Degrees of unsolvability in topological spaces with countable cs-networks

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Workshop on Mathematical Logic and its Application, Kyoto, Sep 17, 2016

Goal

Develop the theory of degrees of unsolvability in topological spaces (including spaces which are non-metrizable, not second-countable, etc.)

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— What is the motivation?

- In previous works [1,2], we utilized a generalization of the theory of degrees of unsolvability to give (partial/complete) solutions to *preexisting open problems in other areas of mathematics*.
- We are looking for more applications — but currently, the theory itself is still far from complete. So many things are yet to be done, even in the very basic part.

[1] V. Gregoriades, T. Kihara, and K. M. Ng, *Turing degrees in Polish spaces and decomposability of Borel functions*, submitted.

[2] T. Kihara, and A. Pauly, *Point degree spectra of represented spaces*, submitted.

Definition

1. An (ω^ω-)representation of a set $\mathcal{X}$ is a partial surjection $\delta :\subseteq \omega^\omega \to \mathcal{X}$.
2. A topological space $\mathcal{X}$ is admissibly represented if it has a universal continuous representation δ, that is,

   ($\forall$ continuous $\rho :\subseteq \omega^\omega \to \mathcal{X}$)($\exists$ continuous $\nu :\subseteq \omega^\omega \to \omega^\omega$) such that $\rho = \delta \circ \nu$.

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Suppose that $\mathcal{X}$ is represented by δ.

- If $\delta(\boldsymbol{p}) = \boldsymbol{x}$, then we think of $\boldsymbol{p}$ as a name of $\boldsymbol{x}$.
- The complexity of $\boldsymbol{x}$ is identified with that of $\delta^{-1}\{\boldsymbol{x}\}$.
- The degree of $\boldsymbol{x}$ is the *degree of difficulty of calling a name of* $\boldsymbol{x}$.

Degrees of difficulty of calling a name

$(\mathcal{X}, \delta_{\mathcal{X}})$, $(\mathcal{Y}, \delta_{\mathcal{Y}})$: represented spaces.

1. A point $x \in \mathcal{X}$ is (Turing) reducible to $y \in \mathcal{Y}$ ($x \leq_T y$) if there is a partial computable function $\Phi :\subseteq \omega^\omega \to \omega^\omega$ s.t.

$$(\forall p)\ [p \text{ is a name of } y \implies \Phi(p) \text{ is a name of } x].$$

2. $\mathbf{deg}(x) = \{z : z \equiv_T x\}$ is called the (Turing) degree of x.

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Example of representation

- Let $(B_n)_{n \in \omega}$ be an open basis of a space $\mathcal{X}$. Then, each point $x \in \mathcal{X}$ is named by an enumeration p of its nbhd basis, that is,
$$\delta(p) = x \iff \mathrm{range}(p) = \{n \in \omega : x \in B_n\}.$$
- The degree of x is the enumeration degree of its nbhd basis.

A network for a space $\mathcal{X}$ is a collection $\mathcal{N}$ of subsets of $\mathcal{X}$ such that

$$(\forall x \in \mathcal{X})(\forall U \text{ open nbhd of } x)(\exists N \in \mathcal{N})\ x \in N \subseteq U.$$

Example of representation (II)

Let $(N_n)_{n\in\omega}$ be a network for a space $\mathcal{X}$. Then, each point $x \in \mathcal{X}$ is named by an enumeration p of a local subnetwork at x, that is,

- $x \in N_{p(n)}$ for any $n \in \omega$,
- $(\forall U \text{ open nbhd of } x)(\exists n)\ x \in N_{p(n)} \subseteq U$.

Fact (Schröder)

For a topological space $\mathcal{X}$, the following are equivalent:

1. $\mathcal{X}$ is admissibly represented.
2. $\mathcal{X}$ is a qcb_0 space.
3. $\mathcal{X}$ has a countable cs-network.

- A space is *qcb_0* if it is T_0, and is a quotient of a countably based space.
- (Michael 1966) A *cs-network* is a network $\mathcal{N}$ such that every convergent sequence converging to a point $x \in U$ with U open, is eventually in $N \subseteq U$ for some $N \in \mathcal{N}$.

T_0	enumeration degrees
T_1	?
Hausdorff	?
$T_{2\frac{1}{2}}$	?
metrizable	continuous degrees
transfinite dimensional	Turing degrees

Table: Degrees of second-countable spaces

Basic idea of "generalized" degree theory

- *Turing degrees* are degrees of calling names of points of *separable metrizable spaces having transfinite inductive dimension*.
- *Continuous degrees* are degrees of calling names of points of *separable metrizable spaces*.
- *Enumeration degrees* are degrees of calling names of points of *second-countable T_0 spaces*.

To develop our theory, we first deal with the following toy problem:

Toy Problem

Given $m < n$, does there exist a "*degree*" of a point of a T_m-space, which CANNOT be a degree of a point of a T_n-space?

T_3-degrees vs. $T_{2\frac{1}{2}}$-degrees.

- A space is T_3 if it is regular Hausdorff, that is, given any point and closed set are separated by nbhds.
- A space is $T_{2\frac{1}{2}}$ if any two distinct points are separated by closed nbhds.

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Example

The Gandy-Harrington topology τ_{GH} is the topology on ω^ω generated by all computably analytic (i.e., lightface Σ^1_1) sets.

- $(\omega^\omega, \tau_{GH})$ is second-countable, $T_{2\frac{1}{2}}$, but not T_3.

Theorem (**3** vs. $\mathbf{2\frac{1}{2}}$)

Let x be a sufficiently complicated point in ω^ω.
$\mathbf{deg}(x)$: the degree of x w.r.t. the Gandy-Harrington topology.

1. $\mathbf{deg}(x)$ is realized as the degree of a point in a $T_{2\frac{1}{2}}$ space.
2. $\mathbf{deg}(x)$ cannot be realized as the degree of a point in a T_3 space.
3. Indeed, $\mathbf{deg}(x)$ cannot be a degree of a point of a Hausdorff space having a countable closed cs-network.

Remark

Regular $\Longrightarrow$ Having a countable closed cs-network.

The converse is not true, e.g., the sequential topology on the Kleene-Kreisel space $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ has a countable closed cs-network, but not regular (Schröder).

$T_{2\frac{1}{2}}$-degrees vs. T_2-degrees.

- A space is $T_{2\frac{1}{2}}$ if any two distinct points are separated by closed nbhds.
- A space is T_2 if any two distinct points are separated by open nbhds.

$T_{2\frac{1}{2}}$-degrees vs. T_2-degrees.

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Example

The relatively prime integer topology is the topology on the positive integers generated by $\{U_b(a) : a \text{ and } b \text{ are relatively prime}\}$ where $U_b(a) = \{a + bn : n \in \mathbb{Z}\}$.

- This is second-countable, Hausdorff, but not $T_{2\frac{1}{2}}$.

Consider the countable product of the relatively prime integer topology:

Theorem ($2\frac{1}{2}$ vs. 2)

Let $x \in \mathbb{Z}^{\omega}_{>0}$ be sufficiently generic w.r.t. Baire topology.
$\mathbf{deg}(x)$: the degree of x w.r.t. the product relatively prime topology

1. $\mathbf{deg}(x)$ is realized as the degree of a point in a T_2 space.
2. $\mathbf{deg}(x)$ cannot be realized as the degree of a point in a $T_{2\frac{1}{2}}$ space.

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1. $\mathbf{deg}(x)$ is realized as the degree of a point in a T_2 space.
2. $\mathbf{deg}(x)$ cannot be realized as the degree of a point in a $T_{2\frac{1}{2}}$ space.

Moreover, even if we know a name of such an x, we cannot get any new information on names of points in a T_3 space...

1. (Medvedev 1955) A point x is *quasi-minimal* if
   - it has no computable name, but
   - it has no nontrivial information on names of points in $\mathbf{2}^\omega$

$$x \not\leq_T \emptyset \text{ and } (\forall y \in \mathbf{2}^\omega)[y \leq_T x \implies y \leq_T \emptyset].$$

2. A point x is *quasi-minimal w.r.t. $\mathcal{P}$* if
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Theorem (3 vs. 2 — the quasi-minimal version)

Let $x \in \mathbb{Z}^\omega_{>0}$ be Cohen **1**-generic w.r.t. Baire topology.
deg(x): the degree of x w.r.t. the product relatively prime topology

1. **deg**(x) is realized as the degree of a point in a T_2 space.
2. **deg**(x) is quasi-minimal w.r.t. $T_{2\frac{1}{2}}$ spaces having countable closed cs-networks.

T_2-degrees vs. T_1-degrees.

- A space is T_2 if the diagonal is closed.
- A space is T_1 if every singleton is closed.

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Example

The cocylinder topology is the topology on ω^ω generated by $\{\omega^\omega \setminus [\sigma] : \sigma \in \omega^{<\omega}\}$, where $[\sigma] = \{x \in \omega^\omega : \sigma \prec x\}$.

- This is second-countable, T_1, but not Hausdorff.

Theorem (2 vs. 1)

Let $x \in \omega^\omega$ be sufficiently fast-growing as a function on ω.
deg(*x*): the degree of x w.r.t. the cocylinder topology.

1. **deg(*x*)** is realized as the degree of a point in a T_1 space.
2. **deg(*x*)** cannot be realized as the degree of a point in a T_2-space.
3. **deg(*x*)** is quasi-minimal w.r.t. T_2 spaces having countable closed cs-networks.

T_1-degrees vs. T_0-degrees.

T_1-degrees vs. T_0-degrees.

Example

The lower topology is the topology on $\mathbb{R}$ generated by $\{(q, \infty) : q \in \mathbb{Q}\}$.

- This is second-countable, T_0, but not T_1.

Theorem (1 vs. 0)

Let $x \in \mathbb{R}$ be neither left- nor right-c.e.
deg(x): the degree of x w.r.t. the lower topology.

1. **deg**(x) is realized as the degree of a point in a T_0 space.
2. **deg**(x) is quasi-minimal w.r.t. T_1 spaces.

[second-countable]-degrees vs. [non-second-countable]-degrees.

Remark

The category of admissibly represented sps. is cartesian closed. Thus, if $\mathcal{X}$ is admissibly represented, then so is the following space:

$$\mathcal{A}_1(\mathcal{X}) = \{f \in C(\mathcal{X}, \mathbb{S}) : f^{-1}\{\bot\} \text{ is singleton}\},$$

where $\mathbb{S} = \{\top, \bot\}$ is the Sierpiński space, whose open sets are $\emptyset$, $\{\top\}$, and $\{\top, \bot\}$.

Roughly speaking, $\mathcal{A}_1(\mathcal{X})$ is the space of closed singletons in $\mathcal{X}$.

Recursion-theoretic view

The degree of difficulty of calling a name of a point $\{x\}$ in $\mathcal{A}_1(\mathcal{X})$ $\approx$ that of finding an oracle z making x be a $\Pi^0_1(z)$ singleton.

One may think of $\mathcal{A}_1(\omega^\omega)$ as one of the easiest non-second-countable spaces.

We say that $x \in \omega^\omega$ is a *lost melody* if there is $z \in \omega^\omega$ such that $\{x\}$ is a $\Pi^0_1(z)$ singleton (i.e., $\{x\} \leq_T z$), but $x \not\leq_T z'$.

Theorem ([second-countable] vs. [non-second-countable])

Let $x \in \omega^\omega$ be a lost melody s.t. $\{x\}$ is not computable.
$\mathbf{deg}(\{x\})$: the degree of $\{x\}$ as a point in $\mathcal{A}_1(\omega^\omega)$.
Then, $\mathbf{deg}(\{x\})$ is quasi-minimal w.r.t. second-countable spaces.

More remarks on $\mathcal{A}_1(\mathcal{X})$

Proposition

1. If $\mathcal{X}$ is Hausdorff, $\{\{x\}\} \mapsto x : \mathcal{A}_1\mathcal{A}_1(\mathcal{X}) \to \mathcal{X}$ is continuous.
2. There is a T_1 space $\mathcal{X}$ such that $\{\{x\}\} \mapsto x : \mathcal{A}_1\mathcal{A}_1(\mathcal{X}) \to \mathcal{X}$ is not continuous (indeed, not Borel).

Proof of Theorem (**3** vs. $\mathbf{2\frac{1}{2}}$)

The degree of a complicated point in the Gandy-Harrington space cannot be a degree of a point of a Hausdorff space having a countable closed cs-network.

Recall: a point x in a space $\mathcal{X}$ with a countable cs-network $\mathcal{N}$ is named by an enumeration p of a local subnetwork at x, that is,

- $x \in N_{p(n)}$ for any $n \in \omega$,
- $(\forall U$ open nbhd of $x)(\exists n)\ x \in N_{p(n)} \subseteq U$.

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- $x \in N_{p(n)}$ for any $n \in \omega$,
- $(\forall U$ open nbhd of $x)(\exists n)\ x \in N_{p(n)} \subseteq U$.

Consider another representation $\bar{\delta}_{\mathcal{N}}$ of $\mathcal{X}$ defined by $\bar{\delta}_{\mathcal{N}}(p) = x$ iff

- $x \in \overline{N_{p(n)}}$ for any $n \in \omega$,
- $(\forall U$ open nbhd of $x)(\exists n)\ x \in N_{p(n)} \subseteq U$.

Proof of Theorem (**3** vs. $\mathbf{2\frac{1}{2}}$)

The degree of a complicated point in the Gandy-Harrington space cannot be a degree of a point of a Hausdorff space having a countable closed cs-network.

Regular $\Longrightarrow$ Having a countable closed cs-network.

Proposition

If $\mathcal{X}$ is a Hausdorff space having a countable closed cs-network $\mathcal{N}$ then $\mathbf{id} : (\mathcal{X}, \bar{\delta}_{\mathcal{N}}) \to (\mathcal{X}, \delta_{\mathcal{N}})$ is continuous.

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Lemma

$(\mathcal{X}, \delta_{\mathcal{N}})$: Hausdorff space having a countable cs-network.
Let $z \in (\omega^\omega, \tau_{\mathbf{GH}})$ and $x \in \mathcal{X}$.
If a $\delta_{\mathcal{N}}$-name of x is computable relative to a $\mathbf{GH}$-name of z, then no $\mathbf{GH}$-name of z is computable relative to a $\bar{\delta}_{\mathcal{N}}$-name of x.

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- S_e: the e-th lightface Σ^1_1 set.
- A $\mathbf{GH}$-name of x is an enumeration of $G_x = \{e : x \in S_e\}$.

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- A $\mathbf{GH}$-name of x is an enumeration of $G_x = \{e : x \in S_e\}$.
- Assume that $x \leq_T z$ via Ψ, that is,
  - $(e, D) \in \Psi$ and $D \subseteq G_x \Longrightarrow z \in N_e$.
  - U open nbhd of $z \Longrightarrow \exists(e, D) \in \Psi\ [D \subseteq G_x$ and $z \in N_e \subseteq U]$

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- $L = \{n : \forall (m, D) \in \Psi \; [D \subseteq G_x \implies N_m \cap N_n \neq \emptyset]\}$.
- If $n \in L$ then $z \in \overline{N_n}$.

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- $L = \{n : \forall (m, D) \in \Psi\ [D \subseteq G_x \implies N_m \cap N_n \neq \emptyset]\}$.
- If $n \in L$ then $z \in \overline{N_n}$.
- Suppose $z \leq_T (x, \bar{\delta}_{\mathcal{N}})$ via an enumeration Γ:

$$e \in G_x \iff (\exists D \text{ finite})[(e, D) \in \Gamma \text{ and } D \subseteq L].$$

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If a $\delta_{\mathcal{N}}$-name of x is computable relative to a $\mathbf{GH}$-name of z, then no $\mathbf{GH}$-name of z is computable relative to a $\bar{\delta}_{\mathcal{N}}$-name of x.

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$$e \in G_x \iff (\exists D \text{ finite})[(e, D) \in \Gamma \text{ and } D \subseteq L].$$

- Since L is $\Pi^1_1(x)$, this gives a $\Pi^1_1(x)$ definition of G_x; however G_x is clearly $\Sigma^1_1(x)$ complete, a contradiction.