De Groot duality in Computability Theory

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Joint Work with

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Background

- The theory of ω^ω-representations makes it possible to develop computability theory on T_0-spaces with countable cs-networks.
- K.-Pauly (201x): Degree theory on ω^ω-represented spaces.

  My original motivation came from my previous works trying to solve an open problem in descriptive set theory; K. (2015) and Gregoriades-K.-Ng (201x).
- K.-Lempp-Ng-Pauly (201x) established classification theory of e-degrees by using degree theory on second-countable spaces.

  This work includes degree-theoretic analysis of topological separation property, submetrizability, G_δ-spaces, etc.

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  This work includes degree-theoretic analysis of topological separation property, submetrizability, G_δ-spaces, etc.
- However, T_0-spaces with countable cs-networks and continuous functions form a cartesian closed category, which is far larger than the category of second-countable T_0 spaces.
- Thus, one can study...
  computability theory on some NON-second-countable spaces
  without using notions from GRT such as α-recursion, E-recursion, ITTM, etc.

Observation

One can study *computability on some NON-2nd-countable spaces*

without using notions from GRT such as α-recursion, ***E***-recursion, ITTM, etc.

Question

Is it worth studying non-2nd-countable computability theory?

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One can study *computability on some NON-2nd-countable spaces*

without using notions from GRT such as α-recursion, $\boldsymbol{E}$-recursion, ITTM, etc.

Question

Is it worth studying non-2nd-countable computability theory?

Answer

Definitely, YES! Because the space of higher type continuous functionals is not second countable:

There is no 2nd-countable topology on $\boldsymbol{C}(\mathbb{N}^{\mathbb{N}}, \mathbb{N})$ with continuous evaluation.

- Kleene, Kreisel ('50s): Computability theory at higher types.
- Hinman, Normann ('70s, '80s):
  Degree theory on higher type continuous functionals.

- $C(\mathbb{N}^{\mathbb{N}}, \mathbb{N})$: the space of continuous functions $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$.
- $p = (\langle \sigma_s, k_s \rangle)_{s \in \omega}$ is a *name* of $f \in C(\mathbb{N}^{\mathbb{N}}, \mathbb{N})$ iff

$$\{f\} = \bigcap_s [\sigma_s, k_s],$$

where $[\sigma, k] = \{g \in C(\mathbb{N}^{\mathbb{N}}, \mathbb{N}) : (\forall x \succ \sigma)\ g(x) = k\}$.

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Write $\delta_{KK}(p) = f$ if p is a name of f. (KK stands for Kleene-Kreisel)
Consider the quotient topology τ_{KK} on $C(\mathbb{N}^{\mathbb{N}}, \mathbb{N})$ given by δ_{KK}.
The evaluation map is continuous w.r.t. τ_{KK}.

Observation (Openness is NOT a basic concept)

$[\sigma, n]$ is closed, but NOT open w.r.t. τ_{KK}.
There is no countable collection of open sets generating τ_{KK}.

Definition (Arhangel'skii 1959)

A network for a space $\mathcal{X}$ is a collection $\mathcal{N}$ of subsets of $\mathcal{X}$ such that

$$(\forall x \in \mathcal{X})(\forall U \text{ open nbhd of } x)(\exists N \in \mathcal{N})\ x \in N \subseteq U.$$

open network = open basis

Example

$([\sigma, k])_{\sigma,k}$ forms a countable (closed) network for $C(\mathbb{N}^{\mathbb{N}}, \mathbb{N})$.

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$([\sigma, k])_{\sigma,k}$ forms a countable (closed) network for $C(\mathbb{N}^{\mathbb{N}}, \mathbb{N})$.

$\mathcal{N}$ is a *local network at x* if $x \in \bigcap \mathcal{N}$, and

$$(\forall U \text{ open nbhd of } x)(\exists N \in \mathcal{N})\ x \in N \subseteq U.$$

Encoding of a space having a countable network

Let $(N_n)_{n\in\omega}$ be a countable network for a space $\mathcal{X}$.
Then, we say that $p \in \mathbb{N}^{\mathbb{N}}$ is a *name* of $x \in \mathcal{X}$ if

$$\{N_{p(n)} : n \in \mathbb{N}\} \text{ is a local network at } x.$$

A number of variants of networks has been extensively studied in general topology, especially in the context of function space topology (e.g. C_p-theory), generalized metric space theory, etc.

> k-network, cs-network, cs*-network, sn-network, Pytkeev network, etc.

However, in such a context, spaces are mostly assumed to be regular T_1.
e.g. cosmic space, $\aleph_0$-space (Michael 1966), etc.

We don't want to assume regularity, eg. $(C(\mathbb{N}^{\mathbb{N}}, \mathbb{N}), \tau_{KK})$ is not regular (Schröder)

Fact (Schröder 2002)

For a T_0-space $\mathcal{X}$, the following are equivalent:

1. $\mathcal{X}$ is admissibly represented.
2. $\mathcal{X}$ has a countable cs-network.

For a sequential T_0 space, these conditions are also equivalent to being qcb_0:
A space is *qcb_0* if it is T_0, and is a quotient of a second-countable (countably based) space.

- (Guthrie 1971) A *cs-network* is a network $\mathcal{N}$ such that every convergent sequence converging to a point $x \in U$ with U open, is eventually in $N \subseteq U$ for some $N \in \mathcal{N}$.

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- *"Cs-network comes first, then topology."*

  In principle, we cannot recover topology from a network, but given a countable cs-network, we can recover the *sequentialization* of the topology.
- (Schröder) Sequential T_0-spaces with countable cs-networks and continuous functions form a cartesian closed category.

  Y^X is topologized by the sequentialization of the cs-open topology.

Claim

The de Groot dual of $\mathbb{N}^{\mathbb{N}}$ is admissibly represented.

Definition (De Groot et al. 1969)

For a topological space X, the *de Groot dual* is the topology on X generated by the complements of saturated compact sets w.r.t. the original topology on X.

We use X^d to denote the de Groot dual of X.

Dual Representation (K.-Pauly)

The category of admissibly represented sps. is cartesian closed. Thus, if X is admissibly represented, then so is the following:

$$\mathcal{A}_1(X) = \{f \in C(X, \mathbb{S}) : f^{-1}\{\bot\} \text{ is singleton}\},$$

where $\mathbb{S} = \{\top, \bot\}$ is the Sierpiński space, whose open sets are $\emptyset$, $\{\top\}$, and $\{\top, \bot\}$.

Roughly speaking, $\mathcal{A}_1(X)$ is the space of closed singletons in X.

Given an adm. rep. δ of X, we get an adm. rep. δ_1 of $\mathcal{A}_1(X)$.
We define the *dual representation* δ^c of δ by:

$$\delta^c(p) = x \iff (\delta_1(p))^{-1}\{\bot\} = \{x\}.$$

Write X^c for the represented space (X, δ^c).

x has a computable name in X^c iff $\{x\}$ is a Π^0_1 singleton in X.

- Defining points in X^c ≈ "implicitly" defining points in X.
- If X is admissibly represented, so is the dual X^c.

Claim

The de Groot dual of $\mathbb{N}^{\mathbb{N}}$ is admissibly represented.

- De Brecht (2014) introduced the notion of a *quasi-Polish space* to develop "*non-metrizable/non-Hausdorff descriptive set theory*".
- Schröder (unpublished) introduced the notion of a *co-Polish space*.

  A space is co-Polish if $C(X, \mathbb{S})$ is quasi-Polish.
- (Schröder) If X is quasi-Polish, so is $C(C(X, \mathbb{S}), \mathbb{S})$.

  If X is Polish, then the topology on $C(X, \mathbb{S})$ is indeed the compact-open topology.
- Therefore, if X is Polish, the sequentialization of the cs-open topology on $C(X, \mathbb{S})$ coincides with the compact-open topology.
- This concludes $X^d \simeq X^c$ whenever X is Polish.

- $X^d \simeq X^c$ whenever X is Polish.
- We do not know whether $X^d \simeq X^c$ for non-Polish X.
- X^c is better-behaved than X^d from the viewpoint of TTE.

  X^c is admissibly represented whenever X is.
- But, it is unclear whether the classical duality results hold for X^c.

De Groot et al., Lawson, and others

- X is a Hausdorff k-space $\Longrightarrow X^{dd} \simeq X$.
- X is stably compact $\Longrightarrow X^{dd} \simeq X$.

Some partial result:

Theorem (K.-Pauly)

X is second-countable and Hausdorff $\Longrightarrow X^{cc} \simeq X$.

Suppose that X is represented by $\delta :\subseteq \mathbb{N}^{\mathbb{N}} \to X$.

- If $\delta(p) = x$, then we think of p as a name of x.
- The complexity of x is identified with that of $\delta^{-1}\{x\}$ (all names of x).
- The degree of x is the *degree of difficulty of calling a name of* x.

Definition (K.-Pauly 201x)

Let X, Y be represented spaces. Write $x : X \leq_T y : Y$ if there is an algorithm which, given a name of y, returns a name of x.

That is, $x : X \leq_T y : Y$ iff

$$(\exists\Phi)(\forall p)\ [p \text{ is a name of } y \implies \Phi(p) \text{ is a name of } x]$$

The degree of difficulty of calling a name of a point x in X^c
$\approx$ that of finding an oracle z making x be a $\Pi^0_1(z)$ singleton in X.

- $\mathbb{S}^{\mathbb{N}}$ is a universal second-countable T_0-space.
- The degrees of points in $\mathbb{S}^{\mathbb{N}}$ = enumeration degrees.

Observation

Given $A \subseteq \mathbb{N}$, define $\chi_A \in \mathbb{S}^{\mathbb{N}}$ by $\chi_A(n) = \top$ iff $n \in A$.

In the theory of e-degrees, $A \subseteq \mathbb{N}$ is called *quasi-minimal* iff

$$(\forall y \in 2^{\mathbb{N}})\ [y : 2^{\mathbb{N}} \leq_T \chi_A : \mathbb{S}^{\mathbb{N}} \implies y : 2^{\mathbb{N}} \leq_T \emptyset].$$

Definition (De Brecht-K.-Pauly)

For represented spaces X, Y, a point $x \in X$ is Y*-quasi-minimal* if

$$(\forall y \in Y)\ [y : Y \leq_T x : X \implies y : Y \leq_T \emptyset].$$

We say that $x \in \mathbb{N}^{\mathbb{N}}$ is a *Π^0_1-lost melody* if there is $z \in \mathbb{N}^{\mathbb{N}}$ s.t.

- x is implicitly Π^0_1 definable relative to z
- x is not explicitly Δ^0_2 definable relative to z.

In other words, $\{x\}$ is a $\Pi^0_1(z)$ singleton, but $x \not\leq_T z'$.

This terminology comes from an analogous concept in the theory of ITTMs.

Theorem (K.-Pauly)

Every Π^0_1-lost melody x is, as a point in the dualspace $(\mathbb{N}^{\mathbb{N}})^c$, $\mathbb{S}^{\mathbb{N}}$-quasiminimal:

$$(\forall Y \in \mathbb{S}^{\mathbb{N}})\ [y : \mathbb{S}^{\mathbb{N}} \leq_T x : (\mathbb{N}^{\mathbb{N}})^c \implies y : \mathbb{S}^{\mathbb{N}} \leq_T \emptyset]$$

This result can be relativized for any oracle A:
Every $\Pi^0_1(A)$-lost melody x is, as a point in the dualspace $(\mathbb{N}^{\mathbb{N}})^c$, quasiminimal w.r.t. all spaces in SC^A_0,
where SC^A_0 is the class of all A-computable second-countable T_0 spaces.