From **Classical Recursion Theory** to **Descriptive Set Theory** via **Computable Analysis**

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Computability and Complexity in Analysis 2013

Main Theme

Application of Recursion Theory to Descriptive Set Theory

- Which Result in Recursion Theory is applied?

- Which Problem in Descriptive Set Theory is solved?

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- $\Rightarrow$ The Shore-Slaman Join Theorem (1999)
  - It was proved by using Kumabe-Slaman forcing.
  - It was used to show that
    The Turing jump is first-order definable in $\mathcal{D}_T$.

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- $\Rightarrow$ The Decomposability Problem of Borel Functions

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  - It was used to show that
    The Turing jump is first-order definable in $\mathcal{D}_T$.

- Which Problem in Descriptive Set Theory is solved?
- ⇒ The Decomposability Problem of Borel Functions
  - The original decomposability problem was proposed by Luzin, and negatively answered by Keldysh (1934).
  - A partial positive result was given by Jayne-Rogers (1982).
  - The modified decomposability problem was proposed by Andretta (2007), Semmes (2009), Pawlikowski-Sabok (2012), Motto Ros (2013).

Decomposing a hard function F into easy functions

Decomposing a discontinuous function F into easy functions

Decomposing a discontinuous function F into continuous functions

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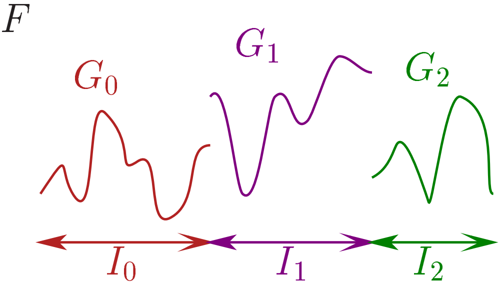
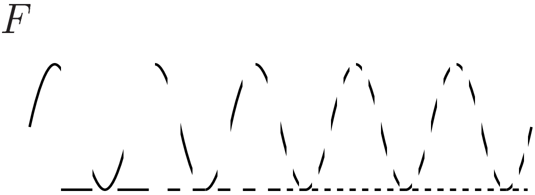


$$F(x) = \begin{cases} G_0(x) & \text{if } x \in I_0 \\ G_1(x) & \text{if } x \in I_1 \\ G_2(x) & \text{if } x \in I_2 \end{cases}$$

Decomposing a discontinuous function into continuous functions



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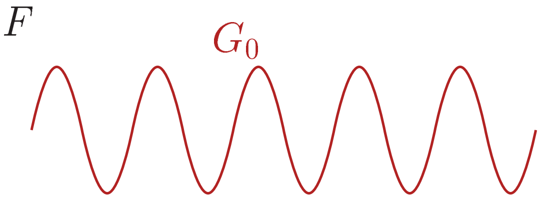


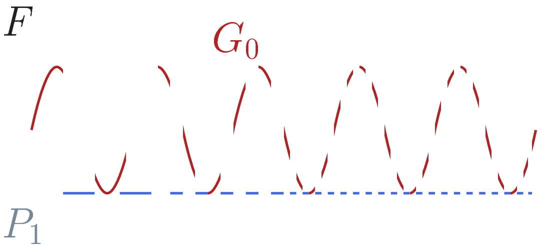
Decomposing a discontinuous function into continuous functions

F

$x \mapsto 0$

P_1

Decomposing a discontinuous function into continuous functions



$$F(x) = \begin{cases} G_0(x) & \text{if } x \notin P_1 \\ 0 & \text{if } x \in P_1 \end{cases}$$

Decomposing a discontinuous function into continuous functions

$$\text{Dirichlet}(x) = \lim_{m\to\infty} \lim_{n\to\infty} \cos^{2n}(m!\pi x)$$

$$\Downarrow$$

$$\text{Dirichlet}(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}. \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

If F is a Borel measurable function on $\mathbb{R}$, then can it be presented by using a countable partition $\{P_n\}_{n\in\omega}$ of $\mathbf{dom}(F)$ and a countable list $\{G_n\}_{n\in\omega}$ of continuous functions as follows?

$$F(x) = \begin{cases} G_0(x) & \text{if } x \in P_0 \\ G_1(x) & \text{if } x \in P_1 \\ G_2(x) & \text{if } x \in P_2 \\ G_3(x) & \text{if } x \in P_3 \\ \vdots & \vdots \end{cases}$$

Luzin's Problem (almost 100 years ago)

Can every Borel function on $\mathbb{R}$ be decomposed into countably many continuous functions?

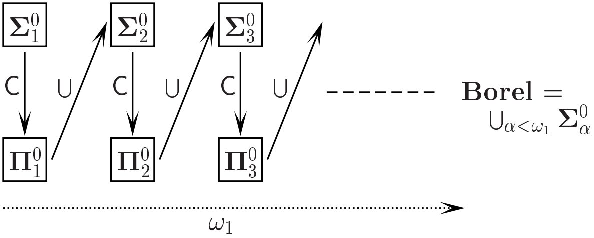
Definition (Baire 1899)

- Baire $\mathbf{0}$ = continuous.
- Baire α = the pointwise limit of a seq. of Baire $< \alpha$ functions.
- Baire function = Baire α for some α.
- The Baire functions = the smallest class closed under taking pointwise limit and containing all continuous functions.

Definition (Borel 1904, Hausdorff 1913)

- $\underset{\sim}{\mathbf{\Sigma}}^0_1$ = open.
- $\underset{\sim}{\mathbf{\Pi}}^0_\alpha$ = the complement of a $\underset{\sim}{\mathbf{\Sigma}}^0_\alpha$ set.
- $\underset{\sim}{\mathbf{\Sigma}}^0_\alpha$ = the countable union of a seq. of $\underset{\sim}{\mathbf{\Pi}}^0_\beta$ sets for some $\beta < \alpha$.
- Borel set = $\underset{\sim}{\mathbf{\Sigma}}^0_\alpha$ for some α.
- The Borel sets = the smallest σ-algebra containing all open sets.

Borel hierarchy



Definition (X, Y: topological spaces, $\mathcal{B} \subseteq \mathcal{P}(X)$)

$f : X \to Y$ is $\mathcal{B}$-measurable if $f^{-1}[A] \in \mathcal{B}$ for every open $A \subseteq Y$.

Lebesgue-Hausdorff-Banach Theorem

$$\text{Baire } \alpha = \Sigma^0_{\alpha+1}\text{-measurable}$$

$$\text{the Baire functions} = \text{the Borel measurable functions}$$

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Can every Borel function on $\mathbb{R}$ be decomposed into countably many continuous functions? $\Longrightarrow$ No! (Keldysh 1934)
An indecomposable Baire **1** function exists!

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Can every Borel function on $\mathbb{R}$ be decomposed into countably many continuous functions? $\Longrightarrow$ No! (Keldysh 1934)
An indecomposable Baire **1** function exists!

Example

The Turing jump $TJ : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is:

$$TJ(x)(n) = \begin{cases} 1, & \text{if the } n\text{-th Turing machine with oracle } x \text{ halts} \\ 0, & \text{otherwise} \end{cases}$$

Then, TJ is Baire **1**, but indecomposable!

Example

Turing jump $TJ : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is indecomposable.

Lemma

For $F : X \to Y$, the following are equivalent:

1. F is decomposable into countably many continuous functions.
2. $(\exists \alpha \in 2^{\mathbb{N}})(\forall x \in 2^{\mathbb{N}})\ F(x) \leq_T x \oplus \alpha$

Here, $(x \oplus y)(2n) = x(n)$ and $(x \oplus y)(2n+1) = y(n)$.

Decomposable $\Longrightarrow (\exists\alpha)(\forall x)\ F(x) \leq_T x \oplus \alpha$

- F is decomposable into continuous functions $F_i : X_i \to Y$.
- (TTE) Since F_i is continuous,
  it must be computable relative to an oracle α_i!
- Hence $(\forall x \in X_i)\ F_i(x) \leq_T x \oplus \alpha_i$
- $(\forall x \in X)\ F(x) \leq_T x \oplus \bigoplus_{i \in \mathbb{N}} \alpha_i$
- Put $\alpha = \bigoplus_{i \in \mathbb{N}} \alpha_i$. □

Decomposable $\Longleftarrow (\exists\alpha)(\forall x)\ F(x) \leq_T x \oplus \alpha$

- Assume $(\forall x \in X)\ F(x) \leq_T x \oplus \alpha$.
- Φ_e: the e-th Turing machine
- $(\forall x \in X)(\exists e \in \mathbb{N})\ \Phi_e(x \oplus \alpha) = F(x)$
- $e[x]$: The least such e for $x \in X$.
- $x \mapsto \Phi_e(x \oplus \alpha)$ is computable relative to α.
- (TTE) $x \mapsto \Phi_e(x \oplus \alpha)$ is continuous.
- For $X_e = \{x \in X : e[x] = e\}$ the restriction $F|_{X_e} = \Phi_e(* \oplus \alpha)$ is continuous $\square$

Hierarchy of Indecomposable Functions

- (Keldysh 1934) For every α there is a Baire α function which is not decomposable into countably many Baire $< \alpha$ functions!
- The α-th Turing jump $x \mapsto x^{(\alpha)}$ is such a function.

Hierarchy of Indecomposable Functions

- (Keldysh 1934) For every α there is a Baire α function which is not decomposable into countably many Baire $< \alpha$ functions!
- The α-th Turing jump $x \mapsto x^{(\alpha)}$ is such a function.

- Which Borel function can we decompose into countably many continuous functions?
- Let's study a finer hierarchy than the Baire hierarchy!

$$\text{Borel} = \bigcup_{\alpha<\omega_1} \underset{\sim}{\mathbf{\Sigma}}^0_\alpha$$

Definition

1. A function $F : X \to Y$ is Borel if
$$A \in \bigcup_{\alpha<\omega_1} \underset{\sim}{\mathbf{\Sigma}}^0_\alpha(Y) \implies F^{-1}[A] \in \bigcup_{\alpha<\omega_1} \underset{\sim}{\mathbf{\Sigma}}^0_\alpha(X).$$
2. A function $F : X \to Y$ is $\underset{\sim}{\mathbf{\Sigma}}^0_\alpha$-measurable if
$$A \in \underset{\sim}{\mathbf{\Sigma}}^0_1(Y) \implies F^{-1}[A] \in \underset{\sim}{\mathbf{\Sigma}}^0_\alpha(X).$$
3. A function $F : X \to Y$ is $\underset{\sim}{\mathbf{\Sigma}}_{\alpha,\beta}$ if
$$A \in \underset{\sim}{\mathbf{\Sigma}}^0_\alpha(Y) \implies F^{-1}[A] \in \underset{\sim}{\mathbf{\Sigma}}^0_\beta(X).$$

A function $F : X \to Y$ is $\underset{\sim}{\Sigma}_{\alpha,\beta}$ if

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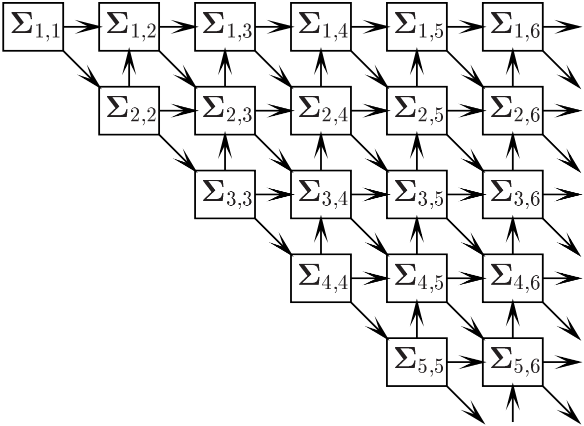
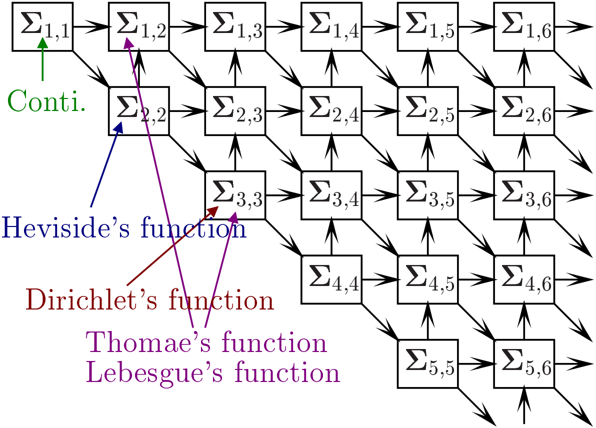


A function $\boldsymbol{F} : \boldsymbol{X} \to \boldsymbol{Y}$ is $\boldsymbol{\Sigma}_{\alpha,\beta}$ if

$$A \in \boldsymbol{\Sigma}^0_{\alpha}(Y) \implies F^{-1}[A] \in \boldsymbol{\Sigma}^0_{\beta}(X).$$



Definition

F: a function from a top. sp. X into a top. sp. Y.

- $F \in \mathrm{dec}(\utilde{\Sigma}_\alpha)$ if it is decomposable into countably many $\utilde{\Sigma}^0_\alpha$-measurable functions.
- $F \in \mathrm{dec}_\beta(\utilde{\Sigma}_\alpha)$ if it is decomposable into countably many $\utilde{\Sigma}^0_\alpha$-measurable functions with $\utilde{\Pi}^0_\beta$ domains,

  that is, there are a list $\{P_n\}_{n\in\omega}$ of Π^0_β subsets of X with $X = \bigcup_n P_n$ and a list $\{G_n\}_{n\in\omega}$ of $\utilde{\Sigma}^0_\alpha$-measurable functions such that $F \restriction P_n = G_n \restriction P_n$ holds for all $n \in \omega$.

$$\boxed{\text{Baire } \alpha} = \boxed{\underset{\sim}{\mathbf{\Sigma}}_{\alpha+1}}$$

- $(f_n)_{n\in\mathbb{N}}$ discretely converges to f if $(\forall x)(\forall^\infty n)\ f(x) = f(x)$.
- $(f_n)_{n\in\mathbb{N}}$ quasi-normally converges to f if $(\exists\varepsilon_n)_{n\in\omega} \to 0)(\forall^\infty n)\ |f(x) - f_n(x)| < \varepsilon_n$.

Theorem (Császár-Laczkovich 1979, 1990)

X: perfect normal, $f : X \to \mathbb{R}$,

$$\boxed{\text{discrete-Baire } \alpha} = \boxed{\mathbf{dec}_\alpha(\underset{\sim}{\mathbf{\Sigma}}_1)}$$

Moreover, if α is successor,

$$\boxed{\text{QN-Baire } \alpha} = \boxed{\mathbf{dec}_\alpha(\underset{\sim}{\mathbf{\Sigma}}_\alpha)}$$

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The Jayne-Rogers Theorem 1982

X, Y: metric separable, X: analytic

For the class of all functions from X into Y,

$$\boxed{\utilde{\Sigma}_{2,2}} = \boxed{\mathbf{dec}_1(\utilde{\Sigma}_1)}$$

Borel Functions and Decomposability

	1	2	3	4	5	6
1	$\utilde{\mathbf{\Sigma}}_1$	$\utilde{\mathbf{\Sigma}}_2$	$\utilde{\mathbf{\Sigma}}_3$	$\utilde{\mathbf{\Sigma}}_4$	$\utilde{\mathbf{\Sigma}}_5$	$\utilde{\mathbf{\Sigma}}_6$
2	–	$\mathbf{dec}_1\utilde{\mathbf{\Sigma}}_1$	?	?	?	?
3	–	–	?	?	?	?
4	–	–	–	?	?	?
5	–	–	–	–	?	?
6	–	–	–	–	–	?

The Jayne-Rogers Theorem 1982

$\boldsymbol{X}, \boldsymbol{Y}$: metric separable, $\boldsymbol{X}$: analytic
For the class of all functions from $\boldsymbol{X}$ into $\boldsymbol{Y}$,

$$\boxed{\utilde{\mathbf{\Sigma}}_{2,2}} = \boxed{\mathbf{dec}_1(\utilde{\mathbf{\Sigma}}_1)}$$

The second level decomposability of Borel functions

	1	2	3	4	5	6
1	$\underset{\sim}{\mathbf{\Sigma}}_1$	$\underset{\sim}{\mathbf{\Sigma}}_2$	$\underset{\sim}{\mathbf{\Sigma}}_3$	$\underset{\sim}{\mathbf{\Sigma}}_4$	$\underset{\sim}{\mathbf{\Sigma}}_5$	$\underset{\sim}{\mathbf{\Sigma}}_6$
2	–	$\mathbf{dec}_1\underset{\sim}{\mathbf{\Sigma}}_1$	$\mathbf{dec}_2\underset{\sim}{\mathbf{\Sigma}}_2$	?	?	?
3	–	–	$\mathbf{dec}_2\underset{\sim}{\mathbf{\Sigma}}_1$	?	?	?
4	–	–	–	?	?	?
5	–	–	–	–	?	?
6	–	–	–	–	–	?

Theorem (Semmes 2009)

For the class of functions on a zero dim. Polish space,

$$\underset{\sim}{\mathbf{\Sigma}}_{2,3} = \mathbf{dec}_2(\underset{\sim}{\mathbf{\Sigma}}_2)$$

$$\underset{\sim}{\mathbf{\Sigma}}_{3,3} = \mathbf{dec}_2(\underset{\sim}{\mathbf{\Sigma}}_1)$$

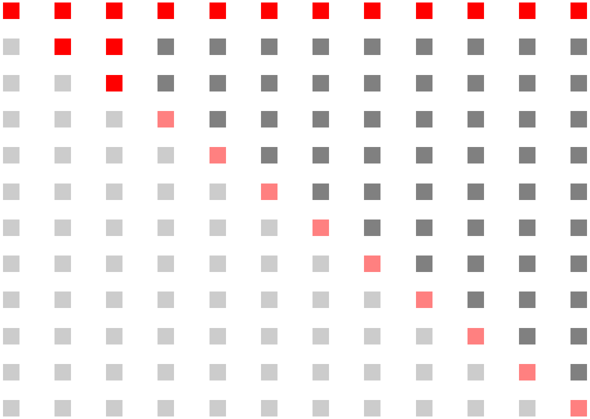
The Decomposability Problem

	1	2	3	4	5	6
1	$\utilde{\Sigma}_1$	$\utilde{\Sigma}_2$	$\utilde{\Sigma}_3$	$\utilde{\Sigma}_4$	$\utilde{\Sigma}_5$	$\utilde{\Sigma}_6$
2	–	$\mathbf{dec}_1\utilde{\Sigma}_1$	$\mathbf{dec}_2\utilde{\Sigma}_2$	$\mathbf{dec}_3\utilde{\Sigma}_3$	$\mathbf{dec}_4\utilde{\Sigma}_4$	$\mathbf{dec}_5\utilde{\Sigma}_5$
3	–	–	$\mathbf{dec}_2\utilde{\Sigma}_1$	$\mathbf{dec}_3\utilde{\Sigma}_2$	$\mathbf{dec}_4\utilde{\Sigma}_3$	$\mathbf{dec}_5\utilde{\Sigma}_4$
4	–	–	–	$\mathbf{dec}_3\utilde{\Sigma}_1$	$\mathbf{dec}_4\utilde{\Sigma}_2$	$\mathbf{dec}_5\utilde{\Sigma}_3$
5	–	–	–	–	$\mathbf{dec}_4\utilde{\Sigma}_1$	$\mathbf{dec}_5\utilde{\Sigma}_2$
6	–	–	–	–	–	$\mathbf{dec}_5\utilde{\Sigma}_1$

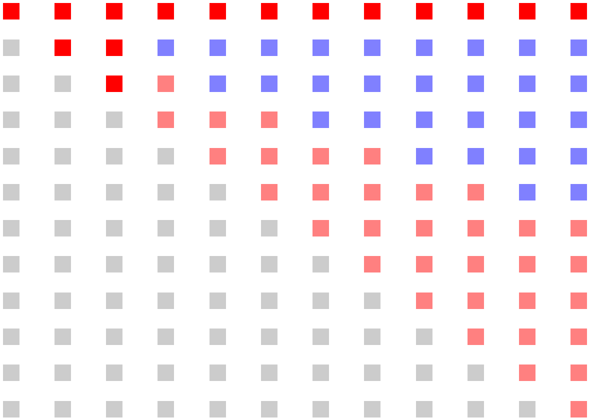
The Decomposability Conjecture (Andretta, Motto Ros et al.)

$$\utilde{\Sigma}_{m+1,n+1} = \mathbf{dec}_n(\utilde{\Sigma}_{n-m+1})$$

Overview of Previous Research



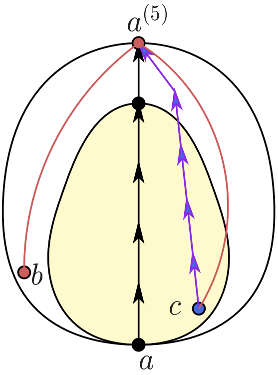
Main Theorem



Shore-Slaman Join Theorem 1999

The following sentence is true in the Turing degree structure.

$$(\forall \boldsymbol{a}, \boldsymbol{b})(\exists \boldsymbol{c} \geq \boldsymbol{a})[((\forall \zeta < \xi)\ \boldsymbol{b} \not\leq \boldsymbol{a}^{(\zeta)}) \to (\boldsymbol{c}^{(\xi)} \leq \boldsymbol{b} \oplus \boldsymbol{a}^{(\xi)} \leq \boldsymbol{b} \oplus \boldsymbol{c})]$$



Shore-Slaman Join Theorem 1999

$$(\forall \mathbf{a}, \mathbf{b})(\exists \mathbf{c} \geq \mathbf{a})[((\forall \zeta < \xi)\ \mathbf{b} \not\leq \mathbf{a}^{(\zeta)}) \rightarrow (\mathbf{c}^{(\xi)} \leq \mathbf{b} \oplus \mathbf{a}^{(\xi)} \leq \mathbf{b} \oplus \mathbf{c})$$

History in Turing degree theory

- Posner-Robinson Join Theorem (1981) is partially generalized if combined with Friedberg Jump Inversion Theorem (1957).
- Jockusch-Shore Problem (1984): Generalize the join theorem to α-REA operators.
- Kumabe and Slaman introduced a forcing notion to solve it.
- Slaman and Woodin showed the first-order definability of the double jump in the Turing universe, by using set theoretic methods such as Levy collapsing and Shoenfield absoluteness, and analyzing the automorphism group of the Turing universe.
- Shore and Slaman showed the join theorem by Kumabe-Slaman forcing, and applied their join theorem to obtain the first-order definability of the Turing jump from the Slaman-Woodin theorem.

Question: $\boldsymbol{\Sigma}_{m+1,n+1} = \mathbf{dec}_n(\boldsymbol{\Sigma}_{n-m+1})$?

Easy direction (Motto Ros 2013)

$$\mathbf{dec}_n(\boldsymbol{\Sigma}_{n-m+1}) \subseteq \boldsymbol{\Sigma}_{m+1,n+1}$$

- Assume that $F \in \mathbf{dec}_n(\boldsymbol{\Sigma}_{n-m+1})$.
- $F_i = F \upharpoonright Q_i$ is $\boldsymbol{\Sigma}^0_{n-m+1}$-measurable, where $Q_i \in \boldsymbol{\Pi}^0_n$.
- If $P \in \boldsymbol{\Sigma}^0_{m+1}$, we have $F_i^{-1}[P] \cap Q_i \in \boldsymbol{\Sigma}^0_{n+1}$.
- Hence, $F^{-1}[P] = \bigcup_i F_i^{-1}[P] \cap Q_i \in \boldsymbol{\Sigma}^0_{n+1}$.

In the above proof, we can uniformly give a $\boldsymbol{\Sigma}^0_{n+1}$-description of $F^{-1}[P]$ from any $\boldsymbol{\Sigma}^0_{m+1}$-description of P.

In the previous proof, we can uniformly give a $\underset{\sim}{\mathbf{\Sigma}}^0_{n+1}$-description of $F^{-1}[P]$ from any $\underset{\sim}{\mathbf{\Sigma}}^0_{m+1}$-description of P.

Definition (de Brecht-Pauly 2012)

- F is $\underset{\sim}{\mathbf{\Sigma}}_{\alpha,\beta}$ iff $F^{-1}[\cdot] \restriction \underset{\sim}{\mathbf{\Sigma}}^0_{\alpha}$ is a function from $\underset{\sim}{\mathbf{\Sigma}}^0_{\alpha}$ into $\underset{\sim}{\mathbf{\Sigma}}^0_{\beta}$.
- F is $\underset{\sim}{\mathbf{\Sigma}}^{\rightarrow}_{\alpha,\beta}$ if $F^{-1}[\cdot] \restriction \underset{\sim}{\mathbf{\Sigma}}^0_{\alpha}$ is continuous, as a function from $\underset{\sim}{\mathbf{\Sigma}}^0_{\alpha}$ into $\underset{\sim}{\mathbf{\Sigma}}^0_{\beta}$.
  Here the space of all $\underset{\sim}{\mathbf{\Sigma}}^0_{\alpha}$ subsets of a topological space is represented by the canonical Borel code up to $\mathbf{\Sigma}^0_{\alpha}$.

Easy direction (Motto Ros 2013)

$$\boxed{\mathbf{dec}_n(\underset{\sim}{\mathbf{\Sigma}}_{n-m+1})} \subseteq \boxed{\underset{\sim}{\mathbf{\Sigma}}^{\rightarrow}_{m+1,n+1}}$$

The Decomposability Problem

$$\boxed{\underset{\sim}{\Sigma}_{m+1,n+1}} = \boxed{\mathbf{dec}_n(\underset{\sim}{\Sigma}_{n-m+1})}$$

Main Theorem (K.)

For functions between Polish spaces with topological dim. $\neq \infty$ and for every $m, n \in \mathbb{N}$,

$$\boxed{\mathbf{dec}_n(\underset{\sim}{\Sigma}_{n-m+1})} \subseteq \boxed{\underset{\sim}{\Sigma}^{\rightarrow}_{m+1,n+1}} \subseteq \boxed{\mathbf{dec}(\underset{\sim}{\Sigma}_{n-m+1})}$$

Moreover, if $2 \leq m \leq n < 2m$ then

$$\boxed{\underset{\sim}{\Sigma}^{\rightarrow}_{m+1,n+1}} = \boxed{\mathbf{dec}_n(\underset{\sim}{\Sigma}_{n-m+1})}$$

The decomposability of continuously Borel functions

	1	2	3	4	5	6
1	$\utilde{\Sigma}_1$	$\utilde{\Sigma}_2$	$\utilde{\Sigma}_3$	$\utilde{\Sigma}_4$	$\utilde{\Sigma}_5$	$\utilde{\Sigma}_6$
2	–	$\mathbf{dec}_1\utilde{\Sigma}_1$	$\mathbf{dec}_2\utilde{\Sigma}_2$	?	?	?
3	–	–	$\mathbf{dec}_2\utilde{\Sigma}_1$	$\mathbf{dec}_3\utilde{\Sigma}_2$	?	?
4	–	–	–	$\mathbf{dec}_3\utilde{\Sigma}_1$	$\mathbf{dec}_4\utilde{\Sigma}_2$	$\mathbf{dec}_5\utilde{\Sigma}_3$
5	–	–	–	–	$\mathbf{dec}_4\utilde{\Sigma}_1$	$\mathbf{dec}_5\utilde{\Sigma}_2$
6	–	–	–	–	–	$\mathbf{dec}_5\utilde{\Sigma}_1$

Main Theorem (K.)

If $2 \leq m \leq n < 2m$ then

$$\utilde{\Sigma}^{\to}_{m+1,n+1} = \mathbf{dec}_n(\utilde{\Sigma}_{n-m+1})$$

Sketch of Proof of $\underset{\sim}{\Sigma}^{\to}_{m+1,n+1} \subseteq \mathbf{dec}(\underset{\sim}{\Sigma}_{n-m+1})$

Lemma (Lightface Analysis)

Let $F : 2^\omega \to 2^\omega$ be a function, and let p, q be oracles.
Assume that the preimage $F^{-1}[A]$ of any lightface $\Sigma^{0,p}_m$ class A under F forms a lightface $\Delta^{0,p\oplus q}_{n+1}$ class, and one can effectively find an index of $F^{-1}[A]$ from an index of A.
Then $(F(x) \oplus p)^{(m)} \leq_T (x \oplus p \oplus q)^{(n)}$ for every $x \in 2^\omega$.

Lemma (Boldface)

$F \in \underset{\sim}{\Sigma}^{\to}_{m+1,n+1}$ iff the preimage of any $\underset{\sim}{\Sigma}^0_m$ class under F forms a $\underset{\sim}{\Delta}^0_{n+1}$ class.

Lemma (Boldface Analysis)

If $F \in \underset{\sim}{\Sigma}^{\to}_{m+1,n+1}$, then there exists $q \in 2^\omega$ such that
$(F(x) \oplus p)^{(m)} \leq_T (x \oplus p \oplus q)^{(n)}$ for all $p \in 2^\omega$.

Sketch of Proof of $\underset{\sim}{\Sigma}^{\rightarrow}_{m+1,n+1} \subseteq \mathbf{dec}(\underset{\sim}{\Sigma}_{n-m+1})$

Shore-Slaman Join Theorem 1999

The following sentence is true in the Turing degree structure.

$$(\forall a, b)(\exists c \geq a)[((\forall \zeta < \xi)\ b \not\leq a^{(\zeta)}) \rightarrow (c^{(\xi)} \leq b \oplus a^{(\xi)} \leq b \oplus c)$$

Lemma (Boldface Analysis; Restated)

If $F \in \underset{\sim}{\Sigma}^{\rightarrow}_{m+1,n+1}$, then there exists $q \in 2^{\omega}$ such that $(F(x) \oplus p)^{(m)} \leq_T (x \oplus p \oplus q)^{(n)}$ for all $p \in 2^{\omega}$.

Decomposition Lemma

$F \in \underset{\sim}{\Sigma}^{\rightarrow}_{m+1,n+1} \Rightarrow (\exists q)\ F(x) \leq_T (x \oplus q)^{(n-m)}$.

Sketch of Proof of $\underset{\sim}{\mathbf{\Sigma}}^{\to}_{m+1,n+1} \subseteq \mathbf{dec}(\underset{\sim}{\mathbf{\Sigma}}_{n-m+1})$

Decomposition Lemma; Restated

$F \in \underset{\sim}{\mathbf{\Sigma}}^{\to}_{m+1,n+1} \Rightarrow (\exists q)\ F(x) \leq_T (x \oplus q)^{(n-m)}$.

Corollary

$F \in \underset{\sim}{\mathbf{\Sigma}}^{\to}_{m+1,n+1} \Rightarrow (\forall x)(\exists e)\ F(x) = \Phi_e((x \oplus q)^{(n-m)})$.

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- $G_e : x \mapsto \Phi_e(x \oplus q)^{(n-m)}$ is $\underset{\sim}{\Sigma}^0_{n-m+1}$-measurable.
- $P_e := \{x \in \mathrm{dom}(G_e) : F(x) = G_e(x)\}$.

Sketch of Proof of $\underset{\sim}{\mathbf{\Sigma}}^{\rightarrow}_{m+1,n+1} \subseteq \mathbf{dec}(\underset{\sim}{\mathbf{\Sigma}}_{n-m+1})$

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- Then $F \upharpoonright P_e = G_e \upharpoonright P_e$, and $\mathbf{dom}(F) = \bigcup_e P_e$.

Sketch of Proof of $\underset{\sim}{\boldsymbol{\Sigma}}^{\to}_{m+1,n+1} \subseteq \mathbf{dec}(\underset{\sim}{\boldsymbol{\Sigma}}_{n-m+1})$

Decomposition Lemma; Restated

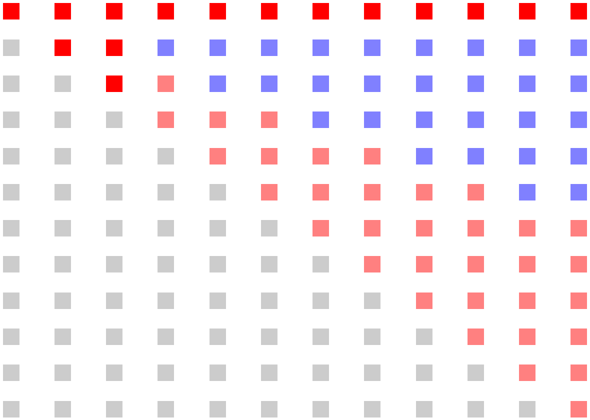
$F \in \underset{\sim}{\boldsymbol{\Sigma}}^{\to}_{m+1,n+1} \Rightarrow (\exists q)\ F(x) \leq_T (x \oplus q)^{(n-m)}$.

Corollary

$F \in \underset{\sim}{\boldsymbol{\Sigma}}^{\to}_{m+1,n+1} \Rightarrow (\forall x)(\exists e)\ F(x) = \Phi_e((x \oplus q)^{(n-m)})$.

- $G_e : x \mapsto \Phi_e(x \oplus q)^{(n-m)}$ is $\underset{\sim}{\boldsymbol{\Sigma}}^0_{n-m+1}$-measurable.
- $P_e := \{x \in \mathbf{dom}(G_e) : F(x) = G_e(x)\}$.
- Then $F \restriction P_e = G_e \restriction P_e$, and $\mathbf{dom}(F) = \bigcup_e P_e$.
- Consequently, $\underset{\sim}{\boldsymbol{\Sigma}}^{\to}_{m+1,n+1} \subseteq \mathbf{dec}(\underset{\sim}{\boldsymbol{\Sigma}}_{n-m+1})$

Main Theorem



The decomposability of continuously Borel functions

	1	2	3	4	5	6
1	$\utilde{\Sigma}_1$	$\utilde{\Sigma}_2$	$\utilde{\Sigma}_3$	$\utilde{\Sigma}_4$	$\utilde{\Sigma}_5$	$\utilde{\Sigma}_6$
2	–	$\mathbf{dec}_1\utilde{\Sigma}_1$	$\mathbf{dec}_2\utilde{\Sigma}_2$	?	?	?
3	–	–	$\mathbf{dec}_2\utilde{\Sigma}_1$	$\mathbf{dec}_3\utilde{\Sigma}_2$	?	?
4	–	–	–	$\mathbf{dec}_3\utilde{\Sigma}_1$	$\mathbf{dec}_4\utilde{\Sigma}_2$	$\mathbf{dec}_5\utilde{\Sigma}_3$
5	–	–	–	–	$\mathbf{dec}_4\utilde{\Sigma}_1$	$\mathbf{dec}_5\utilde{\Sigma}_2$
6	–	–	–	–	–	$\mathbf{dec}_5\utilde{\Sigma}_1$

Main Theorem (K.)

If $2 \leq m \leq n < 2m$ then

$$\utilde{\Sigma}^{\to}_{m+1,n+1} = \mathbf{dec}_n(\utilde{\Sigma}_{n-m+1})$$