Uniform Kurtz randomness

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Abstract

We propose studying uniform Kurtz randomness, which is the uniform relativization of Kurtz randomness. This notion has more natural properties than the usual relativization. For instance, van Lambalgen's theorem holds for uniform Kurtz randomness while not for (the usual relativization of) Kurtz randomness. Another advantage is that lowness for uniform Kurtz randomness has many characterizations, such as those via complexity, martingales, Kurtz tt-traceability, and Kurtz dimensional measure.

Keywords: algorithmic randomness, Kurtz randomness, uniform relativization, van Lambalgen's theorem, lowness for randomness, effective dimension

1 Introduction

In computability theory [29, 26, 27, 6], if one can compute a set $B \in 2^\omega$ given a set $A \in 2^\omega$ as an oracle, we say that B is *computable relative* to A or that B is *reducible* to A. Many reducibilities have been studied such as Turing reducibility (full access to an oracle) and truth-table reducibility (restricted access).

Algorithmic randomness [10, 25] has also studied *relative randomness*. If a set Y is random even if one is allowed to access a set A as an oracle, we say that B is *random relative* to A. Many results have been known relating to Martin-Löf randomness with full access to an oracle (*full relativization*) and it has come to be known that the restriction of the access to an oracle (*partial relativization*) is useful in the study of other randomness notions [1, 2, 7]. Then, how do we find the "proper" relativization for each randomness notion?

One important theorem relating to relative randomness is van Lambalgen's theorem, which says that $A \oplus B$ is Marin-Löf random if and only if A is Martin-Löf random and B is Martin-Löf random relative to A. The "only if" direction is usually called "easy direction" and the "if" direction is called "difficult direction". This theorem is regarded as one reason of the naturalness of Martin-Löf

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randomness. The second author [22] has proposed to use van Lambalgen's theorem as the criterion of the proper relativzation.

Uniform relativization has been proposed in the study of van Lambalgen's theorem for Schnorr randomness. [22, 23]. (Essentially the same notion can be also seen in [15].) In fact, van Lambalgen's theorem holds for uniform Schnorr randomness (the uniform relativization of Schnorr randomness) [22, 23] while it does not hold for the usual relativization of Schnorr randomness [20, 34] (see also [25, Remark 3.5.22]). Furthermore, van Lambalgen's theorem holds for uniformly computable randomness in a weaker sense [23] while it does not hold for computable randomness [20].

Subsequently, van Lambalgen's theorem for $\mathrm{Demuth}_{\mathrm{BLR}}$ randomness (a partial relativization of Demuth randomness) has been shown [7]. As noted in [23], $\mathrm{Demuth}_{\mathrm{BLR}}$ randomness is equivalent to uniform Demuth randomness and the usual relativization of Martin-Löf randomness is equivalent to uniform relativization of Martin-Löf randomness. Thus, the uniform relativization may be the proper relativization for all randomness notions.

The terminology of "full relativization" and "partial relativization" can be confusing because the usual relativization (the full relativization) has a strong connection to partial functions and the uniform relativization (a partial relativization) to total functions. Furthermore, the restriction of the access to an oracle is the demand from the totalness rather than artificially.

In this paper, with such motivation, we study the uniform relativization of Kurtz randomness, which we call *uniform Kurtz randomness*. It has been known that van Lambalgen's theorem does not hold for Kurtz randomness [12]. Later, we will show that van Lambalgen's theorem for uniform Kurtz randomness does hold but in a weaker sense.

Another active topic relating to relative randomness is "lowness". For a given randomness notion R, A is said to be low for R if every R-random set is R-random relative to A, that is, A does not have enough computational power to derandomize a random set. For instance, lowness for ML-randomness has many characterization such as K-triviality, lowness for K and being a base for ML-randomness [24, 17].

Lowness for Schnorr randomness has previously been studied in the literature such as [9, 18, 8]. However, lowness for uniform Schnorr randomness has more natural properties [15, 22, 21]. Similar phenomena have been found for other notions of randomness [1, 2].

To advocate the naturalness of uniform Kurtz randomness (and uniform relativization), we also study lowness for uniform Kurtz randomness. There are already some known results on lowness for Kurtz randomness [11, 30, 16]. Here, we show that lowness for uniform Kurtz randomness has many characterizations.

The overview of this paper is as follows. In Section 3 we introduce uniform Kurtz randomness defined by tests and characterize it via complexity and martingales. In Section 4 we study van Lambalgen's theorem for uniform Kurtz randomness. In Section 5 we introduce the notion of Kurtz h-dimensional measure zero where h is an order, and give characterizations via complexity and martingales. In Section 6 we characterize lowness for uniform Kurtz randomness via Kurtz h-dimensional measure zero and Kurtz tt-traceability. To prove this, we make use of the svelte tree introduced in Greenberg-Miller [16].

2 Preliminaries

We say that $n > 0$ is the index of a finite set $\{x_1, \cdots, x_r\}$ of natural numbers if $n = 2^{x_1} + 2^{x_2} + \cdots + 2^{x_r}$, while 0 is the index of $\emptyset$. In the following we often identify a finite set with its index. For example, $f : \omega \to (2^{<\omega})^{\omega}$ is a computable function if there is a computable function $g : \omega \to \omega$ such that $g(n)$ is the index of $f(n)$ for each n. We also often identify $\sigma \in 2^{<\omega}$ with the natural number n represented by 1σ in binary representation. The length of a string σ is denoted by $|\sigma|$. An *order* is a nondecreasing unbounded function from ω to ω. We denote the empty string by ϵ. For a string σ, let $[\sigma] = \{X : \sigma \preceq X\}$ where $\preceq$ is the prefix relation. For a set $S \subseteq 2^{<\omega}$, let $[\![S]\!] = \bigcup_{\sigma \in S}[\sigma]$.

We recall some results on Kurtz randomness. The reader may refer to [10, 25] for details. Let μ be the uniform measure on the Cantor space 2^{ω}.

Definition 2.1 (Kurtz [19]). *A set $A \in 2^{\omega}$ is* weakly 1-random, *or* Kurtz random, *if it is contained in every c.e. open set with measure 1.*

Definition 2.2 (Wang [31]). *A* Kurtz null test *is a sequence $\{[\![f(n)]\!]\}$ such that $f : \omega \to (2^{<\omega})^{<\omega}$ is a computable function and $\mu([\![f(n)]\!]) \leq 2^{-n}$.*

Theorem 2.3 (Wang [31], after Kurtz [19]). *A set is Kurtz random if and only if it passes all Kurtz null tests.*

Recall that a *martingale* is a function $d : 2^{<\omega} \to \mathbb{R}^+$ to nonnegative reals such that $2d(\sigma) = d(\sigma 0) + d(\sigma 1)$ for each $\sigma \in 2^{<\omega}$.

Theorem 2.4 (Wang [31], Downey et al. [11]). *A set A is not Kurtz random if and only if there are a computable martingale d and a computable order h such that $d(X \upharpoonright n) > h(n)$ for all n.*

Definition 2.5 ([9]). *A* computable measure machine *is a prefix-free machine M such that $\mu([\![\mathrm{dom} M]\!])$ is computable.*

Theorem 2.6 (Downey, Griffiths and Reid [11]). *A set X is not Kurtz random if and only if there is a computable measure machine M and a computable function f such that, for all n, $K_M(X \upharpoonright f(n)) < f(n) - n$.*

3 Uniform Kurtz randomness

The definition of uniform relativization [23] requires some definitions in computable analysis [32, 3, 4, 33]. Let τ be the class of open sets on 2^{ω}. A partial function $f :\subseteq 2^{\omega} \to \tau$ is *computable* if there is a partial computable function $\psi :\subseteq 2^{\omega} \times \omega \to (2^{<\omega})^{<\omega}$ such that $f(Z) = \bigcup_{n \in \omega}[\![\psi(Z, n)]\!]$ for each Z. If such a function ψ is total, then f is also called *total*.

Definition 3.1. *A* uniform Kurtz test *is a total computable function* $f : 2^\omega \to \tau$ *such that* $\mu(f(Z)) = 1$ *for all* $Z \in 2^\omega$*. A set* $B \in 2^\omega$ *is* Kurtz random uniformly relative to $A \in 2^\omega$ *if* $B \in f(A)$ *for each uniform Kurtz test* f*.*

Definition 3.2. *A* uniform Kurtz null test *is a computable function* $f : 2^\omega \times \omega \to (2^{<\omega})^{<\omega}$ *such that, for each* $Z \in 2^\omega$ *and* $n \in \omega$*,* $\mu([\![f(Z,n)]\!]) \le 2^{-n}$*. For a fixed set* X*, we also say that* $\{[\![f(X,n)]\!]\}_{n\in\omega}$ *is* a Kurtz null test uniformly relative to X*.*

Proposition 3.3. *The following are equivalent for* $A \in 2^\omega$ *and* $N \subseteq 2^\omega$*.*

(i) $2^\omega \setminus N = f(A)$ *for a uniform Kurtz test* f*.*

(ii) $N = \bigcap_n [\![g(A,n)]\!]$ *for a uniform Kurtz null test* g*.*

(iii) *A tt-computes a sequence* $\{C_n\}$ *of finite sets of strings such that* $\mu([\![C_n]\!]) \le 2^{-n}$ *and* $N = \bigcap_n [\![C_n]\!]$*.*

Proof. (i) ⇒ (ii): Let f be a uniform Kurtz test. Then there exists $\psi : 2^\omega \times \omega \to (2^{<\omega})^{<\omega}$ such that $f(Z) = \bigcup_n [\![\psi(Z,n)]\!]$. Since $\mu(f(Z)) = 1$ for each $Z \in 2^\omega$, we can effectively calculate $t(Z,n)$ such that $\mu(\bigcup_{m\le t(Z,n)} [\![\psi(Z,m)]\!]) \ge 1 - 2^{-n}$ and $t(Z,n) < t(Z,n+1)$. Let $g : 2^\omega \times \omega \to (2^{<\omega})^{<\omega}$ be a computable function such that

$$[\![g(Z,n)]\!] = 2^\omega \setminus \bigcup_{m\le t(Z,n)} [\![\psi(Z,m)]\!].$$

Then g is a uniform Kurtz null test and

$$N = 2^\omega \setminus f(A) = 2^\omega \setminus \bigcup_n [\![\psi(Z,n)]\!] = \bigcap_n [\![g(Z,n)]\!].$$

(ii) ⇒ (i): Let g be a uniform Kurtz null test. Let $\phi : 2^\omega \times \omega \to (2^{<\omega})^{<\omega}$ be a computable function such that $[\![\phi(Z,n)]\!] = 2^\omega \setminus [\![g(Z,n)]\!]$, and let $f : 2^\omega \to \tau$ be such that $f(Z) = \bigcup_n [\![\phi(Z,n)]\!]$. Then f is a total computable function and $\mu(f(Z)) = 1$ for each $Z \in 2^\omega$, because

$$2^\omega \setminus f(Z) = 2^\omega \setminus \bigcup_n [\![\phi(Z,n)]\!] = \bigcap_n [\![g(Z,n)]\!]$$

is null.

(ii) ⇒ (iii): This is because $\{g(A,n)\}_{n\in\omega}$ is truth-table reducible to A.

(iii) ⇒ (ii): Let Φ be a truth-table functional such that $\Phi(A,n) = C_n$. Then we can effectively check whether or not $\mu([\![\Phi(Z,n)]\!]) \le 2^{-n}$ because $[\![\Phi(Z,n)]\!]$ is clopen. If not, we define $\Psi(Z,n) = \emptyset$ and, otherwise, set $\Psi(Z,n) = \Phi(Z,n)$. Then Ψ is a uniform Kurtz null test. Moreover, $\Psi(A,n) = \Phi(A,n) = C_n$ for each n and $\bigcap_n [\![\Psi(A,n)]\!] = \bigcap_n [\![C_n]\!]$. □

Remark 3.4. *If we drop* totality *from the definition of uniform Kurtz tests, the truth-table reducibility* $\le_{tt}$ *in (iii) is changed into Turing reducibility* $\le_T$*.*

Formally, for every $A \in 2^\omega$ and $N \subseteq 2^\omega$, (i') its complement $2^\omega \setminus N$ is A-c.e. open with measure 1, if and only if (ii') N is the intersection of a Kurtz null test relative to A if and only if (iii') A computes a sequence $\{C_n\}$ of finite set of strings such that $\mu([\![C_n]\!]) \leq 2^{-n}$, and $N = \bigcap_n [\![C_n]\!]$.

We give characterizations of uniform Kurtz randomness via martingales and machines (compare with Theorems 2.4 and 2.6).

Proposition 3.5. *The following are equivalent for sets A and B.*

(i) *A is not Kurtz random uniformly relative to B.*

(ii) *There are a computable function $d : 2^\omega \times 2^{<\omega} \to \mathbb{R}^+$ and a computable order h such that $d(Z, -)$ is a martingale for each $Z \in 2^\omega$ and $d(A \upharpoonright n) > h(n)$ for all $n \in \omega$.*

(iii) *There are a $\mathbb{Q}$-valued martingale $d \leq_{tt} B$ and a computable order h such that $d(A \upharpoonright n) > h(n)$ for all $n \in \omega$.*

(iv) *There are an oracle prefix-free machine M and a computable function h such that $Z \mapsto \mu(\mathrm{dom} M^Z)$ is a computable function and $K_{M^B}(A \upharpoonright h(n)) < h(n) - n$ for all $n \in \omega$.*

Proof. Note that (ii) is equivalent to the following statement:

> (ii') There are a computable function $d : 2^\omega \times 2^{<\omega} \to \mathbb{Q}^+$ to nonnegative reals and a computable order h such that $d(Z, -)$ is a martingale for each $Z \in 2^\omega$ and $d(A \upharpoonright n) > h(n)$ for all $n \in \omega$.

(See [10, Proposition 7.1.2] for the detail.)

(i)$\Rightarrow$(ii'): Suppose that A is not Kurtz random uniformly relative to B. Then there is a uniform Kurtz null test f such that $A \in \bigcap_n [\![f(B, n)]\!]$. Since f is a total computable function, we can assume the existence of a strictly increasing computable function g such that $g(0) = 0$ and $\sigma \in f(Z, n) \Rightarrow |\sigma| = g(n)$. Let k be a computable order such that $k(0) = 0$ and $k(n) \geq g(k(n-1)) + 1$ for all $n \geq 1$.

We construct a $\mathbb{Q}$-valued martingale d^Z as follows. Let $d^Z(\epsilon) = 2$. At stage $n \geq 1$, we define $d(\sigma)$ for $\sigma \in 2^{<\omega}$ such that $g(k(n-1)) < |\sigma| \leq g(k(n))$. Note that $g(k(0)) = 0$. For each $\tau \in f(Z, k(n-1))$, let $a(\tau)$ be the number of strings $\rho \in f(Z, k(n))$ such that $\tau \prec \rho$. For each $\tau \notin f(Z, k(n-1))$, let $a(\tau) = 0$. Note that a is computable from Z. We assume that, for each $\rho \in f(Z, n)$, there is $\tau \in f(Z, n-1)$ such that $\tau \prec \rho$, whence $\sum_{\tau \in f(Z,k(n-1))} a(\tau) = \#f(Z, k(n)) \leq 2^{g(k(n))-k(n)}$ where the last inequality follows from $\mu([\![f(Z, k(n))]\!]) \leq 2^{-k(n)}$. Let $\sigma \in 2^{<\omega}$ be such that $g(k(n-1)) < |\sigma| \leq g(k(n))$. We define a $\mathbb{Q}$-valued martingale $d^Z(\sigma)$ by

$$d^Z(\sigma) = \begin{cases} d^Z(\sigma \upharpoonright g(k(n-1))) & \text{if } a(\sigma \upharpoonright g(k(n-1))) = 0 \\ e^Z(\sigma) & \text{if there is } \tau \in f(Z, k(n)) \text{ such that } \sigma \preceq \tau \\ \frac{d^Z(\sigma \upharpoonright g(k(n-1)))}{2} & \text{otherwise,} \end{cases}$$

where

$$e^Z(\sigma) = d^Z(\sigma \upharpoonright g(k(n-1)))\left(\frac{1}{2} + \frac{2^{|\sigma|-g(k(n-1))}}{2 \cdot a(\sigma \upharpoonright g(k(n)))}\right).$$

Clearly, d^Z is a martingale for each $Z \in 2^\omega$ and is uniformly computable from Z. Let $d = d^B$.

First we show that $d(A \upharpoonright g(k(n))) > \left(\frac{3}{2}\right)^n$ for all $n \in \omega$. If $n = 0$, $d(\epsilon) = 2 > 1 = \left(\frac{3}{2}\right)^0$. By assuming that $d(A \upharpoonright g(k(n-1))) > \left(\frac{3}{2}\right)^{n-1}$, we have

$$\begin{aligned} d(A \upharpoonright g(k(n))) = e^B(A \upharpoonright g(k(n))) &\geq d(A \upharpoonright g(k(n-1)))\left(\frac{1}{2} + \frac{2^{g(k(n))-g(k(n-1))}}{2^{g(k(n))-k(n)+1}}\right) \\ &\geq \frac{3}{2} d(A \upharpoonright g(k(n-1))) > \left(\frac{3}{2}\right)^n. \end{aligned}$$

We define a computable order h by

$$h(m) = \left\lfloor \frac{1}{2} \cdot \left(\frac{3}{2}\right)^{n-1} \right\rfloor \text{ where } g(k(n-1)) \leq m < g(k(n)),$$

where $\lfloor x \rfloor$ denotes the largest integer not greater than x. If $m = 0$, then $d(A \upharpoonright m) = d(\epsilon) = 2 > 1 = h(0) = h(m)$. If m satisfies $g(k(n-1)) < m \leq g(k(n))$, then

$$d(A \upharpoonright m) \geq \frac{d(A \upharpoonright g(k(n-1)))}{2} > \frac{1}{2} \cdot \left(\frac{3}{2}\right)^{n-1} \geq h(m).$$

(ii')$\Rightarrow$(iii): This implication is immediate.

(iii)$\Rightarrow$(i): Assume that $d \leq_{tt} B$. Then there is a truth-table functional Ψ such that Ψ^Z is a $\mathbb{Q}$-valued martingale for each $Z \in 2^\omega$ and $d = \Psi^B$. Let f be a computable order such that $h(f(n)) \geq 2^n$ for all $n \in \omega$. Consider the following clopen set: $C_n^Z = \{\sigma \in 2^{f(n)} \ : \ \Psi^Z(\sigma) \geq 2^n\}$. Then $\mu(C_n^Z) \leq 2^{-n}$ for all $n \in \omega$ and $Z \in 2^\omega$. Since $\Psi^B(A \upharpoonright f(n)) = d(A \upharpoonright f(n)) > h(f(n)) \geq 2^n$ for each n, we have $A \in \bigcap_n C_n^B$.

The proof of (i) $\iff$ (iv) is a straightforward modification of the unrelativized version in [11].

(i)$\Rightarrow$(iv): Suppose that there is a uniform Kurtz null test f such that $A \in \bigcap_n [\![f(B,n)]\!]$. We can assume that, for each n, all the strings in $f(Z,n)$ for $Z \in 2^\omega$ have the same length $g(n)$, where g is a computable order. Let $m : 2^\omega \to \mathbb{R}$ be a function such that

$$m(Z) = \sum_n \sum_{\sigma \in f(Z,2n+2)} 2^{-(|\sigma|-(n+1))} = \sum_n 2^{n+1}\mu([\![f(Z,2n+2)]\!]).$$

Since $\mu([\![f(Z,2n+2)]\!]) \leq 2^{-(2n+2)}$, $m(Z) \leq 1$ and m is computable. By the KC Theorem [10, Theorem 3.6.1], there is an oracle prefix-free machine M such that $\mu(\mathrm{dom} M^Z) = m(Z)$ for each $Z \in 2^\omega$ and $K_{M^Z}(\sigma) \leq |\sigma| - (n+1)$ for each $\sigma \in f(Z, 2n+2)$. Let $h(n) = g(2n+2)$. Since $A \upharpoonright g(2n+2) \in f(B, 2n+2)$, we have

$$K_{M^B}(A \upharpoonright h(n)) \leq h(n) - (n+1)$$

for each n.

(iv)$\Rightarrow$(i): Suppose that the pair A, B satisfies (ii) via an oracle prefix-free machine M and a computable function h. Let

$$f(Z,n) = \{\sigma \in 2^{h(n)} \ : \ K_{M^Z}(\sigma) < h(n) - n.\}.$$

Then $\mu([\![f(Z,n)]\!]) \leq 2^{-n}$. Since $Z \mapsto \mu(\mathrm{dom} M^Z)$ is computable, there is a total computable function $\phi : 2^\omega \to 2^\omega$ such that $\phi(Z)(n) = \mathrm{dom}(M^Z)(n)$ for each $Z \in 2^\omega$ and $n \in \omega$. Then $f(Z,n)$ is finite and f is a computable function. Thus, f is a uniform Kurtz null test. By the definition of M and f, $A \in \bigcap_n [\![f(B,n)]\!]$. □

4 Van Lambalgen's theorem for uniform Kurtz randomness

First recall van Lambalgen's theorem for Kurtz randomness.

Theorem 4.1 ([12, Theorem 3.7,3.8])**.** *If $A \in 2^\omega$ is Kurtz random and $B \in 2^\omega$ is Kurtz random relative to A, then $A \oplus B$ is Kurtz random. There is a pair $A, B \in 2^\omega$ of sets such that $A \oplus B$ is Kurtz random and neither A nor B is Kurtz random relative to the other.*

Thus, the "easy direction" does not hold and the "difficult direction" holds for Kurtz randomness.

4.1 The easy direction

For uniform Kurtz randomness, we can show that the "easy direction" holds.

Theorem 4.2. *If $A \oplus B$ is Kurtz random, then B is Kurtz random uniformly relative to A.*

Proof. Suppose that B is not Kurtz random uniformly relative to A. Then there is a total computable function $f : 2^\omega \to \tau$ such that $B \notin f(A)$ and $\mu(f(Z)) = 1$ for all $Z \in 2^\omega$. We define a c.e. set U by

$$U = \{X \oplus Y \ : \ Y \in f(X)\}.$$

Then $\mu(U) = 1$ and $A \oplus B \notin U$. Hence $A \oplus B$ is not Kurtz random. □

Combined with Theorem 4.1, we can conclude the difference between Kurtz randomness and uniform Kurtz randomness.

Corollary 4.3. *There is a pair $A, B \in 2^\omega$ such that A is Kurtz random uniformly relative to B and not Kurtz random relative to B.*

4.2 The difficult direction

The following is an unexpected result. The "difficult direction" does not hold.

Theorem 4.4. *There is a pair $A, B \in 2^\omega$ of sets such that A and B are mutually uniformly Kurtz random and $A \oplus B$ is not Kurtz random.*

We show this theorem by building such A and B. To make $A \oplus B$ non-Kurtz random, we use the following lemma.

Lemma 4.5. *Let $A, B \in 2^\omega$. If $A(n) = 0$ or $B(n) = 0$ for all n, then $A \oplus B$ is not Kurtz random.*

Proof. Let

$$U_n = \{X \oplus Y \ : \ X(n) = Y(n) = 1\}.$$

Then U_n is a c.e. open set uniformly in n and $\mu(U_n) = \frac{1}{4}$ for all n. Let

$$U = \bigcup_n U_n.$$

Then U is a c.e. open set and

$$\mu(2^\omega \setminus U) = \mu(\bigcap_n (2^\omega \setminus U_n)) = \prod_n \mu(2^\omega \setminus U_n) = 0.$$

Thus $\mu(U) = 1$. If $A(n) = 0$ or $B(n) = 0$ for all n, $A \oplus B \notin U$, whence $A \oplus B$ is not Kurtz random. □

Proof of Theorem 4.4. Let $\{\Phi_i\}_{i\in\omega}$ be an enumeration of all uniform Kurtz tests. Note that any uniform Kurtz test Φ_i can be thought of as a monotone function sending each finite string σ to a clopen set $\Phi_i(\sigma)$ such that $\Phi_i(X) = \bigcup_{\sigma \prec X} \Phi_i(\sigma)$. Note that for every string $\sigma \in 2^{<\omega}$, the open set $\Phi_i(\sigma 0^\omega)$ is dense since it is conull. At stage s, we define strings $\alpha_s \prec A$ and $\beta_s \prec B$ such that $|\alpha_s| = |\beta_s|$.

At stage $s = 2i$, the open density of $\Phi_i(\alpha_s 0^\omega)$ ensures the existence of $\beta \succeq \beta_s$ and m such that $[\![\beta]\!] \subseteq \Phi_i(\alpha_s 0^m)$. Here, we may safely assume that $|\alpha_s 0^m| \geq |\beta|$. Then, define $\alpha_{s+1} = \alpha_s 0^m$ and $\beta_{s+1} = \beta 0^{|\alpha_s| + m - |\beta|}$. At stage $s = 2i + 1$, we define α_s and β_s similarly by replacing α and β. Finally, we set $A = \bigcup_s \alpha_s$ and $B = \bigcup_s \beta_s$.

By construction, $A(n) = 0$ or $B(n) = 0$ for all n, whence $A \oplus B$ is not Kurtz random by Lemma 4.5. Moreover, we can see that B is Kurtz random uniformly relative to A since $B \in [\![\beta_{s+1}]\!] \subseteq \Phi_i(\alpha_{s+1}) \subseteq \Phi_i(A)$ for each i. Similarly, A is Kurtz random uniformly relative to B. □

4.3 Weaker form

We give an explanaion of why the "difficult direction" does not hold for uniform Kurtz randomness. First we give another characterization of Kurtz randomness. Recall the following characterization of Martin-Löf randomness.

Proposition 4.6 (Merkle's criterion of Martin-Löf randomness; [25, Proposition 3.2.17]). *The following are equivalent for a set $X \in 2^\omega$:*

(i) *X is not Martin-Löf random.*

(ii) *$X = x_0 x_1 x_2 \cdots$ for a sequence $\{x_i\}$ of strings such that $K(x_i) \leq |x_i| - 1$ for all i.*

(iii) *There is a prefix-free machine M such that $X = x_0 x_1 x_2 \cdots$ for a sequence $\{x_i\}$ of strings such that $K_M(x_i) \leq |x_i| - 1$ for all i.*

We give a Kurtz-randomness version of this result. For $m, n \in \omega$ such that $m < n$, we write

$$X \upharpoonright [m, n) = X(m)X(m+1)\cdots X(n-1) \in 2^{n-m}.$$

In particular, $X \upharpoonright [0, n) = X \upharpoonright n$.

Theorem 4.7. *The following are equivalent for a set $X \in 2^\omega$.*

(i) *X is not Kurtz random.*

(ii) *There exists a computable order l and a computable measure machine M such that*

$$K_M(X \upharpoonright [l(n), l(n+1))) \leq l(n+1) - l(n) - 1$$

for all n.

(iii) *There exists a computable order l and a prefix-free decidable machine M such that*

$$K_M(X \upharpoonright [l(n), l(n+1))) \leq l(n+1) - l(n) - 1$$

for all n.

Proof. (i)$\Rightarrow$(ii). Suppose X is not Kurtz random. Then there exists a computable function $f : \omega \to (2^{<\omega})^{<\omega}$ and a computable order u such that, for all n,

(i) $f(n) \subseteq 2^{u(n)}$,

(ii) $|f(n)| = 2^{u(n)-n}$,

(iii) $X \upharpoonright u(n) \in f(n)$.

We assume that $u(0) = 0$ and u is strictly increasing.

Let

$$k(0) = 0,\ k(n+1) = u(k(n)) + n + 2 \text{ and } l(n) = u(k(n)).$$

We construct a KC set

$$\langle l(n+1) - l(n) - 1, \sigma \upharpoonright [l(n), l(n+1))\rangle_{n \in \omega,\ \sigma \in f(k(n+1))}.$$

Note that, for $\sigma \in f(k(n+1))$, we have $|\sigma| = u(k(n+1)) = l(n+1)$. The weight of the KC set is

$$\begin{aligned}\sum_n \sum_{\sigma \in f(k(n+1))} 2^{-(l(n+1)-l(n)-1)} &= \sum_n 2^{u(k(n+1))-k(n+1)} \cdot 2^{-u(k(n+1)+u(k(n))+1} \\ &= \sum_n 2^{-n-1} = 1.\end{aligned}$$

Then, the constructed machine M by [10, Theorem 3.6.1] has computable measure. Note that

$$\sigma_n = X \upharpoonright u(n) \in f(n)$$

for each n. Then there is $\tau_n \in 2^{l(n+1)-l(n)-1}$ such that $M(\tau_n) = X \upharpoonright [l(n), l(n+1))$. Thus, $K_M(X \upharpoonright [l(n), l(n+1))) \leq l(n+1) - l(n) - 1$ for each n.

(ii)$\Rightarrow$(iii). This is because a computable measure machine is a prefix-free decidable machine.

(iii)$\Rightarrow$(i). Suppose that X satisfies (iii) for l and M. Let $S_0 = \{\lambda\}$ and

$$S_{n+1} = \{M(\sigma) \ : \ \sigma \in 2^{<l(n+1)-l(n)}, \ |M(\sigma)| = l(n+1) - l(n)\}$$

for each n. Since M is prefix-free, $\mu(\llbracket S_n \rrbracket) \leq 2^{-1}$. Let $f : \omega \to (2^{<\omega})^{<\omega}$ be a computable function such that

$$f(n) = \{x_1 \cdots x_n \ : \ x_i \in S_i \text{ for } i = 1, \cdots, n\}.$$

Then

$$\mu(\llbracket f(n) \rrbracket) = \prod_{i=1}^{n} \mu(\llbracket S_i \rrbracket) \leq 2^{-n}.$$

Hence, f is a Kurtz null test. Since $X \in \llbracket f(n) \rrbracket$ for all n, X is not Kurtz random. □

Theorem 4.7 intuitively says that a set is not Kurtz random if and only if there is a computable separation each of which has some regularity. Thus, even if one can find a computable separation of $A \oplus B$ each of which has some regularity, one may not find such separation in neither of A nor B. However, we can prove van Lambalgen's theorem for uniform Kurtz randomness in a weaker form.

Let $h, g : \omega \to \omega$ be strictly increasing computable functions such that

$$\omega = \{h(n) \ : \ n \in \omega\} \cup \{g(n) \ : \ n \in \omega\}.$$

We write $A \oplus_h B$ to mean the set X such that

$$X(h(n)) = A(n) \text{ and } X(g(n)) = B(n).$$

We call such $\oplus_h$ a computable union.

Theorem 4.8. *The following are equivalent for a set $X \in 2^\omega$:*

(i) *X is Kurtz random.*

(ii) *For each computable union $\oplus_h$, letting $X = A \oplus_h B$, the sets A, B are mutually uniform Kurtz random.*

(iii) *For each computable union $\oplus_h$, letting $X = A \oplus_h B$, at least one of A and B is Kurtz random.*

Proof. The proof of (i)⇒(ii) is essentially the same as the proof of Theorem 4.2.

The direction (ii)⇒(iii) is immediate.

For the proof of (iii)⇒(i), suppose that X is not Kurtz random. Then, by Theorem 4.7, there exists a computable order l and a computable measure machine M such that $K_M(X \upharpoonright [l(n), l(n+1))) \leq l(n+1) - l(n) - 1$. Let $\{X_i\}$ be such that

$$X = X_0 X_1 X_2 \cdots$$

and $|X_n| = l(n+1) - l(n)$ for all n. Then

$$K_M(X_n) \leq |X_n| - 1$$

for all n. Let h be the strictly increasind computable function such that

$$\{h(n) \ : \ n \in \omega\} = \{m \ : \ l(2n) \leq m < l(2n+1)\}$$

and A, B be such that $X = A \oplus_h B$. Then

$$A = X(0)X(2)X(4)\cdots \text{ and } B = X(1)X(3)X(5)\cdots.$$

Hence neither A nor B is Kurtz random. □

5 Kurtz Dimensional Measure

In this section, we introduce and give some characterizations of a notion of effective Hausdorff-like dimension, which will be called Kurtz h-dimensional measure zero. In Section 6, we will use the notion to characterize lowness for uniform Kurtz randomness.

The effectivization of concepts from fractal geometry such as Hausdorff dimension is playing a greater role in the theory of algorithmic randomness (see [10, Section 13]). Hausdorff dimension of a given object is a real number decided by the object. To estimate the exact dimension, researchers sometimes employ the concept of h-dimensional Hausdorff (outer) measure for a real-valued function h rather than that for a real number. Such a function h is called a *dimension function* or a *gauge function* (see Rogers [28] for basic concepts from fractal geometry).

Definition 5.1. *For an order $h : \omega \to \omega$, a set $E \subseteq 2^\omega$ is* Kurtz h-dimensional measure zero *if there is a computable sequence $\{C_n\}_{n\in\omega}$ of finite sets of strings such that*

$$E \subseteq [\![C_n]\!] \text{ and } \sum_{\sigma \in C_n} 2^{-h(|\sigma|)} \leq 2^{-n} \text{ for all } n \in \omega.$$

We also say that $A \in 2^\omega$ is Kurtz h-dimensional measure zero *if $\{A\}$ is Kurtz h-dimensional measure zero.*

We shall call E Kurtz s-dimensional measure zero when $h(n) = sn$ for a real s. Obviously, a set $A \in 2^\omega$ is Kurtz 1-dimensional measure zero if and only if it is not Kurtz random. By replacing $\{C_n\}_{n\in\omega}$ in Definition 5.1 with a sequence of (infinite) sets of strings (respectively, a computable sequence of c.e. sets of strings), we can realize the usual definition of being Hausdorff h-dimensional measure zero (respectively, effective Hausdorff h-dimensional measure zero).

Recall from Theorems 2.4 and 2.6 that Kurtz 1-dimensional measure zero can be characterized by computable martingales and Kolmogorov complexity. It is also well known that the effective Hausdorff dimension is characterized by Kolmogorov complexity and c.e. martingale (see [10, Section 13]). The following theorem provides the analogous characterization for Kurtz dimensional measure.

Theorem 5.2. *Let h be any computable order. Then the following are equivalent for a set A.*

(i) *A is Kurtz h-dimensional measure zero.*

(ii) *There are a computable martingale d and a computable order g such that*

$$(\forall n \in \omega)(\exists k \in [g(n), g(n+1))) \quad d(A \upharpoonright k) \geq 2^n \cdot 2^{k-h(k)}.$$

(iii) *There are a computable measure machine M and a computable order g such that*

$$(\forall n \in \omega)(\exists k \in [g(n), g(n+1))) \quad K_M(A \upharpoonright k) \leq h(k) - n.$$

The statement (ii) of Theorem 5.2 for $h(n) = n$ says that

> there are a computable martingale d and a computable order g such that
> $$(\forall n \in \omega)(\exists k \in [g(n), g(n+1))) \quad d(A \upharpoonright k) \geq 2^n.$$

In this case we can replace $\exists k \in [g(n), g(n+1))$ with $k = g(n)$ by savings lemma ([10, Proposition 6.3.8], [25, Exercise 7.1.14]) or slow-but-sure-winnings lemma ([5, Lemma 2.3]). Then the replaced statement is equivalent to the latter half of Theorem 2.4. The same holds for (iii) of Theorem 5.2 and the latter half of Theorem 2.6.

Proof. (i)⇒(ii): Suppose that A is Kurtz h-dimensional measure zero via a sequence $\{C_n\}_{n\in\omega}$. Find $t(n+1) \geq 2(n+1)+1$ such that all strings contained in $C_{t(n+1)}$ are longer than any strings of $C_{t(n)}$. Here $t(n) \geq 2n+1$ implies that $\sum_{\sigma\in C_{t(n)}} 2^{-h(|\sigma|)} \leq 2^{-2n-1}$. For each σ, let B_σ be a martingale defined by

$$B_\sigma(\tau) = \begin{cases} 2^{|\tau|-h(|\sigma|)} & \text{if } \tau \preceq \sigma \\ 2^{|\sigma|-h(|\sigma|)} & \text{if } \sigma \prec \tau \\ 0 & \text{otherwise.} \end{cases}$$

Then $d = \sum_n \sum_{\sigma\in C_{t(n)}} 2^n B_\sigma$ is a computable martingale with the initial capital

$$\sum_n \sum_{\sigma\in C_{t(n)}} 2^{n-h(|\sigma|)} = \sum_n 2^n \cdot 2^{-2n-1} \leq \sum_n 2^{-n-1} = 1.$$

Define g to be a computable order such that the length of every string in $C_{t(n)}$ is contained in $[g(n), g(n+1))$. Then, for all $n \in \omega$, there is a $k \in [g(n), g(n+1))$ such that $A \upharpoonright k \in C_{t(n)}$, that is,

$$d(A \upharpoonright k) \geq 2^n B_{A\upharpoonright k}(A \upharpoonright k) = 2^n \cdot 2^{k-h(k)}.$$

(ii)⇒(iii): By our assumption, for every $n \in \omega$, there is $k \in [g(n), g(n+1))$ such that $d(A \upharpoonright k) \geq 2^n 2^{k-h(k)}$. Without loss of generality, we may assume that $d(\epsilon) = 1$. Consider the following clopen set:

$$C_n = \{\sigma \in 2^{<\omega} : |\sigma| \in [g(2n), g(2n+1)), \text{ and } d(\sigma) \geq 2^{2n} 2^{|\sigma|-h(|\sigma|)}\}.$$

Let D_n be an antichain generating C_n. Then

$$\sum_{\sigma\in D_n} 2^{n-h(|\sigma|)} \leq \sum_{\sigma\in D_n} 2^{n-h(|\sigma|)} \frac{2^{-2n} d(\sigma)}{2^{|\sigma|-h(|\sigma|)}} = 2^{-n} \sum_{\sigma\in D_n} 2^{-|\sigma|} d(\sigma) \leq 2^{-n}.$$

Here, the last inequality follows from Kolmogorov's inequality (see [10, Theorem 6.3.3] with our assumption $d(\epsilon) = 1$. Thus, by the KC theorem [10, Theorem 3.6.1], we can construct a computable measure machine M such that, for each $n \in \omega$, $K_M(\sigma) \leq h(|\sigma|) - n$ for each $\sigma \in D_n$. In particular, for all $n \in \omega$, there is $k \in [g(2n), g(2n+1))$ such that $K_M(A \upharpoonright k) \leq h(k) - n$.

(iii)⇒(i): Assume that $K_M(A \upharpoonright k) \leq h(k) - n$ for some $k \in [g(n), g(n+1))$. Consider the sequence $\{C_n\}_{n\in\omega}$ of clopen sets defined by

$$C_n = \{\sigma \in 2^{<\omega} : |\sigma| \in [g(n), g(n+1)), \text{ and } K_M(\sigma) \leq h(|\sigma|) - n\}.$$

Then $A \in \bigcap_n C_n$, and

$$\sum_{\sigma\in C_n} 2^{-h(|\sigma|)} \leq 2^{-n} \sum_{\sigma\in C_n} 2^{-K_M(\sigma)} \leq 2^{-n}.$$

Hence, A is Kurtz h-dimensional measure zero. ∎

6 Lowness for Uniform Kurtz Randomness

Greenberg and Miller [16] characterized lowness for Kurtz randomness as being hyperimmune-free with no ability to compute a diagonally noncomputable function. Unfortunately, this kind of characterization seems to be unconformable to lowness for uniform relativizations, because in the tt-degrees $2^\omega/\equiv_{tt}$, the notion of hyperimmune-freeness is meaningless and has no analog. Therefore, although we know a much simpler proof [10, Theorem 12.4.5] of the characterization, for the above reason we are unable to use it to characterize our uniform version of lowness. Luckily, however as we will see in this section, a slight modification of the original proof of Greenberg and Miller [16] is sufficient to give acceptable characterizations of lowness for uniform Kurtz randomness.

A set $A \in 2^\omega$ is said to be *low for uniform Kurtz randomness* if $X \in 2^\omega$ is uniform Kurtz random relative to A whenever X is Kurtz random. A set $A \in 2^\omega$ is said to be *low for uniform Kurtz tests* if $f(A)$ includes a Kurtz test for every uniform Kurtz test f. For a given order p, a *computable trace with bound* p is a computable sequence $\{D_n\}_{n\in\omega}$ of finite sets of strings such that $\#D_n \leq p(n)$ for each $n \in \omega$. A computable trace $\{D_n\}_{n\in\omega}$ *Kurtz-traces* a function $f : \omega \to \omega$ if there is a strictly increasing computable sequence $\{l_n\}_{n\in\omega}$ of natural numbers such that

$$(\forall k \in \omega)(\exists n \in [l_k, l_{k+1})) \quad f(n) \in D_n.$$

A set $A \in 2^\omega$ is said to be *Kurtz tt-traceable* if there is a computable order p such that, for every $f \leq_{tt} A$, there is a computable trace with bound p that Kurtz-traces f.

Theorem 6.1. *The following are equivalent for a set A.*

(i) *A is Kurtz h-dimensional measure zero for every computable order h.*

(ii) *A is low for uniform Kurtz tests.*

(iii) *A is low for uniform Kurtz randomness.*

(iv) *A tt-computes no infinite subset of a Kurtz random set.*

(v) *A is Kurtz tt-traceable.*

Proof. (i)⇒(ii): By Proposition 3.3, every Kurtz null test $\{C_n^A\}_{n\in\omega}$ uniformly relative to A can be thought of as a truth table functional Ψ such that $[\![\Psi^A(n)]\!] = C_n^A$, and $\mu(C_n^A) \leq 2^{-n}$. Then there is a computable order u such that, for all $Z \in 2^\omega$ and all $n \in \omega$, the value $\Psi^{Z\upharpoonright u(n)}(n)$ is determined. In particular, $[\![\Psi^{A\upharpoonright u(n)}(n)]\!] = C_n^A$. Let h be a computable order fulfilling $2^{-h(u(n))} \geq 1/(n+1)$ for all $n \in \omega$. Assume that A is Kurtz h-dimensional measure zero. By our assumption, we have a computable sequence $\{D_n\}_{n\in\omega}$ of finite sets of strings such that $A \in [\![D_n]\!]$ and $\sum_{\sigma\in D_n} 2^{-h(|\sigma|)} < 1/(n+1)$ for all $n \in \omega$. Thus, each $\sigma \in D_n$ has length greater than $u(n)$, and moreover D_n contains at most k strings of length $\leq u(n+k)$, since, otherwise,

$$\sum_{\sigma\in D_n} 2^{-h(|\sigma|)} \geq (k+1)2^{-h(u(n+k))} \geq \frac{k+1}{n+k+1} \geq \frac{1}{n+1}.$$

Hence, D_n can be viewed as a finite sequence $\{\sigma^n_k\}_{k<|D_n|}$ of strings such that the length of each σ^n_k is greater than or equal to $u(n+k)$. Thus, there is $k < |D_n|$ such that $A \upharpoonright u(n+k) = \sigma^n_k \upharpoonright u(n+k)$. Inductively define a computable order r by $r(0) = 0$ and $r(n+1) = r(n) + |D_{r(n)}|$. Now $\rho(k)$ is defined by $\sigma^{r(n)}_{k-r(n)} \upharpoonright u(k)$ for each $k \in [r(n), r(n+1))$. Then

$$(\forall n \in \omega)(\exists k \in [r(n), r(n+1)))\ A \upharpoonright u(k) = \rho(k).$$

For all $n \in \omega$ and $k \in [r(n), r(n+1))$, define $E_k \subseteq 2^\omega$ by

$$E_k = \begin{cases} [\![\Psi^{\rho(k)}(k)]\!], & \text{if } \mu([\![\Phi^{\rho(k)}(k)]\!]) \leq 2^{-k}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Note that $|\rho(k)| = u(k)$ implies that $\Psi^{\rho(k)}(k)$ is defined for all $k \in \omega$ by our assumption for u. Therefore, $\{E_k\}_{k\in\omega}$ is a computable sequence of clopen sets, and we have

$$C^A_n \subseteq \bigcup_{k=r(n)}^{r(n+1)-1} E_k, \text{ and } \mu\left(\bigcup_{k=r(n)}^{r(n+1)-1} E_k\right) \leq 2^{-r(n)+1} \leq 2^{-n+1}.$$

Consequently, for $B_n = \bigcup_{r(n-1)<t\leq r(n)} E_t$, the sequence $\{B_n\}_{n\in\omega}$ is a Kurtz null test such that $\bigcap_n C^A_n \subseteq \bigcap_n B_n$. In other words, A is low for uniform Kurtz null tests.

(ii)$\Rightarrow$(iii): Obvious.

(iii)$\Rightarrow$(iv): Let $I \subseteq \omega^{\leq\omega}$ be the set of (finite or infinite) strings $\sigma \in \omega^{\leq\omega}$ which are strictly increasing, that is, $\sigma(n) < \sigma(n+1)$ for each $n \in \omega$. Let $\mathrm{rng}(\sigma)$ denote the range of $\sigma \in I$, so that $\mathrm{rng}(\sigma) = \{\sigma(n) : n < |\sigma|\}$. From now on, we think of each $B \subseteq \omega$ as a strictly increasing string $B^\star \in I$, where $B^\star(n)$ is the n-th least element contained in B. For any $\sigma \in I$, we denote by $\hat{P}^\sigma$ all supersets of the subset of ω obtained from σ, that is,

$$\hat{P}^\sigma = \{X \in 2^\omega : \mathrm{rng}(\sigma) \subseteq X\}.$$

Lemma 6.2. *A set A tt-computes an infinite subset of a Kurtz random set if and only if there exists an infinite set $B \leq_{tt} A$ such that the class $\hat{P}^{B^\star}$ contains a Kurtz random set. Moreover, if a set A tt-computes an infinite set $B \subseteq \omega$, then $\hat{P}^{B^\star}$ is a Kurtz null test uniformly relative to A.*

Proof. The first equivalence clearly holds. Assume that there is a truth-table functional Ψ such that $\Psi^A = B$. Inductively define a truth-table functional $\Phi^Z(n)$ by $\Phi^Z(0) = \Psi^Z(0)$, and $\Phi^Z(n+1) = \max\{\Psi^Z(n+1), \Phi^Z(n)+1\}$ for each $n \in \omega$. Then, Φ^Z defines an infinite set $B(Z)$, for every $Z \in 2^\omega$. Moreover, if $B(Z)$ is infinite, then $\hat{P}^{B(Z)^\star}$ is null. Therefore, $Z \mapsto \hat{P}^{B(Z)^\star}$ is a uniform Kurtz null test. Hence, $\hat{P}^{B^\star} = \hat{P}^{B(A)^\star}$ is a Kurtz null test uniformly relative to A. $\square$

Now, assume that A is low for uniform Kurtz randomness. By Lemma 6.2, the class $\hat{P}^{B^\star}$ is a Kurtz null test uniformly relative to A for every $B \leq_{tt} A$. By lowness, $\hat{P}^{B^\star}$ contains no Kurtz random set. Again by Lemma 6.2, A *tt*-computes no infinite subset of a Kurtz random set.

(iv)⇒(v): We again use the notation $\hat{P}^f$ for $f \in I$. Moreover, given $f \in I$, define $\hat{f} \leq_{tt} f$ as $\hat{f}(n) = f \upharpoonright n$ for each $n \in \omega$. Then we define $P^f \subseteq 2^\omega$ as follows.

$$P^f = \hat{P}^{\hat{f}} = \{X \in 2^\omega : X(f \upharpoonright n) = 1 \text{ for all } n \in \omega\},$$

where each $f \upharpoonright n$ is identified with a natural number via a fixed bijection between $\omega^{<\omega}$ and ω. As in the proof of Lemma 6.2, for each increasing total function $f \in I \cap \omega^\omega$ we can see that $f \leq_{tt} A$ if and only if $\mathrm{rng}(f) \leq_{tt} A$. We first recall the following property of P^f.

Lemma 6.3 (See Greenberg-Miller [16, Theorem 5.2])**.** *Let $f : \omega \to \omega$ be a strictly increasing function. Then, no Kurtz null test includes P^f if and only if P^f contains a Kurtz random set.*

Proof. Obviously, if P^f contains a Kurtz random set, then there is no Kurtz test including P^f. Conversely, as mentioned in Greenberg-Miller [16, Theorem 5.2], if some nonempty clopen subclass $P^f \cap [\rho]$ is covered by a Kurtz test, then so is all of P^f. Assume that P^f is covered by no Kurtz test. Let $\{Q_n\}_{n\in\omega}$ be a (non-effective) list of all Kurtz tests. We construct an element $X = \lim_n \xi_n \in P^f$ which is contained in no Kurtz test. Let ξ_0 be the empty string. Assume that ξ_n has been already defined, and it is extendible in P^f. Then, $P^f \cap [\xi_n]$ is not covered by a Kurtz test, as mentioned before. Hence, we can find some ξ_{n+1} extending ξ_n in the class $(P^f \cap [\xi_n]) \setminus Q_n$. Then, $X = \lim_n \xi_n$ is Kurtz random, which is contained in P^f. □

The key notion we will use is that of the *svelte tree* introduced by Greenberg-Miller [16]. A finite antichain $A \subseteq \omega^{<\omega}$ is *k-svelte* via a sequence $\{S_n\}_{n\in\omega}$ of finite sets if

$$S_m \subseteq \omega^{k+m},\ \#S_m \leq 2^m, \text{ and } [\![A]\!] \subseteq \bigcup_{m\in\omega} [\![S_m]\!].$$

Here, compared with the original definition in [16], we should mention that some conditions are removed, the antichain A is supposed to be the set of leaves of a tree, the indexing of $\{S_n\}_{n\in\omega}$ is shifted by k, and the value n_m is set to be m for each $m \in \omega$. Indeed, however, the above special case suffices to show our theorem.

Lemma 6.4. *For a finite antichain $A \subseteq \omega^{<\omega}$ and a natural number $k \in \omega$, if $\mu(\bigcup_{f\in[\![A]\!]} P^f) \leq 2^{-(k+1)}$ holds, then one can find a sequence confirming that A is k-svelte, effectively in A and k.*

Proof. Such a sequence exists by Greenberg-Miller [16, Theorem 3.3]. If it exists, by brute-force, we can effectively find such a sequence. □

Given a closed set $Q \subseteq 2^\omega$, let $N_Q \subseteq \omega^\omega$ be the set $\{f \in \omega^\omega \cap I : P^f \subseteq Q\}$.

Lemma 6.5. *If $Q \subseteq 2^\omega$ is clopen, then we can effectively find a finite antichain $A_Q \subseteq \omega^{<\omega}$ such that $N_Q = [\![A_Q]\!]$.*

Proof. See Greenberg-Miller [16, Lemma 4.3 and Remark 4.4]. $\square$

We restrict our attention to a bounded subset of N_Q for a given closed set Q. For each order u, we denote by N_Q^u the set of all $f \in N_Q$ such that $|f(n)| = u(n)$ for each $n \in \omega$, where we think of each $f \in N_Q$ as a function from ω into $2^{<\omega}$.

Lemma 6.6. *Assume that Q is a Kurtz null test. Then, for each order u, there are a computable trace $\{D_n\}_{n\in\omega}$ with bound $n \mapsto 2^n$ and $D_n \subseteq \omega^n$ for each $n \in \omega$ and a computable sequence $\{l_k\}_{k\in\omega}$ of natural numbers such that*

$$N_Q^u \subseteq \bigcup_{n=l_k}^{l_{k+1}-1} [\![D_n]\!], \textit{ for every } k \in \omega.$$

Proof. Assume that a Kurtz null test $\{C_n\}_{n\in\omega}$ with $\mu(C_n) \leq 2^{-n}$ and $Q = \bigcap_n C_n$ is given. By Lemma 6.5, we can effectively find a sequence $\{A_n\}$ of finite antichains generating $\{N_{C_n}\}$. By the definition of N_{C_n}, we have $\bigcup_{g\in[\![A_n]\!]} P^g \subseteq C_n$. Hence, $\mu(\bigcup_{g\in[\![A_n]\!]} P^g) \leq 2^{-n}$. Therefore, by Lemma 6.4, we can effectively find a sequence $\{S_m^n\}_{m\in\omega}$ confirming that A_{n+1} is n-svelte, uniformly in n. In other words,

$$S_m^n \subseteq \omega^{n+m},\ \#S_m^n \leq 2^m, \text{ and } N_{C_{n+1}} = [\![A_{n+1}]\!] \subseteq \bigcup_{m\in\omega} [\![S_m^n]\!].$$

For each computable order u, because $N_{C_{n+1}}^u$ is compact, it is covered by $\bigcup_{m<c(n)} S_m^n$ for some $c(n) \in \omega$. Note that we can effectively find such a $c(n)$, since $N_{C_{n+1}}^u$ and $\bigcup_m S_m^n$ are computable. Inductively define $l_0 = 0$, and $l_{n+1} = l_n + c(l_n)$ for each $n \in \omega$. For each $k \in \omega$ and each $n \in [l_k, l_{k+1})$, we define $D_n = S_{n-l_k}^{l_k} \subseteq \omega^n$, where $\#D_n \leq 2^{n-l_k} \leq 2^n$. We now have

$$N_Q^u \subseteq N_{C_{l_k+1}}^u \subseteq \bigcup_{m<c(l_k)} [\![S_m^{l_k}]\!] = \bigcup_{n=l_k}^{l_{k+1}-1} [\![D_n]\!]$$

for every $k \in \omega$, as desired. $\square$

Now, we assume that A tt-computes no infinite subset of a Kurtz random set. For each $g \leq_{tt} A$, we claim the existence of a computable trace with bound $n \mapsto 2^{n+1}$ that Kurtz-traces g. Let Ψ be a truth-table functional such that $\Psi(A) = g$, and find a computable order u such that $\Psi(Z \upharpoonright u(n), n)$ is defined for all $n \in \omega$. Then, in particular, $\Psi(A \upharpoonright u(n), n) = g(n)$. Define $f(n) = A \upharpoonright u(n)$ for each $n \in \omega$. By Lemmas 6.2 and 6.3, for every order u and every strictly increasing function $f \leq_{tt} A$ with $|f(n)| = u(n)$ for each $n \in \omega$, there is a Kurtz null test $Q \subseteq 2^\omega$ such that $P^f = \hat{P}^{\hat{f}} \subseteq Q$ holds. Note that $P^f \subseteq Q$ if and only if $f \in N_Q^u$. Since Q is a Kurtz null test, we have two sequences $\{D_n\}_{n\in\omega}$ and $\{l_k\}_{k\in\omega}$ in Lemma 6.6. Thus, every $h \in N_Q^u$ is Kurtz traced by $\{D_n\}_{n\in\omega}$ and $\{l_k\}_{k\in\omega}$. For each string $\sigma \in (2^{<\omega})^{<\omega}$, let σ^* denote the last value of σ, that is, $\sigma^* = \sigma(|\sigma| - 1)$. Note that $\sigma \in D_{n+1} \subseteq (2^{<\omega})^{n+1}$ implies that $\Psi(\sigma^*, n)$ is defined, since σ^* is of length $u(n)$. For $E_n = \{\Psi(\sigma, n) : \sigma \in D_{n+1}\}$, the trace $\{E_n\}_{n\in\omega}$ Kurtz-traces $n \mapsto \Psi(h(n), n)$ for all $h \in N_Q^u$. In particular, $g : n \mapsto \Psi(f(n), n)$ is Kurtz-traced.

(v)⇒(i): Assume that A is Kurtz tt-traceable via a computable order $n \mapsto 2^{p(n)}$. Given a computable order h, we can find a computable order $u : \omega \to \omega$ such that $h(u(n)) \geq p(n) + n + 1$ for each $n \in \omega$. By our assumption, we have a computable trace $\{D_n\}_{n\in\omega}$ with $\#D_n \leq 2^{p(n)}$ and a strictly increasing computable sequence $\{l_k\}_{k\in\omega}$ of natural numbers, where, for every $k \in \omega$, there is $n \in [l_k, l_{k+1})$ such that $A \upharpoonright u(n) \in D_n$. Without loss of generality, we may assume that $D_n \subseteq 2^{u(n)}$. Then, define $C_k = \bigcup_{n\in[l_k,l_{k+1})} D_n$, for each $k \in \omega$. Note that $A \in [\![C_k]\!]$ for all $k \in \omega$. To estimate the weight of C_k, we note the following inequality:

$$\begin{aligned}\sum_{\sigma\in C_k} 2^{-h(|\sigma|)} &= \sum_{n=l_k}^{l_{k+1}-1} \#D_n \cdot 2^{-h(u(n))} \\ &\leq \sum_{n=l_k}^{l_{k+1}-1} 2^{p(n)-h(u(n))} \leq \sum_{n=l_k}^{l_{k+1}-1} 2^{-n-1} \leq 2^{-l_k} \leq 2^{-k}.\end{aligned}$$

Hence, A is Kurtz h-dimensional measure zero. □

Corollary 6.7. *If a set is low for uniform Schnorr randomness, then it is low for uniform Kurtz randomness*

Proof. By Franklin-Stephan [15], a set A is low for uniform Schnorr randomness if and only if it is computably tt-traceable, that is, there is a computable order p such that for every $f \leq_{tt} A$, there is a computable trace $\{D_n\}_{n\in\omega}$ with bound p such that $f(n) \in D_n$ for every $n \in \omega$. In particular, $\{D_n\}_{n\in\omega}$ Kurtz-traces f. Hence, by Theorem 6.1, A is turned out to be low for uniform Kurtz randomness. □

Corollary 6.8. *There is a set which is low for uniform Kurtz randomness, but is not low for Kurtz randomness.*

Proof. Franklin [14] constructed a 1-generic set G which is Turing equivalent to a Schnorr trivial set A. Here, recall that a set $G \in 2^\omega$ is 1-generic if it is contained in every c.e. open set dense along it. Moreover, Franklin-Stephan [15] showed that a set A is Schnorr trivial if and only if it is low for uniform Schnorr randomness. By Corollary 6.7, A is low for uniform Kurtz randomness. Suppose for the sake of contradiction that A is low for Kurtz randomness. Then G is also low for Kurtz randomness, since G is Turing equivalent to A. However, every 1-generic set G is obviously Kurtz random, which contradicts that G is low for Kurtz randomness. □

Franklin [13] showed that a set is low for Schnorr randomness if and only if it is low for uniform Schnorr randomness and hyperimmune-free, where recall that A is *hyperimmune-free* if every total A-computable function is dominated by a total computable function. An analogous result also holds for lowness for uniform Kurtz randomness.

Corollary 6.9. *A set is low for Kurtz randomness if and only if it is low for uniform Kurtz randomness and hyperimmune-free.*

Proof. Greenberg-Miller [16] characterized that A is low for Kurtz randomness if and only if it computes no diagonally noncomputable function and is hyperimmune-free. Moreover, if A is low for Kurtz randomness, then it is clearly low for uniform Kurtz randomness. Therefore, one direction is clear. Conversely, assume that A is low for uniform Kurtz randomness and hyperimmune-free. It suffices to show that A computes no diagonally noncomputable function. Suppose for the sake of contradiction that A computes a diagonally noncomputable function. Then A *tt*-computes it since A is hyperimmune-free. Then, A is complex (see [10, Theorem 8.16.5]), that is, there is a computable order g such that $K(A \upharpoonright n) \geq g(n)$ for all $n \in \omega$. Now we note that A is Kurtz h-dimensional dimensional measure zero for every computable order h, by Theorem 6.1. In particular, by Theorem 5.2, there is a computable measure machine M such that $K_M(A \upharpoonright n) < g(n)$ for infinitely many $n \in \omega$. This implies a contradiction. $\square$

7 Kurtz reducibility

By Theorem 6.1, we can give a triviality-type characterization of lowness for uniform Kurtz randomness via the following reducibility, although we hesitate to call this Kurtz reducibility, as it has a rather different form from Schnorr reducibility and K-reducibility. Let $A \leq_{\mathrm{Kur}} B$ denote that, for each computable order h, the fact that B is Kurtz h-dimensional measure zero implies that A is Kurtz h-dimensional measure zero. If $A \leq_{\mathrm{Kur}} B$ and A is Kurtz random, then B is Kurtz random. A set $A \in 2^\omega$ is low for uniform Kurtz randomness if and only if $A \leq_{\mathrm{Kur}} \emptyset$.

We restate this formally.

Definition 7.1. *Let $A, B \in 2^\omega$. We say that A is* Kurtz reducible *to B (denoted by $A \leq_{\mathrm{Kur}} B$) if, for each computable order h, the fact that B is Kurtz h-dimensional measure zero implies that A is Kurtz h-dimensional measure zero.*

Proposition 7.2. *If $A \leq_{\mathrm{Kur}} B$ and A is Kurtz random, then B is Kurtz random.*

Proof. Suppose B is not Kurtz random. Then B is Kurtz id-dimensional measure zero where id is the identity function. By the assumption $A \leq_{\mathrm{Kur}} B$, A is Kurtz id-dimensional measure zero. Thus, A is not Kurtz random. $\square$

Proposition 7.3. *A set $A \in 2^\omega$ is low for uniform Kurtz randomness if and only if $A \leq_{\mathrm{Kur}} \emptyset$.*

Proof. Suppose that A is low for uniform Kurtz randomness. Then A is Kurtz h-dimensional measure zero for all computable orders h. Thus, $A \leq_{\mathrm{Kur}} \emptyset$.

Suppose that $A \leq_{\mathrm{Kur}} \emptyset$. Let h be a computable order. Since $\emptyset$ is Kurtz h-dimensional measure zero, A is Kurtz h-dimensional measure zero. Since h is arbitrary, A is low for uniform Kurtz randomness. $\square$

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