mathbf{\Delta}}^0_3$ sets

The Wadge degrees of $\underset{\sim}{\mathbf{\Delta}}^0_3$ sets are exactly those represented by **forests labeled by trees**.




Tree/Forest-representation of $\underset{\sim}{\mathbf{\Delta}}^0_4$ sets

The Wadge degrees of $\underset{\sim}{\mathbf{\Delta}}^0_4$ sets are exactly those represented by
forests labeled by trees which are labeled by trees.

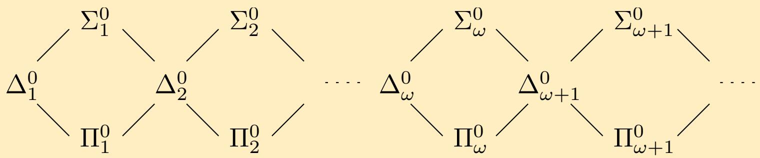
## Term operation vs Ordinal operation

$\to \;\approx\; +$; $\quad \sqcup \;\approx\; \sup$; $\quad \langle \bullet \rangle \;\approx\; \omega_1^{\bullet}$

- The term $\langle 0 \to 1 \rangle = \Sigma^0_2$; $\quad \langle 1 \to 0 \rangle = \Pi^0_2$ $\quad$ (rank $\omega_1$)
- The term $\langle\langle 0 \to 1 \rangle\rangle = \Sigma^0_3$; $\quad \langle\langle 1 \to 0 \rangle\rangle = \Pi^0_3$ $\quad$ (rank $\omega_1^{\omega_1}$)
- The term $\langle\langle\langle 0 \to 1 \rangle\rangle\rangle = \Sigma^0_4$; $\quad \langle\langle\langle 1 \to 0 \rangle\rangle\rangle = \Pi^0_4$ $\quad$ (rank $\omega_1^{\omega_1^{\omega_1}}$)

Let $\omega_1 \uparrow\uparrow n$ be the $n$-th level of the exponential tower over $\omega_1$.

- The term $\langle 0 \to 1 \rangle^n = \Sigma^0_{n+1}$; $\quad \langle 1 \to 0 \rangle^n = \Pi^0_{n+1}$ $\quad$ (rank $\omega_1 \uparrow\uparrow n$)




What's the rank of $\underset{\sim}{\Sigma}^0_\omega$ sets? Is it $\sup_{n<\omega}(\omega_1 \uparrow\uparrow n)$? No!!

- $\varepsilon_{\omega_1+\alpha}$ = the $\alpha$-th solution of "$\omega_1^x = x$".
- $\varepsilon_{\omega_1+1} = \sup_{n<\omega}(\omega_1 \uparrow\uparrow n)$

**Theorem (Wadge)**

The Wadge rank of $\underset{\sim}{\Sigma}^0_\omega$ sets is $\varepsilon_{\omega_1+\omega_1}$.

Why? Term presentation:

- The term $\langle 0^\rightarrow 1\rangle^n$ represents $\Sigma^0_{n+1}$ (rank $\omega_1 \uparrow\uparrow n$)
- The term $\sqcup_{n<\omega}\langle 0^\rightarrow 1\rangle^n$ represents a disjoint union of $\Sigma^0_n$ sets (rank $\varepsilon_{\omega_1+1}$)
- The term $1^\rightarrow \sqcup_{n<\omega} \langle 0^\rightarrow 1\rangle^n$ represents a more complicated set which is still in $\Delta^0_\omega$ (rank $\varepsilon_{\omega_1+1} + 1$)
- The term $\langle 1^\rightarrow \sqcup_{n<\omega} \langle 0^\rightarrow 1\rangle^n\rangle$ corresponds to rank $\omega_1^{\varepsilon_{\omega_1+1}+1}$
- The term $\sqcup_{m<\omega}\langle 1^\rightarrow \sqcup_{n<\omega} \langle 0^\rightarrow 1\rangle^n\rangle^m$ corresponds to rank $\varepsilon_{\omega_1+2}$
- The term $\langle 0^\rightarrow 1\rangle^\omega$ represents $\Sigma^0_\omega$ (rank $\varepsilon_{\omega_1+\omega_1}$)

- The term $\langle 0^\to \langle 1^\to 0\rangle^\omega\rangle$ = rank $\omega_1^{\varepsilon_{\omega_1\cdot 2}+1}$
- The term $\langle 0^\to 1^\to 0\rangle^\omega$ represents d-$\Sigma^0_\omega$ (rank $\varepsilon_{\omega_1\cdot 3}$)
- The term $\langle\langle 0^\to 1\rangle\rangle^\omega$ represents $\Sigma^0_{\omega+1}$ (rank $\varepsilon_{\omega_1^2}$)
- The term $\langle\langle 0^\to 1\rangle^n\rangle^\omega$ represents $\Sigma^0_{\omega+n}$ (rank $\varepsilon_{\omega_1\uparrow\uparrow n}$)
- The term $\langle\langle 0^\to 1\rangle^\omega\rangle^\omega$ represents $\Sigma^0_{\omega\cdot 2}$ (rank $\varepsilon_{\varepsilon_{\omega_1+\omega_1}}$)
- The term $\langle\langle 0^\to 1\rangle^n\rangle^{\omega\cdot m}$ represents $\Sigma^0_{\omega\cdot m+n}$ (rank $\phi_1^{(m)}(\phi_0^{(n)}(0))$)

Here $\phi_0(x) = \omega_1^{1+x}$ and $\phi_1(x) = \varepsilon_{\omega_1+1+x}$.
Define $\phi_2(x)$ as the $x$-th solution of "$\phi_1(x) = x$".

- The term $\sqcup_{n<\omega}\langle 0^\to 1\rangle^{\omega\cdot n}$ = rank $\phi_2(0) = \sup_{n<\omega}\phi_1^{(n)}(0)$
- The term $\langle 1^\to \sqcup_{n<\omega} \langle 0^\to 1\rangle^{\omega\cdot n}\rangle$ = rank $\phi_0(\phi_2(0)+1)$
- The term $\sqcup_{m<\omega}\langle 1^\to \sqcup_{n<\omega} \langle 0^\to 1\rangle^{\omega\cdot n}\rangle^{\omega\cdot m} = \phi_2(1)$
- The term $\langle 0^\to 1\rangle^{\omega^2}$ represents $\Sigma^0_{\omega^2}$ (rank $\phi_2(\omega_1)$)

**Fact (for sets, essentially due to Duparc? K.-Montalbán for more general cases)**

The Wadge degrees of Borel sets
= The terms in the signature $\mathcal{L} = \{0, 1, \to, \sqcup, \langle\cdot\rangle^{\omega^\alpha} : \alpha < \omega_1\}$.

**Term operation vs Ordinal operation**

$\to \;\approx\; +$; $\sqcup \;\approx\; \sup$; $\langle\bullet\rangle \;\approx\; \omega_1^\bullet = \phi_0(\bullet)$; $\langle\bullet\rangle^{\omega^\alpha} \;\approx\; \phi_\alpha(\bullet)$

**Example**

- (Wadge) The Veblen hierarchy of base $\omega_1$:
  $\phi_\alpha(\gamma)$: the $\gamma^{\text{th}}$ ordinal closed under $+$, $\sup_{n\in\omega}$, and $(\phi_\beta)_{\beta<\alpha}$.
- $\phi_0$ enumerates $\omega_1, \omega_1^2, \omega_1^3, \dots, \omega_1^{\omega+1}, \omega_1^{\omega+2}, \dots$
- $\phi_1$ enumerates $\varepsilon_{\omega_1+1}, \varepsilon_{\omega_1+2}, \varepsilon_{\omega_1+3}, \dots$
- $\undertilde{\Sigma}^0_{\omega^\alpha}, \undertilde{\Pi}^0_{\omega^\alpha}$: Wadge-rank $\phi_\alpha(\omega_1)$ $(0 < \alpha < \omega_1)$.
- $\undertilde{\Sigma}^1_1, \undertilde{\Pi}^1_1$: Wadge-rank $\phi_{\omega_1}(0) = \sup_{\xi<\omega_1} \phi_\xi(\omega_1)$.

Beyond Borel:

**The difference hierarchy over $\underset{\sim}{\mathbf{\Pi}}^1_1$**

- The 2nd level: $A_1 \setminus A_0$.
- The 3rd level: $A_2 \setminus (A_1 \setminus A_0)$.
- The 4th level: $A_3 \setminus (A_2 \setminus (A_1 \setminus A_0))$.
- The $n$-th level: $A_{n-1} \setminus (A_{n-2} \setminus (\cdots \setminus (A_1 \setminus A_0)))$
- The finite level: Boolean combination of $\underset{\sim}{\mathbf{\Pi}}^1_1$ sets.

One may assume that $(A_n)$ is an increasing seq. of $\mathbf{\Pi}^1_1$ sets:

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_{n-2} \subseteq A_{n-1} \subseteq A_n \subseteq \omega^\omega$$

$$\mathbf{1} \leftarrow \mathbf{0} \leftarrow \mathbf{1} \leftarrow \cdots \leftarrow \mathbf{1} \leftarrow \mathbf{0} \leftarrow \mathbf{1} \leftarrow \mathbf{0}$$

Its transfinite extension is called the *increasing difference hierarchy* over $\underset{\sim}{\mathbf{\Pi}}^1_1$.

Recall: The Wadge rank of Borel sets = $\phi_{\omega_1}(0)$.

**Theorem (Fournier 2016, AD)**

- The Wadge rank of the $(1+\eta)$-th level of the increasing difference hierarchy over $\underset{\sim}{\Pi}^1_1$ is $\phi_{\omega_1}(\eta)$.
- In particular, the Wadge rank of the increasing difference hierarchy over $\underset{\sim}{\Pi}^1_1$ is $\phi_{\omega_1}(\omega_1)$.

## Theorem (Kechris-Martin 197x?)

Under the axiom of determinacy (AD), the Wadge rank of the $\omega$-th level of the decreasing difference hierarchy over $\underset{\sim}{\mathbf{\Pi}}^1_1$ is $\omega_2$.

Kechris and Martin have located the pointclass $\Gamma$ such that $o(\Delta) = \omega_2$ with the aid of Theorem 1.2. Namely, let $\Gamma$ be the class of $\omega$-$\underset{\sim}{\mathbf{\Pi}}^1_1$ sets, that is, sets of the form

$$A = \bigcup_{n<\omega} A_{2n} - A_{2n+1}$$

where $\langle A_n : n < \omega \rangle$ is a decreasing sequence of $\underset{\sim}{\mathbf{\Pi}}^1_1$ sets. Then $\Gamma$ is nonselfdual, and both $\Gamma$ and $\check{\Gamma}$ are closed under intersections with $\underset{\sim}{\mathbf{\Pi}}^1_1$ sets. By Theorem 1.2 we have $o(\Delta) \geq \omega_2$. By analyzing the ordinal games associated to Wadge games involving sets in $\Delta$, Martin showed $o(\Delta) \leq \omega_2$. Thus $o(\Delta) = \omega_2$.

— J. Steel, Closure properties of pointclasses, In *Wadge Degrees and Projective Ordinals: The Cabal Seminar, Volume II*.

- $D_\eta(\undertilde{\Pi}^1_1)$: The $\eta$-th level of the increasing DH.
- $Diff_{\omega_1}(\undertilde{\Pi}^1_1)$: The whole increasing DH.
- $D^*_\eta(\undertilde{\Pi}^1_1)$: The $\eta$-th level of the decreasing DH.
- $Diff^*_{\omega_1}(\undertilde{\Pi}^1_1)$: The whole decreasing DH.

Define $Diff_\eta(\undertilde{\Pi}^1_1)$ as the class of all $A$ s.t. $A, \neg A \in D_\eta(\undertilde{\Pi}^1_1)$.

$$D_n(\undertilde{\Pi}^1_1) = D^*_n(\undertilde{\Pi}^1_1) \subset \cdots \subset D_\eta(\undertilde{\Pi}^1_1) \subset \cdots \subset Diff_{\omega_1}(\undertilde{\Pi}^1_1) \subset Diff^*_\omega(\undertilde{\Pi}^1_1) \subset \dots$$

- (Fournier 2016, AD) The Wadge rank of $D_{1+\eta}(\undertilde{\Pi}^1_1)$ is $\phi_{\omega_1}(\eta)$.
- (Kechris-Martin, AD) The Wadge rank of $Diff^*_\omega(\undertilde{\Pi}^1_1)$ is $\omega_2$.

**Question (Fournier 2016)**

If weakening AD, is $Diff_{\omega_1}(\undertilde{\Pi}^1_1) = Diff^*_\omega(\undertilde{\Pi}^1_1)$ consistent?

## Two difference hierarchies (DHs)

Increasing difference hierarchy:

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_{n-2} \subseteq A_{n-1} \subseteq A_n \subseteq \omega^\omega$$

$$1 \leftarrow 0 \leftarrow 1 \leftarrow \cdots \leftarrow 1 \leftarrow 0 \leftarrow 1 \leftarrow 0$$

Decreasing difference hierarchy:

$$\omega^\omega \supseteq B_0 \supseteq B_1 \supseteq B_2 \supseteq \cdots \supseteq B_{n-2} \supseteq B_{n-1} \supseteq B_n$$

$$0 \rightarrow 1 \rightarrow 0 \rightarrow 1 \rightarrow \cdots \rightarrow 1 \rightarrow 0 \rightarrow 1$$

- The increasing DH over $\utilde{\mathbf{\Sigma}}^0_\alpha$ = The decreasing DH over $\utilde{\mathbf{\Sigma}}^0_\alpha$.
- Finite levels of increasing DH = finite levels of decreasing DH.
- The increasing DH over $\utilde{\mathbf{\Pi}}^1_1$ $\neq$ The decreasing DH over $\utilde{\mathbf{\Pi}}^1_1$ !!!

## Observation by Gandy, Spector, Kreisel, Sacks, ... (1959~1960s)

$$\Delta^1_1 : \Pi^1_1 : \Sigma^1_1 \quad \approx \quad \text{finite : c.e. : co-c.e.}$$

We call $\Pi^1_1$ hyp c.e. and $\Sigma^1_1$ hyp co-c.e.

## Rough Idea

- $\mathit{Diff}_{\omega_1}(\underset{\sim}{\Pi}^1_1)$: hyp-computability with finite mind-changes, but with a mind-change countdown starting from $< \omega_1$, i.e.,

  change the current guess $\Longrightarrow$ decrease the value of the counter

- $\mathit{Diff}^*_{\omega}(\underset{\sim}{\Pi}^1_1)$: hyp-computability with finite mind-changes

$\mathrm{Diff}^*_\omega(\underset{\sim}{\mathbf{\Pi}}^1_1)$, $\underset{\sim}{\mathbf{\Sigma}}^1_1$-LNP: hyp-computable with finite mind-changes $\quad \omega_2$

$\Uparrow$

$\underset{\sim}{\mathbf{\Pi}}^1_1$-LNP$_{\mathsf{WO}}$: hyp-computable with finite mind-changes along $\omega_1 + 1$ countdown

$\Uparrow$

$\mathrm{Diff}_{\omega_1}(\underset{\sim}{\mathbf{\Pi}}^1_1)$: hyp-computable with finite mind-changes along $\omega_1$ countdown $\quad \phi_{\omega_1}(\omega_1)$

$\Uparrow$

$\mathrm{Diff}_\omega(\underset{\sim}{\mathbf{\Pi}}^1_1)$, $\underset{\sim}{\mathbf{\Pi}}^1_1$-LNP: hyp-computable with finite mind-changes along $\omega$ countdown

$\phi_{\omega_1}(\omega)$

Example of hyp-computability with finite mind-changes:
Let's consider the following principle in a $\beta$-model.

**$\Sigma^1_1$-least number principle**

Given a $\Sigma^1_1$ formula $\varphi(x)$, if $\forall n \neg\varphi(n)$ is false, then there is a least number $n \in \mathbb{N}$ satisfying $\varphi(n)$.

In other words, "if a $\Sigma^1_1$ set $S \subseteq \mathbb{N}$ is nonempty, then $\min S$ exists".

How is it difficult to calculate $\min S$?
The $\Sigma^1_1$-least number principle can be restated as:

"If a hyp co-c.e. set $S \subseteq \mathbb{N}$ is nonempty, then $\min S$ exists".

How is it difficult to calculate $\min S$?

One can calculate $\min S$ by a "*hyp-computation process with finite mind-changes*".

“If a hyp co-c.e. set $S \subseteq \mathbb{N}$ is nonempty, then $\min S$ exists”.

Calculate $\min S$ by a “*hyp-computation process with finite mind-changes*”.

- Fix a hyp-computation process $\Phi$ enumerating $\mathbb{N} \setminus S$.
- Our “hyp-algorithm” $\Psi$ first guess that $0$ is the right answer.
- After some hyp-computation steps, $\Phi$ may enumerate $0$ (so $0 \notin S$).
- In this case, our hyp-algorithm $\Psi$ changes the guess to the least number $n$ which has not been enumerated by $\Phi$ by this stage.
- After some hyp-computation steps, $\Phi$ may enumerate $n$ (so $n \notin S$).
- In this case, our hyp-algorithm $\Psi$ changes the guess to the least number $n'$ which has not been enumerated by $\Phi$ by this stage.

  Continue this procedure...

- Since $S$ is nonempty, $\Psi$'s guess stabilizes after some finite mind-changes.

In a certain sense, the strength of the $\Sigma^1_1$-least number principle is equivalent to “hyp-computability with finite mind-changes”.

### $\Sigma^1_1$-least number principle

"If a hyp co-c.e. set $S \subseteq \mathbb{N}$ is nonempty, then $\min S$ exists".

⇒ This is hyp-computable with finite mind-changes ≈ $Diff^*_\omega(\underset{\sim}{\Pi}^1_1)$

### $\Pi^1_1$-least number principle

"If a hyp c.e. set $S \subseteq \mathbb{N}$ is nonempty, then $\min S$ exists".

⇒ This is hyp-computable with finite mind-changes along an $\omega$-countdown ≈ $Diff_\omega(\underset{\sim}{\Pi}^1_1)$

### $\Pi^1_1$-least number principle on WO

"If a hyp c.e. set $S \subseteq \mathbb{N}$ is nonempty, and if $\prec$ is a well-order on $\mathbb{N}$ then $\min_\prec S$ exists".

⇒ If the order-type of $\prec$ is $\eta$, then this is hyp-computable with finite mind-changes along an $\eta$-countdown ≈ $Diff_\eta(\underset{\sim}{\Pi}^1_1)$

- For $y \in \omega^\omega$ let $P_y$ be the $\underset{\sim}{\mathbf{\Pi}}^1_1$ subset of $\mathbb{N}$ coded by $y$.
- If $x \in WO$ let $<_x$ be the well-order on $\mathbb{N}$ coded by $x$.

**Definition ($\mathbf{\Pi}^1_1$-least number principle on WO)**

For $x, y \in \omega^\omega$

$$\mathbf{\Pi}^1_1\text{-}LNP_{WO}(x \oplus y) = \begin{cases} <_x\text{-smallest element of } P_y & \text{if } x \in WO \\ 0 & \text{if } x \notin WO \end{cases}$$

This way of thinking solves Fournier's question.

- As usual, there exists a hierarchy of hyp-computability with finite mind-changes, but with ordinal mind-change countdowns.
- Surprisingly, a value of a mind-change counter can exceed $\omega_1$!!! (without any set-theoretic assumption)

$\mathbf{\Pi}^1_1$-$LNP_{WO}$ is hyp-computable with finite mind-changes with $(\omega_1 + 1)$-countdown.

$$\Pi^1_1\text{-}LNP_{WO}(x \oplus y) = \begin{cases} \prec_x\text{-smallest element of } P_y & \text{if } x \in WO \\ 0 & \text{if } x \notin WO \end{cases}$$

**Theorem: $\Pi^1_1$-least number principle on WO**

- Begin with any guess and ordinal counter $\omega_1 < \omega_1 + 1$.
- If a given $x$ is found to be **WO**, then change the ordinal counter to the order type of $x$, which is smaller than $\omega_1$.
- When something is first enumerated into $P_y$, we guess the $\prec_x$-least element $\alpha \in P_y$ and change the ordinal counter to $\alpha$.
- If something smaller than the previous guess is enumerated into $P_y$, then change the guess as above. Continue this procedure.
- This procedure is hyp-computable with finite mind changes along ordinal counter $\omega_1 + 1$.
- This is clearly intermediate between $Diff_{\omega_1}(\underset{\sim}{\Pi}^1_1)$ and $Diff^*_\omega(\underset{\sim}{\Pi}^1_1)$.

$\mathrm{Diff}^*_\omega(\underset{\sim}{\mathbf{\Pi}}^1_1)$, $\underset{\sim}{\mathbf{\Sigma}}^1_1$-LNP: hyp-computable with finite mind-changes $\omega_2$

↑

$\underset{\sim}{\mathbf{\Pi}}^1_1$-LNP$_{\mathsf{WO}}$: hyp-computable with finite mind-changes along $\omega_1 + 1$ countdown

↑

$\mathrm{Diff}_{\omega_1}(\underset{\sim}{\mathbf{\Pi}}^1_1)$: hyp-computable with finite mind-changes along $\omega_1$ countdown $\phi_{\omega_1}(\omega_1)$

↑

$\mathrm{Diff}_\omega(\underset{\sim}{\mathbf{\Pi}}^1_1)$, $\underset{\sim}{\mathbf{\Pi}}^1_1$-LNP: hyp-computable with finite mind-changes along $\omega$ countdown

$\phi_{\omega_1}(\omega)$

- $\mathit{Diff}_\alpha(\underset{\sim}{\mathbf{\Pi}}^1_1)$: hyp-computability with finite mind-changes with countdown from $\alpha$.
- $\mathit{Diff}_{\omega_1}(\underset{\sim}{\mathbf{\Pi}}^1_1) \subsetneq \mathit{Diff}^*_\omega(\underset{\sim}{\mathbf{\Pi}}^1_1)$: $\alpha$ is not necessarily a countable ordinal.
- $\mathit{Diff}^*_\beta(\underset{\sim}{\mathbf{\Pi}}^1_1)$: hyp-computability with at most $\beta$ mind-changes.
- Again, is $\beta$ not necessarily a countable ordinal?

## Higher limit lemma (Monin)

The following are equivalent for a set $A \subseteq \omega$:

1. $A$ is hyp-computable with ordinal mind-changes.
2. $A$ is Turing reducible to Kleene's $\mathcal{O}$.

We interpret the second condition as the condition "$\underset{\sim}{\mathbf{\Delta}}^0_1$ relative to a $\underset{\sim}{\mathbf{\Pi}}^1_1$-complete set."

## Question

$\mathit{Diff}^*_{\omega_1}(\underset{\sim}{\mathbf{\Pi}}^1_1)$ = "$\underset{\sim}{\mathbf{\Delta}}^0_1$ relative to $\underset{\sim}{\mathbf{\Pi}}^1_1$"?

- $\underset{\sim}{\Sigma}^0_1(\underset{\sim}{\Pi}^1_1)$: The smallest family containing all $\underset{\sim}{\Pi}^1_1$ and $\underset{\sim}{\Sigma}^1_1$ sets and closed under countable union, finite intersection, and continuous preimage.
- $\underset{\sim}{\Delta}^0_1(\underset{\sim}{\Pi}^1_1)$: The family of all sets $A$ such that both $A$ and its complement belong to $\underset{\sim}{\Sigma}^0_1(\underset{\sim}{\Pi}^1_1)$.

**Theorem**

$Diff^*_{\omega_1}(\underset{\sim}{\Pi}^1_1) \subsetneq \underset{\sim}{\Delta}^0_1(\underset{\sim}{\Pi}^1_1)$.

$$D_n(\underset{\sim}{\Pi}^1_1) = D^*_n(\underset{\sim}{\Pi}^1_1) \subsetneq \cdots \subsetneq D_\alpha(\underset{\sim}{\Pi}^1_1) \subsetneq \cdots \subsetneq Diff_{\omega_1}(\underset{\sim}{\Pi}^1_1) \subsetneq Diff^*_\omega(\underset{\sim}{\Pi}^1_1) \subsetneq$$

$$\subsetneq D^*_\omega(\underset{\sim}{\Pi}^1_1) \subsetneq \cdots \subsetneq D^*_\alpha(\underset{\sim}{\Pi}^1_1) \subsetneq \cdots \subsetneq Diff^*_{\omega_1}(\underset{\sim}{\Pi}^1_1) \subsetneq \underset{\sim}{\Delta}^0_1(\underset{\sim}{\Pi}^1_1).$$

$\underset{\sim}{\mathbf{\Delta}}^0_1$ relative to $\underset{\sim}{\mathbf{\Pi}}^1_1$: hyp-computable with ordinal mind-changes

$\Uparrow$

$\underset{\sim}{\mathbf{\Sigma}}^1_1$-LNP$_{\mathsf{WO}}$: hyp-computable with $\omega_1 + 1$ mind-changes

$\Uparrow$

$\mathrm{Diff}^*_{\omega_1}(\underset{\sim}{\mathbf{\Pi}}^1_1)$: hyp-computable with $\omega_1$ mind-changes

$\Uparrow$

$\mathrm{Diff}^*_{\omega}(\underset{\sim}{\mathbf{\Pi}}^1_1)$, $\underset{\sim}{\mathbf{\Sigma}}^1_1$-LNP: hyp-computable with finite mind-changes $\qquad \omega_2$

$\Uparrow$

$\underset{\sim}{\mathbf{\Pi}}^1_1$-LNP$_{\mathsf{WO}}$: hyp-computable with finite mind-changes along $\omega_1 + 1$ countdown

$\Uparrow$

$\mathrm{Diff}_{\omega_1}(\underset{\sim}{\mathbf{\Pi}}^1_1)$: hyp-computable with finite mind-changes along $\omega_1$ countdown $\qquad \phi_{\omega_1}(\omega_1)$

$\Uparrow$

$\mathrm{Diff}_{\omega}(\underset{\sim}{\mathbf{\Pi}}^1_1)$, $\underset{\sim}{\mathbf{\Pi}}^1_1$-LNP: hyp-computable with finite mind-changes along $\omega$ countdown

$\phi_{\omega_1}(\omega)$

## Summary

- $\mathit{Diff}_{\omega_1}(\boldsymbol{\Pi}^1_1)$: hyp-computability with finite mind-changes, but with a mind-change countdown starting from $< \omega_1$.
- $\mathit{Diff}^*_{\omega}(\boldsymbol{\Pi}^1_1)$: hyp-computability with finite mind-changes.
- The Wadge rank of $\mathit{Diff}_{\omega_1}(\boldsymbol{\Pi}^1_1)$ is $\phi_{\omega_1}(\omega_1)$.
- The Wadge rank of $\mathit{Diff}^*_{\omega}(\boldsymbol{\Pi}^1_1)$ is $\omega_2$.
- $\Pi^1_1$-$LNP_{WO}$ is in between $\mathit{Diff}_{\omega_1}(\boldsymbol{\Pi}^1_1)$ and $\mathit{Diff}^*_{\omega}(\boldsymbol{\Pi}^1_1)$.
- The Wadge rank of $\mathit{Diff}^*_{\omega+1}(\boldsymbol{\Pi}^1_1)$ is much larger than $\omega_2 \cdot \omega_1$. (It seems at least $\omega_2^2$).
- $\Sigma^1_1$-$LNP_{WO}$ is in between $\mathit{Diff}^*_{\omega_1}(\boldsymbol{\Pi}^1_1)$ and $\boldsymbol{\Delta}^0_1(\boldsymbol{\Pi}^1_1)$.

## Question

- What is the Wadge rank of $\mathit{Diff}^*_{\omega+1}(\boldsymbol{\Pi}^1_1)$?
- What is the Wadge rank of $\mathit{Diff}^*_{\omega_1}(\boldsymbol{\Pi}^1_1)$?