Topos-theoretic aspect of the degrees of unsolvability

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- Hyland (1982) introduced the *effective topos* $\mathbf{Eff}$, "*the world of computable mathematics*."
- $\mathbf{Eff}$ has the smallest subtopos (the degenerated topos), "*the world of inconsistent mathematics*."
- $\mathbf{Eff}$ has the second smallest subtopos $\mathbf{Set}$, "*the world of set-theoretic mathematics*."
- For each $\alpha \in 2^\omega$ Hyland found a subtopos $\mathbf{Eff}[\alpha]$ of $\mathbf{Eff}$, "*the world of α-relatively computable mathematics*."

$$\alpha \leq_T \beta \iff \mathbf{Eff}[\beta] \text{ is a subtopos of } \mathbf{Eff}[\alpha]$$

- (Idea) smaller topos ≈ stronger theory
- Let us examine the structure of all subtoposes of a topos!
  - ▷ non-degenerated subtoposes of $\mathbf{Eff}$ ≈ all worlds between "*the computable world*" and "*the set-theoretic world*".
- (Key Idea) "a subtopos of $\mathbf{Eff}$" ≈ "$\mathbf{Eff}$ relative to an oracle".
  - ▷ An oracle changes a world / model / semantics.

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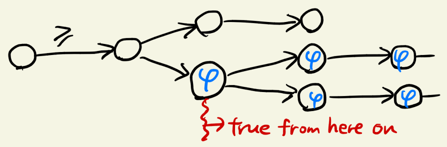
Kleene's realizability interpretation (1945)

- A proof of $A \wedge B$ is a pair of proofs of A and B.
- A proof of $A \vee B$ is a pair of a tag indicating which of A or B is correct and a proof of the formula for the correct side.
- A proof of $A \to B$ is (a code of) a computable function that, given a proof of A, outputs a proof of B .
- A proof of $\exists x \in I.\, A(x)$ is a pair of a code of a witness $c \in I$ of the existence and a proof of the formula $A(c)$.
- A proof of $\forall x \in I.\, A(x)$ is (a code of) a computable function that, given a code of an element $c \in I$, outputs a proof of $A(c)$.

- This interpretation can obviously be made relative to an oracle.
  - ▷ Given an oracle α, replace "computable" with "α-computable".
  - ▷ An oracle α is not necessarily single-valued; e.g. Lifschitz realizability (realizability relative to Π^0_1 classes)
  - ▷ An oracle changes semantics.
- Factors causing changes in semantics:
  - ▷ Coverage: factor causing changes in Kripke semantics
  - ▷ Oracle: factor causing changes in realizability interpretation

Kripke Semantics for Intuitionistic Logic

- Intuitionistic Kripke model is a preorder $(P, \leq)$ with an assignment of which atomic propositions φ are valid at which positions $p \in P$.
  - If φ is true at $p \in P$ then we write $p \Vdash \varphi$.
  - Moreover we assume that $q \leq p \Vdash \varphi$ implies $q \Vdash \varphi$.

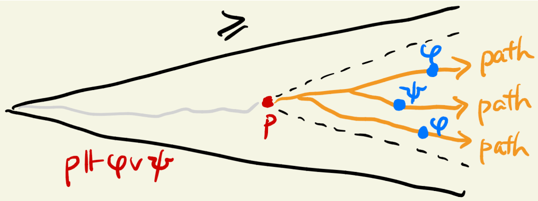


In intuitionistic model, in order to claim that *$\varphi \vee \psi$ is valid at a position $p \in P$*, one must determine whether φ or ψ is valid at the position p.

- $p \Vdash \varphi \wedge \psi \iff p \Vdash \varphi$ and $p \Vdash \psi$.
- $p \Vdash \varphi \vee \psi \iff p \Vdash \varphi$ or $p \Vdash \psi$.
- $p \Vdash \varphi \rightarrow \psi \iff (\forall q \leq p)\ [q \Vdash \varphi$ implies $q \Vdash \psi]$.

Beth Semantics

- To deepen our understanding of models of semi-constructive mathematics, it is also useful to have a model that
  - ▷ does not "*immediately decide which is valid*"
  - ▷ but rather "*postpones the decision of which is valid*".
- In the Beth model, to assert that "$\varphi \vee \psi$ is valid at position p" is to know that "*no matter what path α we take beyond position p, at some point along α either φ or ψ will be determined to be valid*".



- $p \Vdash \varphi \wedge \psi \iff p \Vdash \varphi$ and $p \Vdash \psi$.
- $p \Vdash \varphi \vee \psi \iff (\forall \alpha \ni p \text{ path})(p \geq \exists q \in \alpha)\ [q \Vdash \varphi \text{ or } q \Vdash \psi]$.
- $p \Vdash \varphi \to \psi \iff (\forall q \leq p)\ [q \Vdash \varphi \text{ implies } q \Vdash \psi]$.

Beth Semantics

- $p \Vdash \varphi \wedge \psi \iff p \Vdash \varphi$ and $p \Vdash \psi$.
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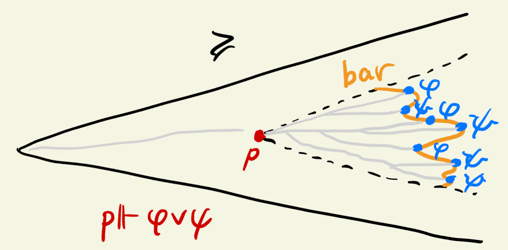
- $p \Vdash \varphi \to \psi \iff (\forall q \leq p)\ [q \Vdash \varphi \text{ implies } q \Vdash \psi]$.

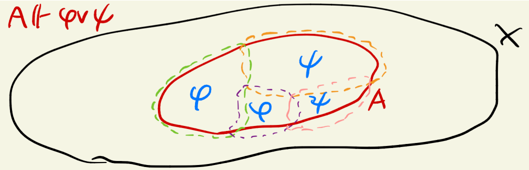
(Observation) $p \Vdash \varphi \vee \psi \iff (\exists B \text{ bar for } p)(\forall b \in B)\ [b \Vdash \varphi \text{ or } b \Vdash \psi]$



Covering Semantics

- There is also a type of model that asserts that *"$\varphi \vee \psi$ is globally valid"* if it is *locally determined whether φ or ψ is valid*.
- E.g., for a topological space, use the complete lattice $(\mathcal{O}(X), \subseteq)$ of open sets in X as a base of Kripke-like model.

- $A \Vdash \varphi \wedge \psi \iff A \Vdash \varphi$ and $A \Vdash \psi$.
- $A \Vdash \varphi \vee \psi \iff (\exists \mathcal{U}$ open cover of $A)(\forall V \in \mathcal{U})\ V \Vdash \varphi$ or $V \Vdash \psi$.
- $A \Vdash \varphi \to \psi \iff (\forall B \subseteq A)\ [B \Vdash \varphi$ implies $B \Vdash \psi]$.



In general, for a complete lattice L, a *cover* of $a \in L$ is a set $U \subseteq L$ s.t. $a \leq \bigvee U$.

(Weak) Forcing Semantics

- Another type of model, with slightly looser conditions than Beth semantics, requires that *the truth need not be determined along all paths, but only that it should be determined along any generic path*.

- $p \Vdash \varphi \wedge \psi \iff p \Vdash \varphi$ and $p \Vdash \psi$.
- $p \Vdash \varphi \vee \psi \iff (\forall q \leq p)(\exists r \leq q)\ [r \Vdash \varphi$ or $r \Vdash \psi]$.
- $p \Vdash \varphi \to \psi \iff (\forall q \geq p)\ [q \Vdash \varphi$ implies $q \Vdash \psi]$.

(Observation) $p \Vdash \varphi \vee \psi \iff (\exists D$ dense below $p)(\forall q \in D)\ [q \Vdash \varphi$ or $q \Vdash \psi]$

Unifying Various Kripke-like Semantics

There are many different definitions of $a \Vdash \varphi \vee \psi$:

- (Kripke) $a \Vdash \varphi$ or $a \Vdash \psi$.
- (Beth) $(\exists B$ bar for $a)(\forall b \in B)$ $[b \Vdash \varphi$ or $b \Vdash \psi]$.
- (Covering) $(\exists U$ cover of $a)(\forall b \in U)$ $[b \Vdash \varphi$ or $b \Vdash \psi]$.
- (Forcing) $(\exists D$ dense below $a)(\forall b \in D)$ $[b \Vdash \varphi$ or $b \Vdash \psi]$.

All these examples can be unified by giving some assignment $a \mapsto J_a$:

- $(\exists V \in J_a)(\forall b \in V)$ $[b \Vdash \varphi$ or $b \Vdash \psi]$.

▷ Without loss of generality, one may assume that J_a is downward closed.

$$J_a^{Kr} = \{\downarrow a\}; \qquad J_a^{Be} = \{\downarrow B : B \text{ bar for } a\};$$
$$J_a^{Cov} = \{\downarrow U : U \text{ cover of } a\}; \qquad J_a^{Fo} = \{\downarrow D : D \text{ dense below } a\}.$$

If an assignment $a \mapsto J_a$ satisfies a certain condition,
then it is called a *Grothendieck topology* (on an underlying poset).

(Example) $J^{Kr}, J^{Be}, J^{Cov}, J^{Fo}$ are Grothendieck topologies.

Table of contents:

$j(\mathcal{U}) := \{A : A \subseteq \bigcup \mathcal{U}\}$ "the collection of all sets covered by $\mathcal{U}$"

1. (inflationary) If $A \in \mathcal{U}$ then A is covered by $\mathcal{U}$:
$$\mathcal{U} \subseteq j(\mathcal{U})$$
2. (downward closed) For $A \subseteq B$, if B is covered by $\mathcal{U}$, then so is A:
$$A \subseteq B \in j(\mathcal{U}) \implies A \in j(\mathcal{U})$$
3. (monotone) For $\mathcal{U} \subseteq \mathcal{V}$, if $\mathcal{U}$ covers A, then so does $\mathcal{V}$:
$$\mathcal{U} \subseteq \mathcal{V} \implies j(\mathcal{U}) \subseteq j(\mathcal{V})$$
4. (idempotent) If A is covered by $\mathcal{U}$, and every $B \in \mathcal{U}$ is covered by $\mathcal{V}$, then A is covered by $\mathcal{V}$:
$$j \circ j(\mathcal{U}) \subseteq j(\mathcal{U})$$
5. (local) If $\mathcal{U}$ covers A, then so does $\mathcal{U} \upharpoonright A := \{U \cap A : U \in \mathcal{U}\}$. Indeed, if $\mathcal{U}$ covers $A \in \mathcal{V}$, then so does $\{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$:
$$j(\mathcal{U}) \cap \mathcal{V} = j(\mathcal{U} \cap \mathcal{V}) \cap \mathcal{V} \quad \text{(for downward closed } \mathcal{U}, \mathcal{V}\text{)}$$

- (inflationary) + (monotone) + (idempotent) = a closure operator.
- A coverage ≈ a local closure operator on downward closed sets.

The downward closed sets form a complete Heyting algebra under $\subseteq$.

- ▷ Let Ω be a complete Heyting algebra:

A map $j\colon \Omega \to \Omega$ is *nucleus* if:

1. (monotone) $x \leq y \implies j(x) \leq j(y)$.
2. (inflationary) $x \leq j(x)$.
3. (idempotent) $j \circ j(x) \leq j(x)$.
4. (local) $j(x) \wedge y = j(x \wedge y) \wedge y$.

- A nucleus is a local closure operator on Ω.
- (local) $\iff$ $(x \leftrightarrow y) \leq (j(x) \leftrightarrow j(y))$
- (mon.) + (local) = (locally monotone) $(x \to y) \leq (j(x) \to j(y))$
  - ▷ (∴) nucleus $\iff$ (loc. mon.) + (infl.) + (idem.)
- nucleus $\implies$ ($\wedge$-preserving) $j(x \wedge y) = j(x) \wedge j(y)$
  - ▷ (∴) A nucleus is a $\wedge$-preserving closure operator on Ω.

- Kripke semantics:
  the semantics of the topos $\mathbf{Set}^{P^{op}}$ of presheaves over a poset P.
- A nucleus j on the downward closed sets Ω in P
  $\approx$ a Grothendieck topology J on P.
- The collection of J-sheaves over P again forms a topos $\mathbf{Sh}_J(P)$, which is a subtopos of $\mathbf{Set}^{P^{op}}$, and indeed:
  - ▷ A subtopos of $\mathbf{Set}^{P^{op}}$ $\approx$ a nucleus on Ω $\approx$ a Gro. topology on P
- Kripke semantics relative to coverage / nucleus / Gro. topology J (a.k.a. Kripke-Joyal semantics):
  the semantics of the topos $\mathbf{Sh}_J(P)$ of J-sheaves over a poset P.
- A Grothendieck topology (coverage) is a factor that causes changes in a presheaf topos.
- What is a factor that causes changes in a topos other than a presheaf topos?
  - ▷ It is a Lawvere-Tierney topology.
  - ▷ A subtopos of a topos $\mathcal{E}$ $\approx$ a Lawvere-Tierney topology on $\mathcal{E}$

A map $j \colon \Omega \to \Omega$ on a complete Heyting algebra Ω is *nucleus* if:

1. (locally monotone) $x \to y \leq j(x) \to j(y)$.
2. (inflationary) $x \leq j(x)$.
3. (idempotent) $j \circ j(x) \leq j(x)$.

A Grothendieck topology on a poset $P \approx$ A nucleus on downsets in P $\approx$ A Lawvere-Tierney topology on the presheaf topos $\mathbf{Set}^{P^{op}}$.

An Lawvere-Tierney topology on $\mathbf{Eff}$ can be explicitly described as:

A map $j \colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ is a *Lawvere-Tierney topology* on $\mathbf{Eff}$ if all of the following formulas are realizable (uniformly in x, y):

1. (locally monotone) $(x \to y) \to (j(x) \to j(y))$.
2. (inflationary) $x \to j(x)$.
3. (idempotent) $j \circ j(x) \to j(x)$.

- This notion should give all the subtoposes of $\mathbf{Eff}$.
- How on earth does this notion relate to oracles?

Table of contents:

1. Kripke-like semantics
2. Grothendieck / Lawvere-Tierney topology
3. Oracle vs. LT-topology
4. Degree theory

Factors causing changes in semantics:

- Coverage: factor causing changes in Kripke semantics
- Oracle: factor causing changes in realizability interpretation

The key point is to notice that the following two are similar:

- A sieve U is a j-cover of an object p.
- With the help of an oracle j, an algorithm p can solve a problem U.

Factors causing changes in semantics:

- Coverage: factor causing changes in Kripke semantics
- Oracle: factor causing changes in realizability interpretation

The key point is to notice that the following three are similar:

- A sieve U is a j-cover of an object p.
- Under a theory j, a formula provable from any assumption $\varphi \in U$ is also provable from the assumption of a formula p.
- With the help of an oracle j, an algorithm p can solve a problem U.

With the help of an oracle F, an algorithm p can solve a problem U.

- $p \in j^1_F(U)$: $\iff$ p is a program that solves a problem U by making exactly one query to an oracle F.
- $p \in j^{\leq 1}_F(U)$: $\iff$ p is a program that solves a problem U by making at most one query to an oracle F.
- $p \in j_F(U)$: $\iff$ p is a program that solves a problem U by making at most finitely many queries to an oracle F.

Under a suitably generalized notion of "oracle" (to be explained later), one can prove the following:

Theorem (K.)

- $j \equiv j^1_F$ for some "oracle" F $\iff$ (locally monotone) for j is realizable.
- $j \equiv j^{\leq 1}_F$ for some "oracle" F $\iff$ (locally monotone) and (inflationary) for j are realizable.
- $j \equiv j_F$ for some "oracle" F $\iff$ (locally monotone), (inflationary) and (idempotent) for j are realizable; that is, j is an LT-topology.

Relative computability with exactly one query	locally monotone $(x \to y) \to (j(x) \to j(y))$
Relative computability with at most one query	loc. mon. + inflationary $x \to j(x)$
Relative computability with finitely many queries	loc. mon. + infl. + idempotent (Lawvere-Tierney topology) $j \circ j(x) \to j(x)$

Table of contents:

- (Key Idea) "a subtopos of $\mathbf{Eff}$" $\approx$ "$\mathbf{Eff}$ relative to an oracle".
  - Here, oracle need not be a decision problem (a total function).

- Turing degrees: degrees for decision problems (total functions)
- partial degrees: degrees for partial functions
- Medvedev degrees: degrees for mass problems
- Weihrauch degrees: degrees for partial multifunctions

- Hyland (1982): The Turing degrees embed into the subtoposes of $\mathbf{Eff}$, by $\alpha \mapsto \mathbf{Eff}[\alpha]$.
- Faber-van Oosten (2014): The partial degrees (in a certain sense) correspond to the realizability subtoposes of $\mathbf{Eff}$.
  - Note: there are a lot of non-realizability subtoposes of $\mathbf{Eff}$.
  - (A realizability topos is the ex/reg-completion of the category of assemblies over a partial combinatory algebra.)

Using a partial function as an oracle

[Hereafter, a function f from some $D \subseteq X$ to Y is written as $f :\subseteq X \to Y$]

Turing-like reducibility for partial functions $f :\subseteq \mathbb{N} \to \mathbb{N}$:

- The major ones are *Kleene-style* (1952) and *Sasso-style* (1971).
  - ▷ [Point] Since an oracle is only partially defined, when making a query to the oracle, the response may not be returned forever!
- Kleene-style: One can access multiple parts of the oracle in parallel.
- Sasso-style: Once we have made a query to the oracle, we have to wait for a response from the oracle.

- Kleene's partial degrees $\simeq$ the enumeration degrees of the graphs.

  (∴) It has been widely believed that the study of partial degrees can be absorbed into the theory of enumeration degrees.
  - ▷ This is not true for Sasso-style partial degree!

Definition (Sasso 1971, van Oosten 1997, Madore 2012)

A partial function f is *subTuring reducible* to a partial function g if there exists a Turing functional Φ which computes $\Phi^g(n) \downarrow = f(n)$ *without making a query outside of* $\mathbf{dom}(g)$ for any input $n \in \mathbf{dom}(f)$.

- Unfortunately, none of the recursion theorists studied this.
- Translating Faber-van Oosten's work (2014) into computability-theoretic terms, it reads as follows:
  - The subTuring degrees $\simeq$ the realizability subtoposes of $\mathbf{Eff}$.

$$f \leq_{subT} g \iff \mathbf{Eff}[g] \text{ is a subtopos of } \mathbf{Eff}[f]$$

Theorem (K.-Ng, last week)

The subTuring degrees form a dense lattice.

Hence, the realizability subtoposes of $\mathbf{Eff}$ form a dense lattice.

- A partial function f is *quasiminimal* $\iff$ f is noncomputable, and every total $g \leq_{subT} f$ is computable.
- A partial function f is *effectively quasiminimal* $\iff$ $f \not\leq_{subT} \emptyset$ and
$$\exists u \leq_{subT} f\ \forall \text{total } g\ (g \leq_{subT} f \text{ via } e \implies g = \varphi_{u(e)}).$$

Theorem (K.-Ng, this week)

There exists an effectively quasiminimal subTuring degree.

- $\mathbf{CT_0}$: every total relation on $\mathbb{N}$ is computable:
$$\forall x \exists y\, A(x,y) \to \exists e \forall x\, A(x, \varphi_e(x))$$
- $\mathbf{ECT_0!}$: every partial function on $\mathbb{N}$ is computable:
$$\forall x(N(x) \to \exists! y\, A(x,y)) \to \exists e \forall x (N(x) \to A(x, \varphi_e(x)))$$
where N is an almost negative formula.

Corollary (K.-Ng, this week)

There exists a realizability subtopos $\mathcal{E}$ of $\mathbf{Eff}$ s.t. $\mathcal{E} \models \mathbf{CT_0} + \neg\mathbf{ECT_0!}$.

Fujiwara (an expert in constructive reverse math) told me that he has never seen a model of $\mathbf{CT_0} + \neg\mathbf{ECT_0}$, so this result may be new in constructive math.

- (Key Idea) "a subtopos of **Eff**" ≈ "**Eff** relative to an oracle".
  - Here, oracle need not be a decision problem (a total function).

- Turing degrees: degrees for decision problems (total functions)
- subTuring degrees: degrees for partial functions
- Medvedev degrees: degrees for mass problems
- Weihrauch degrees: degrees for partial multifunctions

- The subTuring degrees ≃ the realizability subtoposes of **Eff**.
  - Note: there are a lot of non-realizability subtoposes of **Eff**.
- To describe non-realizability subtoposes, a multifunction is needed.
  - Given a partial multifunction g on $\mathbb{N}$, one can construct a subtopos $\mathbf{Eff}[g]$ of **Eff**.
- In fact, more than multifunction is needed.
  - (Bauer 2022) An extended Weihrauch predicate.
  - (K. 2023) A multifunction with "(secret) parameter"
  - (K. 202x) A multifunction on "$\mathbb{N}$-multi-represented spaces".

(Def.) $j \leq k \iff j(x) \to k(x)$ is realizable (uniformly in x)

Theorem (K.)

The subTuring lattice is isomorphic to the lattice of $\bigcup, \bigcap$-preserving LT-topologies.

exactly one query	locally monotone
at most one query	loc. mon. + inflationary
finitely many queries	loc. mon. + infl. + idempotent

Theorem (K.)

- The Weihrauch lattice is isomorphic to the lattice of $\bigcap$-preserving locally monotone maps.
- The pointed Weihrauch lattice is isomorphic to the lattice of $\bigcap$-preserving locally monotone inflationary maps.
- The generalized Weihrauch lattice is isomorphic to the lattice of $\bigcap$-preserving LT-topologies.

We need more than partial multifunctions:

- (Bauer 2022) An extended Weihrauch predicate.
- (K. 2023) A multifunction with "(secret) parameter"
- (K. 202x) A multifunction on "$\mathbb{N}$-multi-represented spaces".

Theorem (K.)

- The extended Weihrauch lattice is isomorphic to the lattice of locally monotone maps.
- The pointed extended Weihrauch lattice is isomorphic to the lattice of locally monotone inflationary maps.
- The generalized extended Weihrauch lattice is isomorphic to the lattice of LT-topologies.

Sasso-style subTuring reducibility can be represented in game form.

(a.k.a. van Oosten's "*dialogue*" 1997):

Opp	***Pro***
$x_0 \in \mathbf{dom}(f)$	
	Query: $z_0 \in \mathbf{dom}(g)$
$x_1 = g(z_0)$	
	Query: $z_1 \in \mathbf{dom}(g)$
$x_2 = g(z_1)$	
$\vdots$	$\vdots$
	Query: $z_n \in \mathbf{dom}(g)$
$x_{n+1} = g(z_n)$	
	Halt: $z_{n+1} = f(x_0)$

- Opp's moves are natural numbers $x_0, x_1, x_2, \dots$
- Pro's moves are of the forms $(\texttt{Query}, z_i)$ or $(\texttt{Halt}, z_i)$.
  - `Query` is a signal to ask a query to oracle.
  - `Halt` is a signal to terminate the computation.
- f is *subT-reducible* to $g \iff$ Pro has a computable winning strategy.

A subTuring-reduction game can be considered for multifunctions.

Hirschfeldt-Jockusch's reduction game (a.k.a. Lee-van Oosten's "*sight*"):

Opp	***Pro***
$x_0 \in \mathbf{dom}(F)$	
	Query: $z_0 \in \mathbf{dom}(G)$
$x_1 \in G(z_0)$	
	Query: $z_1 \in \mathbf{dom}(G)$
$x_2 \in G(z_1)$	
$\vdots$	$\vdots$
	Query: $z_n \in \mathbf{dom}(G)$
$x_{n+1} \in G(z_n)$	
	Halt: $z_{n+1} \in F(x_0)$

- Opp's moves are natural numbers $x_0, x_1, x_2, \dots$
- Pro's moves are of the forms $(\texttt{Query}, z_i)$ or $(\texttt{Halt}, z_i)$.
  - `Query` is a signal to ask a query to oracle.
  - `Halt` is a signal to terminate the computation.
- F is *GW-reducible* to $G \iff$ Pro has a computable winning strategy.

What we need is the following *reduction game* on multi-represented spaces:

Merlin	Arthur	Nimue
$\mathbf{p}_0$ is a name of $x_0 \in \mathbf{dom}(F)$		
	Query: $\mathbf{q}_0$ is a name of	$z_0 \in \mathbf{dom}(G)$
$\mathbf{p}_1$ is a name of $x_1 \in G(z_0)$		
	Query: $\mathbf{q}_1$ is a name of	$z_1 \in \mathbf{dom}(G)$
$\mathbf{p}_2$ is a name of $x_1 \in G(z_1)$		
$\vdots$	$\vdots$	$\vdots$
	Query: $\mathbf{q}_n$ is a name of	$z_n \in \mathbf{dom}(G)$
$\mathbf{p}_{n+1}$ is a name of $x_n \in G(z_n)$		
	Halt: $\mathbf{q}_{n+1}$ is a name of	$z_{n+1} \in F(x_0)$

- Merlin's move is a pair $(\mathbf{p}, x)$; Nimue's move is z; and Arthur's move is either $(\mathtt{Query}, q)$ or $(\mathtt{Halt}, q)$.
  - Arthur cannot see "*points*" and can only see "*names*", and Arthur can only perform computable procedures
  - Merlin and Nimue can see both "*names*" and "*points*", and Merlin and Nimue can perform any procedure.
- F is *LT-reducible* to G $\iff$ Arthur-Nimue has a winning strategy where Arthur's strategy is computable.

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