The Uniform Martin Conjecture and Wadge Degrees

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Main Theorem (K. and Montalbán)

(**AD$^+$**) Let Q be BQO. There is an isomorphism between

the "*natural*" many-one degrees of Q-valued functions on ω

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The assumption **AD**$^+$ can be slightly weakened as:

ZF + **DC** + **AD**+"All subsets of ω^ω are completely Ramsey (that is, every subset of ω^ω has the Baire property w.r.t. the Ellentuck topology)".

Definition

1. Let $A, B \subseteq \omega$. A is many-one reducible to B if there is a computable function $\Phi : \omega \to \omega$ such that
$$(\forall n \in \omega)\ n \in A \iff \Phi(n) \in B.$$
2. Let $A, B \subseteq \omega^\omega$. A is Wadge reducible to B if there is a continuous function $\Psi : \omega^\omega \to \omega^\omega$ such that
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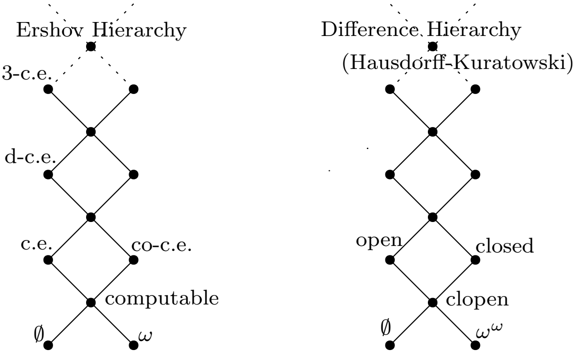
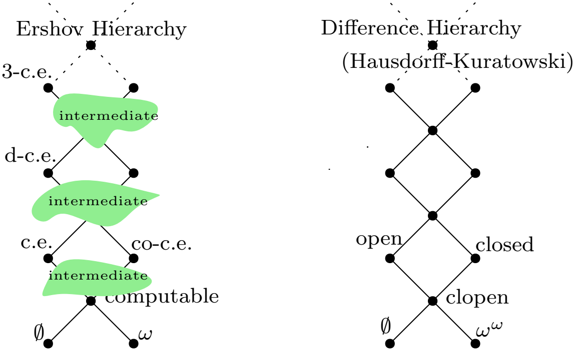


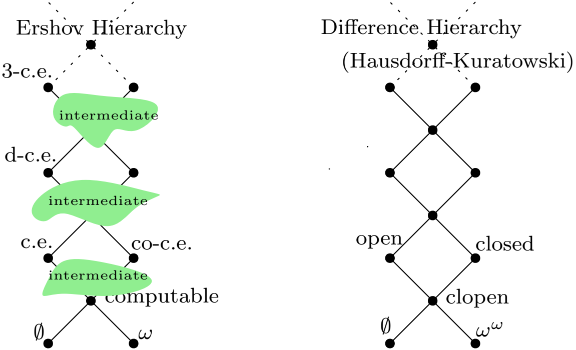
Many-one degrees versus Wadge degrees



Many-one degrees versus Wadge degrees

The structure of the many-one degrees is very complicated:

- There are continuum-size antichains, every countable distributive lattice is isomorphic to an initial segment, etc.
- (Nerode-Shore 1980) The theory of the many-one degrees is computably isomorphic to the true second-order arithmetic.



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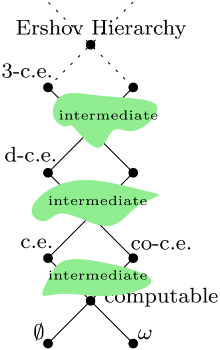

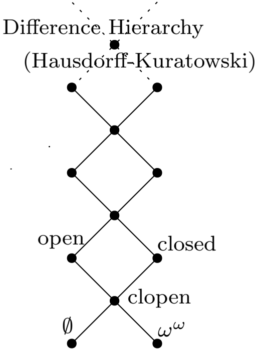

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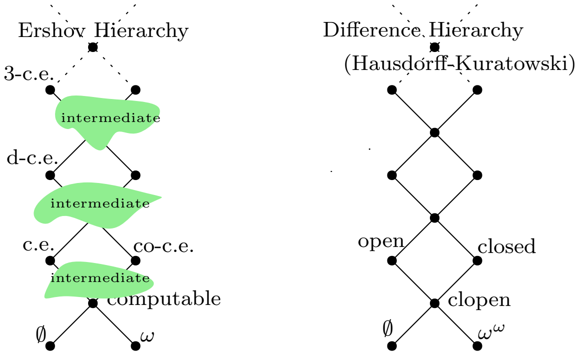

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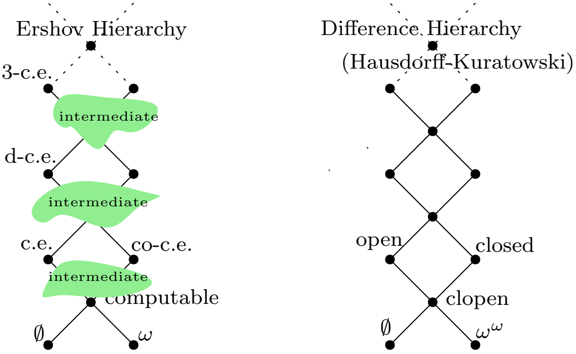

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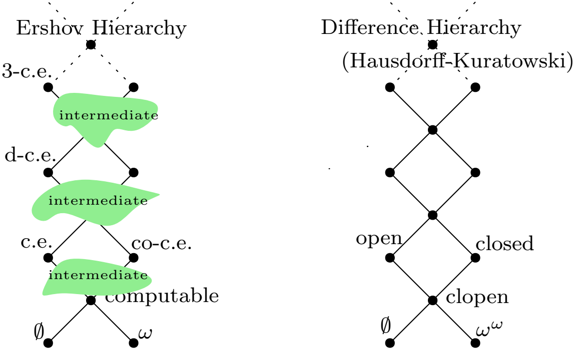

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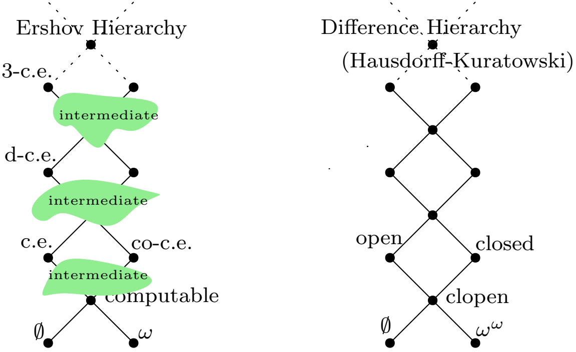

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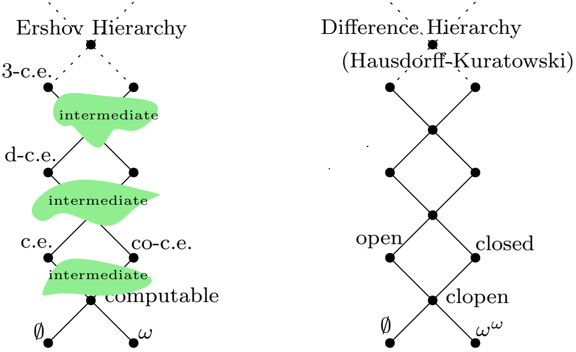

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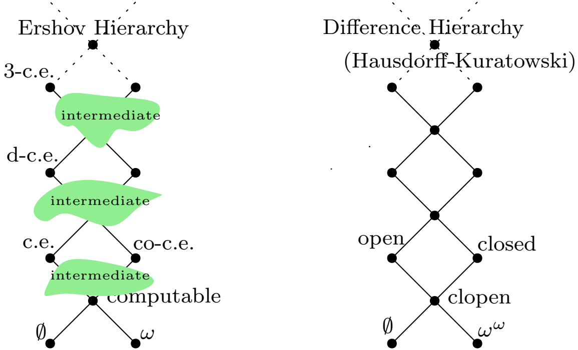

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  In other words, it is induced by a homomorphism f from $\equiv_T$ to $\equiv_T$, that is, $X \equiv_T Y$ implies $f(X) \equiv_T f(Y)$.
- (AD) The Martin measure μ is defined on $\equiv_T$-invariant sets in 2^ω by:

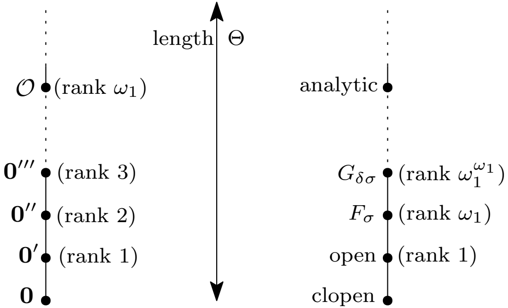
$$\mu(A) = \begin{cases} 1 & \text{if } (\exists x)(\forall y \geq_T x)\ y \in A, \\ 0 & \text{otherwise.} \end{cases}$$

- For homomorphisms f, g from $\equiv_T$ to $\equiv_T$, define

$$f \leq_T^\nabla g \iff f(x) \leq_T g(x),\ \mu\text{-a.e.}$$

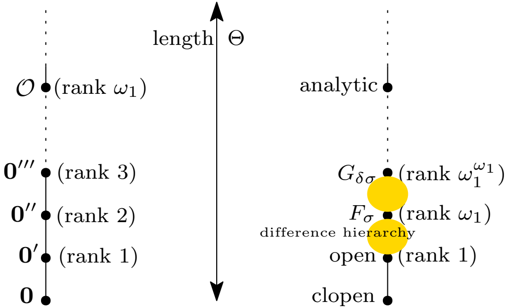
The Martin Conjecture (1960's)

1. For every homomorphism f from $\equiv_T$ to $\equiv_T$
   - either f maps a μ-conull set into a single $\equiv_T$-class
   - or f is increasing, that is, $f(x) \geq_T x$, μ-a.e.
2. The increasing homomorphisms from $\equiv_T$ to $\equiv_T$ are
   - well-ordered by $\leq_T^\nabla$,
   - and each successor rank is given by the Turing jump.



Natural Turing degrees and Wadge degrees

- (Steel, Slaman-Steel 80's) The Martin conjecture is true for uniform homomorphisms!
- In particular, increasing uniform homomorphisms are well-ordered, and each successor rank is given by the Turing jump.
- (Becker 1988) Indeed, increasing uniform homomorphisms form a well-order of type $\mathbf{\Theta}$.



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(Hypothesis) Natural degrees are induced by homomorphisms.

Definition

$f : 2^\omega \to 2^\omega$ is a uniform homomorphism from $\equiv_T$ to $\equiv_m$ (abbreviated as $(\equiv_T, \equiv_m)$-UH) if there is a function $u : \omega^2 \to \omega^2$ such that for all $X, Y \in 2^\omega$,

$$X \equiv_T Y \text{ via } (i, j) \implies f(X) \equiv_m f(Y) \text{ via } u(i, j).$$

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Definition

Given $f, g\colon 2^\omega \to 2^\omega$, we say that f is many-one reducible to g on a cone (written as $f \leq_m^\nabla g$) if

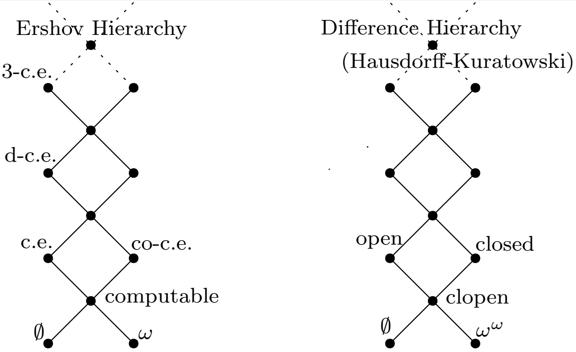
$$(\exists C \in 2^\omega)(\forall X \geq_T C)\ f(X) \leq_m^C g(X).$$

Here $\leq_m^C$ is many-one reducibility relative to C.

Theorem (K. and Montalbán)

($\mathbf{ZF} + \mathbf{DC}_{\mathbb{R}} + \mathbf{AD}$) The $\equiv^{\nabla}_{\mathbf{m}}$-degrees of uniform homomorphisms from $\equiv_T$ to $\equiv_m$ are isomorphic to the Wadge degrees.

(Cor.) The $\equiv^{\nabla}_{\mathbf{m}}$-degrees of $(\equiv_T, \equiv_m)$-UHs form a semi-well-order.



Natural many-one degrees $\simeq$ Wadge degrees

Generalize our result to Q-valued functions for any better-quasi-order (BQO) Q.

Definition

Let Q be a quasi-order.

1. Let $A, B : \omega \to Q$. A is many-one reducible to B if there is a computable function $\Phi : \omega \to \omega$ such that
$$(\forall n \in \omega)\ A(n) \leq_Q B \circ \Phi(n).$$
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What is the motivation of thinking about Q-valued functions?

Theorem (Marks)

1. The many-one equivalence on **2**-valued functions is not a uniformly universal countable Borel equivalence relation.
2. The many-one equivalence on **3**-valued functions is a uniformly universal countable Borel equivalence relation.

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In particular, $\equiv_T$ is uniformly Borel reducible to $\equiv_m$ on $\mathbf{3}^\omega$.
Such a reduction has to be *uniform homomorphism from $\equiv_T$ to $\equiv_m$*!

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Such a reduction has to be *uniform homomorphism from* $\equiv_T$ *to* $\equiv_m$!

Our earlier motivation was to understand why $\mathbf{2} \neq \mathbf{3}$...

- We had conjectured that the structure of natural m-degrees on $\mathbf{2}^\omega$ is too simple to be uniformly universal, while that on $\mathbf{3}^\omega$ has to be sufficiently complicated to be uniformly universal.
- However, we eventually concluded that both structures are very very simple!

Theorem (K. and Montalbán)

(**AD**$^+$) Let Q be BQO.

The $\equiv_{\mathbf{m}}^{\nabla}$-degrees of uniform hom. from $(2^\omega; \equiv_T)$ to $(Q^\omega; \equiv_m)$

are isomorphic to

the Wadge degrees of Q-valued functions on ω^ω.

(Woodin) **AD**$^+$ $=$ **DC**$_{\mathbb{R}}$ $+$ "every set of reals is ∞-Borel" $+$ "$<\Theta$-Ordinal Determinacy".

The assumption **AD**$^+$ can be slightly weakened as:

ZF $+$ **DC** $+$ **AD**$+$"All subsets of ω^ω are completely Ramsey"

(every subset of ω^ω has the Baire property w.r.t. the Ellentuck topology).

Natural Q-many-one degrees = Q-Wadge degrees.

Natural $\mathcal{Q}$-many-one degrees = $\mathcal{Q}$-Wadge degrees.

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How does the structure of $\mathcal{Q}$-Wadge degrees look like?

Natural Q-many-one degrees $=$ Q-Wadge degrees.

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How does the structure of Q-Wadge degrees look like?

- **Tree**(S): The set of all S-labeled well-founded countable trees.
- $^{\sqcup}$**Tree**(S): The set of all forests written as a countable disjoint union of trees in **Tree**(S).

Theorem (extending Duparc's and Selivanov's works)

Let Q be a BQO.

- The Q-Wadge degrees of $\underset{\sim}{\mathbf{\Delta}}^0_2$-functions $\simeq$ $^{\sqcup}\mathbf{Tree}(Q)$.
- The Q-Wadge degrees of $\underset{\sim}{\mathbf{\Delta}}^0_3$-functions $\simeq$ $^{\sqcup}\mathbf{Tree}(\mathbf{Tree}(Q))$.
- The Q-Wadge degrees of $\underset{\sim}{\mathbf{\Delta}}^0_4$-functions $\simeq$ $^{\sqcup}\mathbf{Tree}(\mathbf{Tree}(\mathbf{Tree}(Q)))$.
- The Q-Wadge degrees of $\underset{\sim}{\mathbf{\Delta}}^0_5$-functions $\simeq$ $^{\sqcup}\mathbf{Tree}(\mathbf{Tree}(\mathbf{Tree}(\mathbf{Tree}(Q))))$.
- and so on... (similar results hold for all transfinite ranks)

The Wadge degree of a $\mathcal{Q}$-valued $\underset{\sim}{\mathbf{\Delta}}^0_\omega$-function (hence the $\boldsymbol{m}$-degree of a $\mathcal{Q}$-valued natural $\mathbf{\Delta}^0_\omega$-function) can be described by a *term* in the language consisting of:

1. Constant symbols $\boldsymbol{q}$ (for $\boldsymbol{q} \in \mathcal{Q}$).
2. A $\mathbf{2}$-ary function symbol $\rightarrow$.
3. An ω-ary function symbol $\sqcup$.
4. A unary function symbol $\langle \cdot \rangle$.

We need additional function symbols $\langle\cdot\rangle^{\omega^\alpha}$ to represent all Borel Wadge degrees.

Example

1. The term $\mathbf{0}^{\rightarrow}\mathbf{1}$ represents open sets (c.e. sets).
2. The term $\mathbf{1}^{\rightarrow}\mathbf{0}$ represents closed sets (co-c.e. sets).
3. The term $\mathbf{0} \sqcup \mathbf{1}$ represents clopen sets (computable sets).
4. The term $\mathbf{0}^{\rightarrow}\mathbf{1}^{\rightarrow}\mathbf{0}$ represents differences of two open sets ($\boldsymbol{d}$-c.e. sets).
5. The term $\langle\mathbf{0}^{\rightarrow}\mathbf{1}\rangle$ represents $\boldsymbol{F_\sigma}$ sets ($\boldsymbol{\emptyset'}$-c.e. sets).

Definition

For a term T, define the class Σ_T of functions as follows:

1. Σ_q consists only of the constant function $x \mapsto q$.
2. $f \in \Sigma_{\sqcup_i S_i}$ iff there is a clopen partition $(C_i)_{i\in\omega}$ of ω^ω such that $f \upharpoonright C_i$ is in Σ_{S_i}.
3. $f \in \Sigma_{S\to T}$ iff there is an open set $U \subseteq \omega^\omega$ such that $f \upharpoonright U$ is in Σ_T and $f \upharpoonright (\omega^\omega \setminus U)$ is in Σ_S.
4. $f \in \Sigma_{\langle T\rangle}$ iff it is decomposed as $f = g \circ h$, where g is in Σ_T and h is Baire-one.

1. $\Sigma_{0\to 1} = \utilde{\Sigma}^0_1$, $\Sigma_{1\to 0} = \utilde{\Pi}^0_1$, and $\Sigma_{0\sqcup 1} = \utilde{\Delta}^0_1$.
2. $\Sigma_{0\to 1\to 0}$ = differences of $\utilde{\Sigma}^0_1$ sets.
3. $\Sigma_{\langle 0\to 1\rangle} = \utilde{\Sigma}^0_2$, and $\Sigma_{\langle 1\to 0\rangle} = \utilde{\Pi}^0_2$.
4. No term corresponds to $\utilde{\Delta}^0_2$ (this reflects the fact that there is no $\utilde{\Delta}^0_2$-complete set; $\utilde{\Delta}^0_2$ is divided into unbounded ω_1-many Wadge degrees).

We define a quasi-order $\trianglelefteq$ on terms, which is shown to be isomorphic to the Wadge degrees of finite Borel rank.

Definition of $\trianglelefteq$

We inductively define a quasi-order $\trianglelefteq$ on terms as follows:

$$p \trianglelefteq q \iff p \leq_Q q,$$
$$\langle U \rangle \trianglelefteq \langle V \rangle \iff U \trianglelefteq V,$$

and if S and T are of the form $\langle U \rangle^{\rightarrow} \bigsqcup_i S_i$ and $\langle V \rangle^{\rightarrow} \bigsqcup_j T_j$, then

$$S \trianglelefteq T \iff \begin{cases} (\forall i)\ S_i \trianglelefteq T & \text{if } \langle U \rangle \trianglelefteq \langle V \rangle, \\ (\exists j)\ S \trianglelefteq T_j & \text{if } \langle U \rangle \ntrianglelefteq \langle V \rangle. \end{cases}$$

We can extend this quasi-order $\trianglelefteq$ to terms in the extended language (which has additional function symbols $\langle \cdot \rangle^{\omega^\alpha}$ representing transfinite nests of trees).

This extended version is shown to be isomorphic to the Wadge degrees of all Borel functions.

Theorem (K. and Montalbán)

(ZFC) Let Q be BQO. The following structures are all isomorphic:

1. The $\equiv^{\nabla}_{\mathbf{m}}$-degrees of $\underset{\sim}{\mathbf{\Delta}}^0_{1+\xi}$-measurable $(\equiv_T, \equiv_m)$-uniform homomorphisms from $(2^\omega; \equiv_T)$ to $(Q^\omega; \equiv_m)$.
2. The Wadge degrees of Q-valued $\underset{\sim}{\mathbf{\Delta}}^0_{1+\xi}$-measurable functions.
3. $({}^{\sqcup}\mathrm{Tree}^{\xi}(Q), \trianglelefteq)$.

(Very very rough idea of) proof

- (1) $\iff$ (2): Block's recent work on "very strong BQO" + Game-theoretic argument + degree-theoretic analysis of thin $\mathbf{\Pi}^0_1$ classes.
- (2) $\iff$ (3): Introduce an operation which bridges $\underset{\sim}{\mathbf{\Delta}}^0_n$ and $\underset{\sim}{\mathbf{\Delta}}^0_{n+1}$ by using Montalbán's recent notion of "the *jump operator via true stages*", and then apply the Friedberg jump inversion theorem.

Theorem (K. and Montalbán [1])

1. ($\mathbf{AD} + \mathbf{DC}_{\mathbb{R}}$) There is an isomorphism between the $\equiv^{\nabla}_{\mathbf{m}}$-degrees of UH decision problems and the Wadge degrees of subsets of ω^{ω}.
2. ($\mathbf{AD}^{+}$) For any BQO $\mathcal{Q}$, there is an isomorpshim between the $\equiv^{\nabla}_{\mathbf{m}}$-degrees of UH $\mathcal{Q}$-valued problems and the Wadge degrees of $\mathcal{Q}$-valued functions on ω^{ω}.

AD = The Axiom of Determinacy (every set of reals is determined).

$\mathbf{DC}_{\mathbb{R}}$ = The Dependent Choice on $\mathbb{R}$.

$\mathbf{AD}^{+}$ = $\mathbf{DC}_{\mathbb{R}}$ + "every set of reals is ∞-Borel" + "< Θ-Ordinal Determinacy".

Theorem (K. and Montalbán [2])

$$(\underset{\sim}{\mathbf{\Delta}}^{0}_{1+\xi}\text{-UH}(\omega^{\omega}, \mathcal{Q}^{\omega}), \leq^{\nabla}_{\mathbf{m}}) \simeq (\underset{\sim}{\mathbf{\Delta}}^{0}_{1+\xi}(\omega^{\omega}, \mathcal{Q}), \leq_{\mathbf{w}}) \simeq ({}^{\sqcup}\mathbf{Tree}^{\xi}(\mathcal{Q}), \trianglelefteq).$$

[1] T. Kihara and A. Montalbán, The uniform Martin's conjecture for many-one degrees, submitted (arXiv:1608.05065).

[2] T. Kihara and A. Montalbán, On the structure of the Wadge degrees of BQO-valued Borel functions, in preparation.

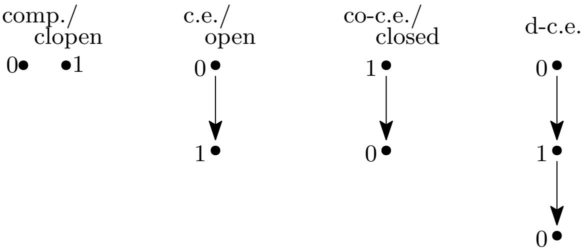
Appendix

Let Q be a quasi-order.

1. Q is a well-quasi-order (WQO) if it has no infinite decreasing seq. and no infinite antichain. It is equivalent to saying that
$$(\forall f : \omega \to Q)(\exists m < n)\ f(m) \leq_Q f(n).$$
2. (Nash-Williams 1965) Q is a better-quasi-order (BQO) if
$$(\forall f : [\omega]^\omega \to Q \text{ continuous})(\exists X \in [\omega]^\omega)\ f(X) \leq_Q f(X^-).$$
where X^- is the shift of X, that is, $X^- = X \setminus \{\min X\}$.

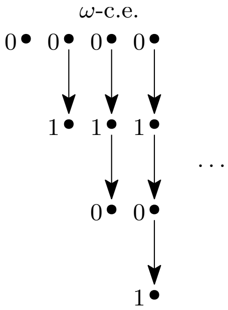
BQO $\Longrightarrow$ WQO.
(Example) A finite quasi-order is a BQO. A well-order is a BQO.

Tree/Forest-representation of various Δ^0_2 sets:



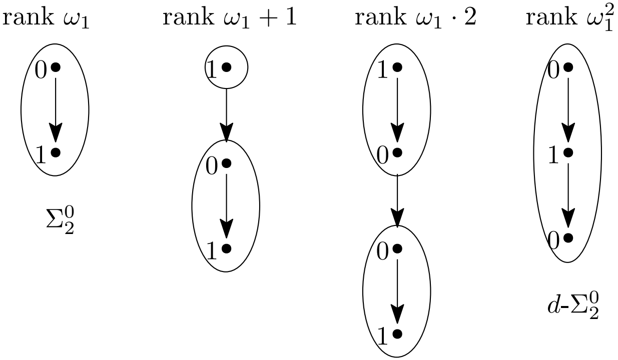
- (computable/clopen) Given an input x, effectively decide $x \notin A$ (indicated by 0) or $x \in A$ (indicated by 1).
- (c.e./open) Given an input x, begin with $x \notin A$ (indicated by 0) and later x can be enumerated into A (indicated by 1).
- (co-c.e./closed) Given an input x, begin with $x \in A$ (indicated by 1) and later x can be removed from A (indicated by 0).
- (d-c.e.) Begin with $x \notin A$ (indicated by 0), later x can be enumerated into A (indicated by 1), and x can be removed from A again (indicated by 0).

Forest-representation of a complete ω-c.e. set:



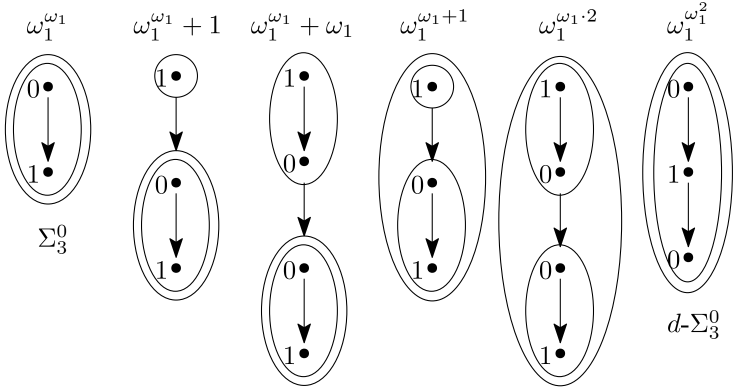
(ω-c.e.) The representation of "ω-c.e." is a forest consists of linear orders of finite length (a linear order of length $n + 1$ represents "n-c.e.").

- Given an input x, effectively choose a number $n \in \omega$ giving a bound of the number of times of mind-changes until deciding $x \in A$.



Tree/Forest-representation of $\underset{\sim}{\mathbf{\Delta}}^0_3$ sets

The Wadge degrees of $\underset{\sim}{\mathbf{\Delta}}^0_3$ sets are exactly those represented by **forests labeled by trees**.



Tree/Forest-representation of $\underset{\sim}{\mathbf{\Delta}}_4^0$ sets

The Wadge degrees of $\underset{\sim}{\mathbf{\Delta}}_4^0$ sets are exactly those represented by

forests labeled by trees which are labeled by trees.

Definition

1. We say that $A \subseteq [\omega]^\omega$ is Ramsey if there is $X \in [\omega]^\omega$ such that either $[X]^\omega \subseteq A$ or $[X]^\omega \cap A = \emptyset$.
2. **Γ-Det** is the hypothesis "*every* **Γ** *set of reals is determined*".
3. **Γ-Ramsey** is the hypothesis "*every* **Γ** *set of reals is Ramsey*".

Remark

What we really need is the hypothesis

"every **Γ** set of reals is completely Ramsey"

(i.e., every **Γ** set has the Baire property w.r.t. Ellentuck topology)

but for most natural pointclasses **Γ**, this hypothesis is known to be equivalent to **Γ-Ramsey** (Brendle-Löwe (1999)).

Definition

1. We say that $A \subseteq [\omega]^\omega$ is Ramsey if there is $X \in [\omega]^\omega$ such that either $[X]^\omega \subseteq A$ or $[X]^\omega \cap A = \emptyset$.
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3. **Γ-Ramsey** is the hypothesis "*every* Γ *set of reals is Ramsey*".

- (Martin 1975) **ZF** + **DC** ⊢ **Borel-Det**.
- (Galvin-Prikry 1973; Silver 1970) **ZF** + **DC** ⊢ $\boldsymbol{\Sigma}^1_1$-**Ramsey**.
- (Harrington-Kechris 1981) **PD** implies **Projective-Ramsey**.
  - Indeed, they showed that $\boldsymbol{\Delta}^1_{2n+2}$-**Det** implies $\boldsymbol{\Pi}^1_{2n+2}$-**Ramsey**.
  - (Fang-Magidor-Woodin 1992) $\boldsymbol{\Sigma}^1_1$-**Det** implies $\boldsymbol{\Sigma}^1_2$-**Ramsey**.
- (Open Problem) Does **AD** imply that every set of reals is Ramsey?
- (Solovay; Woodin) **AD**$^+$ implies that every set of reals is Ramsey.
  - **AD**$^+$ = **DC**$_\mathbb{R}$ + "every set of reals is ∞-Borel" + "$< \Theta$-Ordinal Determinacy".

Why **Γ-Ramsey**? Because we need the following lemma:

Lemma (ZF + $DC_{\mathbb{R}}$ + Γ-Det + Γ-Ramsey)

Let Q be a BQO.

1. The Q-Wadge degrees of **Γ**-functions form a BQO.
2. A Q-Wadge degree of **Γ**-functions is self-dual if and only if it is σ-join-reducible.

Proof

1. Louveau-Simpson (1982) showed that if a function f from $[\omega]^\omega$ into a metric space has the Baire property w.r.t. Ellentuck topology, then there is an infinite set X such that the restriction $f \upharpoonright [X]^\omega$ is continuous w.r.t. Baire topology. Combine this result with van Engelen-Miller-Steel (1987).
2. For $Q = (2, =)$, it has been shown by Steel-van Wesep (1978) (without **Γ-Ramsey**). Recently Block (2014) introduced the notion of vsBQO and extended the Steel-van Wesep Theorem to vsBQO. Analyze Block's proof, and combine it with Louveau-Simpson (1982).