Topological aspects of enumeration degrees

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Joint Work with

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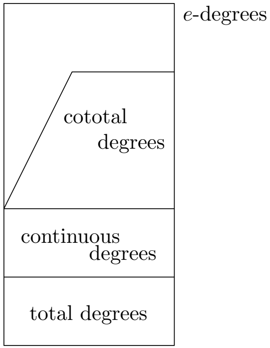
Observation

The enumeration degrees
= The degrees of points in second-countable T_0 spaces.



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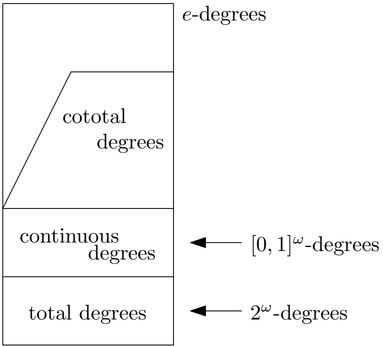
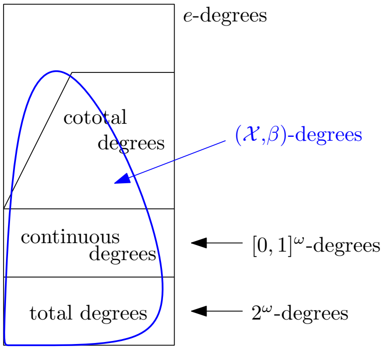


- Total degrees = degrees of points in $\mathbf{2}^\omega$.
- Continuous degrees = degrees of points in $[\mathbf{0}, \mathbf{1}]^\omega$.

Observation

The enumeration degrees
= The degrees of points in second-countable T_0 spaces.



To each T_0 space $\mathcal{X}$ with an enumeration β of a countable basis,
one can assign a substructure $\mathcal{D}(\mathcal{X}, \beta)$ of the *e*-degrees.

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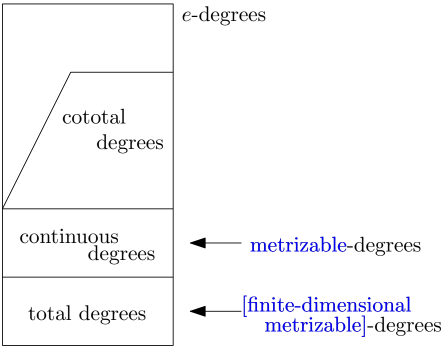
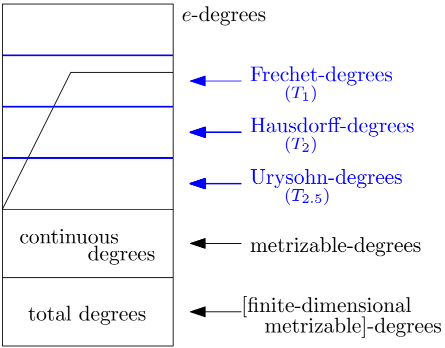


- Total degrees = degrees of points in finite-dimensional metrizable spaces.
- Continuous degrees = degrees of points in metrizable spaces.

Observation

The enumeration degrees
= The degrees of points in second-countable T_0 spaces.



The ***e***-degrees can be classified in terms of general topology!

Observation

The enumeration degrees
= The degrees of points in second-countable T_0 spaces.

The ***e***-degrees can be classified in terms of general topology!

- Total degrees = finite dimensional metrizable ***e***-degrees.
- Continuous degrees = metrizable ***e***-degrees.
- (with Madison) Cototal degrees = ***e***-degrees in G_δ-spaces.
- Graph-cototal degrees = ***e***-degrees in $(\omega_{\mathrm{cof}})^\omega$, where ω_{cof} is the set ω equipped with the cofinite topology.
- Semi-recursive degrees = ***e***-degrees in $\mathbb{R}$ with the lower topology.

To each T_0 space $\mathcal{X}$ with a countable basis β,
one can assign a substructure $\mathcal{D}(\mathcal{X}, \beta)$ of the e-degrees.

Example (Hausdorff e-degrees)

An e-degree **d** is *double-origin* if **d** contains a set of the form:

$$(X \oplus \overline{X}) \oplus (A \cup P) \oplus (B \cup N),$$

where P and N are X-c.e., $A \cup B$ is X-co-c.e.,
and $A, B, P,$ and N are pairwise disjoint.

Remark: every **3**-c.e. e-degree is double-origin.

Let $\mathcal{X}$ be the rational disk endowed with the double origin topology.
The degree structure of $\mathcal{X}^\omega$ = the double-origin e-degrees.
Since $\mathcal{X}^\omega$ is Hausdorff, all double-origin e-degrees are Hausdorff.

Project 1

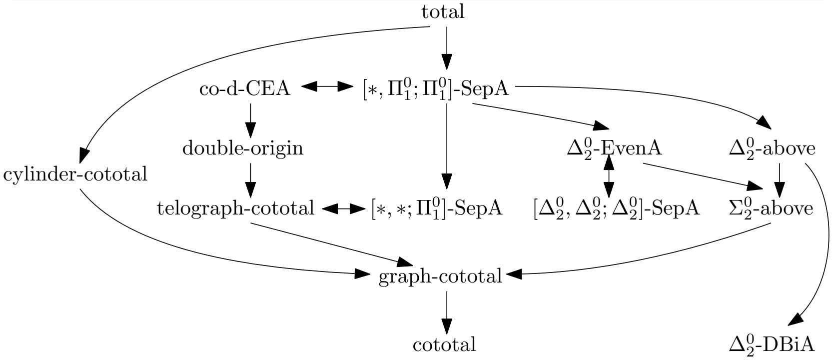
Determine the degree structures of second-countable T_0-spaces which appear in the book "*Counterexamples in Topology* [1] (*CiT*)."

For most second-countable T_0 spaces $\mathcal{X} \in$ ***CiT***,

- \+ $\mathcal{X}$ is very very effective.
- – The degree structure of $\mathcal{X}$ itself is not so interesting.
- \+ However, that of its countable product $\mathcal{X}^\omega$ is interesting!

[1] L. A. Steen and J. A. Seebach, Jr., Counterexamples in Topology. Springer-Verlag, New York, 1978.

Current Status of Project 1



$T_{2.5}$: **irregular lattice space** (co-d-CEA), **Arens square** ($\boldsymbol{\Delta^0_2}$-DBiA), **Roy's lattice space** ($\boldsymbol{\Delta^0_2}$-EvenA).

T_2: **double origin topology** (double-origin).

T_1: **cofinite topology** (graph-cototal), **cocylinder topology** (cylinder-cototal), **telophase topology** (telograph-cototal).

To each T_0 space $\mathcal{X}$ with a countable basis $\beta = (B_e)_{e \in \omega}$, one can assign a substructure $\mathcal{D}(\mathcal{X}, \beta)$ of the e-degrees.

Definition

The degree of $x \in \mathcal{X}$ is defined by the e-degree of its coded neighborhood filter:

$$\mathrm{Nbase}_\beta(x) = \{e \in \omega : x \in B_e\}.$$

Then, the degree structure of $\mathcal{X}$ (relative to β) is defined by

$$\mathcal{D}(\mathcal{X}, \beta) = \{\deg_e(\mathrm{Nbase}_\beta(x)) : x \in \mathcal{X}\}.$$

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One can assign degree structures to certain non-second-countable spaces (only using computability on ω, without using α-recursion, E-recursion, ITTM, etc)

[E.g. Arhangel'skii (1959) introduced the notion of a network in general topology. Use a countable cs-network to define the degree structure as in Schröder (2002)]

But, if a space is second-countable, then it coincides with the above definition.

$$\mathrm{Nbase}_\beta(x) = \{e \in \omega : x \in B_e\}.$$
$$\mathcal{D}(\mathcal{X}, \beta) = \{\mathbf{deg_e}(\mathrm{Nbase}_\beta(x)) : x \in \mathcal{X}\}.$$

Example (Hausdorff e-degrees)

The relatively prime integer topology on the positive integers $\mathbb{Z}_{>0}$ is generated by

$$U_b(a) = \{a + bt : t \in \mathbb{Z}\},$$

where a and b are relatively prime. Then, for $x \in \mathbb{Z}^\omega_{>0}$,

$$\mathrm{Nbase}(x) = \{\langle n, a, b \rangle : (\exists t \in \mathbb{Z})\ x(n) = a + bt\}.$$

$$\mathrm{Nbase}_\beta(x) = \{e \in \omega : x \in B_e\}.$$
$$\mathcal{D}(\mathcal{X},\beta) = \{\mathbf{deg}_e(\mathrm{Nbase}_\beta(x)) : x \in \mathcal{X}\}.$$

Basic Idea (De Brecht-K.-Pauly; at Dagstuhl)

$\mathcal{P}$: a topological property (e.g. metrizable, Hausdorff, regular)

1. An e-degree $\boldsymbol{d}$ is $\mathcal{P}$ if $\boldsymbol{d} \in \mathcal{D}(\mathcal{X},\beta)$ for some "effective $\mathcal{P}$" space $(\mathcal{X},\beta)$.
2. An e-degree $\boldsymbol{d}$ is *$\mathcal{P}$-quasiminimal* if for any effective $\mathcal{P}$ space $(\mathcal{X},\beta)$, $(\forall \boldsymbol{a})\ [\boldsymbol{a} \le \boldsymbol{d}\ \&\ \boldsymbol{a} \in \mathcal{D}(\mathcal{X},\beta) \Longrightarrow \boldsymbol{a} = \mathbf{0}]$.

T_3: **Cantor space, Euclidean space, Hilbert cube.**

$T_{2.5}$: **irregular lattice space, Arens square, Roy's lattice space, Gandy-Harrington topology.**

T_2: **double origin topology, relatively prime integer topology.**

T_1: **cofinite topology, cocylinder topology, telophase topology.**

T_0: **lower topology, Sierpiński space.**

Project 2

Given $m < n$, construct a T_n-degree which is T_m-quasiminimal!

T_3-degrees vs. $T_{2.5}$-degrees.

- A space is T_3 if it is regular Hausdorff, that is, given any point and closed set are separated by nbhds.
- A space is $T_{2.5}$ if any two distinct points are separated by closed nbhds.

T_3: **Cantor space, Euclidean space, Hilbert cube.**

$T_{2.5}$: **irregular lattice space, Arens square, Roy's lattice space, Gandy-Harrington topology.**

- Let $\mathcal{L}$ be the irregular lattice space.

$$\mathcal{D}(\mathcal{L}^\omega) = \text{"}\mathbf{3}\text{-c.e. above total degrees"}$$

- (Folklore) There is a quasiminimal **3**-c.e. ***e***-degree.
- (Corollary) There is a $T_{2.5}$-degree which is (T_3-)quasiminimal.

$T_{2.5}$-degrees vs. T_2-degrees.

- A space is $T_{2.5}$ if any two distinct points are separated by closed nbhds.
- A space is T_2 if any two distinct points are separated by open nbhds.

$T_{2.5}$: **irregular lattice space, Arens square, Roy's lattice space, Gandy-Harrington topology.**

T_2: **double origin topology, relatively prime integer topology.**

Theorem

Let $\mathcal{P}$ be the set $\mathbb{Z}_{>0}$ endowed with the relatively prime integer topology. $(\mathcal{X}_n, \beta_n)_{n\in\omega}$: a countable collection of $T_{2.5}$-spaces.

1. $\mathcal{D}(\mathcal{P}^\omega) \nsubseteq \bigcup_{n\in\omega} \mathcal{D}(\mathcal{X}_n, \beta_n)$.
2. A sufficiently generic point in $\mathcal{P}^\omega$ is (T_3-)quasiminimal.

(Open Question): Does there exist a $T_{2.5}$-quasiminimal T_2-degree?

T_2-degrees vs. T_1-degrees.

- A space is T_2 if the diagonal $\{(x, y) : x = y\}$ is closed.
- A space is T_1 if every singleton is closed.

T_2: **double origin topology, relatively prime integer topology.**

T_1: **cofinite topology, cocylinder topology, telophase topology.**

Theorem

Let $\mathcal{T}$ be the set $\omega \cup \{\infty, \infty^*\}$ endowed with the telophase topology. $(\mathcal{X}_n, \beta_n)_{n\in\omega}$: a countable collection of effective Hausdorff spaces. Then, there is $x \in \mathcal{T}^\omega$ which is $(\mathcal{X}_n, \beta_n)$-quasiminimal for any n.

If our definition of an "effective T_2 space" satisfies that

only countably many effective T_2 space exists,

then the above shows that $\mathcal{T}^\omega$ contains a T_2-quasiminimal degree.

In particular, there exists a T_2-quasiminimal T_1-degree.

T_2, T_1, and T_D-degrees.

- A space is T_2 if the diagonal $\{(x, y) : x = y\}$ is Π^0_1.
- A space is uniformly T_D if the diagonal $\{(x, y) : x = y\}$ is Δ^0_2.
- A space is T_1 if every singleton is Π^0_1.
- A space is T_D if every singleton is Δ^0_2.

The T_D-separation axiom was introduced by Aull-Thron (1963).

Observation (Independently by de Brecht?)

- T_2-degrees $=$ Uniform T_D-degrees.
- T_1-degrees $=$ T_D-degrees.

T_1-degrees vs. T_0-degrees.

T_1: **cofinite topology, cocylinder topology, telophase topology.**

T_0: **lower topology, Sierpiński space.**

- Define $\mathbf{Name}_{\subseteq}(\mathcal{X}) = \{A \subseteq \omega : (\exists x \in \mathcal{X})\ A \subseteq \mathbf{Nbase}(x)\}$, etc.
- $\mathcal{X}$ is $T_1 \implies \mathbf{Name}_{=}(\mathcal{X}) = \mathbf{Name}_{\subseteq}(\mathcal{X}) \cap \mathbf{Name}_{\supseteq}(\mathcal{X})$.
- A T_1 space $\mathcal{X}$ is *strongly Γ-named* if there are Γ sets P, N s.t.

  $$\mathbf{Name}_{\subseteq}(\mathcal{X}) \subseteq N,\ \mathbf{Name}_{\supseteq}(\mathcal{X}) \subseteq P, \text{ and } \mathbf{Name}_{=}(\mathcal{X}) = P \cap N.$$

- $\mathbb{R}_<$: the set of reals equipped with the lower topology.
- (Theorem) If $x \in \mathbb{R}_<$ is not Δ^0_n,
  then x is quasiminimal w.r.t. strongly Π^0_n-named T_1-spaces.

T_3: Cantor space, Euclidean space, Hilbert cube.

$T_{2.5}$: irregular lattice space, Arens square, Roy's lattice space, Gandy-Harrington topology.

T_2: double origin topology, relatively prime integer topology.

T_1: cofinite topology, cocylinder topology, telophase topology.

T_0: lower topology, Sierpiński space.

Current Status of Project 2

1. There is a ($\boldsymbol{T_3}$-)quasiminimal $\boldsymbol{T_{2.5}}$-degree.
2. There is a, non-$\boldsymbol{T_{2.5}}$, $\boldsymbol{T_2}$-degree.
3. There is a $\boldsymbol{T_2}$-quasiminimal $\boldsymbol{T_1}$-degree.
4. There is a $\boldsymbol{T_1}$-quasiminimal $\boldsymbol{e}$-degree.

Here we have assumed that "there are only countably many effective spaces."

Open Question

Does there exist a $T_{2.5}$-quasiminimal T_2-degree?

$T_{2.5}$: **irregular lattice space, Arens square, Roy's lattice space, Gandy-Harrington topology.**

T_2: **double origin topology, relatively prime integer topology.**

- A pointclass Γ is *lightface* if it is relativizable, and Γ^x is countable for any x.
- An e-degree is *Γ-above-$\mathcal{X}$* ($\Gamma^{\oplus}\mathcal{X}$) if it contains a set of the form $A \oplus \mathrm{Nbase}(x)$ for some $x \in \mathcal{X}$ and $A \in \Gamma^x$.

Proposition

For any lightface pointclass Γ, there is a $T_{2.5}$-space $(\mathcal{X}, \beta)$ s.t.

$$\mathcal{D}(\mathcal{X}, \beta) = \Gamma^{\oplus}[0,1]^{\omega}.$$

"Above-Continuous" Conjecture

For any $T_{2.5}$-space $(\mathcal{X}, \beta)$, there is a lightface pointclass Γ s.t.

$$\mathcal{D}(\mathcal{X}, \beta) \subseteq \Gamma^{\oplus}[0,1]^{\omega}.$$

Project 3

Develop degree theory on non-second-countable spaces!

There are many interesting spaces which are not second-countable, but have countable cs-networks. For instance,

- The Kleene-Kreisel space (the space of higher type functionals).
- The hyperspace of closed singletons.

The degree structure of the former one has been studied by Hinman, Normann, and others from 1970s. The degree structure of the latter one is connected to the degree-theoretic study on Π^0_1 singletons.