Counterexamples in Computable Continuum Theory

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Local Computability:

- Every nonempty open set in $\mathbb{R}^n$ has a computable point.
- Not every nonempty co-c.e. closed set in $\mathbb{R}^n$ has a computable point (Kleene, Kreisel, etc. 1940’s–50’s).

Global Computability:

- If a co-c.e. closed set is homeomorphic to an $n$-sphere, then it is computable (Miller 2002).
- If a co-c.e. closed set is homeomorphic to an arc, then it is "almost" computable, i.e., every co-c.e. arc is approximated from the inside by computable arcs.

Let us study the computable content of Continuum Theory! Here, "Continuum Theory" is a branch of topology studying connected compact spaces.
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- If a nonempty co-c.e. closed subset $F \subseteq \mathbb{R}^1$ has no computable points, then $F$ must be disconnected.
- Does there exist a nonempty (simply) connected co-c.e. closed set in $\mathbb{R}^n$ without computable points?

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Computability Theory

Definition

\{B_e\}_{e \in \mathbb{N}}: \text{an effective enumeration of all rational open balls.}

1. \(x \in \mathbb{R}^n\) is \textit{computable} if \(\{e \in \mathbb{N} : x \in B_e\}\) is c.e.
   Equivalently, \(x = (x_1, \ldots, x_n) \in \mathbb{R}^n\) is computable iff \(x_i\) is computable for each \(i \leq n\).
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### Definition

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### Remark

- **\( F \) is co-c.e. closed \( \iff \) \( F \) is a computable point in the hyperspace \( \mathcal{A}_-(\mathbb{R}^n) \) of closed subsets of \( \mathbb{R}^n \) under lower Fell topology.**

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Fact

1. (Kleene, Kreisel, etc.) There exists a nonempty co-c.e. closed set $P \subseteq \mathbb{R}^1$ which has no computable point.

2. Every nonempty connected co-c.e. closed subset $P \subseteq \mathbb{R}^1$ contains a computable point.
Connected co-c.e. closed Sets

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Fact
1. There exists a nonempty connected co-c.e. closed subset $P^{(2)} \subseteq \mathbb{R}^2$ which has no computable point.
2. There exists a nonempty simply connected co-c.e. closed subset $P^{(3)} \subseteq \mathbb{R}^3$ which has no computable point.
- \( X \) is \( n \)-connected \iff \( \) the first \( n + 1 \) homotopy groups vanish identically.

- \( X \) is path-connected \iff \( X \) is 0-connected.

- \( X \) is simply connected \iff \( X \) is 1-connected.

- \( X \) is contractible \iff \( \) the identity map on \( X \) is null-homotopic.

- \( X \) is contractible \implies \( X \) is \( n \)-connected for any \( n \).

**Observation**

Not every nonempty \( n \)-connected co-c.e. closed set in \( \mathbb{R}^{n+2} \) contains a computable point, for any \( n \in \mathbb{N} \).
Observation (Restated)

- **Not** every nonempty *n*-connected co-c.e. closed set in $\mathbb{R}^{n+2}$ contains a computable point, for any $n \in \mathbb{N}$.
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- Every nonempty *n*-connected co-c.e. closed set in \( \mathbb{R}^{n+1} \) contains a computable point, for \( n = 0 \).
Observation (Restated)

- Not every nonempty $n$-connected co-c.e. closed set in $\mathbb{R}^{n+2}$ contains a computable point, for any $n \in \mathbb{N}$.
- Every nonempty $n$-connected co-c.e. closed set in $\mathbb{R}^{n+1}$ contains a computable point, for $n = 0$.

Question

1. (Le Roux-Ziegler) Does every simply connected planar co-c.e. closed set contain a computable point?
2. Does every contractible Euclidean co-c.e. closed set contain a computable point?
\( A \in P \) is \( \varepsilon \)-approximated from the inside by \( B \in Q \)

\[ d_H(A, B) < \varepsilon \]

**Definition**

1. The *Hausdorff distance* between nonempty closed subsets \( A_0, A_1 \) of a metric space \( (X, d) \) is defined by:
   \[ d_H(A_0, A_1) = \max_{i<2} \sup_{x \in A_i} \inf_{y \in B_{1-i}} d(x, y). \]

2. \( P, Q \): classes of continua.
   \( P \) is *approximated (from the inside)* by \( Q \) if
   \[ (\forall A \in P) \ \inf \{ d_H(B, A) : A \supseteq B \in Q \} = 0. \]
Proposition

Arc-Connected Continua is approximated by Locally Connected Continua.

Proof

1. By compactness, $X$ has an $\varepsilon$-net $\{x_i\}_{i<n} \subseteq X$ for any $\varepsilon > 0$. (i.e., $\bigcup_{i<n} B(x_i; \varepsilon)$ covers $X$)
2. Let $\gamma_{ij} \subseteq X$ be an arc with end points $x_i$ and $x_j$.
3. $Y = \bigcup_{i,j<n} \gamma_{ij} \subseteq X$, and $d_H(Y, X) \leq \varepsilon$.
4. $\gamma_{ij}^*$ is inductively defined as:
   - $\gamma_{ij}^* \subseteq \gamma_{ij} \cup \bigcup_{(k,l)<(i,j)} \gamma_{kl}$.
   - If $\gamma_{ij}$ intersects with $\bigcup_{(k,l)<(i,j)} \gamma_{kl}$, then $\gamma_{ij}^* \cap \bigcup_{(k,l)<(i,j)} \gamma_{kl}$ is an arc.
5. $Y^* = \bigcup_{i,j<n} \gamma_{ij}^*$ is locally connected, $Y^* \subseteq Y \subseteq X$, and $d_H(Y^*, X) \leq \varepsilon$. 
If a continua in a class \( C \) has no computable point, then \( C \) is not approximated by Computable Closed Sets.

Theorem (Miller 2002; Iljazović 2009)

1. Every Euclidean co-c.e. \( n \)-sphere is computable. Hence, Every co-c.e. Jordan curve is computable.
2. Co-c.e. Arcs is approximated by Computable Arcs. In this sense, every co-c.e. arc is “almost” computable.
Let $S$ be a topological space.

1. $S$ is **connected** if it is not union of disjoint open sets.
2. $S$ is **locally connected** if it has a base of connected sets.
3. A **continuum** is a connected compact metric space.
4. A **dendroid** is a continuum $S$ such that $(\forall x, y \in S)$
   
   $$S[x, y] = \min\{Y \subseteq X : x, y \in Y & Y \text{ is connected}\}$$
   
   exists, and such $S[x, y]$ is an arc.
5. A **dendrite** is a locally connected dendroid.
6. A **tree** is a dendrite with finitely many ramification points.
Example

We plot a tree \( T \subseteq 2^{<\omega} \) on the Euclidean plane \( \mathbb{R}^2 \). Then the plotted picture \( \Psi(T) \subseteq \mathbb{R}^2 \) is a dendrite.

\[ \Psi(2^{<\mathbb{N}}) \]

Protting \( 2^{<\mathbb{N}} \) on \( \mathbb{R}^2 \).

- \( \Psi(T) \) is a tree if \( T \) is finite.
- However \( \Psi(T) \) is not a tree if \( T \) is infinite.
- Thus, \( \Psi(2^{<\omega}) \) is a dendrite which is not a tree.
A Cantor fan and a harmonic comb are dendroids, but not dendrites.

Here a harmonic comb is defined by:

$$
\left( [0, 1] \times \{0\} \right) \cup \left( \left( \{0\} \cup \{1/n : n \in \mathbb{N}\} \right) \times [0, 1] \right).
$$
Remark

Dendroids is approximated by Trees.
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Main Theorem

**Computable Dendrites** is **not** approximated by **Co-c.e. Trees**.
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2. **Co-c.e. Dendrites** is not approximated by **Computable Dendrites**.
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Not every contractible planar co-c.e. dendroid contains a computable point. This is the solution to Question of Le Roux, and Ziegler.
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5. **Not** every contractible planar co-c.e. dendroid contains a computable point.
   - This is the solution to *Question of Le Roux, and Ziegler*. 
**Theorem**

**Co-c.e. Dendrites is not approximated by Computable Dendrites.**

---

**Basic Dendrite** has \(2^n\) many *n-risings* of height \(2^{-n}\).
Fix a non-computable c.e. set $A \subseteq \mathbb{N}$.

The Basic construction around an $n$-rising is following:

- If $n \in A$, then an $n$-rising will be a cut point.
- If $n \notin A$, then an $n$-rising will be a ramification point.
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Lemma

1. Every subdendrite of $\Psi(2^{<\omega})$ is homeomorphic to $\Psi(T)$ for a subtree $T \subseteq 2^{<\omega}$.

2. $T \subseteq 2^{<\omega}$ is co-c.e. closed (c.e., computable, resp.) tree iff $\Psi(T) \subseteq \mathbb{R}^2$ is co-c.e. closed (c.e., computable, resp.) dendrite.

3. Every computable subdendrite $D \subseteq \Psi(2^{<\omega})$ there exists a computable subtree $T \subseteq 2^{<\omega}$ such that $D \subseteq \Psi(T)$ holds, and $D$ and $\Psi(T)$ has same paths.
Definition (Cenzer-K.-Weber-Wu 2009)

A co-c.e. closed subset $P$ of Cantor space is **tree-immune** if a co-c.e. tree $T_P \subseteq 2^{<\omega}$ has no infinite computable subtree. Here $T_P = \{\sigma \in 2^{<\omega} : (\exists f \supset \sigma) f \in P\}$.

Example

The set of all consistent complete extensions of Peano Arithmetic is tree-immune.

Lemma

Let $P$ be a tree-immune co-c.e. closed subset of Cantor space, and $D \subseteq \Psi(T_P)$ be any computable subdendrite. Then $D$ contains no path.
Now we start *True Construction*.

An $n$-rising has a copy of a tree-immune co-c.e. closed set of scale $2^{-n}$.
Fix a non-computable $\Sigma_1^0$ set $A \subseteq \mathbb{N}$.

The True Construction around an $n$-rising is following:

- $n \in A \implies$ any top of an $n$-rising will be a cut point.
- $n \notin A \implies$ any top of an $n$-rising will be inaccessible by computable dendrites.

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Fix a non-computable $\Sigma_1^0$ set $A \subseteq \mathbb{N}$.

The True Construction around an $n$-rising is following:

$n \in A \implies$ If a dendrite $D$ passes this $n$-rising, then $D$ contains a top of this $n$-rising.

$n \notin A \implies$ Any computable dendrite contains no top of $n$-rising.
The construction of the co-c.e. closed dendrite $H$ is completed.
Theorem (Restated)

Co-c.e. Dendrites is not approximated by Computable Dendrites.

- The construction of the co-c.e. closed dendrite $H$ is completed.
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- It suffices to show that $D$ cannot pass 2 distinct risings.
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Let \( D \subseteq H \) be any computable dendrite.

It suffices to show that \( D \) cannot pass 2 distinct risings.

If \( D \) passes \( m, n \)-risings, then \( D \) passes a \( k \)-rising for all \( k \geq \min\{m, n\} \).
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- If $D$ passes $m, n$-risings, then $D$ passes a $k$-rising for all $k \geq \min\{m, n\}$.
- Since $D$ is co-c.e. closed, we can enumerate all $k$ such that $D$ contains no top of any $k$-rising.
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- This enumeration yields the complement of a c.e. set $A$. 

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- If $D$ passes $m, n$-risings, then $D$ passes a $k$-rising for all $k \geq \min\{m, n\}$.
- Since $D$ is co-c.e. closed, we can enumerate all $k$ such that $D$ contains no top of any $k$-rising.
- This enumeration yields the complement of a c.e. set $A$.
- This contradicts non-computability of $A$. 
Theorem

**Computable Dendroids is not approximated by Co-c.e. Dendrites.**
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- We will use *harmonic combs* in place of *the Basic Dendrite.*
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- We will use harmonic combs in place of the Basic Dendrite.
- Before starting the construction, we take account of the fact that topologist’s sine curve is not path-connected.
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- We will use harmonic combs in place of the Basic Dendrite.
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- It means that we cannot cut-pointize infinite many risings, on one harmonic comb.
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- Before starting the construction, we take account of the fact that *topologist's sine curve* is not path-connected.
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- Our idea is using a computable approximation of a certain *limit computable function*. 
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- It means that we cannot *cut-pointize* infinite many risings, on one harmonic comb.
- Our idea is using a computable approximation of a certain *limit computable function*.
- One harmonic comb replaces one rising.
- *The Basic Dendroid* will be constructed by connecting infinitely many harmonic combs.
Basic Dendroid has $2^n$ many $n$-harmonic combs of height $2^{-n}$. Each $n$-harmonic comb has infinitely many risings.
Basic Dendroid has $2^n$ many \textit{$n$-harmonic combs} of height $2^{-n}$. Each \textit{$n$-harmonic comb} has $(\omega + 1)$-many risings; They are $(n, \alpha)$-risings for $\alpha < \omega + 1$. 
To prove the theorem, we need the following lemma.

There exists a limit computable function $p$ such that, for every uniformly c.e. sequence $\{U_n\}$ of cofinite c.e. sets, it holds that $p(n) \in U_n$ for almost all $n$.

Proof. $\{V_e\}$: an effective enumeration of uniformly c.e. decreasing sequence of c.e. sets. $(e; x) = \{i \leq e: x \in (V_i)_e\}$: The $e$-state of $x$. $p(e)$ chooses $x$ to maximize the $e$-state.
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**Lemma**

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**Proof**

- \( \{V_e\} \): an effective enumeration of uniformly c.e. decreasing sequence of c.e. sets.
- \( \sigma(e, x) = \{i \leq e : x \in (V_i)_e\} \): *The e-state of x*.
- \( p(e) \) chooses \( x \) to maximize the e-state.
\( p = \lim_s \rho_s \): a limit computable function in the previous lemma.

*The construction on an \( n \)-harmonic comb* is following:

\[
\rho_s(n) = m
\]

\((n, m)\)-rising

\[
\rho_s(n) \neq m
\]

\((\exists s) \rho_s(n) = m \Rightarrow \text{an } (n, m)\text{-rising will be a cut point.} \]

\((\forall s) \rho_s(n) \neq m \Rightarrow \text{an } (n, m)\text{-rising will be a ramification point.} \)
\( p = \lim_s p_s \): a limit computable function in the previous lemma. 

The construction on an \( n \)-harmonic comb is following:

\[
\begin{align*}
\forall m \in \mathbb{N} & , \\
\lim_{s \to m} p_s(n) & = m \\
\lim_{s \to m} p_s(n) & \neq m
\end{align*}
\]

Since \( p(n) = \lim s p_s(n) \) changes his mind at most finitely often, he cut-pointizes only finitely many risings on an \( n \)-harmonic comb.
\( p = \lim_{s} p_s \): a limit computable function in the previous lemma. 

*The construction on an \( n \)-harmonic comb* is following:

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p_s(n) = m
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\[
(n, m) \text{-rising}
\]

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p_s(n) \neq m
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Thus each \( n \)-harmonic comb, actually, will be homeomorphic to a harmonic comb. The construction yields *computable dendroid* \( K \).
Recall that a dendrite is a *locally connected* dendroid. On a harmonic comb, any top of almost all rising must be *inaccessible* by a dendrite.

\[ \supseteq \]

Locally connected \( D \)

\( n \)-harmonic comb

\( D \) contains tops of only three risings; \((n, 1)\)-rising; \((n, 4)\)-rising; \((n, \omega)\)-rising
$(\exists s)\ p_s(n) = m \implies$ any top of an $(n, m)$-rising will be a cut point.

Meanwhile, any top of almost all risings will be inaccessible by a given dendrite.
If a dendrite $D$ passes an $(n, m)$-rising, then $D$ contains a top of an $(n, m)$-rising. Meanwhile, any dendrite contains no top of almost all risings.
$p_s(n) = m$

$\begin{align*}
\text{(n, m)-rising} \\
\downarrow \\
p_s(n) &\neq m
\end{align*}$

$U_D^n$: the set of all $(n, m)$-risings whose top is not accessed by a dendrite $D$. Then $U_D^n$ is cofinite for all $n$. If $D$ passes $n$-harmonic comb then $p(n) \notin U_D^n$. 
Theorem (Restated)

**Computable Dendroids** is not approximated by **Co-c.e. Dendrites.**

- \( K \): the computable dendroid in the construction.
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- $U_n$ is cofinite by previous observation.
- $\{U_n\}$ is uniformly c.e., since $D$ is co-c.e. closed.
- It suffices to show that $D$ cannot pass 2 distinct combs.
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Observation (Restated)

- Not every nonempty $n$-connected co-c.e. closed set in $\mathbb{R}^{n+2}$ contains a computable point, for any $n \in \mathbb{N}$. 
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- Every nonempty $n$-connected co-c.e. closed set in $\mathbb{R}^{n+1}$ contains a computable point, for $n = 0$. 

1. (Le Roux-Ziegler) Does every simply connected planar co-c.e. closed set contain a computable point?
2. Does every contractible Euclidean co-c.e. closed set contain a computable point?

Main Theorem

Not every nonempty contractible planar co-c.e. closed set contains a computable point.
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A fat approximation of Cantor set:

- $P$: a co-c.e. closed subset of Cantor set.
- $P_s$: a fat approximation of $P$ at stage $s$.
- $l_s, r_s$: the leftmost and rightmost of $P_s$. 

We call these intervals $I_{l_s}; I_{r_s} \subseteq P_s \setminus P_s$ free blocks.
A fat approximation of Cantor set:

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Prepare a stretched co-c.e. closed class $D_0^- = P \times [0, 1]$.

- $P \subseteq \mathbb{R}^1$: a co-c.e. closed set without computable points.
- $P_s$: a fat approximation of $P$ (Note that $P = \bigcap_s P_s$).
- $D_0^- = [0, 1] \times P_0$. 
$D_0$ is the following connected closed set.

The desired co-c.e. closed set $D$ will be obtained by carving $D_0$. 
\( \alpha \in \mathbb{R} \): an incomputable left-c.e. real.

There is a computable sequence \( \{J_s\} \) of rational open intervals s.t.

1. \( \min J_s \to \alpha \) as \( s \to \infty \).
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\( \text{diam}(J_s) \to 0 \) as \( s \to \infty \).
3. Either \( J_{s+1} \subset J_s \) or \( \max J_s < \min J_{s+1} \), for each \( s \).
Our construction starts with $D_0$. 

![Diagram showing a stretched block with free blocks on the sides.](image-url)
By carving free blocks, stretch $P_0$ toward $\max J_0$. 

max $J_0$
By carving free blocks, stretch $P_0$ toward $\min J_0$. 
Proceed one step with a fat approximation of $P$. 

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Only the problem is the case of $J_{s+1} \not\subset J_s$!
In the case of $J_{s+1} \not\subset J_s$:

Overview of $D_s$ (above $D_p$)

Pick the greatest $p \leq s$ such that $J_{s+1} \subset J_p$. 
In the case of $J_{s+1} \not\subseteq J_s$:

Overview of $D_s$ (above $D_p$)

Go back to $D_p$ by caving free blocks into the shape of $P$. 
Overview of $D_s$ (above $D_p$)

By caving free blocks on $D_p$ into the shape of $P$, stretch toward $\max J_{s+1}$ and back to $\min J_{s+1}$.
Main Theorem (Restated)

Not every nonempty contractible planar co-c.e. closed set contains a computable point.

\[ D = \bigcap_s D_s \] is co-c.e. closed.
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Stretching \([0, 1] \times P\) cannot introduce new computable points.

Of course, \((\alpha, y)\) is also incomputable.

Hence, \(D\) has no computable points.
Main Theorem (Restated)

1. **Computable Dendrites** is not approximated by **Co-c.e. Trees**.
2. **Co-c.e. Dendrites** is not approximated by **Computable Dendrites**.
3. **Computable Dendroids** is not approximated by **Co-c.e. Dendrites**.
4. **Co-c.e. Dendroids** is not approximated by **Computable Dendroids**.
5. **Not** every contractible planar co-c.e. dendroid contains a computable point.

   - This is the solution to *Question of Le Roux, and Ziegler*. 

Not every contractible planar co-c.e. dendroid contains a computable point.
Corollary

There is a contractible & locally contractible & computable closed set, which is \( \text{computable Path} \)-connected, but not \( \text{co-c.e. Arc} \)-connected.

\( \text{Connected & Locally Connected & Computable Closed Sets} \) is not approximated by \( \text{Connected & Locally Connected & computable Closed Sets} \).

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Not every contractible planar co-c.e. closed set contains a computable point. This is the solution to Question of Le Roux, and Ziegler.

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Thank you!