Counterexamples in Computable Continuum Theory

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1. Local Computability:
   - Every nonempty open set in $\mathbb{R}^n$ has a computable point.
   - Not every nonempty co-c.e. closed set in $\mathbb{R}^n$ has a computable point (Kleene, Kreisel, etc. 1940's–50's).

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   - If a nonempty co-c.e. closed subset $F \subseteq \mathbb{R}^1$ has no computable points, then F must be *disconnected*.
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3. Let us study the computable content of *Continuum Theory*!

   Here, "Continuum Theory" is a branch of topology studying connected compact spaces.

Definition

$\{B_e\}_{e\in\mathbb{N}}$: an effective enumeration of all rational open balls.

1. $x \in \mathbb{R}^n$ is *computable* if $\{e \in \mathbb{N} : x \in B_e\}$ is c.e.
   Equivalently, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ is computable iff x_i is computable for each $i \leq n$.

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Remark

- F is co-c.e. closed $\iff$ F is a computable point in the hyperspace $\mathcal{A}_-(\mathbb{R}^n)$ of closed subsets of $\mathbb{R}^n$ under lower Fell topology.
- F is computable closed $\iff$ F is a computable point in the hyperspace $\mathcal{A}(\mathbb{R}^n)$ of closed subsets of $\mathbb{R}^n$ under Fell topology.

Connected co-c.e. closed Sets

Fact

1. (Kleene, Kreisel, etc.) There exists a nonempty co-c.e. closed set $P \subseteq \mathbb{R}^1$ which has no computable point.
2. Every nonempty *connected* co-c.e. closed subset $P \subseteq \mathbb{R}^1$ contains a computable point.

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Fact

1. There exists a nonempty *connected* co-c.e. closed subset $P^{(2)} \subseteq \mathbb{R}^2$ which has no computable point.
2. There exists a nonempty *simply connected* co-c.e. closed subset $P^{(3)} \subseteq \mathbb{R}^3$ which has no computable point.

- X is n-connected $\iff$ the first $n+1$ homotopy groups vanish identically.
- X is path-connected $\iff$ X is 0-connected.
- X is simply connected $\iff$ X is 1-connected.
- X is contractible $\iff$ the identity map on X is null-homotopic.
- X is contractible $\implies X$ is n-connected for any n.

Observation

Not every nonempty *n-connected* co-c.e. closed set in $\mathbb{R}^{n+2}$ contains a computable point, for any $n \in \mathbb{N}$.

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- Not every nonempty *n-connected* co-c.e. closed set in $\mathbb{R}^{n+2}$ contains a computable point, for any $n \in \mathbb{N}$.

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- Not every nonempty ***n*-connected** co-c.e. closed set in $\mathbb{R}^{n+2}$ contains a computable point, for any $n \in \mathbb{N}$.
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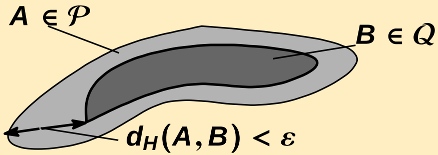
Question

1. (Le Roux-Ziegler) Does every simply connected planar co-c.e. closed set contain a computable point?
2. Does every contractible Euclidean co-c.e. closed set contain a computable point?

$A \in \mathcal{P}$ is ε-approximated from the inside by $B \in \mathcal{Q}$



Definition

1. The *Hausdorff distance* between nonempty closed subsets A_0, A_1 of a metric space (X, d) is defined by:
$d_H(A_0, A_1) = \max_{i<2} \sup_{x \in A_i} \inf_{y \in B_{1-i}} d(x, y)$.
2. $\mathcal{P}, \mathcal{Q}$: classes of continua.
$\mathcal{P}$ is *approximated (from the inside)* by $\mathcal{Q}$ if
$(\forall A \in \mathcal{P})\ \inf\{d_H(B, A) : A \supseteq B \in \mathcal{Q}\} = 0$.

Proposition

ARC-CONNECTED CONTINUA is approximated by LOCALLY CONNECTED CONTINUA.

Proof

1. By compactness, X has an ε-net $\{x_i\}_{i<n} \subseteq X$ for any $\varepsilon > 0$. (i.e., $\bigcup_{i<n} B(x_i;\varepsilon)$ covers X)
2. Let $\gamma_{ij} \subseteq X$ be an arc with end points x_i and x_j.
3. $Y = \bigcup_{i,j<n} \gamma_{ij} \subseteq X$, and $d_H(Y,X) \leq \varepsilon$.
4. γ^*_{ij} is inductively defined as:
   - $\gamma^*_{ij} \subseteq \gamma_{ij} \cup \bigcup_{(k,l)<(i,j)} \gamma_{kl}$.
   - If γ_{ij} intersects with $\bigcup_{(k,l)<(i,j)} \gamma_{kl}$, then $\gamma^*_{ij} \cap \bigcup_{(k,l)<(i,j)} \gamma_{kl}$ is an arc.
5. $Y^* = \bigcup_{i,j<n} \gamma^*_{ij}$ is locally connected, $Y^* \subseteq Y \subseteq X$, and $d_H(Y^*,X) \leq \varepsilon$.

If a continua in a class $\mathcal{C}$ has no computable point, then $\mathcal{C}$ is not approximated by Computable Closed Sets.

Theorem (Miller 2002; Iljazović 2009)

1. Every Euclidean co-c.e. n-sphere is computable. Hence, Every co-c.e. Jordan curve is computable.
2. Co-c.e. Arcs is approximated by Computable Arcs. In this sense, every co-c.e. arc is "*almost*" computable.

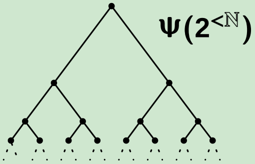
Definition

Let S be a topological space.

1. S is *connected* if it is not union of disjoint open sets.
2. S is *locally connected* if it has a base of connected sets.
3. *A continuum* is a connected compact metric space.
4. *A dendroid* is a continuum S such that $(\forall x, y \in S)$ $S[x, y] = \min\{Y \subseteq X : x, y \in Y \ \& \ Y \text{ is connected}\}$ exists, and such $S[x, y]$ is an arc.
5. *A dendrite* is a locally connected dendroid.
6. *A tree* is a dendrite with finitely many ramification points.

Example

We plot a tree $T \subseteq 2^{<\omega}$ on the Euclidean plane $\mathbb{R}^2$.
Then the plotted picture $\Psi(T) \subseteq \mathbb{R}^2$ is a dendrite.

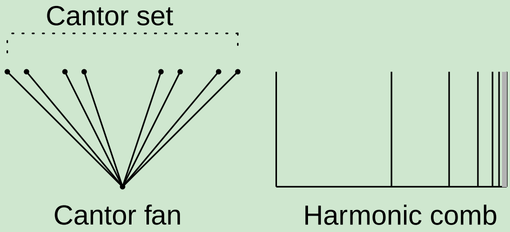


Protting $2^{<\mathbb{N}}$ on $\mathbb{R}^2$.

- $\Psi(T)$ is a tree if T is finite.
- However $\Psi(T)$ is not a tree if T is infinite.
- Thus, $\Psi(2^{<\omega})$ is a dendrite which is not a tree.

Example

A Cantor fan and *a harmonic comb* are dendroids, but not dendrites.



Here a harmonic comb is defined by:
$\big([0,1]\times\{0\}\big)\cup\big((\{0\}\cup\{1/n : n\in\mathbb{N}\})\times[0,1]\big)$.

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Effectiveness for Tree-Like Continua

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5. Not every contractible planar co-c.e. dendroid contains a computable point.
   - This is the solution to *Question of Le Roux, and Ziegler*.

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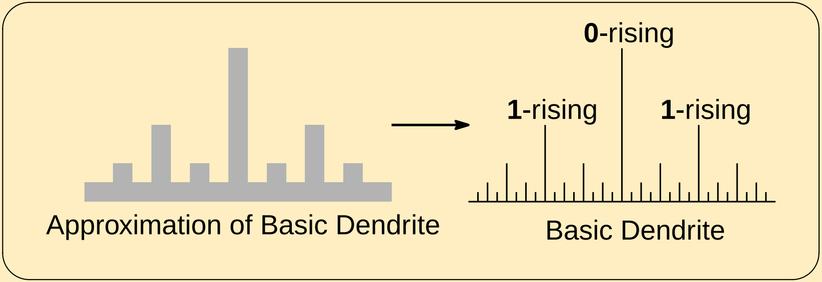
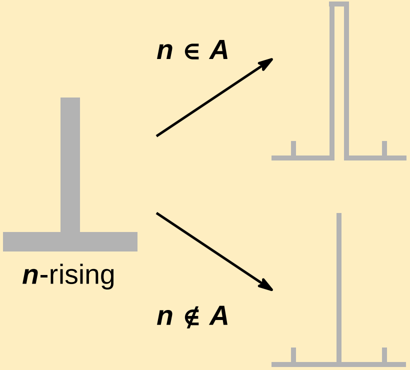


Basic Dendrite has $\mathbf{2}^{n}$ many $\boldsymbol{n}$*-risings* of height $\mathbf{2}^{-n}$.

Fix a non-computable c.e. set $A \subseteq \mathbb{N}$.

*The Basic construction around an **n**-rising* is following:



$n \in A \Longrightarrow$ an n-rising will be *a cut point*.

$n \notin A \Longrightarrow$ an n-rising will be *a ramification point*.

To prove the theorem, we need to prepare some tools.

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Lemma

1. *Every subdendrite of* $\Psi(2^{<\omega})$ *is homeomorphic to* $\Psi(T)$ *for a subtree* $T \subseteq 2^{<\omega}$.
2. $T \subseteq 2^{<\omega}$ *is co-c.e. closed (c.e., computable, resp.) tree iff* $\Psi(T) \subseteq \mathbb{R}^2$ *is co-c.e. closed (c.e., computable, resp.) dendrite.*
3. *Every computable subdendrite* $D \subseteq \Psi(2^{<\omega})$ *there exists a computable subtree* $T \subseteq 2^{<\omega}$ *such that* $D \subseteq \Psi(T)$ *holds, and* D *and* $\Psi(T)$ *has same paths.*

Definition (Cenzer-K.-Weber-Wu 2009)

A co-c.e. closed subset P of Cantor space is *tree-immune* if a co-c.e. tree $T_P \subseteq 2^{<\omega}$ has no infinite computable subtree.
Here $T_P = \{\sigma \in 2^{<\omega} : (\exists f \supset \sigma)\ f \in P\}$

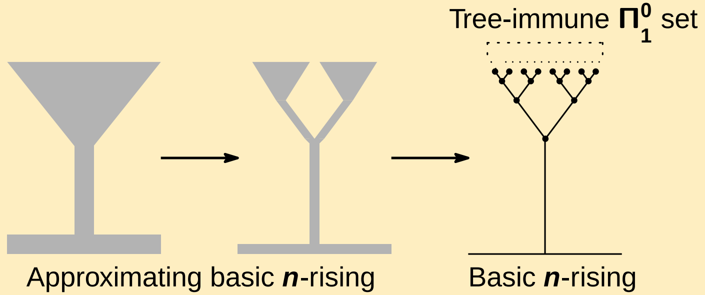
Example

The set of *all consistent complete extensions of Peano Arithmetic* is tree-immune.

Lemma

Let P be a tree-immune co-c.e. closed subset of Cantor space, and $D \subseteq \Psi(T_P)$ be any computable subdendrite. Then D contains no path.

Now we start *True Construction*.



An n-rising has a copy of *a tree-immune co-c.e. closed set* of scale 2^{-n}.

Fix a non-computable Σ^0_1 set $A \subseteq \mathbb{N}$.

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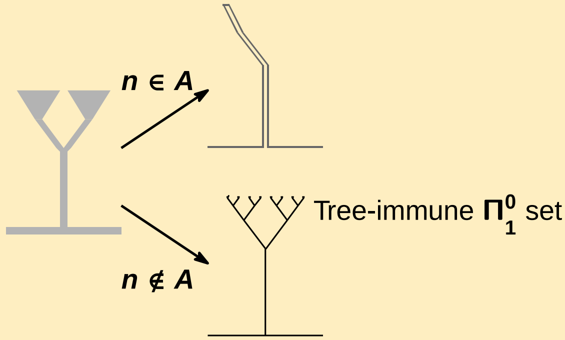

$n \in A \implies$ any top of an n-rising will be *a cut point*.

$n \notin A \implies$ any top of an n-rising will be *inaccessible* by computable dendrites.

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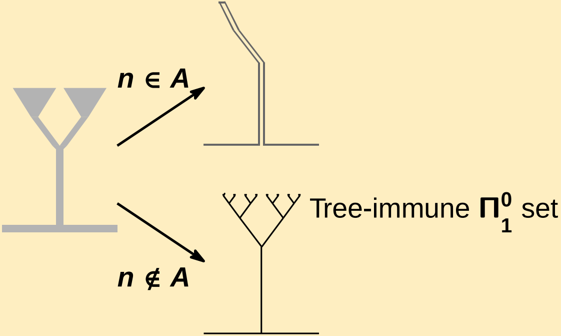

$n \in A \Longrightarrow$ If a dendrite D passes this n-rising, then D contains a top of this n-rising.

$n \notin A \Longrightarrow$ Any computable dendrite contains no top of n-rising.

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- This contradicts non-computability of A.

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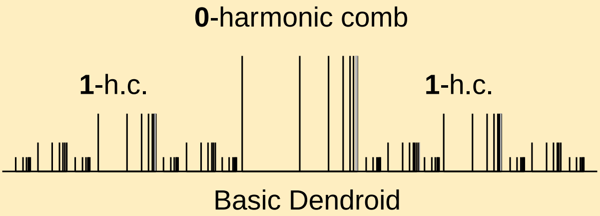
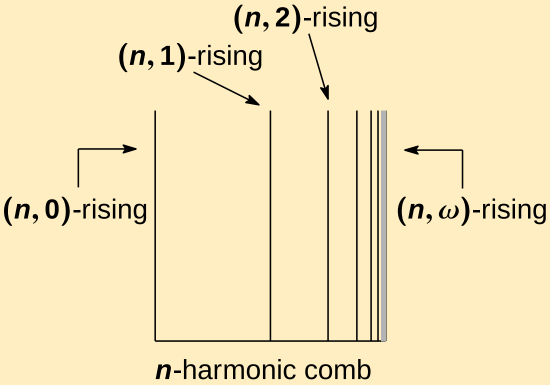
- It means that we can*not cut-pointize* infinite many risings, on one harmonic comb.
- Our idea is using a computable approximation of a certain *limit computable function*.
- One harmonic comb replaces one rising.
- *The Basic Dendroid* will be constructed by connecting infinitely many harmonic combs.



Basic Dendroid has $\mathbf{2}^{n}$ many ***n**-harmonic combs* of height $\mathbf{2}^{-n}$.
Each ***n**-harmonic comb* has infinitely many *risings*.



Basic Dendroid has 2^n many *n-harmonic combs* of height 2^{-n}.
Each *n-harmonic comb* has $(\omega + 1)$*-many risings*;
They are (n, α)-risings for $\alpha < \omega + 1$.

To prove the theorem, we need the following lemma.

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Lemma

There exists a limit computable function p such that, for every uniformly c.e. sequence $\{U_n\}$ of cofinite c.e. sets, it holds that $p(n) \in U_n$ for almost all n.

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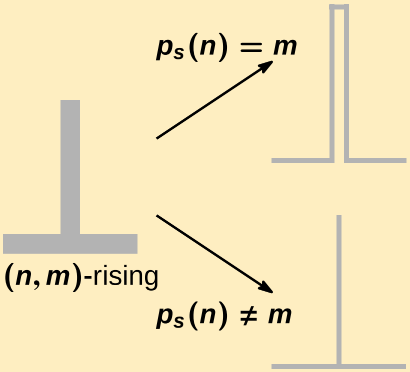
Proof

- $\{V_e\}$: an effective enumeration of uniformly c.e. decreasing sequence of c.e. sets.
- $\sigma(e,x) = \{i \le e : x \in (V_i)_e\}$: *The e-state of x*.
- $p(e)$ chooses x to maximize the e-state.

$p = \lim_s p_s$: a limit computable function in the previous lemma.
*The construction on an **n**-harmonic comb* is following:



$(\exists s)\ p_s(n) = m \Longrightarrow$ an (n, m)-rising will be *a cut point*.
$(\forall s)\ p_s(n) \neq m \Longrightarrow$ an (n, m)-rising will be *a ramification point*.

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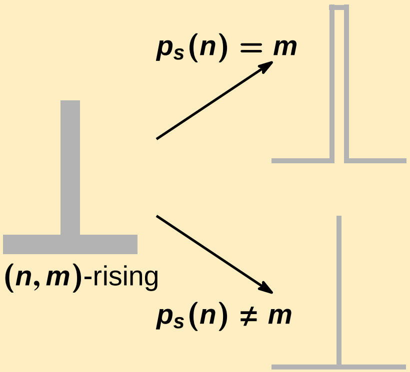
The construction on an n-harmonic comb is following:



Since $p(n) = \lim_s p_s(n)$ changes his mind at most finitely often, he *cut-pointizes only finitely many risings* on an n-harmonic comb.

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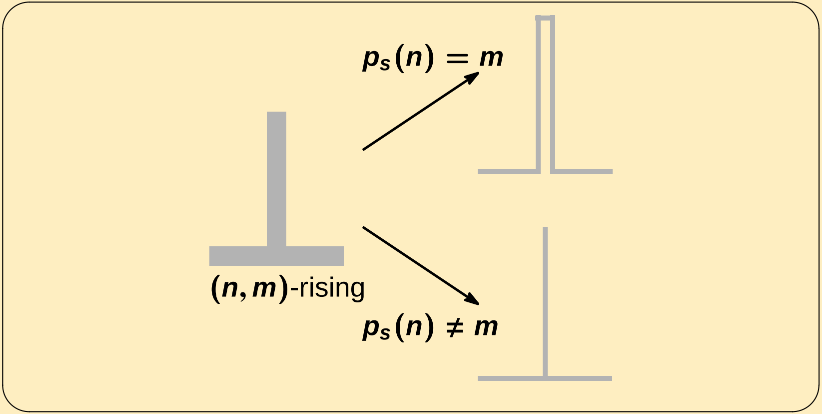
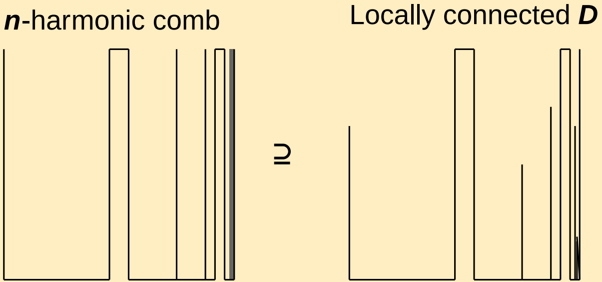
The construction on an n-harmonic comb is following:

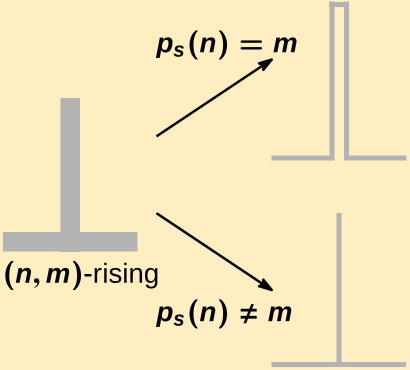


Thus each n-harmonic comb, actually, will be homeomorphic to a harmonic comb. The construction yields *computable dendroid K*.

Recall that a dendrite is a *locally connected* dendroid.
On a harmonic comb, any top of almost all rising must be *inaccessible* by a dendrite.

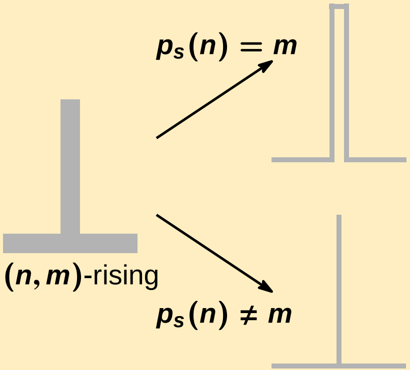


$\boldsymbol{D}$ contains tops of only three risings;
$(\boldsymbol{n}, \mathbf{1})$-rising; $(\boldsymbol{n}, \mathbf{4})$-rising; $(\boldsymbol{n}, \omega)$-rising



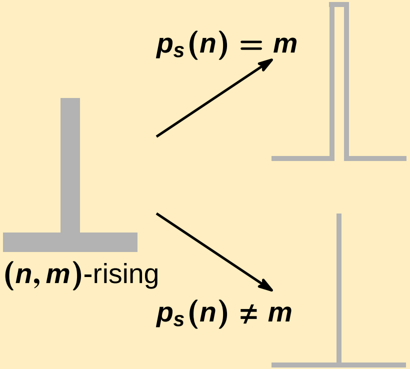
$(\exists s)\ p_s(n) = m \Longrightarrow$ any top of an (n, m)-rising will be *a cut point*.

Meanwhile, any top of almost all risings will be *inaccessible* by a given dendrite.



$(\exists s)\ p_s(n) = m \Longrightarrow$ If a dendrite D passes an (n, m)-rising, then D contains a top of an (n, m)-rising.

Meanwhile, any dendrite contains no top of almost all risings.



U_n^D: the set of all (n,m)-risings whose top is not accessed by a dendrite D. Then U_n^D is *cofinite* for all n.

If D passes n-harmonic comb then $p(n) \notin U_n^D$.

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- If D passes an n-comb, it must hold that $p(n) \notin U_n$.
- It contradicts our choice of p which satisfies $p(n) \in U_n$ for almost all n.

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Question

1. (Le Roux-Ziegler) Does every simply connected planar co-c.e. closed set contain a computable point?
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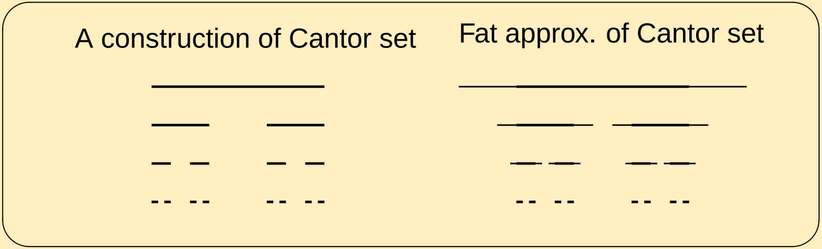
Main Theorem

Not every nonempty contractible planar co-c.e. closed set contains a computable point.

A fat approximation of Cantor set:



- P: a co-c.e. closed subset of Cantor set.
- P_s: a fat approximation of P at stage s.
- l_s, r_s: the leftmost and rightmost of P_s.

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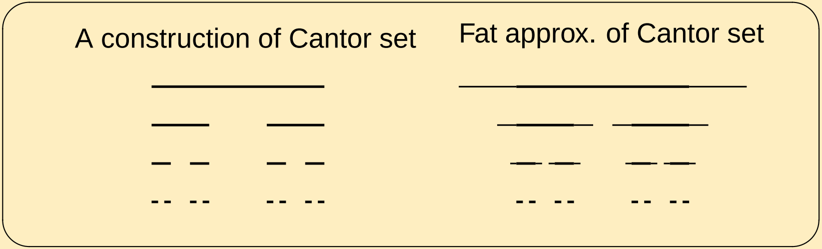

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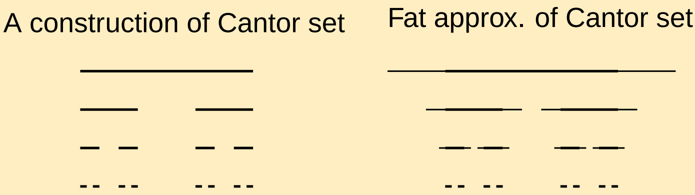

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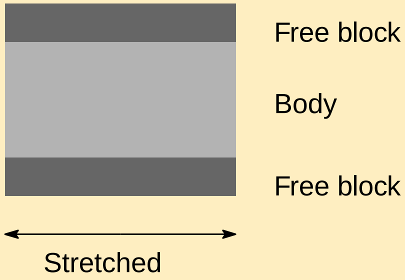
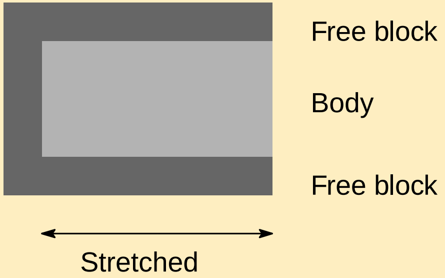
- l_s, r_s: the leftmost and rightmost of P_s.
- $[l_s, l_{s+1}] \cap P_s, [r_{s+1}, r_s] \cap P_s$ contains intervals I^l_s, I^r_s.
- We call these intervals $I^l_s, I^r_s \subseteq P_s \setminus P_{s+1}$ free blocks.

Prepare a stretched co-c.e. closed class $D_0^- = P \times [0,1]$.



- $P \subseteq \mathbb{R}^1$: a co-c.e. closed set without computable points.
- P_s: a fat approximation of P (Note that $P = \bigcap_s P_s$).
- $D_0^- = [0,1] \times P_0$.

D_0 is the following connected closed set.

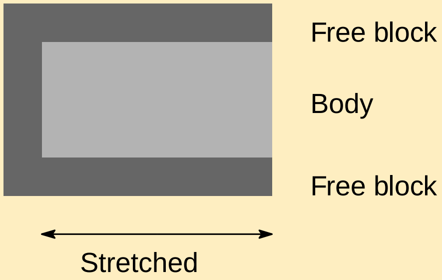


The desired co-c.e. closed set D will be obtained by carving D_0.

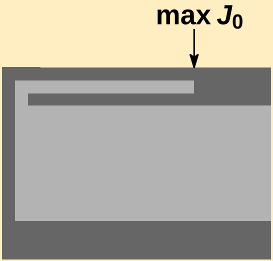
Destination

- $\alpha \in \mathbb{R}$: an incomputable left-c.e. real.
- There is a computable sequence $\{J_s\}$ of rational open intervals s.t.
  - $\min J_s \to \alpha$ as $s \to \infty$.
  - $\mathrm{diam}(J_s) \to 0$ as $s \to \infty$.
  - Either $J_{s+1} \subset J_s$ or $\max J_s < \min J_{s+1}$, for each s.

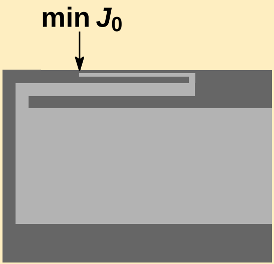
Our construction starts with $\boldsymbol{D_0}$.



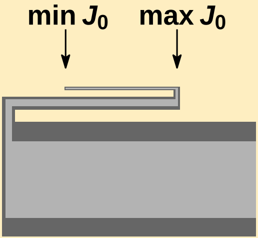
By carving free blocks, stretch $\boldsymbol{P_0}$ toward $\mathbf{max}\,\boldsymbol{J_0}$.



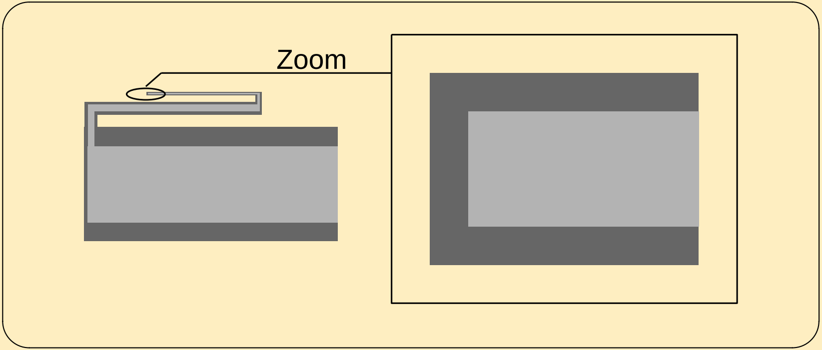
By carving free blocks, stretch P_0 toward $\min J_0$.



Proceed one step with a fat approximation of $\boldsymbol{P}$.



D_1 is defined by this,



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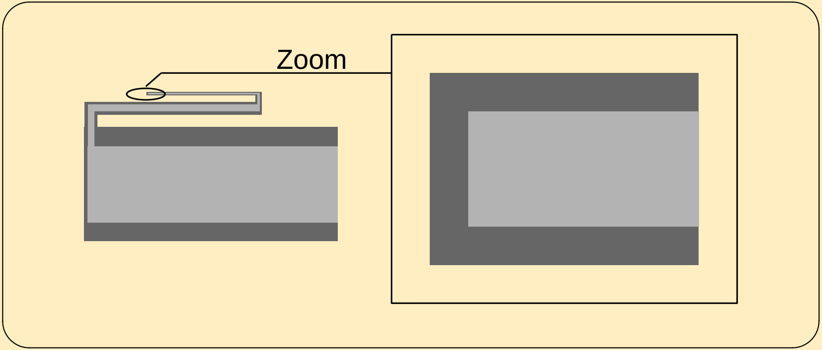

- If $J_1 \subset J_0$, then the construction of D_2 is similar as that of D_1.
- i.e., on the top block, stretch toward $\max J_1$ and back to $\min J_1$, by caving free blocks.

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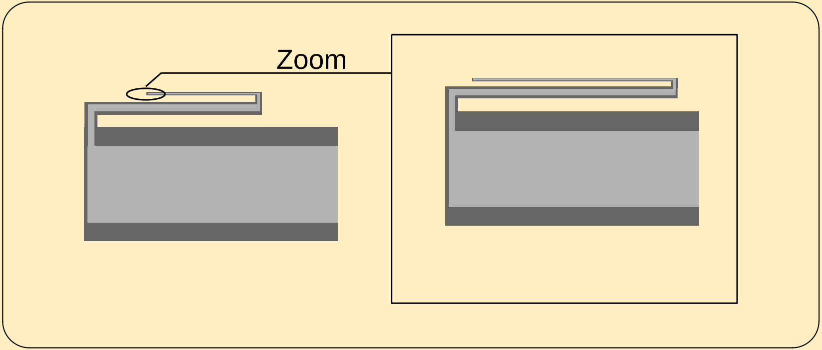

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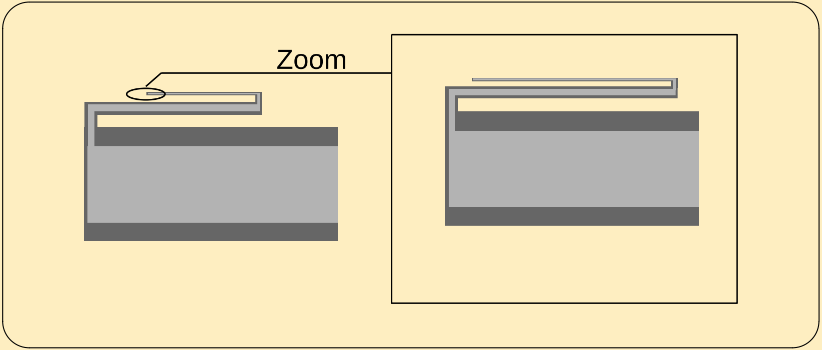

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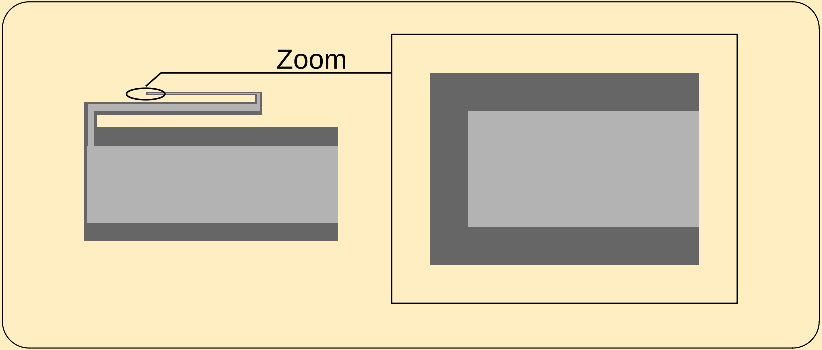

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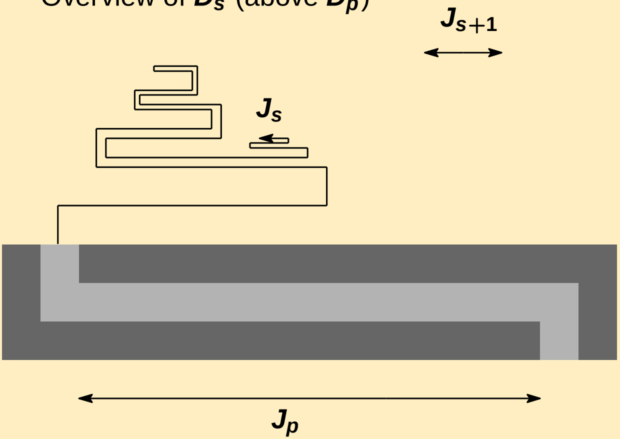
- Only the problem is the case of $J_{s+1} \not\subset J_s$!

In the case of $J_{s+1} \not\subset J_s$:

Overview of D_s (above D_p)



Pick the greatest $p \le s$ such that $J_{s+1} \subset J_p$.

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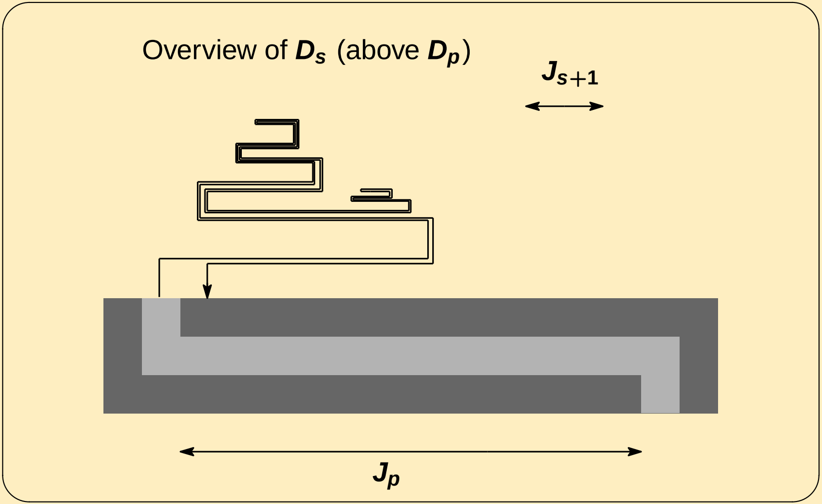

Go back to D_p by caving free blocks into the shape of P.

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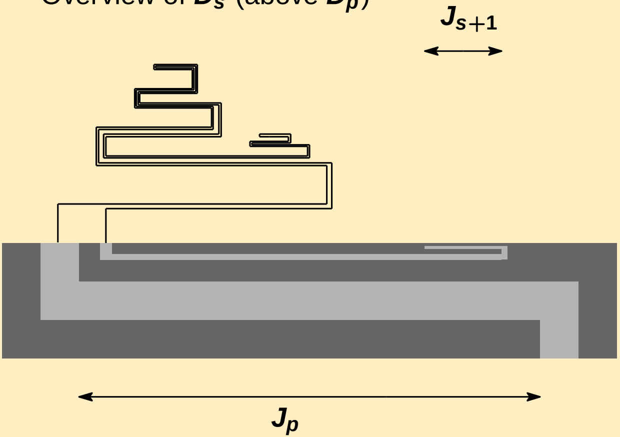

By caving free blocks on D_p into the shape of P, stretch toward **max** J_{s+1} and back to **min** J_{s+1}.

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    - This is the solution to *Question of Le Roux, and Ziegler*.

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Thank you!