Martin-like phenomena in the classification of real-valued functions

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Day-Downey-Westrick (DDW) recently introduced $m$-, $tt$-, and $T$-reducibility for real-valued functions (cf. Linda’s talk on Thursday).

We give a full description of the structures of DDW’s $m$- and $T$-degrees of real-valued functions.

**Caution:** Without mentioning, we always assume Woodin’s $\text{AD}^+$. But, of course:

- If we restrict our attention to Borel sets and Baire functions, every result presented in this talk is provable within $\text{ZFC}$.
- If we restrict our attention to projective sets and functions, every result presented in this talk is provable within $\text{ZF+DC+PD}$.
- We even have $L(\mathbb{R}) \models \text{AD}^+$, assuming that there are arbitrarily large Woodin cardinals.
1st Main Theorem (K.)

The structure of the DDW-$m$-degrees of real-valued functions looks like the following figure:

More precisely...

- The DDW-$m$-degrees form a semi-well-order of height $\Theta$.
- The ranks $\alpha + j + 3n$ and $\alpha + j + 3n + 1$ consist of single equivalence classes $\Delta_{\alpha+n}^{jr}$ and $\Delta_{\alpha+n}$, respectively.
- The rank $\alpha + j + 3n + 2$ consists of two equiv. classes $\Sigma_{\alpha+n}$ and $\Pi_{\alpha+n}$.
- Here, $j = 0$ if $\alpha = 1$; $j = -1$ if $\text{cf}(\alpha) = \omega$; $j = -2$ if $\text{cf}(\alpha) > \omega$.

$\Delta_{1}^{jr}$ = constant functions; $\Delta_{1}$ = continuous functions;
$\Sigma_{1}$ = lower semi-continuous; $\Pi_{1}$ = upper semi-continuous. (cf. Linda’s talk)
Theorem (K.-Montalbán; 201x)
The Wadge degrees $\approx$ the “natural” many-one degrees.

DDW defined $T$-reducibility for $\mathbb{R}$-valued functions as parallel continuous (strong) Weihrauch reducibility (cf. Linda’s talk on Thursday).

2nd Main Theorem (K.)
The parallel continuous (strong) Weihrauch degrees of total single-valued functions are exactly the “natural” Turing degrees.

(Steel ’82; H. Becker ’88) The “natural” Turing degrees form a well-order of type $\Theta$.

Corollary
The parallel continuous (strong) Weihrauch degrees of total single-valued functions form a well-order of length $\Theta$.

Consequently, the parallel continuous (strong) Weihrauch degrees gives us a natural extension of the Baire hierarchy to non-Baire functions
(Note: the $\widehat{cW}$-rank of a Baire function coincides $2^+$ its Baire rank)
Introduction to descriptive set theory
• Pointclass: $\Gamma \subseteq \omega^\omega$
• Dual: $\check{\Gamma} = \{\omega^\omega \setminus A : A \in \Gamma\}$.
• A pointclass $\Gamma$ is selfdual iff $\Gamma = \check{\Gamma}$.
• For $A, B \subseteq \omega^\omega$, $A$ is Wadge reducible to $B$ ($A \leq_w B$) if $(\exists \theta$ continuous$()(\forall X \in \omega^\omega) X \in A \iff \theta(X) \in B$.
• $A \subseteq \omega^\omega$ is selfdual if $A \equiv_w \omega^\omega \setminus A$.
• $A \subseteq \omega^\omega$ is selfdual iff $\Gamma_A = \{B \in \omega^\omega : B \leq_w A\}$ is selfdual.

$\Delta^i_\alpha$ is selfdual, but $\Sigma^i_\alpha$ and $\Pi^i_\alpha$ are nonselfdual.

Theorem (Wadge; Martin-Monk 1970s)
The Wadge degrees are semi-well-ordered.

In particular, nonselfdual pairs are well-ordered, say $(\Gamma_\alpha, \check{\Gamma}_\alpha)_{\alpha<\Theta}$ where $\Theta$ is the height of the Wadge degrees.
A pointclass $\Gamma$ has the **separation property** if

$$(\forall A, B \in \Gamma) [A \cap B = \emptyset \implies (\exists C \in \Gamma \cap \check{\Gamma}) A \subseteq C \& B \cap C = \emptyset]$$

**Example (Lusin 1927, Novikov 1935, and others)**

- $\Pi^0_\alpha$ has the separation property for any $\alpha < \omega_1$.
- $\Sigma^1_1$ and $\Pi^1_2$ have the separation property.
- (PD) $\Sigma^1_{2n+1}$ and $\Pi^1_{2n+2}$ have the separation property.
Nonselfdual pairs are well-ordered, say \((\Gamma_\alpha, \check{\Gamma}_\alpha)_{\alpha<\Theta}\).

**Theorem (Van Wasep 1978; Steel 1981)**

Exactly one of \(\Gamma_\alpha\) and \(\check{\Gamma}_\alpha\) has the separation property.

- \(\Pi_\alpha\): the one which has the separation property
- \(\Sigma_\alpha\): the other one
- \(\Delta_\alpha = \Sigma_\alpha \cap \Pi_\alpha\)

**Example**

\(\Delta_1 = \text{clopen } (\Delta_1^0)\); \(\Sigma_1 = \text{open } (\Sigma_1^0)\); \(\Pi_1 = \text{closed } (\Pi_1^0)\);

\(\Delta_\alpha, \Sigma_\alpha, \Pi_\alpha (\alpha < \omega_1)\): the \(\alpha^{\text{th}}\) level of the Hausdorff difference hierarchy

\(\Sigma_{\omega_1} = F_{\sigma^0} (\Sigma_2^0)\); \(\Pi_{\omega_1} = G_\delta (\Pi_2^0)\)
Example

$\Sigma_{\omega_1} = F_\sigma$ (i.e. $\Sigma^0_2$); $\Pi_{\omega_1} = G_\delta$ (i.e. $\Pi^0_2$)

$\Sigma^2_{\omega_1}$ = the difference of two $G_\delta$ sets; $\Pi^2_{\omega_1}$ = the diff. of two $F_\sigma$ sets

$\Delta_{\omega_1^\alpha}, \Sigma^\alpha_{\omega_1}, \Pi^\alpha_{\omega_1}$ ($\alpha < \omega_1$): the $\alpha^{th}$ level of the diff. hierarchy over $F_\sigma$

$a_0 \to \cdots \to a_n$: an ordered space endowed with a Sierpiński-like representation;
circle $\bigcirc$: the jump of an inner represented space.
(Ex. $A \subseteq \omega^\omega$ is $F_\sigma$ (i.e. $\Sigma^0_2$) iff $\chi_A : \omega^\omega \to \mathcal{S}'$ is continuous)
Example

$\Sigma_{\omega_1}^{\omega_1} = G_{\delta\sigma} (\Sigma_3^0)$; $\Pi_{\omega_1}^{\omega_1} = F_{\sigma\delta} (\Pi_3^0)$

$\Sigma_{\omega_1}^{\omega_2} = \text{the difference of two } F_{\sigma\delta} \text{ sets};$ $\Pi_{\omega_1}^{\omega_2} = \text{the diff. of two } G_{\delta\sigma} \text{ sets}$

$\Delta_{\omega_1}^{\alpha}, \Sigma_{\omega_1}^{\alpha}, \Pi_{\omega_1}^{\alpha} (\alpha < \omega_1): \text{the } \alpha^{\text{th}} \text{ level of the diff. hierarchy over } G_{\delta\sigma}$
Example

- $\Sigma_2^0$, $\Pi_2^0$: Wadge-rank $\omega_1$.
- $\Sigma_3^0$, $\Pi_3^0$: Wadge-rank $\omega_1^{\omega_1}$.
- $\Sigma_n^0$, $\Pi_n^0$: Wadge-rank $\omega_1 \uparrow\uparrow n$ (the $n$th level of the superexp. hierarchy).
- $\varepsilon_0[\omega_1] := \lim_{n \to \infty} (\omega_1 \uparrow\uparrow n)$: Its cofinality is $\omega$. Hence, the class of rank $\varepsilon_0[\omega_1]$ is selfdual. Moreover, $\Delta_{\varepsilon_0}[\omega_1]$ is far smaller than $\Delta_0^\omega$. (Because we can use arbitrarily deep nests of circles to define a $\Delta_0^\omega$ set.)
- (Duparc 2001) $\varepsilon_{\omega_1}[\omega_1]$: the $\omega_1$th fixed point of the exp. of base $\omega_1$. $\Sigma_\omega^0$, $\Pi_\omega^0$: Wadge-rank $\varepsilon_{\omega_1}[\omega_1]$. 

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(Duparc 2001) The Veblen hierarchy of base $\omega_1$:
- $\phi_\alpha(\gamma)$: the $\gamma^{th}$ ordinal closed under $+$, $\sup_{n \in \omega}$, and $(\phi_\beta)_{\beta < \alpha}$.
- $\phi_0$ enumerates $1, \omega_1, \omega_1^2, \omega_1^3, \ldots, \omega_1^{\omega+1}, \omega_1^{\omega+2}, \ldots$
- $\phi_1$ enumerates $1, \varepsilon_{\omega_1}[\omega_1], \ldots$
- W-rank $\phi_0(\gamma) \approx$ a well-founded nest of circles of rank $\gamma$.
- W-rank $\phi_\alpha(\gamma) \approx$ a well-founded nest of $\omega^\alpha$-circles of rank $\gamma$.

An $\omega^\alpha$-circle corresponds to the $\omega^\alpha$-th jump of a representation.

Thus, every Borel Wadge degree is characterized in terms of representation

- $\Sigma_0^0, \Pi_0^0$: Wadge-rank $\phi_\alpha(1)$ ($0 < \alpha < \omega_1$).
- $\Sigma_1^1, \Pi_1^1$: Wadge-rank $\sup_{\xi < \omega_1} \phi_\xi(1)$.
Definition

Let $Q$ be a partial order. For $A, B : \omega^\omega \to Q$, $A$ is $Q$-Wadge reducible to $B$ ($A \leq^w_Q B$) if

$$(\exists \theta \text{ continuous})(\forall X \in \omega^\omega) A(X) \leq_Q B \circ \theta(X).$$

- $2 = \{0, 1\}$: 0 and 1 are incomparable.
- $\mathbb{T} = \{0, 1, \bot\}$: Plotkin's order; $\bot < 0, 1$.
- Wadge studied the $2$- and $\mathbb{T}$-Wadge degrees.
- (Wadge's Lemma) The $\mathbb{T}$-Wadge degrees are semi-linear-ordered.
- (Van Engelen-Miller-Steel 1987; Block 2014) If $Q$ is BQO, so is the $Q$-Wadge degrees.
  In particular, the $\mathbb{T}$-Wadge degrees are semi-well-ordered.

(K.-Montalbán 201x) If $Q$ is BQO, every Borel $Q$-Wadge degree is characterized in terms of represented spaces as before.
(using Sierpiński-like representations of $Q$-labeled orders and $\omega^\alpha$-circles.)
Embedding Lemma for the DDW-\(m\)-degrees

1st Main Theorem (K.)

The structure of the DDW-\(m\)-degrees of real-valued functions looks like the following figure:
Let $Q$ be a partial order. For $A, B : \omega^\omega \to Q$, $A$ is $Q$-$m$-Wadge reducible to $B$ ($A \leq_{mw}^Q B$) if there are $\psi : \omega \to \omega$ and a continuous function $\theta : \omega^\omega \to \omega^\omega$ such that

$$(\forall X \in \omega^\omega) \ A(m \upharpoonright X) \leq_Q B(\psi(m) \upharpoonright \theta(m, X)).$$

- $A \leq_{Lip}^Q B \implies A \leq_{mw}^Q B \implies A \leq_w^Q B$.
- $A$ is $m$-$\sigma$-join-reducible ($m$-$\sigma$-jr) if $A \upharpoonright n <_w A$ for any $n \in \omega$.

**Lemma**

The structure of the $2$-$m$-Wadge degrees in $\omega^\omega$ looks like the following:

That is, each successor selfdual Wadge degree splits into two degrees (which are linearly ordered), and other Wadge degrees remain the same.
For a function \( f : \omega^\omega \to \mathbb{R} \), define \( S_f : \omega^\omega \to \mathbb{T} \) as follows:

For any \( p, q \in \mathbb{Q} \) with \( p < q \)

\[
S_f((p, q) \triangle X) = \begin{cases} 
1 & \text{if } q \leq f(X), \\
0 & \text{if } f(X) \leq p, \\
\bot & \text{if } p < f(X) < q.
\end{cases}
\]

A pair \( (p, q) \) is identified with a natural number in an effective manner.

**Definition (Day-Downey-Westrick; cf. Linda’s talk on Thursday)**

For \( f, g : 2^\omega \to \mathbb{R} \), \( f \) is \textit{m}-reducible to \( g \) (\( f \leq_m g \)) if

\[(\exists \psi)(\exists \theta \text{ cont.}) S_f((p, q) \triangle X) \leq_T S_g((\psi(p, q) \triangle \theta(X))).\]

where \( S_g((n, p, q) \triangle \oplus_i X_i) = S_g((p, q) \triangle X_n). \)

**Remark:** \( f \leq_m g \) iff \( S_f \leq_T^{mw} S_g. \)

In particular, the DDW-\( m \)-degrees form a substructure of the \( \mathbb{T} \)-\( mw \)-degrees.
The image of the embedding of the DDW-$m$-degrees

1st Main Theorem (K.)

The structure of the DDW-$m$-degrees of real-valued functions looks like the following figure:
\[ \Lambda^T_\alpha := \{ A : \omega^\omega \to T \mid (\exists S \in \Gamma_\alpha) \ A \leq_w S \} \text{ for } \Lambda \in \{ \Sigma, \Pi, \Delta \}. \]
\[ \Delta^T_\alpha \neq \Sigma^T_\alpha \cap \Pi^T_\alpha. \]
- For a Wadge degree \( d \), \( \Gamma_d = \{ B : \deg_w(B) \leq d \} \).
- A \( T \)-Wadge degree \( d \) is proper if \( \Gamma_d \neq \Lambda^T_\alpha \) for any \( \Lambda \) and \( \alpha \), that is, it is not the \( T \)-Wadge degree of a 2-valued function.

**Surjectivity Lemma**

For every non-proper \( T \)-Wadge degree \( d \), there is \( f : 2^\omega \to \mathbb{R} \) such that \( S_f \) is \( \Gamma_d \)-complete.

**Lemma**

Let \( A \) be \( \Delta^T_\alpha \)-complete.

1. If either \( \alpha \) is successor or \( A \) is \( m \)-\( \sigma \)-join-reducible, then there is \( f : 2^\omega \to \mathbb{R} \) s.t. \( S_f \equiv_{mw}^T A \).
2. Otherwise, there is no \( f : 2^\omega \to \mathbb{R} \) s.t. \( S_f \equiv_{mw}^T A \).

Consequently, the DDW-\( m \)-degrees subsume all non-proper \( T \)-\( mw \)-degrees.
Lemma

If $d$ is a proper $\mathbb{T}$-Wadge degree, then

$$(\exists \alpha < \Theta) \Delta^\mathbb{T}_\alpha \subseteq \Gamma_d \subseteq \Sigma^\mathbb{T}_\alpha \cap \Pi^\mathbb{T}_\alpha.$$

Proof

- $\alpha < \Theta$: Least ordinal s.t. $\Gamma_d \subseteq \Sigma^\mathbb{T}_\alpha \cap \Pi^\mathbb{T}_\alpha$.
- Claim: $(\forall \beta < \alpha) \Sigma^\mathbb{T}_\beta \cup \Pi^\mathbb{T}_\beta \subseteq \Gamma_d$.
- $A, B_0, B_1$: $\Gamma_d$, $\Sigma^\mathbb{T}$, and $\Pi^\mathbb{T}$-complete.
- $A \not\equiv_w B_i$ for some $i < 2$, since $\Gamma_d \not\subseteq \Sigma^\mathbb{T} \cap \Pi^\mathbb{T}$.
- By Wadge’s Lemma, $B_{1-i} \equiv_w \neg B_i \leq_w A$.
- Since $d$ is proper, $B_{1-i} \not\equiv_w A$, and thus $A \not\leq_w B_{1-i}$.
- By Wadge’s Lemma, $\neg B_{1-i} \leq_w A$. 
Lemma

If $d$ is a proper $\mathbb{T}$-Wadge degree, then there is no $f : \omega^\omega \to \mathbb{R}$ such that $S_f$ is $\Gamma_d$-complete.

Proof

- $S_f \in \Gamma_d$; Then there is $\alpha$ s.t. $\Delta^\mathbb{T}_\alpha \subseteq \Gamma_d \subseteq \Sigma^\mathbb{T}_\alpha \cap \Pi^\mathbb{T}_\alpha$.
- Claim: $S_f \restriction \langle p, q \rangle$ is $\Delta^\mathbb{T}_\alpha$ for any $p < q$.
- $A_0, A_1 \subseteq \omega^\omega$: $\Sigma_\alpha$-, and $\Pi_\alpha$-complete; $p_0 < q_0 < p_1 < q_1$.
- Since $S_f \in \Sigma^\mathbb{T}_\alpha \cap \Pi^\mathbb{T}_\alpha$, $S_f \restriction \langle p_i, q_i \rangle \leq_w A_i$ via $\tau_i$.
- Define $B_0 = \tau_0^{-1}[-A_0]$ and $B_1 = \tau_1^{-1}[A_1]$; these are $\Pi_\alpha$.
  \[ f(X) \leq p_0 \implies X \in B_0 \implies f(X) < q_0. \]
  \[ f(X) \geq p_1 \implies X \in B_1 \implies f(X) > q_1. \]
- Since $B_0 \cap B_1 = \emptyset$, by the separation property of $\Pi_\alpha$, there is a $\Delta_\alpha$ set $C$ s.t. $B_1 \subseteq C$ and $B_0 \cap C = \emptyset$.
  \[ f(X) \geq q \implies X \in C \implies f(X) > p. \]
- This means that $S_f \leq_w C$, that is, $S_f \in \Delta^\mathbb{T}_\alpha$. 

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By the previous lemmas, we conclude:

1st Main Theorem (K.)

The structure of the DDW-m-degrees of real-valued functions looks like the following figure:

That is, each successor selfdual Wadge degree splits into two degrees (which are linearly ordered), and other Wadge degrees remain the same.

\[ \Delta_{jr}^1, \Delta_1, \Delta_{jr}^2, \Delta_2, \ldots, \Delta_{jr}^\omega, \Delta_\omega, \ldots, \Delta_{jr}^\! \]  

\[ \Pi_1, \Pi_1, \Pi_2, \Pi_2, \ldots, \Pi_\omega, \Pi_\omega, \ldots, \Pi_\! \]  

\[ \Sigma_1, \Sigma_1, \Sigma_2, \Sigma_2, \ldots, \Sigma_\omega, \Sigma_\omega, \ldots, \Sigma_\! \]  

\[ \Pi_1, \Pi_1, \Pi_2, \Pi_2, \ldots, \Pi_\omega, \Pi_\omega, \ldots, \Pi_\! \]  

\[ \Delta_{jr}^1 = \text{constant functions}; \Delta_1 = \text{continuous functions}; \Sigma_1 = \text{lower semi-continuous}; \Pi_1 = \text{upper semi-continuous}. \text{ (cf. Linda’s talk) } \]
Historical background on the classification of Baire functions
Baire (1899): Baire hierarchy.
Bourgain (1980), many others: Classifying Baire-one functions by Cantor-Bendixson-like ranks, mind-changes, etc.
Pawlak 2000 and others: Hierarchy of Baire-one-star functions.
first level function = \(\Delta^0_2\)-measurable = Baire-one-star = discrete-Baire one
Weihrauch (around 1990), Hertling, and others, Carroy (2013): Continuous strong Weihrauch reducibility \(\leq^c_{SW}\).
Carroy (2013): CB-rank analysis of functions under \(\leq^c_{SW}\). The continuous functions with compact domain and countable range form a well-order of type \(\omega_1\) under \(\leq^c_{SW}\).
Day-Downey-Westrick (201x):
\(\leq_T := \text{Parallel continuous strong Weihrauch reducibility } \leq^c_{SW}\).
Introduction to Martin’s conjecture

2nd Main Theorem (K.)
The parallel continuous (strong) Weihrauch degrees of total single-valued functions are exactly the natural Turing degrees.
Natural Solution to Post’s Problem:
Is there a “natural” intermediate c.e. Turing degree?

Natural degrees should be relativizable and degree invariant:

- (Relativizability) It is a function \( f : 2^\omega \rightarrow 2^\omega \).
- (Degree-Invariance) \( X \equiv_T Y \) implies \( f(X) \equiv_T f(Y) \).

(Sacks 1963) Is there a degree invariant c.e. operator which always gives us an intermediate Turing degree?

(Lachlan 1975) There is no uniformly degree invariant c.e. operator which always gives us an intermediate Turing degree.

(The Martin Conjecture; a.k.a. the 5th Victoria-Delfino problem)

- Degree invariant increasing functions are well-ordered,
  and each successor rank is given by the Turing jump.

(Cabal) The VD problems 1-5 appeared in 1978; the VD problems 6-14 in 1988. Only the 5th and 14th are still unsolved (the 14th asks whether \( \text{AD}^+ = \text{AD} \)).

(Steel 1982) The Martin Conjecture holds true for uniformly degree invariant functions.
(Hypothesis) Natural degrees are relativizable and degree-invariant.

- \( f : 2^\omega \to 2^\omega \) is uniformly degree invariant (UI) if there is a function \( u : \omega^2 \to \omega^2 \) such that for all \( X, Y \in 2^\omega \),
  \[ X \equiv_T Y \text{ via } (i, j) \implies f(X) \equiv_T f(Y) \text{ via } u(i, j). \]

- \( f : 2^\omega \to 2^\omega \) is uniformly order preserving (UOP) if there is a function \( u : \omega \to \omega \) such that for all \( X, Y \in 2^\omega \),
  \[ X \leq_T Y \text{ via } e \implies f(X) \leq_T f(Y) \text{ via } u(e). \]

- \( f \) is Turing reducible to \( g \) on a cone (\( f \leq_T^v g \)) if
  \[ (\exists C \in 2^\omega)(\forall X \geq_T C) f(X) \leq_T g(X) \oplus C. \]

**Theorem** (Steel 1982; Slaman-Steel 1988; Becker 1988)

- The \( \equiv_T^v \)-degree of UI functions form a well-order of length \( \Theta \).
- Every UI function is \( \equiv_T^v \)-equivalent to a UOP function.
Proof of the 2nd Main Theorem

2nd Main Theorem (K.)
The parallel continuous (strong) Weihrauch degrees of total single-valued functions are exactly the natural Turing degrees.

Indeed, the identity map induces an isomorphism between the $\leq_T^V$-degrees of UOP functions and the $\widehat{CW}$-degrees!
Embedding Lemma

Assume that \( f \) and \( g \) are UOP functions. Then,

\[
f \leq_T g \iff f \leq_{SW} g \iff f \leq_W \widehat{g}.
\]

A uniformly pointed perfect tree (u.p.p. tree) is a perfect tree \( T \subseteq 2^{<\omega} \) s.t. \( X \oplus T \leq_T T[X] \) uniformly in \( X \), where \( T[X] \) is the \( X \)-th path through \( T \).

Martin’s Cone Lemma (1968)

Any countable partition of \( 2^\omega \) has a part containing a u.p.p. tree.

Proof of \( f \leq_T g \implies f \leq_{SW} g \)

- Assume that \( f(X) \leq_T g(X) \) on a cone.
- By MCL, there is a u.p.p. tree \( T \) s.t. \( f(T[X]) \leq_T g(T[X]) \) via a \( \Phi \).
- Since \( f \) is UOP and \( X \leq_T T[X] \) uniformly, \( f(X) \leq_T f(T[X]) \) via a \( \Psi \).
- Hence, \( f = \Psi \circ \Phi \circ g \circ T \).
Embedding Lemma

Assume that $f$ and $g$ are UOP operators. Then,

$$f \leq_T g \iff f \leq_{\text{sw}}^c g \iff f \leq_W^c \tilde{g}.$$ 

It remains to show the following:

Surjectivity Lemma

Every function is $\overline{cW}$-equivalent to a UOP operator.

That is, for any $f : 2^\omega \to \mathbb{R}$, there is a UOP operator $g$ s.t. $\tilde{f} \equiv_W^c \tilde{g}$. 

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Lemma (H. Becker 1988)
For any reasonable pointclass $\Gamma$ and its indexing $U$, the pointclass jump $J^U_\Gamma$ is a UOP jump operator. Moreover, the $\equiv^\forall_\Gamma$-degree of $J^U_\Gamma$ is independent of the choice of $U$.

Lemma (H. Becker 1988)
Every nonselfdual Wadge class is the relativization of a reasonable pointclass.

Surjectivity Lemma (Nonselfdual)
For any reasonable pointclass $\Gamma$,

\[ S_f \text{ is } \Gamma\text{-complete} \iff \hat{f} \equiv^c_{SW} J_\Gamma. \]

Corollary
If $S_f$ is nonselfdual, then there is a UOP function $g$ s.t. $\hat{f} \equiv^c_{SW} \hat{g}$. 

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Recall that the cofinality of a selfdual Wadge rank is at most $\omega$.

**Lemma (Successor)**

If $S_f \in \Delta_{\alpha+1}$ and $S_g$ is $\Sigma_\alpha$-complete, then $f \leq^c_{sw} \widehat{g}$.

**Lemma (Limit of cofinality $\omega$)**

If $\alpha$ is a limit ordinal of cofinality $\omega$, then there is a UOP function $g$ such that $\widehat{f} \equiv^c_{sw} \widehat{g}$.

Consequently,

**Surjectivity Lemma**

Every $\mathbb{R}$-valued function is $\widehat{cW}$-equivalent to a UOP function.
1st Main Theorem (K.)
The structure of the DDW-$m$-degrees of real-valued functions looks like the following figure:

2nd Main Theorem (K.)
The parallel continuous (strong) Weihrauch degrees of total single-valued functions are exactly the natural Turing degrees.

Indeed, the identity map induces an isomorphism between the $\leq^v_T$-degrees of UOP functions and the $\text{cW}$-degrees!