Wadge-like classifications of real valued functions
(The Day-Downey-Westrick reducibilities for $\mathbb{R}$-valued functions)

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Aspects of Computation: in celebration of the research work of Professor Rod Downey

Institute for Mathematical Sciences, NUS, Singapore, Sep 4th, 2017

- Day-Downey-Westrick (DDW) recently introduced m-, tt-, and T-reducibility for real-valued functions.
- We give a full description of the structures of DDW's m- and T-degrees of real-valued functions.

Caution: Without mentioning, we always assume Woodin's $\mathbf{AD}^+$.

But, of course:

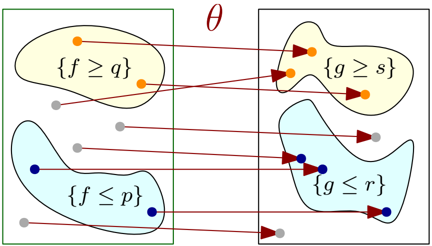
- If we restrict our attention to Borel sets and Baire functions, every result presented in this talk is provable within ZFC.
- If we restrict our attention to projective sets and functions, every result presented in this talk is provable within ZF+DC+PD.
- We even have $L(\mathbb{R}) \models \mathbf{AD}^+$, assuming that there are arbitrarily large Woodin cardinals.

Day-Downey-Westrick's *m*-reducibility

For $f, g : 2^\omega \to \mathbb{R}$, say f is ***m*-reducible** to g (written $f \leq_m g$) if given $p < q$, there are $r < s$ and continuous $\theta : 2^\omega \to 2^\omega$ s.t.

if Z separates the level sets $\{x : g(x) \leq r\}$ and $\{x : g(x) \geq s\}$,

$\theta^{-1}[Z]$ separates the level sets $\{x : f(x) \leq p\}$ and $\{x : f(x) \geq q\}$.



Definition (Bourgain 1980)

Let $f : 2^\omega \to \mathbb{R}$, $p < q$, and $S \subseteq 2^\omega$.
Define the (f, p, q)-derivative $D_{f,p,q}(S)$ of S as follows.

$$S \setminus \bigcup \{x \in S : (\exists U \ni x)\ f[U] \subseteq (-\infty, q) \text{ or } f[U] \subseteq (p, \infty)\},$$

where U ranges over open sets. Consider the derivation procedure

$$P^0_{f,p,q} = 2^\omega,\ P^{\xi+1}_{f,p,q} = D_{f,p,q}(P^\xi_{f,p,q}),\ P^\lambda_{f,p,q} = \bigcap_{\xi<\lambda} P^\xi_{f,p,q} \text{ for } \lambda \text{ limit}$$

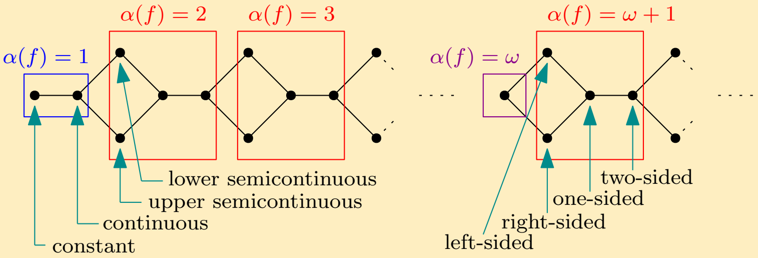
The Bourgain rank $\alpha(f)$ is defined as follows:

$$\alpha(f) = \min\{\alpha : (\forall p < q)\ P^\alpha_{f,p,q} = \emptyset\}.$$

- $\alpha(f) = 1$ iff f is continuous.
- The rank $\alpha(f)$ exists iff f is a Baire-one function.

Theorem (Day-Downey-Westrick)

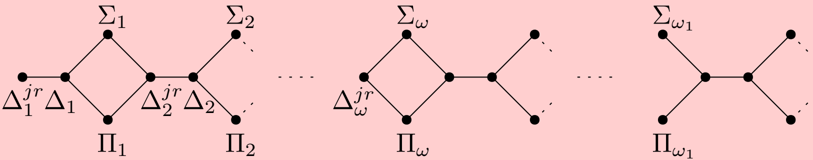
- For Baire-one functions, $\alpha(f) \leq \alpha(g)$ implies $f \leq_m g$.
- The α-rank **1** consists of two m-degrees.
- Each successor α-rank $>$ **1** consists of four m-degrees.
- Each limit α-rank consists of a single m-degree.



This gives a full description of the m-degrees of the Baire-one functions.

1st Main Theorem (K.)

The structure of the DDW-m-degrees of real-valued functions looks like the following figure:

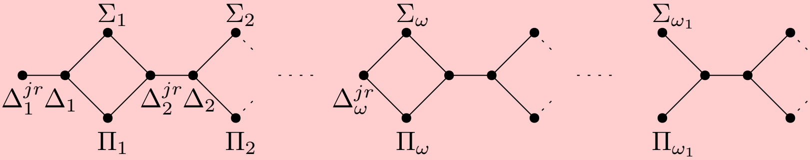


The DDW-m-degrees form a semi-well-order of height Θ.
For a limit ordinal $\xi < \Theta$ and finite $n < \omega$,

- the DDW-m-rank $\xi + 3n + c_\xi$ consists of two incomparable degrees
- each of the other ranks consists of a single degree.

Here, $c_\xi = 2$ if $\xi = 0$; $c_\xi = 1$ if $\mathbf{cf}(\xi) = \omega$; and $c_\xi = 0$ if $\mathbf{cf}(\xi) \geq \omega_1$.

Δ_1^{jr} = constant functions; Δ_1 = continuous functions;
Σ_1 = lower semi-continuous; Π_1 = upper semi-continuous.



- $\mathbf{\Pi}_\xi$: The ξ^{th} nonselfdual pointclass with the separation prop.
- $\mathbf{\Sigma}_\xi$: The ξ^{th} nonselfdual pointclass with no separation prop.
- $\mathbf{\Delta}_\xi$: The ξ^{th} selfdual pointclass.
- $\mathbf{\Delta}^{jr}_\xi$: Lipschitz σ-join-reducibles in the ξ^{th} selfdual pointclass.

Relationship with DDW's result (i.e., the structure below ω_1)

- The α-rank 1 consists of $\{\mathbf{\Delta}^{jr}_1, \mathbf{\Delta}_1\}$.
- The α-rank $\xi + 1$ $(\xi > 0)$ consists of $\{\mathbf{\Sigma}_\xi, \mathbf{\Pi}_\xi, \mathbf{\Delta}^{jr}_{\xi+1}, \mathbf{\Delta}_{\xi+1}\}$.
- The limit α-rank ξ consists of $\{\mathbf{\Delta}^{jr}_\xi\}$.
- $\mathbf{\Sigma}_\xi \approx$ left-sided; $\mathbf{\Pi}_\xi \approx$ right-sided; $\mathbf{\Delta}^{jr}_\xi \approx$ one-sided; $\mathbf{\Delta}_\xi \approx$ two-sided.

Theorem (K.-Montalbán; 201x)

The Wadge degrees ≈ the “natural” many-one degrees.

DDW defined T-reducibility for $\mathbb{R}$-valued functions as parallel continuous (strong) Weihrauch reducibility ($f \leq_T g$ iff $f \leq^c_{sW} \widehat{g}$).

2nd Main Theorem (K.)

The DDW T-degrees ≈ the “natural” Turing degrees.

(Steel ’82; Becker ’88) The “natural” Turing degrees form a well-order of type Θ.
Hence, the DDW T-degrees (of nonconst. functions) form a well-order of type Θ.
(The DDW T-rank of a Baire class function coincides with $2+$ its Baire rank)

More Theorems... (with Westrick)

There are many other characterizations of DDW T-degrees, e.g., relative computability w.r.t. point-open topology on the space $\mathbb{R}^{(2^\omega)}$.

Introduction to descriptive set theory

- Pointclass: $\Gamma \subseteq \omega^\omega$
- Dual: $\check{\Gamma} = \{\omega^\omega \setminus A : A \in \Gamma\}$.
- A pointclass Γ is selfdual iff $\Gamma = \check{\Gamma}$.
- For $A, B \subseteq \omega^\omega$, A is Wadge reducible to B ($A \leq_w B$) if

$$(\exists \theta \text{ continuous})(\forall X \in \omega^\omega)\ X \in A \iff \theta(X) \in B.$$

- $A \subseteq \omega^\omega$ is selfdual if $A \equiv_w \omega^\omega \setminus A$.
- $A \subseteq \omega^\omega$ is selfdual iff $\Gamma_A = \{B \in \omega^\omega : B \leq_w A\}$ is selfdual.

$\undertilde{\Delta}^i_\alpha$ is selfdual, but $\undertilde{\Sigma}^i_\alpha$ and $\undertilde{\Pi}^i_\alpha$ are nonselfdual.

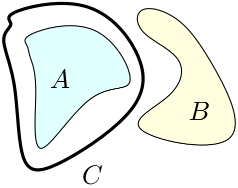
Theorem (Wadge; Martin-Monk 1970s)

The Wadge degrees are semi-well-ordered.

In particular, nonselfdual pairs are well-ordered, say $(\Gamma_\alpha, \check{\Gamma}_\alpha)_{\alpha<\Theta}$ where Θ is the height of the Wadge degrees.

A pointclass Γ has the separation property if

$$(\forall A, B \in \Gamma)\ [A \cap B = \emptyset \implies (\exists C \in \Gamma \cap \check{\Gamma})\ A \subseteq C \ \&\ B \cap C = \emptyset]$$



Example (Lusin 1927, Novikov 1935, and others)

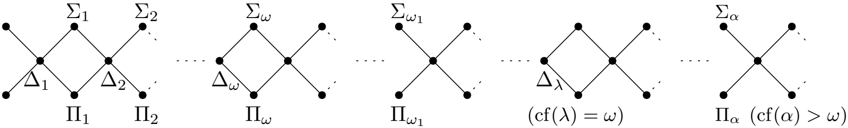
- $\underset{\sim}{\Pi}^0_\alpha$ has the separation property for any $\alpha < \omega_1$.
- $\underset{\sim}{\Sigma}^1_1$ and $\underset{\sim}{\Pi}^1_2$ have the separation property.
- (PD) $\underset{\sim}{\Sigma}^1_{2n+1}$ and $\underset{\sim}{\Pi}^1_{2n+2}$ have the separation property.

Nonselfdual pairs are well-ordered, say $(\mathbf{\Gamma}_\alpha, \check{\mathbf{\Gamma}}_\alpha)_{\alpha<\Theta}$.

Theorem (Van Wasep 1978; Steel 1981)

Exactly one of $\mathbf{\Gamma}_\alpha$ and $\check{\mathbf{\Gamma}}_\alpha$ has the separation property.

- $\mathbf{\Pi}_\alpha$: the one which has the separation property
- $\mathbf{\Sigma}_\alpha$: the other one
- $\mathbf{\Delta}_\alpha = \mathbf{\Sigma}_\alpha \cap \mathbf{\Pi}_\alpha$

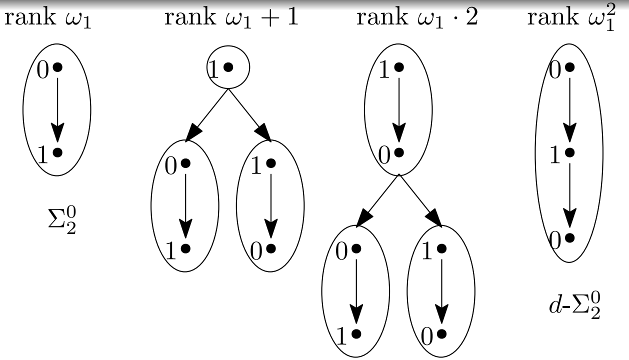


Example

$\mathbf{\Delta}_1$ = clopen ($\underset{\sim}{\mathbf{\Delta}}^0_1$); $\mathbf{\Sigma}_1$ = open ($\underset{\sim}{\mathbf{\Sigma}}^0_1$); $\mathbf{\Pi}_1$ = closed ($\underset{\sim}{\mathbf{\Pi}}^0_1$);

$\mathbf{\Delta}_\alpha, \mathbf{\Sigma}_\alpha, \mathbf{\Pi}_\alpha$ $(\alpha < \omega_1)$: the α^{th} level of the Hausdorff difference hierarchy

$\mathbf{\Sigma}_{\omega_1} = F_\sigma$ ($\underset{\sim}{\mathbf{\Sigma}}^0_2$); $\mathbf{\Pi}_{\omega_1} = G_\delta$ ($\underset{\sim}{\mathbf{\Pi}}^0_2$)



Example

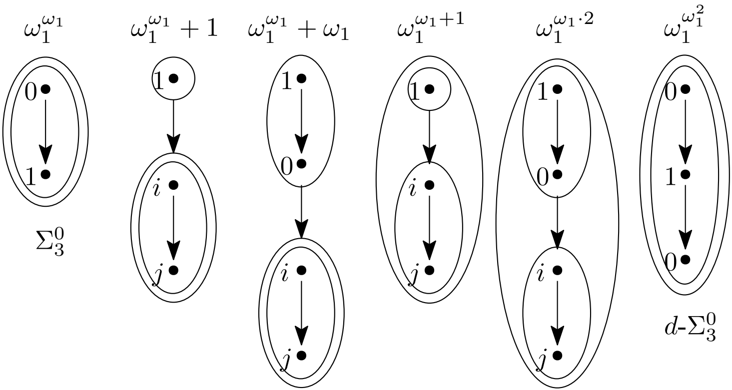
$\Sigma_{\omega_1} = F_\sigma$ (i.e. $\underset{\sim}{\Sigma}^0_2$); $\Pi_{\omega_1} = G_\delta$ (i.e. $\underset{\sim}{\Pi}^0_2$)

$\Sigma_{\omega_1^2}$ = the difference of two G_δ sets; $\Pi_{\omega_1^2}$ = the diff. of two F_σ sets

$\Delta_{\omega_1^\alpha}, \Sigma_{\omega_1^\alpha}, \Pi_{\omega_1^\alpha}$ $(\alpha < \omega_1)$: the α^{th} level of the diff. hierarchy over F_σ

$a_0 \to \cdots \to a_n$: an ordered space endowed with a Sierpiński-like representation;
circle ◯: the jump of an inner represented space.

(Ex. $A \subseteq \omega^\omega$ is F_σ (i.e. $\underset{\sim}{\Sigma}^0_2$) iff $\chi_A : \omega^\omega \to \mathbb{S}'$ is continuous)



Example

$\Sigma_{\omega_1^{\omega_1}} = G_{\delta\sigma}(\underset{\sim}{\Sigma}^0_3)$; $\Pi_{\omega_1^{\omega_1}} = F_{\sigma\delta}(\underset{\sim}{\Pi}^0_3)$

$\Sigma_{\omega_1^{\omega_1^2}}$ = the difference of two $F_{\sigma\delta}$ sets; $\Pi_{\omega_1^{\omega_1^2}}$ = the diff. of two $G_{\delta\sigma}$ sets

$\Delta_{\omega_1^{\omega_1^\alpha}}, \Sigma_{\omega_1^{\omega_1^\alpha}}, \Pi_{\omega_1^{\omega_1^\alpha}}$ ($\alpha < \omega_1$): the α^{th} level of the diff. hierarchy over $G_{\delta\sigma}$

Example

- $\utilde{\mathbf{\Sigma}}^0_2$, $\utilde{\mathbf{\Pi}}^0_2$: Wadge-rank ω_1.
- $\utilde{\mathbf{\Sigma}}^0_3$, $\utilde{\mathbf{\Pi}}^0_3$: Wadge-rank $\omega_1^{\omega_1}$.
- $\utilde{\mathbf{\Sigma}}^0_n$, $\utilde{\mathbf{\Pi}}^0_n$: Wadge-rank $\omega_1 \uparrow\uparrow n$ (the n^{th} level of the superexp hierarchy)
- $\varepsilon_0[\omega_1] := \lim_{n\to\infty}(\omega_1 \uparrow\uparrow n)$: Its cofinality is ω.
  Hence, the class of rank $\varepsilon_0[\omega_1]$ is selfdual.
  Moreover, $\mathbf{\Delta}_{\varepsilon_0[\omega_1]}$ is far smaller than $\utilde{\mathbf{\Delta}}^0_\omega$.
  (Because we can use arbitrarily deep nests of circles to define a $\utilde{\mathbf{\Delta}}^0_\omega$ set.)
- (Duparc 2001) $\varepsilon_{\omega_1}[\omega_1]$: the ω_1^{th} fixed point of the exp. of base ω_1.
  $\utilde{\mathbf{\Sigma}}^0_\omega$, $\utilde{\mathbf{\Pi}}^0_\omega$: Wadge-rank $\varepsilon_{\omega_1}[\omega_1]$.

Example

- (Duparc 2001) The Veblen hierarchy of base ω_1:
  $\phi_\alpha(\gamma)$: the γ^{th} ordinal closed under $+$, $\sup_{n\in\omega}$, and $(\phi_\beta)_{\beta<\alpha}$.
- ϕ_0 enumerates $1, \omega_1, \omega_1^2, \omega_1^3, \dots, \omega_1^{\omega+1}, \omega_1^{\omega+2}, \dots$
- ϕ_1 enumerates $1, \varepsilon_{\omega_1}[\omega_1], \dots$
- W-rank $\phi_0(\gamma) \approx$ a well-founded nest of circles of rank γ.
- W-rank $\phi_\alpha(\gamma) \approx$ a well-founded nest of ω^α-circles of rank γ.

  An ω^α-circle corresponds to the ω^α-th jump of a representation.

  Thus, every Borel Wadge degree is characterized in terms of representation
- $\utilde{\Sigma}^0_{\omega^\alpha}$, $\utilde{\Pi}^0_{\omega^\alpha}$: Wadge-rank $\phi_\alpha(1)$ ($0 < \alpha < \omega_1$).
- $\utilde{\Sigma}^1_1$, $\utilde{\Pi}^1_1$: Wadge-rank $\sup_{\xi<\omega_1} \phi_\xi(1)$.

Definition

Let $\boldsymbol{Q}$ be a partial order. For $\boldsymbol{A}, \boldsymbol{B} : \omega^\omega \to \boldsymbol{Q}$,
$\boldsymbol{A}$ is **$\boldsymbol{Q}$-Wadge reducible** to $\boldsymbol{B}$ ($\boldsymbol{A} \leq_{\boldsymbol{w}}^{\boldsymbol{Q}} \boldsymbol{B}$) if

$$(\exists \theta \text{ continuous})(\forall X \in \omega^\omega)\ A(X) \leq_Q B \circ \theta(X).$$

- $\mathbf{2} = \{0, 1\}$: $\mathbf{0}$ and $\mathbf{1}$ are incomparable.
- $\mathbb{T} = \{0, 1, \perp\}$: Plotkin's order; $\perp < 0, 1$.
- Wadge studied the **2**- and $\mathbb{T}$-Wadge degrees.
- (Wadge's Lemma) The $\mathbb{T}$-Wadge degrees are semi-linear-ordered.
- (Van Engelen-Miller-Steel 1987; Block 2014)
  If $\boldsymbol{Q}$ is BQO, so is the $\boldsymbol{Q}$-Wadge degrees.
  In particular, the $\mathbb{T}$-Wadge degrees are semi-well-ordered.

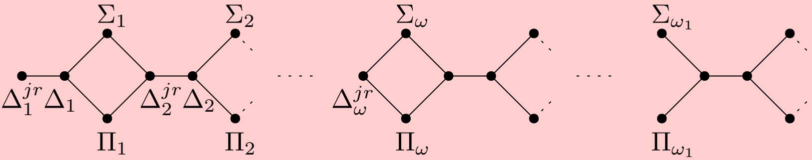
(K.-Montalbán 201x) If $\boldsymbol{Q}$ is BQO, every Borel $\boldsymbol{Q}$-Wadge degree is characterized in terms of represented spaces as before.

(using Sierpiński-like representations of $\boldsymbol{Q}$-labeled trees and ω^α-circles.)

Embedding Lemma for the DDW-***m***-degrees

1st Main Theorem (K.)

The structure of the DDW-***m***-degrees of real-valued functions looks like the following figure:



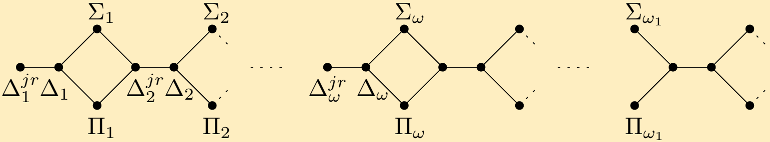
Let Q be a partial order. For $A, B : \omega^\omega \to Q$,
A is Q-m-Wadge reducible to B ($A \leq^Q_{mw} B$) if there are $\psi : \omega \to \omega$ and a continuous function $\theta : \omega^\omega \to \omega^\omega$ such that

$$(\forall X \in \omega^\omega)\ A(m^\frown X) \leq_Q B(\psi(m)^\frown\theta(m, X)).$$

- $A \leq^Q_{Lip} B \implies A \leq^Q_{mw} B \implies A \leq^Q_{w} B$.
- A is Lipschitz σ-join-reducible if $A \upharpoonright n <_w A$ for any $n \in \omega$.

Lemma

The structure of the **2**-m-Wadge degrees in ω^ω looks like the following:



That is, each successor selfdual Wadge degree splits into two degrees (which are linearly ordered), and other Wadge degrees remain the same.

For a function $f : \omega^\omega \to \mathbb{R}$, define $S_f : \omega^\omega \to \mathbb{T}$ as follows:
For any $p, q \in \mathbb{Q}$ with $p < q$

$$S_f(\langle p, q\rangle^\frown X) = \begin{cases} 1 & \text{if } q \le f(X), \\ 0 & \text{if } f(X) \le p, \\ \bot & \text{if } p < f(X) < q. \end{cases}$$

A pair $\langle p, q\rangle$ is identified with a natural number in an effective manner.

Recall that f is m-reducible to g (written $f \le_m g$) if
given $p < q$, there are $r < s$ and continuous $\theta : 2^\omega \to 2^\omega$ s.t.

if Z separates the level sets $\{x : g(x) \le r\}$ and $\{x : g(x) \ge s\}$,

$\theta^{-1}[Z]$ separates the level sets $\{x : f(x) \le p\}$ and $\{x : f(x) \ge q\}$.

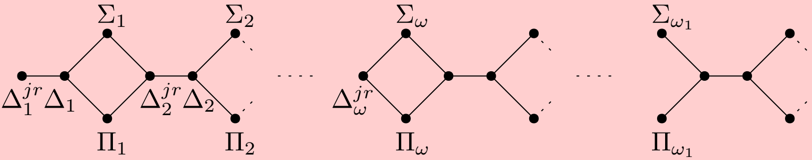
Remark: $f \le_m g$ iff $S_f \le^{\mathbb{T}}_{mw} S_g$.

In particular, the DDW-m-degrees form a substructure of the $\mathbb{T}$-mw-degrees.

The image of the embedding of the DDW-***m***-degrees

1st Main Theorem (K.)

The structure of the DDW-***m***-degrees of real-valued functions looks like the following figure:



- $\Lambda^{\mathbb{T}}_{\alpha} := \{A : \omega^{\omega} \to \mathbb{T} \mid (\exists S \in \Gamma_{\alpha})\ A \leq_w S\}$ for $\Lambda \in \{\Sigma, \Pi, \Delta\}$.
- $\Delta^{\mathbb{T}}_{\alpha} \neq \Sigma^{\mathbb{T}}_{\alpha} \cap \Pi^{\mathbb{T}}_{\alpha}$.
- For a Wadge degree d, $\Gamma_d = \{B : \deg_w(B) \leq d\}$.
- A $\mathbb{T}$-Wadge degree d is proper if $\Gamma_d \neq \Lambda^{\mathbb{T}}_{\alpha}$ for any Λ and α, that is, it is not the $\mathbb{T}$-Wadge degree of a $\mathbf{2}$-valued function.

Surjectivity Lemma

For every non-proper $\mathbb{T}$-Wadge degree d, there is $f : \mathbf{2}^{\omega} \to \mathbb{R}$ such that S_f is Γ_d-complete.

Lemma

Let A be $\Delta^{\mathbb{T}}_{\alpha}$-complete.

1. If either α is successor or A is Lip-σ-join-reducible, then there is $f : \mathbf{2}^{\omega} \to \mathbb{R}$ s.t. $S_f \equiv^{\mathbb{T}}_{mw} A$.
2. Otherwise, there is no $f : \mathbf{2}^{\omega} \to \mathbb{R}$ s.t. $S_f \equiv^{\mathbb{T}}_{mw} A$.

Consequently, the DDW-m-degrees subsume all non-proper $\mathbb{T}$-mw-degrees in $\mathbf{2}^{\omega}$

Lemma

If d is a proper $\mathbb{T}$-Wadge degree, then

$$(\exists\alpha < \Theta)\ \Delta^{\mathbb{T}}_{\alpha} \subseteq \Gamma_d \subseteq \Sigma^{\mathbb{T}}_{\alpha} \cap \Pi^{\mathbb{T}}_{\alpha}.$$

Proof

- $\alpha < \Theta$: Least ordinal s.t. $\Gamma_d \subseteq \Sigma^{\mathbb{T}}_{\alpha} \cap \Pi^{\mathbb{T}}_{\alpha}$.
- Claim: $(\forall\beta < \alpha)\ \Sigma^{\mathbb{T}}_{\beta} \cup \Pi^{\mathbb{T}}_{\beta} \subseteq \Gamma_d$.
- A, B_0, B_1: Γ_d-, $\Sigma^{\mathbb{T}}_{\beta}$-, and $\Pi^{\mathbb{T}}_{\beta}$-complete.
- $A \not\leq_w B_i$ for some $i < 2$, since $\Gamma_d \not\subseteq \Sigma^{\mathbb{T}}_{\beta} \cap \Pi^{\mathbb{T}}_{\beta}$.
- By Wadge's Lemma, $B_{1-i} \equiv_w \neg B_i \leq_w A$.
- Since d is proper, $B_{1-i} \not\equiv_w A$, and thus $A \not\leq_w B_{1-i}$.
- By Wadge's Lemma, $B_i \equiv_w \neg B_{1-i} \leq_w A$.

Lemma

If d is a proper $\mathbb{T}$-Wadge degree, then there is no $f : \omega^\omega \to \mathbb{R}$ such that S_f is Γ_d-complete.

Proof

- $S_f \in \Gamma_d$; Then there is α s.t. $\Delta^{\mathbb{T}}_\alpha \subseteq \Gamma_d \subseteq \Sigma^{\mathbb{T}}_\alpha \cap \Pi^{\mathbb{T}}_\alpha$.
- Claim: $S_f \upharpoonright \langle p, q \rangle$ is $\Delta^{\mathbb{T}}_\alpha$ for any $p < q$.
- A_0, A_1: Σ_α-, and Π_α-complete; $p = p_0 < q_0 < p_1 < q_1 = q$.
- Since $S_f \in \Sigma^{\mathbb{T}}_\alpha \cap \Pi^{\mathbb{T}}_\alpha$, $S_f \upharpoonright \langle p_i, q_i \rangle \leq_w A_i$ via τ_i.
- Define $B_0 = \tau_0^{-1}[\neg A_0]$ and $B_1 = \tau_1^{-1}[A_1]$; these are Π_α.

$$f(X) \leq p_0 \implies X \in B_0 \implies f(X) < q_0.$$
$$f(X) \geq q_1 \implies X \in B_1 \implies f(X) > p_1.$$

- Since $B_0 \cap B_1 = \emptyset$, by the separation property of Π_α, there is a Δ_α set C s.t. $B_1 \subseteq C$ and $B_0 \cap C = \emptyset$.

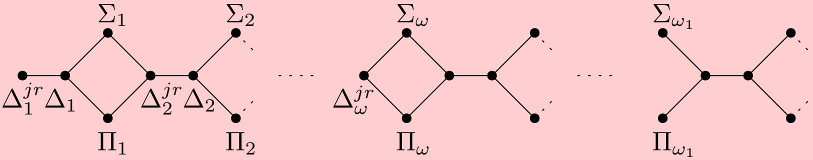
$$f(X) \geq q = q_1 \implies X \in C \implies f(X) > p = p_0.$$

- This means that $S_f \leq_w C$, that is, $S_f \in \Delta^{\mathbb{T}}_\alpha$.

By the previous lemmas, we conclude:

1st Main Theorem (K.)

The structure of the DDW-***m***-degrees of real-valued functions looks like the following figure:



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That is, each successor selfdual Wadge degree splits into two degrees (which are linearly ordered), and other Wadge degrees remain the same.

Historical background on the classification of Baire functions

- Baire (1899): Baire hierarchy.
- Jayne (1974): level (m, n) Baire/Borel functions.
- O'Malley (1977): Baire-one-star functions.
- Császár-Laczkovich (1979): Baire hierarchy by discrete limit.
- Bourgain (1980), many others: Classifying Baire-one functions by Cantor-Bendixson-like ranks, mind-changes, etc.
- Kechris-Louveau (1990): Ranks on Baire-one functions.
- Pawlak 2000 and others: Hierarchy of Baire-one-star functions.
- first level function = $\underset{\sim}{\mathbf{\Delta}}^0_2$-measurable = Baire-one-star = discrete-Baire one
- Weihrauch (around 1990), Hertling, and others, Carroy (2013): Continuous strong Weihrauch reducibility $\leq^c_{sW}$.
  Carroy (2013): CB-rank analysis of functions under $\leq^c_{sW}$.
  The continuous functions with compact domain and countable range form a well-order of type ω_1 under $\leq^c_{sW}$.
- Day-Downey-Westrick (201x):
  $\leq_T$:= Parallel continuous strong Weihrauch reducibility $\leq^c_{\widehat{sW}}$.

Introduction to Martin's conjecture

2nd Main Theorem (K.)

The DDW $\boldsymbol{T}$-degrees $\approx$ the "natural" Turing degrees.

- Natural Solution to Post's Problem:
  Is there a "*natural*" intermediate c.e. Turing degree?
- Natural degrees should be relativizable and degree invariant:
  - (Relativizability) It is a function $f : 2^\omega \to 2^\omega$.
  - (Degree-Invariance) $X \equiv_T Y$ implies $f(X) \equiv_T f(Y)$.
- (Sacks 1963) Is there a degree invariant c.e. operator which always gives us an intermediate Turing degree?
- (Lachlan 1975) There is no uniformly degree invariant c.e. operator which always gives us an intermediate Turing degree.
- (The Martin Conjecture; a.k.a. the $\mathbf{5}^{\text{th}}$ Victoria-Delfino problem)
  - Degree invariant increasing functions are well-ordered,
  - and each successor rank is given by the Turing jump.
- (Cabal) The VD problems **1**-**5** appeared in 1978; the VD problems **6**-**14** in 1988. Only the $\mathbf{5}^{\text{th}}$ and $\mathbf{14}^{\text{th}}$ are still unsolved (the $\mathbf{14}^{\text{th}}$ asks whether $\mathbf{AD}^+ = \mathbf{AD}$).
- (Steel 1982) The Martin Conjecture holds true for uniformly degree invariant functions.

(Hypothesis) Natural degrees are relativizable and degree-invariant.

- $f : 2^\omega \to 2^\omega$ is uniformly degree invariant (UI) if there is a function $u : \omega^2 \to \omega^2$ such that for all $X, Y \in 2^\omega$,

$$X \equiv_T Y \text{ via } (i,j) \implies f(X) \equiv_T f(Y) \text{ via } u(i,j).$$

- $f : 2^\omega \to 2^\omega$ is uniformly order preserving (UOP) if there is a function $u : \omega \to \omega$ such that for all $X, Y \in 2^\omega$,

$$X \leq_T Y \text{ via } e \implies f(X) \leq_T f(Y) \text{ via } u(e).$$

- f is Turing reducible to g on a cone ($f \leq_T^\nabla g$) if

$$(\exists C \in 2^\omega)(\forall X \geq_T C)\ f(X) \leq_T g(X) \oplus C.$$

Theorem (Steel 1982; Slaman-Steel 1988; Becker 1988)

- The $\equiv_T^\nabla$-degree of UI functions form a well-order of length Θ.
- Every UI function is $\equiv_T^\nabla$-equivalent to a UOP function.

Proof of the 2nd Main Theorem

2nd Main Theorem (K.)

The DDW $\boldsymbol{T}$-degrees ≈ the "natural" Turing degrees.

Indeed, the identity map induces an isomorphism between the $\leq_{\boldsymbol{T}}^{\nabla}$-degrees of UOP functions and the DDW $\boldsymbol{T}$-degrees!

Embedding Lemma

Assume that f and g are UOP functions. Then,

$$f \leq_T^{\nabla} g \iff f \leq_{sW}^{c} g \iff f \leq_{W}^{c} \widehat{g}.$$

A uniformly pointed perfect tree (u.p.p. tree) is a perfect tree $T \subseteq 2^{<\omega}$ s.t. $X \oplus T \leq_T T[X]$ uniformly in X, where $T[X]$ is the X-th path through T.

Martin's Cone Lemma (1968)

Any countable partition of 2^{ω} has a part containing a u.p.p. tree.

Proof of $f \leq_T^{\nabla} g \implies f \leq_{sW}^{c} g$

- Assume that $f(X) \leq_T g(X)$ on a cone.
- By MCL, there is a u.p.p. tree T s.t. $f(T[X]) \leq_T g(T[X])$ via a Φ.
- Since f is UOP and $X \leq_T T[X]$ uniformly, $f(X) \leq_T f(T[X])$ via a Ψ.
- Hence, $f = \Psi \circ \Phi \circ g \circ T$.

Embedding Lemma

Assume that f and g are UOP functions. Then,

$$f \leq_T^\nabla g \iff f \leq_{sW}^c g \iff f \leq_W^c \widehat{g}.$$

It remains to show the following:

Surjectivity Lemma

Every function is $\widehat{cW}$-equivalent to a UOP function.

That is, for any $f : 2^\omega \to \mathbb{R}$, there is a UOP function g s.t. $\widehat{f} \equiv_W^c \widehat{g}$.

Lemma (Becker 1988)

For any reasonable pointclass Γ and its indexing U, the pointclass jump J^U_Γ is a UOP jump operator.

Moreover, the $\equiv^\nabla_T$-degree of J^U_Γ is independent of the choice of U.

Lemma (Becker 1988)

Every nonselfdual Wadge class is the relativization of a reasonable pointclass.

Surjectivity Lemma (Nonselfdual)

For any reasonable pointclass Γ,

$$S_f \text{ is } \Gamma\text{-complete} \Longrightarrow \widehat{f} \equiv^c_{sW} \widehat{J_\Gamma}.$$

Corollary

If S_f is nonselfdual, then there is a UOP function g s.t. $\widehat{f} \equiv^c_{sW} \widehat{g}$.

Recall that the cofinality of a selfdual Wadge rank is at most ω.

Lemma (Successor)

If $S_f \in \mathbf{\Delta}_{\alpha+1}$ and S_g is $\mathbf{\Sigma}_\alpha$-complete, then $f \leq^c_{sW} \widehat{g}$.

Lemma (Limit of cofinality ω)

If α is a limit ordinal of cofinality ω, then there is a UOP function g such that $\widehat{f} \equiv^c_{sW} \widehat{g}$.

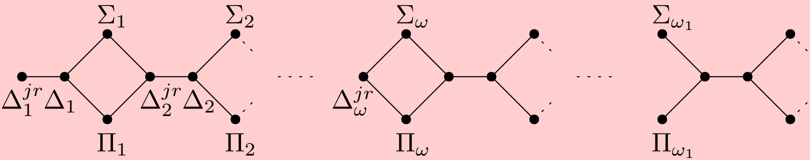
Consequently,

Surjectivity Lemma

Every $\mathbb{R}$-valued function is $\widehat{cW}$-equivalent to a UOP function.

1st Main Theorem (K.)

The structure of the DDW-***m***-degrees of real-valued functions looks like the following figure:



2nd Main Theorem (K.)

The DDW ***T***-degrees ≈ the “natural” Turing degrees.

Indeed, the identity map induces an isomorphism between the $\leq_T^\nabla$-degrees of UOP functions and the DDW ***T***-degrees!