Lawvere-Tierney topologies for computability theorists

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Background

- Hyland (1982) introduced the *effective topos* $\mathbf{Eff}$, "*the world of computable mathematics*."
- A *(Lawvere-Tiernery) topology* j on a topos $\mathcal{E}$ yields a subtopos $\mathcal{E}_j$.

Example

- The largest LT-topology $\mathbf{ff}$ collapses everything, $\mathbf{Eff}_{\mathbf{ff}}$, "*the world of inconsistent mathematics*."
- The maximal LT-topology $\neg\neg$ changes the world $\mathbf{Eff}$ to $\mathbf{Eff}_{\neg\neg} \simeq \mathbf{Set}$, "*the world of set-theoretic mathematics*."
- The smallest LT-topology $\mathbf{id}$ changes nothing: $\mathbf{Eff}_{\mathbf{id}} \simeq \mathbf{Eff}$.

$$\mathbf{id} < \neg\neg < \mathbf{ff} \qquad\qquad \mathbf{Eff} \hookleftarrow \mathbf{Set} \hookleftarrow \mathbf{Eff}_{\mathbf{ff}}$$

LT-Topologies between $\mathbf{id}$ and $\neg\neg$ on the effective topos
$\approx$ Toposes between "*the computable world*" and "*the set-theoretic world*".

LT-Topologies between $\mathbf{id}$ and $\neg\neg$ on the effective topos
≈ Toposes between "*the computable world*" and "*the set-theoretic world*".

An LT-topology on the effective topos is a kind of data that indicate how much non-computability to add to the world. In other words,

an LT-topology plays the same role as an *oracle*.

- Hyland (1982) found a topology j_d on $\mathbf{Eff}$, for each Turing degree d, which induces a topos $\mathbf{Eff}_{j_d}$ that corresponds to "*the world of d-computable mathematics*".
- Such a topology j_d is called a *Turing degree topology*.
- Pitts (1981) found a topology j_{Pitts} on $\mathbf{Eff}$ which is not equivalent to any Turing degree topology. Indeed,

$$\mathbf{id} < j_{\mathrm{Pitts}} < \neg\neg \qquad \text{and} \qquad (\forall d)\quad j_{\mathrm{Pitts}} \not\leq j_d$$

- Phoa (1989) showed that for any topology j on the effective topos,

$$[(\forall d)\ \ j_d \leq j] \implies \neg\neg \leq j.$$

"*If a world dominates all d-computable worlds, then it must be the set-theoretic world (or inconsistent)*"

Question (Lee 2011)

Does there exist the least LT-topology on $\mathbf{Eff}$ strictly above $\mathbf{id}$; that is, "*a world of non-computable mathematics which is closest to computable mathematics*?"

To solve this problem, it is necessary to have a vague grasp of the overall structure of what LT-topologies are.

A LT-topology is also called a *local operator* or a *geometric modality*.

"*Whereas local operators/subtoposes of Grothendieck toposes can be neatly described in terms of Grothendieck topologies, for realizability toposes the study of local operators is not so easy [...] The lattice of local operators in* $\mathbf{Eff}$ *is vast and notoriously difficult to study.*"

— S. Lee and J. van Oosten, *Basic subtoposes of the effective topos*, APAL 164 (2013).

So, how can we study LT-topologies on $\mathbf{Eff}$?

How can we study LT-topologies on $\mathbf{Eff}$?

- An LT-topology plays the same role as an *oracle*.
- Thanks to Hyland, we know that each Turing degree yields a topology.
- Indeed, one can use a partial function on $\mathbb{N}$ as an oracle.
- Not only that, but even a partial multi-valued function on $\mathbb{N}$ can be used as an oracle, and has a corresponding topology on the effective topos!!!

What does it mean to use a partial multifunction as an oracle?

- Our model is the same as that of an ordinary programming language,
- except that a program $\mathbf{P}$ can contain a special instruction $\mathbf{b} := \square(\mathbf{a})$.
- $\mathbf{P}$ accepts a number n as input and a partial multifunction f on $\mathbb{N}$ as oracle.
- The instruction $\mathbf{b} := \square(\mathbf{a})$ assigns *one of the values* of $f(\mathbf{a})$ to the variable $\mathbf{b}$.
- However, if $f(\mathbf{a})$ is undefined, the computation will never terminate.
- If f is multi-valued, this generally produces a *nondeterministic* computation.

Yes, this yields the $\mathbb{N}$-version of the generalized Weihrauch degrees.

Observation

There exists an embedding of

the $\mathbb{N}$-version of the generalized Weihrauch degrees

into

the lattice of Lawvere-Tierney topologies on the effective topos.

- Note that this holds for most of relative realizability toposes (where $\mathbb{N}$ is replaced with the corresponding partial combinatory algebra.)
- Hence, the generalized Weihrauch degrees (in the usual sense) embed into the lattice of LT-topologies on the Kleene-Vesley topos.
- So, it is possible to position the study of the structure of

  LT-topologies on relative realizability toposes

  as an extension of the Weihrauch-style reverse mathematics.

Anyway, any other LT-topologies besides generalized Weih. degrees?

LT-topologies besides generalized Weihrauch degrees:

- Consider the following probabilistic computation:
- A program $\mathbf{P}$ is given an oracle α at random, and for an input n, the oracle computation $\mathbf{P}^\alpha(n)$ halts with probability at least $1-\varepsilon$, i.e.,

$$\mu(A) \geq 1-\varepsilon \text{ and } (\forall \alpha \in A)\ \mathbf{P}^\alpha(n)\downarrow$$

  for some set $A \subseteq 2^{\mathbb{N}}$.
- This probabilistic computation yields a multifunction such that the value $\mathbf{P}^\alpha(n)$ for each $\alpha \in A$ is a possible output.
- Let us write $\mathtt{ProbError}_\varepsilon\mathbf{P}$ for this.
- If one wants to make explicit a parameter A for an input n, we use the notation $\mathtt{ProbError}_\varepsilon\mathbf{P}(n \mid A)$, that is,

$$\mathtt{ProbError}_\varepsilon\mathbf{P}(n \mid A)\downarrow \iff \mu(A) \geq 1-\varepsilon \ \wedge\ (\forall \alpha \in A)\ \mathbf{P}^\alpha(n)\downarrow$$
$$y \in \mathtt{ProbError}_\varepsilon\mathbf{P}(n \mid A) \iff \exists \alpha \in A\ [\mathbf{P}^\alpha(n) = y]$$

- Although the roles of n and A are entirely different, it can be regarded as a partial multifunction

$$\mathtt{ProbError}_\varepsilon\mathbf{P} :\subseteq \mathbb{N} \times \mathcal{P}(2^{\mathbb{N}}) \rightrightarrows \mathbb{N}.$$

The procedure of giving an oracle randomly to the program $\mathbf{P}$ and having it perform a computation with error probability at most ε:

$$\mathtt{ProbError}_\varepsilon\mathrm{P} :\subseteq \mathbb{N} \times \mathcal{P}(2^{\mathbb{N}}) \rightrightarrows \mathbb{N}.$$

- $\mathtt{ProbError}_\varepsilon\mathrm{P}(n \mid A)$, where n is an input given by us, while A is a witness that the computation halts except for probability at most ε.
- n is an input that is disclosed during the computation, while A is an unknown input that cannot be accessed during the computation.
- Hence, we call n a *public input*, and A a *secret input*.

Definition

A partial multifunction $g :\subseteq \mathbb{N} \times \Lambda \rightrightarrows \mathbb{N}$ is called an *LT-problem*. We write an input for g as $(n \mid c)$. We call n a *public input* and c a *secret input*.

What does it mean to use an LT-problem as an oracle?

⇒ A secret input for an oracle acts like an *advice* string.

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Example

- $\mathtt{ProbError}_\varepsilon\mathtt{P}$ is an LT-problem.
- Any partial multifunction g can be thought of as the LT-problem defined by $\hat{g}(n \mid *) = g(n)$.
- $\mathtt{Advice}_\mathbb{N} : \{*\} \times \mathbb{N} \to \mathbb{N}$ defined by $\mathtt{Advice}_\mathbb{N}(* \mid n) = n$ is an LT-problem.
- In the Kleene-Vesley topos, $\mathtt{Advice}_\mathbb{N} : \{*\} \times \mathbb{N} \to \mathbb{N}$ can be used to deal with *nonuniform computability*.
- However, $\mathtt{Advice}_\mathbb{N} \simeq \neg\neg$ in the effective topos.

What does it mean to use an LT-problem as an oracle?

$\Rightarrow$ A secret input for an oracle acts like an *advice* string.

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What does it mean to use an LT-problem as an oracle?
⇒ A secret input for an oracle acts like an *advice* string.

- In the $\mathbb{N}^{\mathbb{N}}$-context, *one-query* relative computation with advices has been studied by Ziegler, Brattka, Pauly, and others.
- Coincidentally, one-query relative computation for LT-problems has been studied by Bauer (2021) under the name "*extended Weihrauch reducibility*".
- (Be careful that this new definition of *extended Weihrauch reducibility* by Bauer (2021) is a much wider concept than the old definition by Bauer-Yoshimura (2014).)
- However, we need *many-query* relative computation with advices.

Idea of our many-query relative computation model

- During a computation with an LT-problem oracle f, when the program makes a query n to f, the advisor chooses a parameter c.
- However, the information of c chosen by the advisor is not given to the machine,
- but only the information of one of the possible values of $f(n \mid c)$ is given.
- If this process computes a partial multifunction g when the advisor secretly makes the best choice, then we declare that *g is LT-reducible to f*.

Theorem

The structure of LT-degrees of LT-problems is isomorphic to the lattice of Lawvere-Tierney topologies on the effective topos.

(This has almost been proven by Lee and van Oosten (2013), although their language is completely different from ours, and in particular they do not give any computational interpretation of their notions)

The definition of LT-reducibility can be understood by describing it as an imperfect information game between three players, Merlin, Arthur, and Nimue.

Computable reduction game with imperfect information

Merlin:	$(x_0 \mid c_0)$			x_1			x_2			$\dots$
Arthur:		y_0			y_1			y_2		$\dots$
Nimue:			z_0			z_1			z_2	$\dots$

- The player Merlin makes a public input x_0 and a secret input c_0 on his first move.
- Here, among the moves of Merlin, only the secret input c_0 is invisible to Arthur.
- All of Nimue's moves are visible to Merlin, but not to Arthur, a mere human being.
- The players Merlin and Nimue, who are not mere humans, can see all the previous moves at each round.

Computable reduction game with imperfect information

Merlin:	$(x_0 \mid c_0)$			x_1			x_2			$\dots$
Arthur:		y_0			y_1			y_2		$\dots$
Nimue:			z_0			z_1			z_2	$\dots$

For LT-problems f and g, the reduction game for $f \leq_{LT} g$ proceeds as:

- First, Merlin chooses $(x_0 \mid c_0) \in \mathbf{dom}(f)$.
- At the nth round, Arthur reacts with $y_n = \langle j, u_n \rangle$.
    - The choice $j = 0$ indicates that Arthur makes a new query u_n to g.
    - The choice $j = 1$ indicates that Arthur declares termination of the game with u_n.
- At the nth round, Nimue makes an advice parameter z_n, i.e., $(u_n \mid z_n) \in \mathbf{dom}(g)$.
- At the $(n+1)$th round, Merlin responds to the query made by Arthur and Nimue at the previous stage. This means that $x_{n+1} \in g(u_n \mid z_n)$.

Computable reduction game with imperfect information

Merlin:	$(x_0 \mid c_0)$			x_1			x_2			$\dots$
Arthur:		y_0			y_1			y_2		$\dots$
Nimue:			z_0			z_1			z_2	$\dots$

- Arthur *and* Nimue *win this game* if either Merlin violates the rule before Arthur or Nimue violates the rule, or both Arthur and Nimue obey the rule and Arthur declares termination with $u_n \in f(x_0 \mid c_0)$.
- Arthur can only read the public moves $x_0, x_1, x_2, \dots$, and the other players can see all the moves.
- So, Arthur's strategy is a partial function $\tau \subseteq: \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$, which reads Merlin's public moves $x_0, \dots, x_n$ and then returns y_n.

$f \leq_{LT} g \iff$ there exists a pair of Arthur's computable strategy and Nimue's strategy which is winning for the reduction game for $f \leq_{LT} g$.

Yes, this is exactly a generalized Weihrauch reduction game, except for the existence of *secret moves*!

Lee-van Oosten's *dedicated sight* is essentially the same, but not game-theoretic.

- It is known that a Weihrauch problem has the so-called *diamond operator*.
- An LT-problem also has the diamond operator.

Definition

Given an LT-problem h, define the new LT-problem $h^\diamond$ as follows:

- An input for $h^\diamond$ is an Arthur-Nimue strategy $(\tau \mid \eta)$, where Arthur's computable strategy τ is a public input, and Nimue's strategy η is a secret input.
- $h^\diamond(\tau \mid \eta)$ is defined only if, along any play following the strategy $(\tau \mid \eta)$, either Merlin violates the rule before Arthur or Nimue violates the rule, or both Arthur and Nimue obey the rule and Arthur declares termination.
- $u \in h^\diamond(\tau \mid \eta)$ if and only if there is a play that follows the strategy $(\tau \mid \eta)$ such that Arthur declares termination with u at some round.

$f \leq_{LT} g \iff$ "f is reducible to $g^\diamond$ with one query."

Let $\mathcal{E}$ be a topos, with subobject classifier $\top \colon \mathbf{1} \to \Omega$. A *Lawvere-Tierney topology* on $\mathcal{E}$ is a morphism $j \colon \Omega \to \Omega$ such that

$$j \circ \top = \top, \qquad j \circ \wedge = \wedge \circ (j \times j), \qquad j \circ j = j.$$

- In the effective topos, $\Omega = (\mathcal{P}(\mathbb{N}), \Leftrightarrow)$.
- For $p \in \Omega$, define $g^{\diamond\to}(p)$ as the set of Arthur's computable winning strategies for $\dot{p} \leq_{LT} g$, where $\dot{p}(* \mid *) = p$.

Observation

If g is an LT-problem, $g^{\diamond\to} \colon \Omega \to \Omega$ is a Lawvere-Tierney topology.

- If one can solve a problem p without any help, it is clear that one can also solve the problem p with the help of g.
- If one can solve problems p and q with the help of g, then by running these strategies in parallel, one can also solve $p \wedge q$ with the help of g.
- The last condition follows from transitivity of $\leq_{LT}$.

$f : \Omega \to \Omega$ is *computably monotone* $\iff$ the following is realizable:

$$\forall p, q\ [(p \to q) \to (f(p) \to f(q))].$$

$f \leq g \iff$ "$\forall p\ [f(p) \to g(p)]$" is realizable.

Proposition

The structure of one-query LT-degrees of LT-problems is isomorphic to the ordering of computably monotone functions.

[Note] One-query LT-reducibility = Bauer ('21)'s extended Weihrauch reducibility = the $\mathbb{N}$-version of Weihrauch reducibility, with two inner reductions called a public inner reduction and a secret inner reduction.

- For an LT-problem g, define $g^{\to} : \Omega \to \Omega$ as follows:

  $$\langle n, e \rangle \in g^{\to}(p) \iff e \text{ realizes } g(n \mid c) \to p \text{ for some } c.$$

- Roughly speaking, $g^{\to}(p)$ is a problem that asks us to solve a problem p with one-query help of g.
- Of the solutions n and e to $g^{\to}(p)$, we sometimes call n an *inner reduction* and e an *outer reduction*. (And, c is a secret inner reduction.)
- Note: $g^{\to} : \Omega \to \Omega$ is computably monotone.

Proposition

The structure of one-query LT-degrees of LT-problems is isomorphic to the ordering of computably monotone functions.

- Claim: g is one-query reducible to $h \iff g^{\to} \leq h^{\to}$.

  (⇒) If one can solve g by using h, and $\dot{p}$ by using g, then by transitivity one can also solve $\dot{p}$ by using h. Here this transitivity is computably witnessed.

  (⇐) Given input $(n \mid c)$, consider $p = g(n \mid c)$.
  A trivial algorithm Φ_n depending on n solves $\dot{p}$ by using g.
  By $g^{\to} \leq h^{\to}$, one can effectively transform Φ_n into a new algorithm Ψ_n which solves $\dot{p}(* \mid *) = g(n \mid c)$ by using h.
  Clearly $n \mapsto \Psi_n$ witnesses that g is one-query reducible to h.

- For a computably monotone $f : \Omega \to \Omega$, define an LT-problem $f^{\leftarrow}$ as follows:

$$\mathbf{dom}(f^{\leftarrow}) = \left\{(n \mid c) \in \mathbb{N} \times \mathcal{P}(\mathbb{N}) : n \in j(c)\right\}, \qquad f^{\leftarrow}(n \mid c) = c.$$

- $f^{\leftarrow\to} \equiv g$ is due to Lee-van Oosten (2013).

Theorem

The structure of LT-degrees of LT-problems is isomorphic to the lattice of Lawvere-Tierney topologies on the effective topos.

- We claim that $g \mapsto g^{\diamond\to}$ yields a desired isomorphism.
- $j^{\leftarrow} \leq_{LT} j^{\leftarrow\diamond}$ with one-query, so $j \equiv j^{\leftarrow\to} \leq j^{\leftarrow\diamond\to}$.
- Lee-van Oosten considered

$$L(f)(p) := \forall q\ [[(p \to q) \land (f(q) \to q)] \to q],$$

and showed that $L(f)$ is the $\leq$-least topology above f.
- It suffices to show that $f^{\leftarrow\diamond\to} \leq L(f)$.

$$L(f)(p) := \forall q\, [[(p \to q) \land (f(q) \to q)] \to q],$$

Want to show: $f^{\leftarrow\diamond\rightarrow} \leq L(f)$.

- A realizer for $f^{\leftarrow\diamond\rightarrow}(p)$ is a pair $\langle \mathbf{d}, \mathbf{e}\rangle$ of an inner reduction $\mathbf{d}$ and an outer reduction $\mathbf{e}$ for $\dot{p} \leq_{1T} f^{\leftarrow\diamond}$, i.e., $\mathbf{e}$ realizes $f^{\leftarrow\diamond}(\mathbf{d} \mid c) \to p$ for some c.

  Note that $(\mathbf{d} \mid c)$ is an Arthur-Nimue strategy for the game $\mathfrak{G}(f^{\leftarrow})$.
- Given $\mathbf{a} \vdash p \to q$ and $\mathbf{b} \vdash f(q) \to q$, independent of q.
- If $(n \mid z)$ is Arthur and Nimue's queries made at some round, then $(n \mid z) \in \mathrm{dom}(f^{\leftarrow})$, which means that $n \in f(z)$ and $f^{\leftarrow}(n \mid z) = z$.
- Since $\mathbf{b}$ realizes $f(z) \to z$, we have $\mathbf{b} \cdot n \in f^{\leftarrow}(n \mid z)$.

  Hence, $\mathbf{b}$ yields Merlin's strategy.
- Therefore, one can simulate one of the plays of the game $\mathfrak{G}(f^{\leftarrow})$ from the information in $\mathbf{d}$, c, and $\mathbf{b}$.
- In particular, can compute Arthur's final move in this play, which yields some $m \in f^{\leftarrow\diamond}(\mathbf{d} \mid c)$.
- Then $\mathbf{e} \cdot m \in p$, and thus $\mathbf{a} \cdot \mathbf{e} \cdot m \in q$.

The lattice structure of Lawvere-Tierney topologies

(or equivalently the LT-degree structure of LT-problems)

Concrete examples of LT-problems:

A problem such that m of the k choices are wrong:

$$\mathbf{dom}(\mathtt{Error}_{m/k}) = \left\{(* \mid A) : A \subseteq \{0, \dots, k-1\} \ \wedge \ |A| = m\right\},$$

$$\mathtt{Error}_{m/k}(* \mid A) = \{0, \dots, k-1\} \setminus A$$

Lessor limited principle of omniscience relative to α:

$$\mathbf{dom}(\mathtt{LLPO}^{\alpha}_{m/k}) = \left\{e \in \mathbb{N} : |\{j < k : \varphi^{\alpha}_{e}(j)\downarrow\}| \leq m\right\},$$

$$\mathtt{LLPO}^{\alpha}_{m/k}(e) = \{0, \dots, k-1\} \setminus \{j < k : \varphi^{\alpha}_{e}(j)\downarrow\}.$$

(Essentially the same notion as $\mathtt{Error}_{m/k}$ is called O^{k}_{m} in Lee-van Oosten (2013))

Observation

- $\mathtt{LLPO}^{\alpha}_{m/k} \leq_{LT} \mathtt{Error}_{m/k}$ for any oracle α.
- $\mathtt{LLPO}_{m/k} \leq_{LT} \mathtt{Error}_{m/k+1}$, but $\mathtt{LLPO}_{m/k} \not\leq_{LT} \mathtt{Error}_{m/k+2}$.
- $\mathtt{LLPO}^{\emptyset'}_{m/k} \not\leq_{LT} \mathtt{Error}_{m/k+1}$.

The following solves Lee-van Oosten's question on O^k_m.

Theorem (K.)

$\mathtt{Error}_{m/k} \equiv_{LT} \mathtt{Error}_{1/\ell}$, *where* $\ell = \lceil \frac{k}{m} \rceil$.

The proof is an analogue of Cenzer-Hinman's related result on Medvedev degrees.

D. Cenzer and P. G. Hinman, *Degrees of difficulty of generalized r.e. separating classes.* Arch. Math. Logic, 46 (2008), pp. 629–647.

Other concrete examples of LT-problems:

Probabilistic computation with error probability ε:

$$\mathtt{ProbError}_\varepsilon(\langle e,n\rangle \mid A)\downarrow \iff A \subseteq 2^{\mathbb{N}} \text{ is compact}$$
$$\wedge\ \mu(A) \geq 1-\varepsilon\ \wedge\ (\forall \alpha \in A)\ \varphi_e^\alpha(n)\downarrow.$$
$$\mathtt{ProbError}_\varepsilon(\langle e,n\rangle \mid A) = \{\varphi_e^\alpha(n) : \alpha \in A\}.$$

Weak weak König's lemma:

$$\mathtt{WWKL}_\varepsilon(\langle e,n,i\rangle)\downarrow \iff \mu(P_i) \geq 1-\varepsilon\ \wedge\ (\forall \alpha \in P_i)\ \varphi_e^\alpha(n)\downarrow.$$
$$\mathtt{WWKL}_\varepsilon(\langle e,n,i\rangle) = \{\varphi_e^\alpha(n) : \alpha \in P_i\}.$$

Theorem (K.)

For any $p,q \in \mathbb{N}$ with $p \leq q$, $\mathtt{ProbError}_{p/q} \equiv_{LT} \mathtt{Error}_{p/q}$.

The proof is nontrivial. This is a phenomenon specific to the effective topos. If we consider another (relative) realizability topos, such as the Kleene-Vesley topos, the situation would be completely different.

An LT-problem is *basic* if it has no public input.

Theorem (Lee-van Oosten 2013)

$\mathtt{Error}_{1/\omega}$ is the least basic LT-problem which is strictly $\leq_{LT}$-above the identity.

However, if non-basic LT-problems are included, then $\mathtt{Error}_{1/\omega}$ is not the smallest.

All-or-counique choice relative to α:

$$\mathbf{dom}(\mathtt{ACC}^{\alpha}) = \mathbb{N}, \qquad \mathtt{ACC}^{\alpha}(e) = \begin{cases} \mathbb{N} \setminus \{\varphi_e^{\alpha}(e)\} & \text{if } \varphi_e^{\alpha}(e) \downarrow \\ \mathbb{N} & \text{if } \varphi_e^{\alpha}(e) \uparrow \end{cases}$$

Obviously, $\mathtt{ACC}^{\alpha} <_{LT} \mathtt{Error}_{1/\omega}$ for any oracle α.

Question (Lee 2011)

Does there exist the least LT-topology on $\mathbf{Eff}$ strictly above $\mathbf{id}$?

We solve Lee's question:

Theorem (K.)

There exists no $\leq_{LT}$-minimal LT-problem which is strictly $\leq_{LT}$-above the identity:

$$(\forall f >_{LT} \mathtt{Id})(\exists g) \quad \mathtt{Id} <_{LT} g <_{LT} f.$$

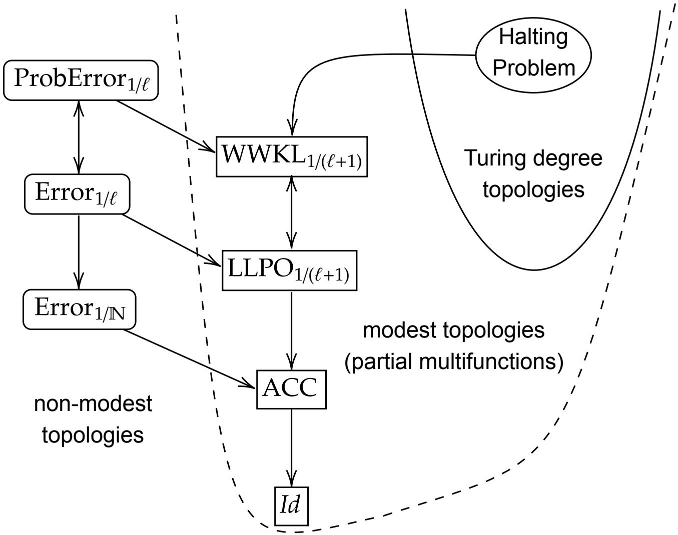

Figure: Lower parts on Lawvere-Tierney topologies on the effective topos

Pitts (1985) considered the following basic LT-problem:

$$\mathbf{dom}(\mathtt{Cofinite}) = \{*\} \times \mathbb{N}, \qquad \mathtt{Cofinite}(* \mid n) = \{m \in \mathbb{N} : m \geq n\}.$$

Theorem (van Oosten 2014)

A total function $f : \mathbb{N} \to \mathbb{N}$ is hyperarithmetic $\iff f \leq_{LT} \mathtt{Cofinite}$.

For a set $A \subseteq \mathbb{N}$, the *lower asymptotic density* of A is defined by

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{|A \cap n|}{n}.$$

Define the basic LT-problem $\mathtt{DenError}_\varepsilon$ as follows:

$$\mathbf{dom}(\mathtt{DenError}_\varepsilon) = \{(* \mid A) : A \subseteq \mathbb{N} \text{ and } \underline{d}(A) \geq 1 - \varepsilon\}$$

$$\mathtt{DenError}_\varepsilon(* \mid A) = A.$$

Observation

$\mathtt{Cofinite} \leq_{LT} \mathtt{DenError}_\varepsilon$ and $\mathtt{Error}_{1/\ell} \leq_{LT} \mathtt{DenError}_{1/\ell}$.

Theorem (K.)

Let $\mathtt{P}$ *be a partial multifunction whose codomain is* $\ell \in \mathbb{N}$ *with* $\ell > 0$*.*
For any $\varepsilon < 1/(\ell + 1)$*, if* $\mathtt{P} \leq_{hLT} \mathtt{DenError}_\varepsilon$*, then* $\mathtt{P}$ *is hyperarithmetic.*

Here, $\leq_{hLT}$ is hyperarithmetic LT-reducibility.

Corollary (K.)

- $f : \mathbb{N} \to \mathbb{N}$ is hyperarithmetic $\iff f \leq_{LT} \mathtt{DenError}_\varepsilon$ for $\varepsilon < 1/2$.
- $\Pi^1_1\text{-}\mathtt{LLPO}_{1/\ell} \leq_{hLT} \mathtt{DenError}_{1/(\ell+1)}$.
- $\Pi^1_1\text{-}\mathtt{LLPO}_{1/\ell} \not\leq_{hLT} \mathtt{DenError}_{1/(\ell+2)}$.

Theorem (K.)

$\mathtt{Cofinite} <_{LT} \mathtt{DenError}_0$.

$$\mathtt{Cofinite} <_{LT} \mathtt{DenError}_0 <_{LT} \cdots <_{LT} \mathtt{DenError}_{1/(\ell+1)} <_{LT} \mathtt{DenError}_{1/\ell} <_{LT} \cdots$$

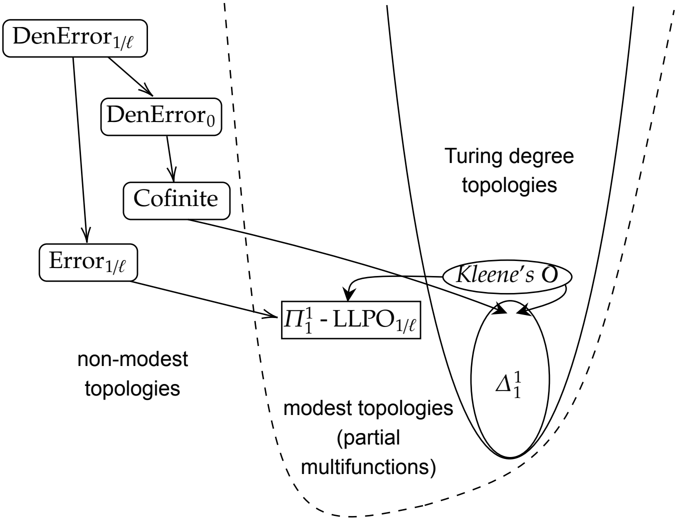

Figure: Higher parts on Lawvere-Tierney topologies on the effective topos

Summary

- It is possible to position the study of the structure of

  Lawvere-Tierney topologies on relative realizability toposes

  as an extension of the Weihrauch-style reverse mathematics.
- In this way, we solved all problems mentioned in

  - Sori Lee, **Subtoposes of the Effective Topos**, preprint, 2011, arXiv:1112.5325
  - Sori Lee and Jaap van Oosten, **Basic subtoposes of the effective topos**, *Annals of Pure and Applied Logic* 164 (2013), pp. 866-883

- There are many other toposes that are related to computability theory and (effective) descriptive set theory.
  - Any $\boldsymbol{\Sigma}^*$-pointclass yields a (relative) realizability topos.
  - For instance, if the pointclass $\underset{\sim}{\boldsymbol{\Pi}}^1_1$ is used as a seed, a topos corresponding to “the world of Borel mathematics” will be created.

Takayuki Kihara, **Lawvere-Tierney topologies for computability theorists**, preprint, 35 pages, available at arXiv:2106.03061