The Second Level Borel Isomorphism Problem

— An Encounter of Recursion Theory and Infinite Dimensional Topology —

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Joint Work with

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Let $\boldsymbol{B}_\alpha(\boldsymbol{X})$ be the Banach space of bounded real valued Baire class α functions on $\boldsymbol{X}$ w.r.t. the supremum norm.

Main Problem (Motto Ros)

Suppose that $\boldsymbol{X}$ is a Polish space which cannot be written as a union of countably many finite dimensional subspaces.
Then, is $\mathcal{B}_n(X)$ linearly isometric to $\mathcal{B}_n([0,1]^{\mathbb{N}})$ for some $n \in \mathbb{N}$?

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- We apply Recursion Theory (a.k.a. Computability Theory) to solve Motto Ros' problem!
- More specifically, an invariant which we call *degree co-spectrum*, a collection of Turing ideals realized as lower Turing cones of points of a Polish space, plays a key role.
- The key idea is measuring the quantity of all possible Scott ideals (ω-models of **RCA + WKL**) realized within the degree co-spectrum (on a cone) of a given space.

Background in Abstract Banach Space Theory

- The basic theory on the Banach spaces $\mathcal{B}_\alpha(X)$ has been studied by Bade, Dachiell, Jayne and others in 1970s.
- Suppose that X is an uncountable Polish space:
  - $\mathcal{B}_\alpha([0,1]) \simeq_{\mathrm{li}} \mathcal{B}_\alpha(X)$ for $\alpha \geq \omega$.
  - If X is a union of countably many finite dim. subspaces $\mathcal{B}_n([0,1]) \simeq_{\mathrm{li}} \mathcal{B}_n(X) \not\simeq_{\mathrm{li}} \mathcal{B}_n([0,1]^{\mathbb{N}})$ for $2 \leq n < \omega$,
  - (Motto Ros) Does there exist an X such that $\mathcal{B}_n([0,1]) \not\simeq_{\mathrm{li}} \mathcal{B}_n(X) \not\simeq_{\mathrm{li}} \mathcal{B}_n([0,1]^{\mathbb{N}})$ for $2 \leq n < \omega$?

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(Jayne) An *α-th level Borel isomorphism* is a bijection $f : X \to Y$ s.t. $E \subseteq X$ is of additive Borel class α iff $f[E] \subseteq Y$ is of additive Borel class α.

By Jayne's theorem (1974), Motto Ros' problem is reformulated as:

The Second-Level Borel Isomorphism Problem

Find an uncountable Polish space which is second-level Borel isomorphic neither to $[0,1]$ nor to $[0,1]^{\mathbb{N}}$.

Consequently, Motto Ros' problem is the problem on the second level Borel isomorphic classification of Polish spaces.

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"We show that any two uncountable Polish spaces that are countable unions of sets of finite dimension are Borel isomorphic at the second level, and consequently at all higher levels. Thus the first level and zero-th level (i.e. homeomorphisms) appear to be the only levels giving rise to nontrivial classifications of Polish spaces."

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- At that time, almost no nontrivial proper infinite dimensional Polish spaces had been discovered yet.
- Therefore, it had been expected that the structure of proper infinite dim. Polish spaces is simple — this conclusion was too hasty!
- By using Recursion Theory, we reveal that the second level Borel isomorphic classification of Polish spaces is highly nontrivial!

Main Theorem (K. and Pauly)

There exists a $2^{\aleph_0}$ collection $(X_\alpha)_{\alpha<2^{\aleph_0}}$ of topological spaces s.t.

1. X_α is an infinite dimensional Cantor manifold for any $\alpha < 2^{\aleph_0}$, i.e., X_α is compact metrizable, and if $X_\alpha \setminus C = U_1 \sqcup U_2$ for some nonempty open U_1, U_2, then C must be infinite dimensional.

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2. X_α possesses Haver's property C (hence, weakly infinite dimensional) for any $\alpha < 2^{\aleph_0}$.
3. If $\alpha \neq \beta$, then $(X_\alpha, \utilde{\Sigma}^0_n(X_\alpha))$ is not isomorphic to $(X_\beta, \utilde{\Sigma}^0_n(X_\beta))$ for any $n \in \omega$, i.e., X_α is not n-th level isomorphic to X_β.

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2. X_α possesses Haver's property C (hence, weakly infinite dimensional) for any $\alpha < 2^{\aleph_0}$.
3. If $\alpha \neq \beta$, then $(X_\alpha, \undertilde{\Sigma}^0_n(X_\alpha))$ is not isomorphic to $(X_\beta, \undertilde{\Sigma}^0_n(X_\beta))$ for any $n \in \omega$, i.e., X_α is not n-th level isomorphic to X_β.
4. If $\alpha \neq \beta$, then the Banach space $\mathcal{B}_n(X_\alpha)$ is not linearly isometric to $\mathcal{B}_n(X_\beta)$ for any $n \in \omega$.

Decomposition Theorem (K.; Gregoriades and K.; K. and Ng)

Let X be a Souslin space and Y be a Polish space.
If $f : X \to Y$ is a function s.t.

$$A \subseteq \underset{\sim}{\mathbf{\Sigma}}^0_m(Y) \;\Rightarrow\; f^{-1}[A] \in \underset{\sim}{\mathbf{\Sigma}}^0_n(X)$$

then, there exists a countable partition $(X_i)_{i\in\omega}$ of X such that the restriction $f|_{X_i}$ is $\underset{\sim}{\mathbf{\Sigma}}^0_{n-m+1}$-measurable for every $i \in \omega$.

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- Later, Gregoriades and K. showed a finite dimensional version of this theorem by combining Louveau's separation theorem (the Gandy-Harrington topology).
- Eventually, K. and Ng showed the complete version of this theorem by extending the Shore-Slaman join theorem to infinite dimensional Polish spaces.

Let X and Y be topological spaces.

1. X is piecewise homeomorphic to Y (written as $X \simeq_{pw} Y$) if there are countable covers $\{X_i\}_{i\in\omega}$ and $\{Y_i\}_{i\in\omega}$ of X and Y such that X_i is homeomorphic to Y_i for every $i \in \omega$.
2. X is piecewise embedded into Y (written as $X \preceq_{pw} Y$) if X is piecewise homeomorphic to a subspace of Y.

By the Decomposition Theorem:

Let X and Y be Polish spaces. Then, the following are equivalent:

1. $B_n(X)$ is linearly isometric to $B_n(Y)$ for some $n \geq 2$.
2. X is second level Borel isomorphic to Y.
3. X is piecewise homeomorphic to Y.

Theorem (Hurewicz; Hurewicz-Wallman 1941)

Let X be an uncountable Polish space. Then,

$$X \simeq_{pw} 2^\omega \iff \dim(X) < \infty$$

(Urysohn 1922) $\dim(\emptyset) = -1$; $\dim(X) \leq \alpha$ iff for every point $x \in X$, there are arbitrarily small open neighborhoods $U \ni x$ with $\dim(\partial U) < \alpha$; $\dim(X) < \infty$ iff there is an ordinal α such that $\dim(X) = \alpha$.

The Piecewise Embeddability Problem

Does there exist an uncountable Polish space X such that

$$2^{\mathbb{N}} \prec_{pw} X \prec_{pw} [0,1]^{\mathbb{N}}?$$

The above problem is equivalent to the 2^{nd} level Borel isomorph. problem.

- The *Borel isomorphism problem* on Souslin spaces was able to be reduced to the same problem on *zero-dimensional* Souslin spaces.
- The *second-level Borel isomorphism problem* is inescapably tied to *infinite dimensional* topology.

- (Alexandrov 1948) X is *weakly infinite dimensional* (w.i.d.) if for each sequence (A_i, B_i) of pairs of disjoint closed sets in X there are partitions L_i in X separating A_i and B_i s.t. $\bigcap_i L_i = \emptyset$.
- (Haver 1973, Addis-Gresham 1978) X is a C-space ($S_c(\mathcal{O}, \mathcal{O})$) if for each sequence $(\mathcal{U}_i)$ of open covers of X there is a pairwise disjoint open family $(\mathcal{V}_i)$ refining $(\mathcal{U}_i)$ s.t. $\bigcup_i \mathcal{V}_i$ covers X.

$$X \leq_{pw} 2^{\mathbb{N}} \iff \dim(X) < \infty \Rightarrow X \text{ is } C \Rightarrow X \text{ is w.i.d.}$$

- (Alexandrov 1951) $\exists$ a w.i.d. metrizable compactum $X >_{pw} 2^{\mathbb{N}}$?
- (R. Pol 1981) There exists a metrizable C-compactum $X >_{pw} 2^{\mathbb{N}}$.
- (E. Pol 1997) There exists an infinite dimensional C-Cantor manifold, i.e., a C-compactum which cannot be separated by any hereditarily weakly infinite dimensional closed subspaces.
- (Chatyrko 1999) There is a collection $\{X_\alpha\}_{\alpha<2^{\aleph_0}}$ of continuum many infinite dimensional C-Cantor manifolds such that X_α cannot be embedded into X_β whenever $\alpha \neq \beta$.

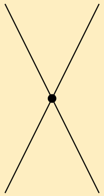
Main Lemma (K. and Pauly)

Let $\mathfrak{M}_\infty$ be the class of all infinite dimensional C-Cantor manifolds. Then, there is an order embedding of $([\aleph_1]^\omega, \subseteq)$ into $(\mathfrak{M}_\infty, \leq_{pw})$.

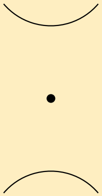
- This solves Motto Ros' problem (and the second level Borel isomorphism problem).
- This strengthen R. Pol's theorem and Chatyrko's theorem in infinite dimensional topology.

To show Main Lemma, we again use Recursion Theory!

Idea of Proof: Upper/Lower Approximation by Zero Dim Spaces



(a) Any point in $\mathbb{R}^n$



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- By *approximating* each point in a space X *by a zero-dim space*, we measure "*how similar the space X is to a zero-dim space*".
- (a) Upper and lower approximations by a zero-dim space *meet*.
- (b) There is a *gap* between upper and lower approximations by a zero-dim space

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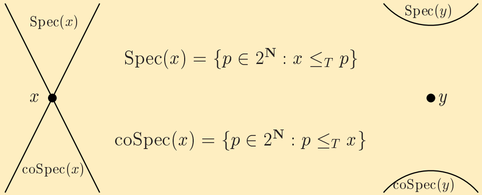

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- $\mathrm{Spec}(x) = \{p \in 2^{\mathbb{N}} : x \leq_T p\}$.
- $\mathrm{coSpec}(x) = \{p \in 2^{\mathbb{N}} : p \leq_T x\}$.

Key Idea

Classification of topological spaces by degrees of unsolvability:

1. The Turing degrees $\simeq$ the degree structure on Cantor space $\mathbf{2}^{\mathbb{N}}$ and Euclidean spaces $\mathbb{R}^n$.
2. The enumeration degrees $\simeq$ the degree structure on the Scott domain $\mathcal{P}(\mathbb{N})$.
3. Hinman (1973): degrees of unsolvability of continuous functionals $\simeq$ the degree structure on the space $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ of Kleene-Kreisel continuous functionals.
4. J. Miller (2004): continuous degrees $\simeq$ the degree structure on the function space $C([0,1])$ and the Hilbert cube $[0,1]^{\mathbb{N}}$.

Definition

Let X and Y be second-countable T_0 spaces with fixed countable open basis $\{B_n^X\}_{n\in\omega}$ and $\{B_n^Y\}_{n\in\omega}$.
A point $x \in X$ is "*Turing reducible*" to a point $y \in Y$ ($x \leq_T y$) if

$$\{n \in \omega : x \in B_n^X\} \leq_e \{n \in \omega : y \in B_n^Y\}.$$

In other words, we identify the "*Turing degree*" of $x \in X$ with the *enumeration degree of the (coded) neighborhood filter* of x.

Example

- The degree structure of *Cantor space* is exactly the same as the *Turing degrees*.
- The degree structure of *Hilbert cube* (a universal Polish space) is exactly the same as the *continuous degrees*.
- The degree structure of *the Scott domain* $\mathcal{O}(\mathbb{N})$ (a universal quasi-Polish space) is exactly the same as the *enumeration degrees*.

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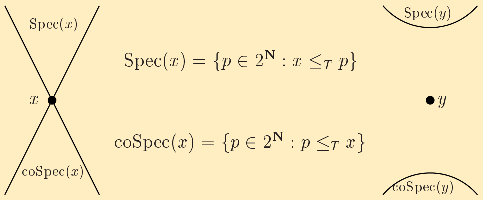

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Lemma (K. and Pauly)

$X \simeq_{pw} Y \Longrightarrow \mathrm{Spec}^r(X) = \mathrm{Spec}^r(Y)$ for some oracle $r \in 2^\omega$.
$\Longrightarrow \mathrm{coSpec}^r(X) = \mathrm{coSpec}^r(Y)$ for some oracle $r \in 2^\omega$.

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$\Longrightarrow \mathbf{coSpec}^r(X) = \mathbf{coSpec}^r(Y)$ for some oracle $r \in 2^\omega$.

1. A Turing ideal $\mathcal{J} \subseteq 2^\omega$ is *realized* by x if $\mathcal{J} = \mathrm{coSpec}(x)$.
2. A countable set $\mathcal{J} \subseteq \mathcal{P}(\omega) \simeq 2^\omega$ is a *Scott ideal* $\iff (\omega, \mathcal{J}) \models \mathbf{RCA} + \mathbf{WKL}$.

Realizability of Scott ideals (J. Miller 2004)

1. $2^\omega \simeq_{pw} \omega^\omega \simeq_{pw} \mathbb{R}^n \simeq_{pw} \bigoplus_{n\in\omega} \mathbb{R}^n$. (*Turing degrees*.)
   No Scott ideal is realized in these spaces!
2. $[0,1]^\omega \simeq_{pw} \mathcal{C}([0,1]) \simeq_{pw} \ell^2$. (*full continuous degrees*.)
   Every countable Scott ideal is realized in these spaces!

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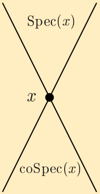

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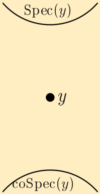

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- **Spec** determines the pw-homeomorphism type of a space, and **coSpec** is invariant under pw-homeomorphism.
- The **coSpec** of any point in a space of **dim** $< \infty$ has to be a principal Turing ideal.
- (Miller) Every countable Scott ideal is realized as **coSpec** of a point in Hilbert cube.

Definition

$\Gamma : 2^{\mathbb{N}} \to [0,1]^{\mathbb{N}}$ is *ω-left-CEA operator* if the infinite sequence $\Gamma(y) = (x_0, x_1, x_2, \dots)$ is generated in a uniformly left-computably enumerable manner by a single Turing machine, that is, there is a left-c.e. operator γ such that for all i,

$$x_i := \Gamma(y)(i) = \gamma(y, i, x_0, x_1, \dots, x_{i-1}).$$

An ω-left-CEA operator $\Gamma : \mathbb{N} \times 2^{\mathbb{N}} \to [0,1]^{\mathbb{N}}$ is *universal* if for every ω-left-CEA operator Ψ, there is e such that $\Psi = \lambda y.\Gamma(e, y)$.

Let ω**CEA** denote the graph of a universal ω-left-CEA operator.

Theorem (K.-Pauly)

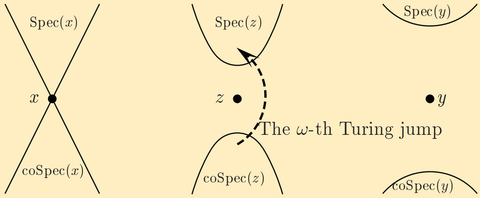
The space ω**CEA** (as a subspace of Hilbert cube) is an intermediate Polish space:

$$\mathbf{2}^{\mathbb{N}} \prec_{pw} \omega\mathbf{CEA} \prec_{pw} [\mathbf{0},\mathbf{1}]^{\mathbb{N}}$$

Remark

Furthermore, ω**CEA** is pw-homeomorphic to the following:

- Rubin-Schori-Walsh (1979)'s strongly infinite dimensional totally disconnected Polish space.
- Roman Pol (1981)'s weakly infinite dimensional compactum which is not decomposable into countably many finite-dim subspaces (a solution to Alexandrov's problem).



(a) $\mathbf{2}^{\mathbb{N}}$ (b) $\omega\mathbf{CEA}$ (c) $[\mathbf{0},\mathbf{1}]^{\mathbb{N}}$

- (a) **coSpec** is principal, and *meets* with **Spec**.
- (b) **coSpec** is not always principal, but the "*distance*" between **Spec** and **coSpec** has to be at most *the ω-th Turing jump*.
- (c) **coSpec** can realize an arbitrary countable Scott ideal, hence **Spec** and **coSpec** can be separated by *an arbitrary distance*.

Proof Sketch of $2^{\mathbb{N}} <_{pw} \omega\mathrm{CEA} <_{pw} [0,1]^{\mathbb{N}}$

$$\omega\mathbf{CEA} = \{(e,p,x_0,x_1,\dots) \in \omega \times 2^\omega \times [0,1]^\omega : (\forall i)\ x_i \text{ is the } e\text{-th left-c.e. real in } (p,x_0,x_1,\dots,x_{i-1}).\}$$

Lemma

For any $p \in 2^\omega$, the following Scott ideal is not realized in ω**CEA**:

$$\mathcal{J}^p = \{z \in 2^\omega : (\exists n)\ z \leq_T p^{(\omega\cdot n)}\}.$$

- Pick $z = (e,p,x_0,x_1,\dots) \in \omega$**CEA**.
- Then, $p \in \mathbf{coSpec}(z)$ and $p^{(\omega)} \in \mathbf{Spec}(z)$.
- Clearly, $p^{(\omega+1)} \notin \mathbf{coSpec}(z)$.

Since **coSpec** (up to an oracle) is invariant under pw-homeomorphism, we have ω**CEA** $<_{pw} [0,1]^{\mathbb{N}}$.

Another separation is based on Kakutani's fixed point theorem.

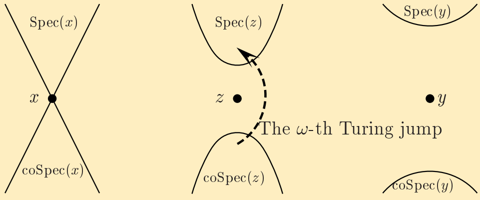
Theorem (J. Miller 2004)

There is a nonempty convex-valued computable function $\Psi : [0,1]^{\mathbb{N}} \to \mathcal{P}([0,1]^{\mathbb{N}})$ with a closed graph such that for every fixed point $\langle x_0, x_1, \dots \rangle \in \mathrm{Fix}(\Psi)$,

$$\mathrm{coSpec}(\langle x_0, x_1, x_2, \dots \rangle) = \{x_0, x_1, x_2, \dots\}.$$

Moreover, such an x realizes a Scott ideal.

- $\mathrm{Fix}(\Psi)$ is a Π^0_1 subset of $[0,1]^\omega$.
- Inductively find $(x_0, x_1, \dots) \in \mathrm{Fix}(\Psi)$, where x_{i+1} is the "*leftmost*" value s.t. $(x_0, x_1, \dots, x_{i+1})$ is extendible in $\mathrm{Fix}(\Psi)$.
- Then, x_{i+1} is left-c.e. in $(x_0, x_1, \dots, x_i)$, uniformly.
- x_{i+1} does not depend on the choice of a name of $(x_0, \dots, x_i)$.



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- (a) **coSpec** is principal, and *meets* with **Spec**.
- (b) **coSpec** is not always principal, but the "*distance*" between **Spec** and **coSpec** has to be at most *the ω-th Turing jump*.
- (c) **coSpec** can realize an arbitrary countable Scott ideal, hence **Spec** and **coSpec** can be separated by *an arbitrary distance*.

1. $\mathbf{coSpec}(2^{\mathbb{N}})$ = all principal Turing ideals.
2. $\mathbf{coSpec}([0,1]^{\mathbb{N}})$ = all principal Turing ideals and Scott ideals.
3. What do we know about $\mathbf{coSpec}(\omega\mathbf{CEA})$?
    - It cannot realize an ω-jump ideal.
    - It realizes a non-principal Turing ideal.
    - We know absolutely nothing about what kind of Turing ideals it realizes; even whether it realizes a jump ideal or not.

How can we control **coSpec** of a Polish space?

For instance, given $\alpha << \beta < \omega_1$, we need a technique for constructing a Polish space such that

- it cannot realize a β-jump ideal,
- it realizes an α-jump ideal.

We say that $\mathcal{G} : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is an *oracle Π^0_2 singleton* if it has a Π^0_2 graph.
For instance, the α-th Turing jump operator TJ^{α} is an oracle Π^0_2 singleton.

Definition (Modified ωCEA Space)

The space $\omega\mathbf{CEA}(\mathcal{G})$ consists of $(d, e, r, x) \in \mathbb{N}^2 \times 2^{\mathbb{N}} \times [0,1]^{\mathbb{N}}$ such that for every i,

1. either $x_i = \mathcal{G}^i(r)$, or
2. there are $u \leq v \leq i$ such that $x_i \in [0,1]$ is the e-th left-c.e. real in $\langle r, x_{<i}, x_{l(u)} \rangle$ and $x_{l(u)} = \mathcal{G}^{l(u)}(r)$, where $l(u) = \Phi_d(u, r, x_{<v})$.

Here: $\mathcal{G}^0(x) = x$ and $\mathcal{G}^{n+1}(x) = \mathcal{G}^n(x) \oplus \mathcal{G}(\mathcal{G}^n(x))$.

We define $\mathbf{Ref}(\mathcal{G}) = \omega\mathbf{CEA}(\mathcal{G}) \cap (\mathbb{N}^2 \times \mathbf{Fix}(\Psi))$.
The subspace $\mathbf{Ref}(\mathcal{G})$ (as a subspace of $[0,1]^{\mathbb{N}}$) is Polish whenever $\mathcal{G}$ is an oracle Π^0_2 singleton.

Suppose that $\mathcal{G}$ is an oracle Π^0_2-singleton. For every oracle $r \in 2^{\mathbb{N}}$, consider two Turing ideals defined as

$$\mathcal{J}_T(\mathcal{G}, r) = \{z \in 2^{\mathbb{N}} : (\exists n \in \mathbb{N})\ x \leq_T \mathcal{G}^n(r)\},$$
$$\mathcal{J}_a(\mathcal{G}, r) = \{z \in 2^{\mathbb{N}} : (\exists n \in \mathbb{N})\ x \leq_a \mathcal{G}^n(r)\}.$$

Here: $\leq_a$ is the arithmetical reducibility.

Main Lemma (**coSpec**-Controlling)

1. For every $x \in \mathbf{Ref}(\mathcal{G})$, there is $r \in 2^{\mathbb{N}}$ such that
$$\mathbf{coSpec}(x) \subseteq \mathcal{J}_a(\mathcal{G}, r).$$
2. For every $r \in 2^{\mathbb{N}}$, there is $x \in \mathbf{Ref}(\mathcal{G})$ such that
$$J_T(\mathcal{G}, r) \subseteq \mathbf{coSpec}(x).$$

If $\mathcal{G} = \mathbf{TJ}^{\alpha}$ is the α-th Turing jump operator for $\alpha \geq \omega$,

1. $\mathbf{coSpec}(\mathbf{Ref}(\mathbf{TJ}^{\alpha}))$ realizes no β-jump ideal for $\beta \geq \alpha \cdot \omega$,
2. $\mathbf{coSpec}(\mathbf{Ref}(\mathbf{TJ}^{\alpha}))$ realizes an α-jump ideal.

1. By **coSpec**-Controlling Lemma, given an oracle $\mathbf{\Pi}^0_2$ singleton $\mathcal{G}$ we can construct a Polish space which realizes all Turing ideals closed under $\mathcal{G}$.
2. $\mathbf{Ref}(\mathcal{G})$ is strongly infinite dimensional and totally disconnected.
3. Hence, its compactification $\gamma\mathbf{Ref}(\mathcal{G})$ (in the sense of Lelek) is a "*Pol-type space*", hence, a metrizable $\boldsymbol{C}$-compacta.
4. Note that Lelek's compactification preserves **Spec** and **coSpec**.

Main Lemma (K. and Pauly)

Let $\mathfrak{M}_\infty$ be the class of all infinite dimensional $\boldsymbol{C}$-Cantor manifolds. Then, there is an order embedding of $([\aleph_1]^\omega, \subseteq)$ into $(\mathfrak{M}_\infty, \leq_{pw})$.

Main Theorem (K. and Pauly)

There exists a $2^{\aleph_0}$ collection $(X_\alpha)_{\alpha<2^{\aleph_0}}$ of topological spaces s.t.

1. X_α is an infinite dimensional Cantor manifold for any $\alpha < 2^{\aleph_0}$,
2. X_α possesses Haver's property C for any $\alpha < 2^{\aleph_0}$.
3. If $\alpha \neq \beta$, then X_α is not n-th level isomorphic to X_β for any $n \in \omega$.
4. If $\alpha \neq \beta$, then the Banach space $\mathcal{B}_n(X_\alpha)$ is not linearly isometric to $\mathcal{B}_n(X_\beta)$ for any $n \in \omega$.

Summary of This Work

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3. Proving **coSpec**-Controlling Lemma.
4. Solving the second-level Borel isomorpshim problem.

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