TOPOLOGICAL REDUCIBILITIES FOR DISCONTINUOUS FUNCTIONS AND THEIR STRUCTURES

TAKAYUKI KIHARA

ABSTRACT. In this article, we give a full description of the topological many-one degree structure of real-valued functions, recently introduced by Day-Downey-Westrick. We also clarify the relationship between the Martin conjecture and Day-Downey-Westrick's topological Turing-like reducibility, also known as parallelized continuous strong Weihrauch reducibility, for single-valued functions: Under the axiom of determinacy, we show that the continuous Weihrauch degrees of parallelizable single-valued functions are well-ordered; and moreover, if f has continuous Weihrauch rank α, then f' has continuous Weihrauch rank $\alpha+1$, where $f'(x)$ is defined as the Turing jump of $f(x)$.

1. INTRODUCTION

1.1. **Summary.** The notion of Wadge degrees provides us an ultimate measure to analyze the topological complexity of subsets of a zero-dimensional Polish space (see [1, 2]). Under the axiom of determinacy, the induced structure forms a semi-well-order of the height Θ, and thus it enables us to assign an ordinal rank to each subset of such a space. Our main question is whether one can introduce a similar ultimate measure which induces a semi-well-ordering of *real-valued functions* on a Polish space. A somewhat related question is also proposed by Carroy [7]. In this article, we give some sort of solution to this kind of problem.

(1) Topological many-one reducibility. Recently, Day-Downey-Westrick [8] introduced a "many-one"-like ordering $\leq_{\mathbf{m}}$ on real-valued functions on Cantor space. Their ordering $\leq_{\mathbf{m}}$ measures the topological complexity of sets separating the lower level sets from the upper level sets of a function. One of our main results in this article is to show that their notion $\leq_{\mathbf{m}}$ behaves like a Wadge ordering, and in particular, it semi-well-orders real-valued functions.

Definition 1.1 (Day-Downey-Westrick [8]). For $f, g\colon 2^\omega \to \mathbb{R}$, we say that f is $\mathbf{m}$-reducible to g (written $f \leq_{\mathbf{m}} g$) if for any rationals p and $\varepsilon > 0$, there are rationals r and $\delta > 0$ and a continuous function $\theta\colon 2^\omega \to 2^\omega$ such that, for any $x \in 2^\omega$,

$$\begin{cases} g(\theta(x)) < r + \delta \implies f(x) < p + \varepsilon, \\ g(\theta(x)) > r - \delta \implies f(x) > p - \varepsilon. \end{cases}$$

One of Day-Downey-Westrick's main discoveries is the connection between their notion of the $\mathbf{m}$-degree and the Bourgain rank (also known as the separation rank [13]) of a Baire-one function. The latter notion is introduced by Bourgain [5] to prove a refinement of the Odell-Rosenthal theorem in Banach space theory: The ℓ^1-index of a separable Banach space is related to the degrees of discontinuity (the Bourgain rank) of double-dual elements as Baire-one functions. Day-Downey-Westrick [8] showed the following:

- The Bourgain rank 1 consists of exactly two **m**-degrees, those of constant functions and continuous functions.
- Every successor Bourgain rank ≥ 2 consists of exactly four **m**-degrees, where the first two **m**-degrees are incomparable, and the others are comparable. For instance, the first two **m**-degrees of Bourgain rank 2 are those of lower semicontinuous and upper semicontinuous functions.
- For any infinite limit ordinal λ, the rank λ consists of exactly one **m**-degree.

Their result completely characterizes the structure of the **m**-degrees of the Baire-one functions, that is, the **m**-degrees of rank below ω_1. In this article, we will give a full description of the structure of the **m**-degrees of all real-valued functions under the axiom of determinacy AD (or all Baire-class functions under ZFC).

Theorem 1.2 (AD)**.** *The* **m***-degrees of real-valued functions on* 2^ω *form a semi-well-order of length* Θ*, where* Θ *is the least nonzero ordinal* α *such that there is no surjection from the reals onto* α*.*

For a limit ordinal $\alpha < \Theta$ *and finite* $n < \omega$*, the* **m***-rank* $\alpha + 3n + c_\alpha$ *consists of two incomparable degrees, and each of the other ranks consists of a single degree, where* $c_\alpha = 2$ *if* $\alpha = 0$*;* $c_\alpha = 1$ *if the cofinality of* α *is* ω*; and* $c_\alpha = 0$ *if the cofinality of* α *is uncountable.*

Here, recall that the axiom of determinacy, AD, is a *locally true* principle under the standard axiomatic system which is adopted by most mathematicians. More precisely, AD always holds in the inner model $L(\mathbb{R})$ under ZFC plus a large cardinal assumption [31]. Thus, even if we work in ZFC (plus extra axioms), one can understand any result under AD as *a statement that is true if we restrict our attention to definable ones* in a certain sense. For instance, Theorem 1.2 restricted to Baire class functions is provable within ZFC.

(2) Topological reducibility for other spaces. So far, we have only mentioned functions of Cantor domain. Now we would like to extend our results to more general domains. The difficulty arises here by the fact that the structure of Wadge degrees of subsets of a nonzero-dimensional Polish space is ill-behaved (cf. Ikegami et al. [12] and Schlicht [23]).

Fortunately, Pequignot [22] has overcome this difficulty by modifying the definition of Wadge reducibility using the theory of an admissible representation, and then, showed that the modified Wadge degree structure of subsets of a second-countable space is semi-well-ordered. Day-Downey-Westrick [8] adopted a similar idea to consider the notion of **m**-reducibility for functions of compact metrizable domain.

By integrating their ideas, we introduce the notion of **m**-reducibility for real-valued functions of (quasi-)Polish domain as follows. Let δ be a total open admissible representation of a Polish space $\mathcal{X}$ (see Lemma 4.1). Then, we introduce the **m**-degree of a function $f\colon \mathcal{X} \to \mathbb{R}$ as that of $f \circ \delta\colon \omega^\omega \to \mathcal{X}$. As in Pequignot [22], this notion is easily seen to be well-defined (see Section 4). Then we will conclude that the **m**-degrees of real-valued functions on Polish spaces form a semi-well-order of length Θ (Observation 4.3).

(3) Bourgain rank. Pequignot's insightful idea also turns out to be very useful for the Wadge-like analysis of the Bourgain rank. We discuss the Bourgain rank in non-compact spaces (which is also considered by Elekes-Kiss-Vidnyánszky [11] via the change of topology, in order to generalize the notion of ranks to Baire class ξ functions, and then to study a cardinal invariant associated with systems of difference equations). Then, based on the notion of sidedness conditions introduced by Day-Downey-Westrick [8], we can classify real-valued functions on a (possibly non-compact) Polish space into (ordered) 5 types (see Definition 4.4). We then generalize the main result in [8] to arbitrary Polish domains as follows: Let $\mathcal{X}$ and $\mathcal{Y}$ be Polish spaces and let $f\colon \mathcal{X} \to \mathbb{R}$ and $g\colon \mathcal{Y} \to \mathbb{R}$ be Baire-one functions. Then, $f \leq_{\mathbf{m}} g$ if and only if either $\alpha(f) < \alpha(g)$ holds or both $\alpha(f) = \alpha(g)$ and $\mathrm{type}(f) \leq \mathrm{type}(g)$ hold, where $\alpha(h)$ is the Bourgain rank of h. We also give the precise connection between the Bourgain rank and the Wadge rank.

(4) Topological Turing reducibility. Finally, we will clarify the relationship between the uniform Martin conjecture and Day-Downey-Westrick's $\mathbf{T}$-degrees of real-valued functions. They defined $\mathbf{T}$-reducibility for real-valued functions as parallelized continuous strong (p.c.s.) Weihrauch reducibility [8], that is, *f is $\mathbf{T}$-reducible to g* if there are continuous functions H, K such that $f = K \circ \widehat{g} \circ H$, where $\widehat{g}$ is the parallelization of g (see Section 5.1).

On the one hand, the notion of Weihrauch reducibility has been one of the most important concepts in modern computable analysis, cf. [6]. On the other hand, the Martin conjecture is one of the most prominent open problems in computability theory (see [18, 19]), which generalizes Sacks' question on a natural solution to Post's problem. To be precise, what is more relevant in this article is the uniform version of this conjecture, which has been solved by [27, 25]. The notion of $\mathbf{T}$-degree (p.c.s. Weihrauch degree) is seemingly unrelated to this conjecture; nevertheless the solution to the uniform Martin conjecture plays a key role in the proofs of several results on $\mathbf{T}$-degrees in this article.

We show that the p.c.s. Weihrauch degrees are exactly the *natural* Turing degrees in the context of the uniform Martin conjecture. More precisely, we will see that the p.c.s. Weihrauch degrees of real-valued functions is isomorphic to the Turing-degrees-on-a-cone of the uniformly Turing degree invariant operators. Indeed, the identity map induces an isomorphism between the Turing-ordering-on-a-cone of the uniformly $\leq_T$-preserving operators and the p.c.s. Weihrauch degrees of real-valued functions. Therefore, by Steel's theorem [27], we finally conclude the following.

Theorem 1.3 (AD). *The p.c.s. Weihrauch degrees of single-valued functions on 2^ω are well-ordered, whose order type is Θ. If $f\colon 2^\omega \to 2^\omega$ is parallelizable (see Section 5), and has p.c.s. Weihrauch rank $\alpha > 0$, then f' is also parallelizable, and has p.c.s. Weihrauch rank $\alpha+1$, where $f'(x)$ is defined as the Turing jump of $f(x)$.*

To our surprise, we can conclude that the p.c.s. Weihrauch degrees are a universal measure that allows us to assign a rank to "all functions".

1.2. **Conventions and notations.** In Sections 1.3, 1.4, 2, 3 and 5, we assume $\mathsf{ZF}+\mathsf{DC}+\mathsf{AD}$ (where DC stands for the axiom of dependent choice). As mentioned above, the axiom of determinacy, AD, is a locally true principle under the standard axiomatic system which is adopted by most mathematicians. As it is locally true, if we restrict our attention to Borel sets and Baire class functions, every result presented in this article is provable within ZFC. If we restrict our attention to projective sets and functions, every result presented in this article is provable within $\mathsf{ZF}+\mathsf{DC}+\mathsf{PD}$ (where PD stands for the axiom of projective determinacy).

We assume that 2^ω is always embedded into $\mathbb{R}$ as a Cantor set. For finite strings $\sigma,\tau\in\omega^{<\omega}$, we write $\sigma\prec\tau$ if τ extends σ. Similarly, for $X\in\omega^\omega$ we write $\sigma\prec X$ if X extends σ. For a string σ, $[\sigma]$ denotes the set of all $X\in\omega^\omega$ extending σ, i.e., $\sigma\prec X$. Let $X\upharpoonright n$ be the initial segment of length n. Let $\sigma^\frown\tau$ denote the concatenation of σ and τ.

1.3. **The structure of Wadge degrees.** We here review classical results in the Wadge degree theory [30, 2]. For sets $A,B\subseteq\omega^\omega$, we say that A is Wadge reducible to B (written $A\leq_w B$) if there exists a continuous function $\theta\colon\omega^\omega\to\omega^\omega$ such that $A=B\circ\theta$, where we often identify a set with its characteristic function.

Given a pointclass Γ (of subsets of ω^ω), let $\check{\Gamma}$ denote its dual, that is, $\check{\Gamma}=\{\omega^\omega\setminus A : A\in\Gamma\}$, and define $\Delta=\Gamma\cap\check{\Gamma}$. A pointclass Γ has the *separation property* if

$$(\forall A,B\in\Gamma)\;[A\cap B=\emptyset\implies(\exists C\in\Delta)\;A\subseteq C\text{ and }B\cap C=\emptyset].$$

The separation property will play a key role in the proof of our main theorem.

A pointclass Γ is *self-dual* if $\Gamma=\check{\Gamma}$. We say that $A\subseteq\omega^\omega$ is *self-dual* if there is a continuous function $\theta\colon\omega^\omega\to\omega^\omega$ such that $A(X)\neq A\circ\theta(X)$ for any $X\in\omega^\omega$. It is equivalent to saying that $A\leq_w\neg A$. Note that A is self-dual if and only if the pointclass $\Gamma_A=\{B : B\leq_w A\}$ is self-dual.

By Wadge [30] and Martin-Monk, non-self-dual pairs are well-ordered, say $(\Gamma_\alpha,\check{\Gamma}_\alpha)_{\alpha<\Theta}$, where Θ is the height of the Wadge degrees. We will use the following beautiful fact to show our Main Theorem 1.2.

Fact 1 (Van Wesep [29] and Steel [26])**.** *Exactly one of Γ_α or $\check{\Gamma}_\alpha$ has the separation property.*

By Π_α, we denote the one which has the separation property, and by Σ_α, we denote the other one (which has the weak reduction property). Then define $\Delta_\alpha=\Sigma_\alpha\cap\Pi_\alpha$.

A set $A\subseteq\omega^\omega$ is Γ-*complete* if $A\in\Gamma$ and $B\leq_w A$ for any $B\in\Gamma$. By definition, a Σ_α-complete set and a Π_α-complete set exist for all $\alpha<\Theta$.

Fact 2 (see [28, 2])**.** *A Δ_α-complete set exists if and only if the cofinality of α is countable.*

We denote the Borel hierarchy by $(\mathbf{\Sigma}^0_\alpha,\mathbf{\Pi}^0_\alpha,\mathbf{\Delta}^0_\alpha)_{\alpha<\omega_1}$. More precisely, a set is in $\mathbf{\Sigma}^0_1$ if it is open, and a set is in $\mathbf{\Sigma}^0_\alpha$ if it is a countable union of sets in $\bigcup_{\beta<\alpha}\mathbf{\Pi}^0_\beta$, where $\mathbf{\Pi}^0_\alpha$ is the dual of $\mathbf{\Sigma}^0_\alpha$. Then, $\mathbf{\Delta}^0_\alpha=\mathbf{\Sigma}^0_\alpha\cap\mathbf{\Pi}^0_\alpha$.

Example 1.4 (Wadge [30, Sections V.E and V.F])**.** The Wadge ranks of sets of finite Borel ranks are calculated as follows.

- Δ_1 = clopen sets ($=\mathbf{\Delta}^0_1$), Σ_1 = open sets ($=\mathbf{\Sigma}^0_1$), and Π_1 = closed sets ($=\mathbf{\Pi}^0_1$).
- For $\alpha<\omega_1$, Δ_α, Σ_α, and Π_α correspond to the α-th level of the Hausdorff difference hierarchy.
- $\Sigma_{\omega_1}=F_\sigma$ ($=\mathbf{\Sigma}^0_2$), and $\Pi_{\omega_1}=G_\delta$ ($=\mathbf{\Pi}^0_2$).
- For $\alpha<\omega_1$, $\Delta_{\omega_1^\alpha}$, $\Sigma_{\omega_1^\alpha}$, and $\Pi_{\omega_1^\alpha}$ correspond to the α-th level of the difference hierarchy over F_σ.
- $\Sigma_{\omega_1^{\omega_1}}=G_{\delta\sigma}$ ($=\mathbf{\Sigma}^0_3$), and $\Pi_{\omega_1^{\omega_1}}=F_{\sigma\delta}$ ($=\mathbf{\Pi}^0_3$).

- Generally, $\Sigma_{\omega_1\uparrow\uparrow n} = \mathbf{\Sigma}^0_n$, where $\omega_1 \uparrow\uparrow n$ is the n-th level of the superexponential hierarchy of base ω_1.

Note that the Wadge rank of $\mathbf{\Sigma}^0_\omega$-complete set is not the first fixed point, but the ω_1-th fixed point, of the exponential tower of base ω_1. In general, Wadge [30, Sections V.E and V.F] has also determined the Wadge ranks of sets of infinite Borel ranks, which are described by using the Veblen hierarchy of base ω_1.

Let $\mathcal{X}$ and $\mathcal{Y}$ be topological spaces. For $A \subseteq \mathcal{X}$ and $B \subseteq \mathcal{Y}$, we write $A/\mathcal{X} \leq_w B/\mathcal{Y}$ if there is a continuous function $\theta\colon \mathcal{X} \to \mathcal{Y}$ such that $X \in A$ if and only if $\theta(X) \in B$.

Lemma 1.5. *For any non-self-dual $A \subseteq \omega^\omega$, there is $B \subseteq 2^\omega$ such that $A/\omega^\omega \equiv_w B/2^\omega$.*

Proof. If A is non-self-dual, then $\neg A \not\leq_w A$, and therefore, Player I wins in the Wadge game $G_w(\neg A, A)$. Put $\hat{\omega} = \omega \cup \{\mathsf{pass}\}$. A winning strategy for Player I gives a continuous function $\theta\colon \hat{\omega}^\omega \to \omega^\omega$ such that for any $X \in \hat{\omega}^\omega$, $\neg A(\theta(X)) \neq A(X^{\mathsf{p}})$, that is, $A(\theta(X)) = A(X^{\mathsf{p}})$. Here, X^{p} is the result of removing all occurrences of passes from X. Define $\eta\colon 2^\omega \to \hat{\omega}^\omega$ by

$$\eta(0^{n_0}10^{n_1}1\dots) = \mathsf{pass}^{n_0}n_0\mathsf{pass}^{n_1}n_1\dots$$

Then, define $B(X) = A(\theta\circ\eta(X))$. We claim that $A/\omega^\omega \equiv_w B/2^\omega$. Clearly $\theta\circ\eta$ witnesses that $B \leq_w A$. To see $A \leq_w B$, let $X \in \omega^\omega$ be a given sequence. Then, for $\tau(X) = 0^{X(0)}10^{X(1)}1\dots$, we have $(\eta\circ\tau(X))^{\mathsf{p}} = X$. Thus,

$$B(\tau(X)) = A(\theta\circ\eta\circ\tau(X)) = A((\eta\circ\tau(X))^{\mathsf{p}}) = A(X),$$

where the second equality follows from our choice of θ. This concludes that $A \leq_w B$. □

1.4. **$\mathcal{Q}$-Wadge degrees.** A set $A \subseteq \omega^\omega$ can be identified with its characteristic function $\chi_A\colon \omega^\omega \to 2$. Thus, the Wadge degrees of subsets of ω^ω can be viewed as the degrees of 2-valued functions on ω^ω. The Wadge degrees have been extended in various directions. For instance, there are various works on the Wadge degrees of partial 2-valued functions on ω^ω (Wadge [30]), ordinal-valued functions on ω^ω (Steel, cf. Duparc [10]), and k-valued functions on ω^ω (Hertling, cf. Selivanov [24]). We can encapsulate all those extensions within the following framework (see also Kihara-Montalbán [15, 16]):

Definition 1.6. Let $(\mathcal{Q};\leq_\mathcal{Q})$ be a quasi-ordered set. For $\mathcal{Q}$-valued functions $\mathcal{A},\mathcal{B}\colon \omega^\omega \to \mathcal{Q}$, we say that *$\mathcal{A}$ is $\mathcal{Q}$-Wadge reducible to $\mathcal{B}$* (written $\mathcal{A} \leq_w \mathcal{B}$) if there is a continuous function $\theta\colon \omega^\omega \to \omega^\omega$ such that

$$(\forall X \in \omega^\omega)\ \mathcal{A}(X) \leq_\mathcal{Q} \mathcal{B}(\theta(X)).$$

As a special case, one can study Wadge's notion of degrees of inseparability of pairs, which will turn out to be a key tool for analyzing the **m**-degrees. In his PhD thesis [30, Section I.E], Wadge introduced the notion of reducibility for pairs of subsets of ω^ω. For $A, B, C, D \subseteq \omega^\omega$, we say that (A, B) *is Wadge reducible to* (C, D) if there exists a continuous function $\theta\colon \omega^\omega \to \omega^\omega$ such that for any $x \in \omega^\omega$,

$$(x \in A \implies \theta(x) \in C) \text{ and } (x \in B \implies \theta(x) \in D).$$

Roughly speaking, this reducibility estimates how inseparable a given pair is. Note that Wadge reducibility for pairs is equivalent to Wadge reducibility for

$\{\top, 0, 1, \bot\}$-valued functions by identifying a pair (A, B) with a function $f_{A,B}$ defined by

$$f_{A,B}(x) = \begin{cases} \top & \text{if } x \in A \cap B, \\ 0 & \text{if } x \in A \setminus B, \\ 1 & \text{if } x \in B \setminus A, \\ \bot & \text{if } x \notin A \cup B, \end{cases}$$

where $\bot < 0, 1 < \top$, and 0 and 1 are incomparable. It is easy to see that the Wadge degrees of $\{\top, 0, 1, \bot\}$-valued functions consist exactly of the Wadge degrees of $\{0, 1, \bot\}$-valued functions plus a greatest degree, where the greatest degree consists of functions containing $\top$ in their ranges.

Hereafter we use the symbols $\mathbf{2}$ and $\mathbf{2}_\bot$ to denote $\{0, 1\}$ and $\{0, 1, \bot\}$, respectively, where $\mathbf{2}$ is considered as a discrete ordered set, and $\mathbf{2}_\bot$ is ordered by $\bot < 0, 1$ as mentioned above (i.e., the flat domain of boolean values).

Wadge determined the structure of the first few Wadge degrees of inseparability of pairs (equivalently those of $\{0, 1, \bot\}$-valued functions). For $\mathcal{A}\colon \omega^\omega \to \mathbf{2}_\bot$, we define $\neg\mathcal{A}\colon \omega^\omega \to \mathbf{2}_\bot$ by $\mathcal{A}(X) = 1 - \mathcal{A}(X)$ if $\mathcal{A}(X) \in \{0, 1\}$; otherwise $\mathcal{A}(X) = \bot$. If $\mathcal{A}$ is $\mathbf{2}$-valued, then $\neg\mathcal{A}$ is obviously the complement of $\mathcal{A}$. Under the axiom of determinacy, Wadge showed that the semilinear ordering principle holds for $\mathbf{2}_\bot$-valued functions.

Fact 3 (Wadge [30, Theorem II.E2])**.** *For any $\mathcal{A}, \mathcal{B}\colon \omega^\omega \to \mathbf{2}_\bot$, either $\mathcal{B} \leq_w \mathcal{A}$ or $\neg\mathcal{A} \leq_w \mathcal{B}$ holds.*

For a function $\mathcal{A}\colon \omega^\omega \to \mathcal{Q}$ and a finite string $\sigma \in \omega^{<\omega}$, by $\mathcal{A} \upharpoonright [\sigma]$ we denote the restriction of $\mathcal{A}$ up to $[\sigma]$, that is, $(\mathcal{A} \upharpoonright [\sigma])(X) = \mathcal{A}(\sigma^\frown X)$. If σ is a string of length 1, $\sigma = \langle n\rangle$ say, then we also write $\mathcal{A} \upharpoonright n$ to denote $\mathcal{A} \upharpoonright [\langle n\rangle]$.

Definition 1.7. We say that a $\mathcal{Q}$-Wadge degree $\mathbf{a}$ is *σ-join-reducible* if $\mathbf{a}$ is the least upper bound of a countable collection $(\mathbf{b}_i)_{i\in\omega}$ of $\mathcal{Q}$-Wadge degrees such that $\mathbf{b}_i <_w \mathbf{a}$. Otherwise, we say that $\mathbf{a}$ is *σ-join-irreducible.*

Given a function $\mathcal{A}$ we use the following notation:

$$\mathcal{F}(\mathcal{A}) = \{X : (\forall n)\ \mathcal{A} \upharpoonright [X \upharpoonright n] \equiv_w \mathcal{A}\}.$$

The following fact gives a better way to characterize σ-join-reducibility, which is a straightforward consequence of the well-foundedness of the Wadge degrees (cf. [2, 28]).

Fact 4. *A set $\mathcal{A} \subseteq \omega^\omega$ is σ-join-irreducible if and only if $\mathcal{F}(\mathcal{A})$ is nonempty.*

A set $\mathcal{A} \subseteq \omega^\omega$ is σ-join-reducible if and only if it is Wadge equivalent to a function of the form $\bigoplus_{n\in\omega} \mathcal{A}_n$, where each $\mathcal{A}_n$ is σ-join-irreducible and $\mathcal{A}_n <_w \mathcal{A}$, and where $\bigoplus_{n\in\omega} \mathcal{A}_n$ is defined by $(\bigoplus_{n\in\omega} \mathcal{A}_n)(n^\frown X) = \mathcal{A}_n(X)$.

We also need Steel–van Wesep's theorem [28].

Fact 5. *A subset of ω^ω is self-dual if and only if it is σ-join-reducible.*

Facts 4 and 5 have $\mathcal{Q}$-analogues; see [15, 16].

2. The structure of $\{0,1,\bot\}$-valued functions

2.1. **The proper $\mathbf{2}_\bot$-Wadge degrees.** For $\Gamma \in \{\Sigma, \Pi, \Delta\}$, define $\Gamma^\diamondsuit_\alpha$ to be the class of all $\mathbf{2}_\bot$-valued functions which are Wadge reducible to a Γ_α subset of ω^ω, that is,

$$\Gamma^\diamondsuit_\alpha = \{\mathcal{A} : \omega^\omega \to \mathbf{2}_\bot \mid (\exists S \in \Gamma_\alpha)\ \mathcal{A} \leq_w S\},$$

where recall that a set S is always identified with a $\mathbf{2}$-valued function. Note that, in general, $\Delta^\diamondsuit_\alpha = \Sigma^\diamondsuit_\alpha \cap \Pi^\diamondsuit_\alpha$ does *not* hold anymore.

Example 2.1. Consider the comparison of reals, viewed as a function $c\colon (2^\omega)^2 \to \mathbf{2}_\bot$ taking (x, y) to 0, $\bot$ or 1 depending on if $x < y$, $x = y$, or $x > y$, where 2^ω is identified with a Cantor set in $\mathbb{R}$. The identity map witnesses $c \leq_w \{(x, y) : x > y\}$ and $c \leq_w \{(x, y) : x \geq y\}$, so $c \in \Sigma^\diamondsuit_1 \cap \Pi^\diamondsuit_1$. However, one can easily see that there exists no total continuous $\{0, 1\}$-valued extension of c, where $c(x) = \bot$ is interpreted as $x \notin \mathrm{dom}(c)$; hence $c \notin \Delta^\diamondsuit_1$. Although c is a function on 2^ω, it can be lifted to a function $c^\star$ on ω^ω by setting $c^\star(x) = \bot$ for any $x \notin 2^\omega$.

Let $\mathcal{D}^\diamondsuit_w$ be the set of all Wadge degrees of $\mathbf{2}_\bot$-valued functions. Then, we define $\mathcal{D}_w \subseteq \mathcal{D}^\diamondsuit_w$ as the set of all Wadge degrees which contain $\mathbf{2}$-valued functions. A Wadge degree $\mathbf{d}$ is called a *proper* $\mathbf{2}_\bot$*-Wadge degree* if $\mathbf{d} \in \mathcal{D}^\diamondsuit_w \setminus \mathcal{D}_w$. For a Wadge degree $\mathbf{d} \in \mathcal{D}^\diamondsuit_w$, let $\Gamma_\mathbf{d}$ be the collection of all $\mathcal{A}\colon \omega^\omega \to \mathbf{2}_\bot$ such that $\mathcal{A} \leq_w \mathcal{B}$ for some $\mathcal{B} \in \mathbf{d}$. Note that any $\mathcal{B} \in \mathbf{d}$ is $\Gamma_\mathbf{d}$-complete.

Lemma 2.2. *For any proper* $\mathbf{2}_\bot$*-Wadge degree* $\mathbf{d}$*, there is* $\alpha < \Theta$ *such that*

$$\Delta^\diamondsuit_\alpha \subseteq \Gamma_\mathbf{d} \subseteq \Sigma^\diamondsuit_\alpha \cap \Pi^\diamondsuit_\alpha.$$

Proof. Let $\alpha < \Theta$ be the least ordinal such that $\Gamma_\mathbf{d} \subseteq \Sigma^\diamondsuit_\alpha \cap \Pi^\diamondsuit_\alpha$. Then, $\Gamma_\mathbf{d} \not\subseteq \Sigma^\diamondsuit_\beta$ or $\Gamma_\mathbf{d} \not\subseteq \Pi^\diamondsuit_\beta$ for any $\beta < \alpha$. Let $\mathcal{A}$ be a $\Gamma_\mathbf{d}$-complete function, and $B_0, B_1 \subseteq \omega^\omega$ be Σ_β- and Π_β-complete sets, respectively. Then, $\mathcal{A} \not\leq_w B_i$ for some $i < 2$. By Fact 3, we have $B_{1-i} \equiv_w \neg B_i \leq_w \mathcal{A}$. Since $\mathbf{d}$ is a proper $\mathbf{2}_\bot$-Wadge degree, $\mathcal{A} \not\equiv_w B_{1-i}$, and therefore, we also have $\mathcal{A} \not\leq_w B_{1-i}$. Again, by Fact 3, we get $B_i \leq_w \neg B_{1-i} \leq_w \mathcal{A}$. Therefore, $B_0, B_1 \leq_w A$, and hence, we conclude that $\Sigma^\diamondsuit_\beta \cup \Pi^\diamondsuit_\beta \subseteq \Gamma_\mathbf{d}$.

If the cofinality of α is uncountable, then $\Delta_\alpha = \bigcup_{\beta<\alpha} \Sigma_\beta$ since there is no Δ_α-complete set by Fact 2 (see also [28]). Therefore, $\Delta_\alpha \subseteq \Gamma_\mathbf{d}$ since $\Sigma_\beta \subseteq \Gamma_\mathbf{d}$ for any $\beta < \alpha$. Thus, $\Delta^\diamondsuit_\alpha \subseteq \Gamma_\mathbf{d}$. Assume that the cofinality of α is countable. Then, by Fact 2, there is a Δ_α-complete set $C \subseteq \omega^\omega$. Since Δ_α is a selfdual pointclass, C is σ-join-reducible by Fact 5, and thus by Fact 4, one can assume that C is of the form $\bigoplus_n C_n$, where for any $n \in \omega$, there is $\beta < \alpha$ such that $C_n \in \Sigma_\beta \cup \Pi_\beta$. If $\mathcal{A}$ is a $\Gamma_\mathbf{d}$-complete function, then $C_n \leq_w \mathcal{A}$ for any n since $\Sigma_\beta \cup \Pi_\beta \subseteq \Gamma_\mathbf{d}$. By combining these Wadge reductions, we obtain $\bigoplus_n C_n \leq_w \mathcal{A}$. Thus, $\Delta^\diamondsuit_\alpha \subseteq \Gamma_\mathbf{d}$. □

2.2. **The non-proper m-Wadge degrees.** We say that $\mathcal{A}\colon \omega^\omega \to \mathcal{Q}$ *is* $\mathcal{Q}$*-*m*-Wadge reducible to* $\mathcal{B}\colon \omega^\omega \to \mathcal{Q}$ ($\mathcal{A} \leq_{mw} \mathcal{B}$) if for any $n \in \omega$, there are $m \in \omega$ and a continuous function $\theta\colon \omega^\omega \to \omega^\omega$ such that for any $X \in \omega^\omega$,

$$\mathcal{A}(n^\frown X) \leq_\mathcal{Q} \mathcal{B}(m^\frown \theta(X)).$$

Note that θ may depend on n, and thus the above reduction involves countably many functions $(\theta_n)_{n\in\omega}$ and $n \mapsto m$. Clearly, this is an intermediate notion between Lipschitz reducibility and Wadge reducibility. The game associated to $\mathcal{Q}$-m-Wadge reducibility has been studied by Kihara-Montalbán [15, Section 4.2] (which is a simpler version of Steel's degree invariant game [27]).

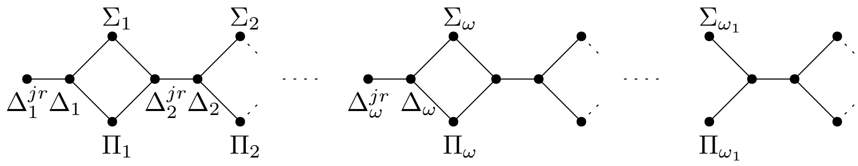

FIGURE 1. The structure of m-Wadge degrees of subsets of ω^ω (The structure of *non-proper* m-Wadge degrees of $\mathbf{2}_\perp$-valued functions of ω^ω)

We consider a $\mathcal{Q}$-analogue of Definition 1.7. We say that $\mathcal{A}\colon \omega^\omega \to \mathcal{Q}$ is *m-σ-join-reducible* (m-σ-jr) if $\mathcal{A} \upharpoonright n <_w \mathcal{A}$ for every $n \in \omega$. Otherwise, we say that $\mathcal{A}$ is *m-σ-join-irreducible* (m-σ-ji).

Lemma 2.3. *Assume that $\mathcal{B}\colon \omega^\omega \to \mathcal{Q}$ is m-σ-ji. Then, for any $\mathcal{A}\colon \omega^\omega \to \mathcal{Q}$,*

$$\mathcal{A} \leq_{mw} \mathcal{B} \iff \mathcal{A} \leq_w \mathcal{B}.$$

Proof. It is clear that $\mathcal{A} \leq_{mw} \mathcal{B}$ implies $\mathcal{A} \leq_w \mathcal{B}$. For the reverse implication, since $\mathcal{B}$ is m-σ-ji, there is m such that $\mathcal{A} \leq_w \mathcal{B} \leq_w \mathcal{B} \upharpoonright m$. Let θ witness $\mathcal{A} \leq_w \mathcal{B} \upharpoonright m$. Then, we have $\mathcal{A}(n^\frown X) \leq_{\mathcal{Q}} \mathcal{B}(m^\frown\theta(n^\frown X))$. This clearly implies that $\mathcal{A} \leq_{mw} \mathcal{B}$. □

As Σ_α and Π_α are non-self-dual, by Fact 5, Σ_α- and Π_α-complete sets are σ-ji (in particular, m-σ-ji). Thus, by Lemma 2.3, $\Sigma_\alpha^\Diamond$- and $\Pi_\alpha^\Diamond$-complete functions are also complete w.r.t. m-Wadge reducibility.

Lemma 2.4. *Let $\alpha < \Theta$ be an ordinal of countable cofinality. Then, there are exactly two $\mathbf{2}_\perp$-m-Wadge degrees of the $\Delta_\alpha^\Diamond$-complete functions.*

Proof. As a Δ_α-complete set is σ-jr, by Fact 4, it is Wadge equivalent to a set of the form $\mathcal{A} = \bigoplus_n \mathcal{A}_n$, which is clearly m-σ-jr, and $\Delta_\alpha^\Diamond$-complete. Then, it is also clear that $0^\frown\mathcal{A} = \{0^\frown X : X \in \mathcal{A}\}$ is an m-σ-ji $\Delta_\alpha^\Diamond$-complete set. Obviously, $\mathcal{A} <_{mw} 0^\frown\mathcal{A}$. Now, $\Delta_\alpha^\Diamond$-complete functions split into m-σ-jr ones and m-σ-ji ones. By Lemma 2.3, if $\mathcal{A}$ and $\mathcal{B}$ are m-σ-ji $\Delta_\alpha^\Diamond$-complete functions, then $\mathcal{A} \equiv_{mw} \mathcal{B}$.

Assume that $\mathcal{A}$ and $\mathcal{B}$ are m-σ-jr $\Delta_\alpha^\Diamond$-complete functions. First consider the case that α is a successor ordinal, $\alpha = \beta + 1$ say. By our assumption, we have $\mathcal{A} = \bigoplus_n (\mathcal{A} \upharpoonright n)$ where $\mathcal{A} \upharpoonright n <_w \mathcal{A}$, which implies that $\mathcal{A} \upharpoonright n$ is not $\Delta_\alpha^\Diamond$-complete. If $\mathcal{A} \upharpoonright n$ has a proper $\mathbf{2}_\perp$-Wadge degree $\mathbf{d}$, then as $\Delta_\alpha^\Diamond \not\subseteq \Gamma_{\mathbf{d}}$, by Lemma 2.2, we have $\mathcal{A} \upharpoonright n \in \Gamma_{\mathbf{d}} \subseteq \Sigma_\beta^\Diamond \cap \Pi_\beta^\Diamond$. The join of such functions is also in $\Sigma_\beta^\Diamond \cap \Pi_\beta^\Diamond$. This implies that $\mathcal{A}$ is Wadge equivalent to the join of functions of non-proper $\mathbf{2}_\perp$-Wadge degrees; that is, there are $m, n \in \omega$ such that $\mathcal{A} \upharpoonright m$ and $\mathcal{A} \upharpoonright n$ are $\Sigma_\beta^\Diamond$- and $\Pi_\beta^\Diamond$-complete, respectively. Again, by Lemma 2.2, for any $n \in \omega$, $\mathcal{B} \upharpoonright n$ is either $\Sigma_\beta^\Diamond$ or $\Pi_\beta^\Diamond$ since $\mathcal{B}$ is m-σ-jr. This shows that $\mathcal{B} \leq_{mw} \mathcal{A}$. By a symmetric argument, we also have $\mathcal{A} \leq_{mw} \mathcal{B}$, and conclude that $\mathcal{A} \equiv_{mw} \mathcal{B}$. For a limit ordinal α, for any $n \in \omega$, $\mathcal{B} \upharpoonright n$ is in $\Sigma_\beta^\Diamond$ for some $\beta < \alpha$ since $\mathcal{B}$ is m-σ-jr. Then there are $m \in \omega$ and γ such that $\beta \leq \gamma < \alpha$ and every $\Sigma_\gamma^\Diamond$ function is $\mathbf{2}_\perp$-Wadge reducible to $\mathcal{A} \upharpoonright m$ since α is limit and $\mathcal{A} = \bigoplus_n (\mathcal{A} \upharpoonright n)$ is $\Delta_\alpha^\Diamond$-complete. Hence, $\mathcal{B} \upharpoonright n \leq_w \mathcal{A} \upharpoonright m$, which implies that $\mathcal{B} \leq_{mw} \mathcal{A}$. By a symmetric argument, we also have $\mathcal{A} \leq_{mw} \mathcal{B}$, and conclude that $\mathcal{A} \equiv_{mw} \mathcal{B}$. □

The usual Wadge degrees form a semi-well-order in which a selfdual degree and a nonselfdual pair appear alternately. Here, by Fact 5 (see also [28, 2]), at a limit

Wadge rank α, a selfdual degree appears first if the cofinality of α is countable, and a nonselfdual pair appears if either $\alpha = 0$ or the cofinality of α is uncountable. Thus, Lemmas 2.3 and 2.4 give the complete description of the *non-proper* $\mathbf{2}_\perp$-m-Wadge degree structure, hence the m-Wadge degree structure of subsets of ω^ω. Each selfdual Wadge degree splits into two degrees (which are linearly ordered), and nonselfdual Wadge degrees remain the same. See Figure 1, where Δ^{jr}_α denotes the class of all sets m-Wadge reducible to an m-σ-jr Δ_α set.

3. Many-one reducibility for real-valued functions

3.1. **Reducibility for real-valued functions.** Day-Downey-Westrick [8] introduced the notion of $\mathbf{m}$-reducibility for real-valued functions. Let $[\mathbb{Q}]^2$ be the set of all pairs (p, q) of rationals such that $p < q$. For $f\colon \omega^\omega \to \mathbb{R}$, we define $\mathrm{Lev}_f\colon [\mathbb{Q}]^2 \times \omega^\omega \to \mathbf{2}_\perp$ as follows.

$$\mathrm{Lev}_f(\langle p,q\rangle^\frown X) = \begin{cases} 0 & \text{if } f(X) \leq p, \\ 1 & \text{if } q \leq f(X), \\ \perp & \text{if } p < f(X) < q. \end{cases}$$

For $f, g\colon \omega^\omega \to \mathbb{R}$, we say that f *is* $\mathbf{m}$*-reducible to* g (written $f \leq_{\mathbf{m}} g$) if for any pair of rationals $p < q$, there are a pair of rationals $r < s$ and a continuous function $\theta\colon \omega^\omega \to \omega^\omega$ such that for any $X \in \omega^\omega$,

$$\mathrm{Lev}_f(\langle p,q\rangle^\frown X) \leq_{\mathbf{2}_\perp} \mathrm{Lev}_g(\langle r,s\rangle^\frown \theta(X)).$$

We denote by $\{f \leq p\}$ and $\{f \geq q\}$ the upper and lower level sets $\{X : f(X) \leq p\}$ and $\{X : f(X) \geq q\}$, respectively. We also define $\{f < p\}$ and $\{f > q\}$ in a similar manner. In the context of Wadge's pair reducibility ([30, Section I.E]; see also Section 1.4), $f \leq_{\mathbf{m}} g$ if and only if for any $p < q$ there are $r < s$ such that $(\{f \leq p\}, \{f \geq q\})$ is Wadge reducible to $(\{g \leq r\}, \{g \geq s\})$.

Observation 3.1. *The above definition of* $\leq_{\mathbf{m}}$ *coincides with Definition 1.1.*

Proof. It is clear that $f \leq_{\mathbf{m}} g$ in the sense of Definition 1.1 if and only if for any p, ε there are r, δ such that $(\{f \leq p-\varepsilon\}, \{f \geq p+\varepsilon\}) \leq_w (\{g \leq r-\delta\}, \{g \geq r+\delta\})$. □

We often identify $[\mathbb{Q}]^2$ and ω via a fixed bijection. Under this identification, Lev_f is thought of as a function from $\omega \times \omega^\omega$ to $\mathbf{2}_\perp$, and thus, the $\mathbf{m}$-degree structure embeds into the $\mathbf{2}_\perp$-m-Wadge degree structure, that is,

Observation 3.2. *For a function* $f\colon \omega^\omega \to \mathbb{R}$*,* $f \leq_{\mathbf{m}} g$ *if and only if* $\mathrm{Lev}_f \leq_{mw} \mathrm{Lev}_g$*.*

We will show that the map $f \mapsto \mathrm{Lev}_f$ induces an isomorphism between the $\mathbf{m}$-degrees on real-valued functions on ω^ω and the m-Wadge degrees on nonempty proper subsets of ω^ω.

Theorem 3.3. *The map* $f \mapsto \mathrm{Lev}_f$ *induces an isomorphism between the quotients of* $(\mathcal{F}(\omega^\omega, \mathbb{R}), \leq_{\mathbf{m}})$ *and* $(\mathcal{P}(\omega^\omega) \setminus \{\emptyset, \omega^\omega\}, \leq_{mw})$*.*

The proof of Theorem 3.3 will be given in the rest of this section. As a consequence of Theorem 3.3, the structure of $\mathbf{m}$-degrees of real-valued functions on ω^ω looks like Figure 1. We will also show that the structure of $\mathbf{m}$-degrees of real-valued functions on 2^ω looks like Figure 2. That is, it is almost isomorphic to the m-Wadge degrees of nonempty proper subsets of ω^ω except that if α is a limit

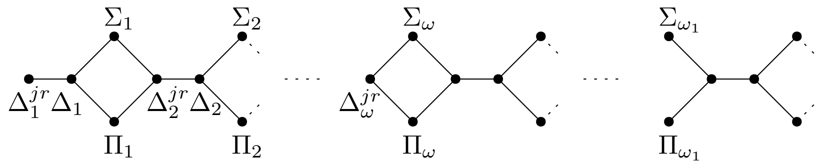

FIGURE 2. The structure of m-degrees of real-valued functions on 2^ω

ordinal of countable cofinality, the m-Wadge degree of an m-σ-ji Δ_α-complete set cannot be realized.

3.2. **Non-proper Wadge degrees.** We first characterize non-proper $\mathbf{2}_\perp$-Wadge degrees realized as real-valued functions.

Lemma 3.4.

(1) *For any nonzero ordinal $\alpha < \Theta$, there are functions $f, g\colon 2^\omega \to \{0,1\}$ such that* Lev_f *and* Lev_g *are $\Sigma^\diamondsuit_\alpha$- and $\Pi^\diamondsuit_\alpha$-complete, respectively.*

(2) *For any nonzero ordinal $\alpha < \Theta$ of countable cofinality, there is a function $f\colon 2^\omega \to [0,1]$ such that* Lev_f *is m-σ-jr and $\Delta^\diamondsuit_\alpha$-complete.*

(3) *For any successor ordinal $\alpha < \Theta$, there is a function $f\colon 2^\omega \to \{0,1\}$ such that* Lev_f *is m-σ-ji and $\Delta^\diamondsuit_\alpha$-complete.*

(4) *For any limit ordinal $\alpha < \Theta$ of countable cofinality, there is a function $f\colon \omega \times 2^\omega \to \{0,1\}$ such that* Lev_f *is m-σ-ji and $\Delta^\diamondsuit_\alpha$-complete.*

(5) *For any limit ordinal $\alpha < \Theta$ of countable cofinality, there is no function $f\colon 2^\omega \to \mathbb{R}$ such that* Lev_f *is m-σ-ji and $\Delta^\diamondsuit_\alpha$-complete.*

(6) *There is no $f\colon \omega^\omega \to \mathbb{R}$ such that* $\mathrm{Lev}_f \equiv_w \emptyset$ *or* $\mathrm{Lev}_f \equiv_w \omega^\omega$.

Proof. (1) Let $A \subseteq 2^\omega$ be a Σ_α-complete set. Such a set exists by Lemma 1.5. Define $\chi_A\colon 2^\omega \to \{0,1\}$ by $\chi_A(X) = 1$ if $X \in A$; otherwise $\chi_A(X) = 0$. Clearly $A \leq_w \mathrm{Lev}_A := \mathrm{Lev}_{\chi_A}$ since $A(X) = \mathrm{Lev}_A(\langle 0,1\rangle^\frown X)$. For $\mathrm{Lev}_A \leq_w A$, since the Wadge rank of A is nonzero, $\emptyset, \omega^\omega \leq_w A$, and therefore, there is $Y_j \in 2^\omega$ such that $A(Y_j) = j$ for each $j \in \{0,1\}$. Given p, q, if $p < 0$, define $\theta_{p,q}(X) = Y_0$, and if $q > 1$, define $\theta_{p,q}(X) = Y_1$. If $0 \leq p < q \leq 1$, then we have $A(X) = \mathrm{Lev}_A(\langle p,q\rangle^\frown X)$, and thus define $\theta_{p,q}(X) = X$. The function θ witnesses that $\mathrm{Lev}_A \leq_w A$. Consequently, Lev_A is $\Sigma^\diamondsuit_\alpha$-complete. By a similar argument, one can construct g such that Lev_g is $\Pi^\diamondsuit_\alpha$-complete.

(2) First assume that α is a limit ordinal, say $\alpha = \sup_i \beta_i$. By (1), we have a function $f_i\colon 2^\omega \to \{0,1\}$ such that Lev_{f_i} is $\Sigma^\diamondsuit_{\beta_i}$-complete for each $i < \omega$. Let $(a_n)_{n\in\omega}$ be strictly decreasing sequence of positive reals converging to 0. We define $g(0^\omega) = 0$ and $g(0^n 1X) = a_{2n+1-f_n(X)}$. Let $A \subseteq \omega^\omega$ be a Δ_α-complete set of the form $\bigoplus_i A_i$ such that A_i is Σ_{β_i}-complete. Then, clearly A is m-σ-jr. We claim that $\mathrm{Lev}_g \equiv_{mw} A$.

For $A \leq_{mw} \mathrm{Lev}_g$, given n we know that $A_n \leq_w f_n$ via some continuous function θ_n. Then,

$$A_n(X) = i \implies f_n(\theta_n(X)) = i \implies g(0^n 1\theta_n(X)) = a_{2n+1-i}.$$

Thus, $X\rangle \mapsto 0^n 1\theta_n(X)$ witnesses $A \restriction n \leq_w \mathrm{Lev}_g \restriction \langle a_{2n+1}, a_{2n}\rangle$. Hence, $A \leq_{mw} \mathrm{Lev}_g$. To see $\mathrm{Lev}_g \leq_{mw} A$, let $p < q$ be rationals. First consider the case that the open interval (p,q) intersects with (a_{2n+2}, a_{2n+1}) for some $n \in \omega$. By our definition of g, if X extends 0^{n+1} then $g(X) \leq a_{2n+2}$, and if X extends $0^m 1$ for some $m \leq n$

then $g(X) \geq a_{2n+1}$. Since the Wadge rank of A is nonzero, $\emptyset, \omega^\omega \leq_w A$, and therefore, there is Y_j such that $A(Y_j) = j$ for each $j < 2$. Define $\theta_{p,q}(X) = Y_0$ if X extends 0^{n+1}; otherwise $\theta_{p,q}(X) = Y_1$. Then,

$$\begin{aligned}\mathrm{Lev}_g(\langle p, q\rangle^\frown X) = 0 &\implies g(X) \leq p < a_{2n+1} \implies g(X) \leq a_{2n+2} \\ &\implies X \succ 0^{n+1} \implies A \circ \theta_{p,q}(X) = A(Y_0) = 0.\end{aligned}$$

Similarly, $\mathrm{Lev}_g(\langle p, q\rangle^\frown X) = 1$ implies that $A \circ \theta_{p,q}(X) = A(Y_1) = 1$. This shows that $\mathrm{Lev}_g \restriction \langle p, q\rangle \leq_w A \restriction \langle 0, 1\rangle$.

Now, assume that (p, q) does not intersects with (a_{2n+2}, a_{2n+1}) for any $n \in \omega$. If $a_0 < p$ then $\mathrm{Lev}_g(\langle p, q\rangle^\frown X) = 0$ for any X. If $q < 0$, then, since $(a_n)_{n\in\omega}$ converges to 0, we have $\mathrm{Lev}_g(\langle p, q\rangle^\frown X) = 1$ for any X. In these cases, it is clear that $\mathrm{Lev}_g \restriction \langle p, q\rangle \leq_w A \restriction \langle 0, 1\rangle$.

Otherwise, $a_1 \leq p \leq a_0$ or $a_{2n+1} \leq p < q \leq a_{2n}$ for some $n \in \omega$. Assume that $a_{2n+1} \leq p < q \leq a_{2n}$. Since $g(X) \in \{a_n\}_{n\in\omega}$ we have $(\{g \leq a_{2n+1}\}, \{g \geq a_{2n+2}\}) = (\{g \leq p\}, \{g \geq q\})$. If X extends 0^m1 for some $m < n$, then $g(X) \geq a_{2n-1} > p$, and thus define $\theta_{p,q}(X) = Y_1$. If X extends 0^{n+1}, then $g(X) \leq a_{2n+2} < p$, and thus define $\theta_{p,q}(X) = Y_0$. If X is of the form 0^n1Z, then $g(X) \in \{a_{2n+1}, a_{2n+2}\}$. One can see that

$$\mathrm{Lev}_g(\langle p, q\rangle^\frown X) = (\{g \leq p\}, \{g \geq q\})(X) = (\{g \leq a_{2n+1}\}, \{g \geq a_{2n+2}\})(X) = f_n(Z).$$

Then, define $\theta_{p,q}(X) = n^\frown \tau_n(Z)$, where τ_n is a continuous function witnesses that $f_n \leq_w A_n$. This shows that $\mathrm{Lev}_g \restriction \langle p, q\rangle \leq_{mw} A_n = A \restriction n$ via $\theta_{p,q}$. For $a_1 \leq p \leq a_0$, a similar argument applies. Thus, we conclude that $\mathrm{Lev}_g \equiv_{mw} A$, that is, Lev_g is m-σ-jr and $\Delta^\diamondsuit_\alpha$-complete.

For the case that α is a successor ordinal, say $\alpha = \beta + 1$. Assume that $\alpha > 1$, so $\beta > 0$. By (1) we have a function $f_0, f_1 \colon 2^\omega \to \{0, 1\}$ such that Lev_{f_0} is $\Sigma^\diamondsuit_\beta$-complete and Lev_{f_1} is $\Pi^\diamondsuit_\beta$-complete. Then define $g \colon 2^\omega \to \{0, 1, 2, 3\}$ by $g(0X) = f_0(X)$ and $g(1X) = 2 + f_1(X)$. By a similar argument as above, one can easily show that Lev_g is m-σ-jr and $\Delta^\diamondsuit_\alpha$-complete. If $\alpha = 1$, then let $f \colon 2^\omega \to \{0, 1\}$ be a constant function. Then, Lev_f is continuous and not constant, so Lev_f is $\Delta^\diamondsuit_1$-complete. For any $p < q$, one can easily see that $\mathrm{Lev}_f \restriction \langle p, q\rangle$ is Wadge reducible to a constant function. Hence, we get $\mathrm{Lev}_f \restriction \langle p, q\rangle <_w \mathrm{Lev}_f$, which means that Lev_f is m-σ-jr.

(4) Assume that $\alpha = \sup_i \beta_i$. By (1), we have a function $f_i \colon 2^\omega \to \{0, 1\}$ such that Lev_{f_i} is $\Sigma^\diamondsuit_{\beta_i}$-complete for each $i < \omega$. Then, define $g(iX) = f_i(X)$. It is clear that Lev_g is $\Delta^\diamondsuit_\alpha$-complete. Moreover, one can see that $\mathrm{Lev}_g \leq_w \mathrm{Lev}_g \restriction \langle 0, 1\rangle$. Hence, Lev_g is m-σ-ji.

(3) A similar argument as in the item (4) also verifies the item (3) except for $\alpha = 1$. Thus, assume that $\alpha = 1$, and let f be a nonconstant continuous function. We claim that Lev_f is m-σ-ji and $\Delta^\diamondsuit_1$-complete. Since f is nonconstant, there are X_0, X_1 such that $f(X_0) < f(X_1)$. Choose rationals $r < s$ such that $f(X_0) \leq r < s \leq f(X_1)$. It suffices to show that $\mathrm{Lev}_f \restriction \langle p, q\rangle \leq_w \mathrm{Lev}_f \restriction \langle r, s\rangle$ for any $p < q$. Since f is continuous, $\{f \leq p\}$ and $\{f \geq q\}$ are both closed. Since Π^0_1 has the separation property, there is a clopen set $C \subseteq \omega^\omega$ separating $\{f \leq p\}$ from $\{f \geq q\}$. Define $\theta_{p,q}(X) = X_0$ if $X \in C$; otherwise $\theta_{p,q}(X) = X_1$. Then, $\theta_{p,q}$ is continuous since C is clopen. It is easy to see that $\theta_{p,q}$ witnesses that $\mathrm{Lev}_f \restriction \langle p, q\rangle \leq_w \mathrm{Lev}_f \restriction \langle r, s\rangle$.

(5) Assume that Lev_f is m-σ-ji and $\Delta^{\diamondsuit}_{\alpha}$-complete. Since Lev_f is m-σ-ji, there are p, q such that $\mathrm{Lev}_f \upharpoonright \langle p,q\rangle$ is $\Delta^{\diamondsuit}_{\alpha}$-complete. However, since $\mathrm{Lev}_f \upharpoonright \langle p,q\rangle$ is a function on 2^ω, and α is limit, it is impossible by compactness: As $\mathcal{A} := \mathrm{Lev}_f \upharpoonright \langle p,q\rangle$ is $\Delta^{\diamondsuit}_{\alpha}$-complete, $\mathcal{A}$ is Wadge equivalent to a function of the form $\bigoplus_n \mathcal{A}_n$, where $\mathcal{A}_n \in \Sigma^{\diamondsuit}_{\beta_n}$ for some $\beta_n < \alpha$. Let $\theta\colon 2^\omega \to \omega^\omega$ witness $\mathcal{A} \leq_w \bigoplus_n \mathcal{A}_n$. As θ is continuous and 2^ω is compact, the image of θ is also compact. Hence, there is $k \in \omega$ such that θ witnesses $\mathcal{A} \leq_w \bigoplus_{n<k} \mathcal{A}_n$. This implies that $\mathcal{A} = \mathrm{Lev}_f \upharpoonright \langle p,q\rangle$ is in $\Sigma^{\diamondsuit}_{\beta}$, where $\beta = \max_{n<k} \beta_n < \alpha$, a contradiction.

(6) Fix $X \in \omega^\omega$. Then, there are p, q such that $p < f(X) < q$. Thus, $\mathrm{Lev}_f(\langle p-1, p\rangle^\frown X) = 1$ and $\mathrm{Lev}_f(\langle q, q+1\rangle^\frown X) = 0$. Therefore, Lev_f is not Wadge reducible to constant functions such as (the characteristic functions of) $\emptyset$ and ω^ω. □

3.3. **Proper Wadge degrees.** We finally show that, as a consequence of the van Wesep-Steel Theorem (Fact 1), proper Wadge degrees disappear in the $\mathbf{m}$-degrees of real-valued functions.

Lemma 3.5. *For any proper $\mathbf{2}_\perp$-Wadge degree $\mathbf{d}$, there is no $f\colon \omega^\omega \to \mathbb{R}$ such that Lev_f is $\Gamma_{\mathbf{d}}$-complete.*

Proof. Let $\alpha < \Theta$ be an ordinal in Lemma 2.2, that is, $\Delta^{\diamondsuit}_{\alpha} \subseteq \Gamma_{\mathbf{d}} \subseteq \Sigma^{\diamondsuit}_{\alpha} \cap \Pi^{\diamondsuit}_{\alpha}$. Assume that $\mathrm{Lev}_f \in \Gamma_{\mathbf{d}}$. We claim that for any $p < q$, $\mathrm{Lev}_f \upharpoonright \langle p,q\rangle$ is $\Delta^{\diamondsuit}_{\alpha}$. Let $U, V \subseteq \omega^\omega$ be Σ_α- and Π_α-complete sets, respectively. Choose rationals $p < r < s < q$. Since $\mathrm{Lev}_f \in \Sigma^{\diamondsuit}_{\alpha} \cap \Pi^{\diamondsuit}_{\alpha}$, there are continuous functions τ_0, τ_1 such that

$$\mathrm{Lev}_f(\langle p,r\rangle^\frown X) \leq_{\mathbf{2}_\perp} U \circ \tau_0(X), \text{ and } \mathrm{Lev}_f(\langle s,q\rangle^\frown X) \leq_{\mathbf{2}_\perp} V \circ \tau_1(X).$$

Consider $A = \{X : U \circ \tau_0(X) = 0\}$ and $B = \{X : V \circ \tau_1(X) = 1\}$, that is, $A = \tau_0^{-1}[\omega^\omega \setminus U]$ and $B = \tau_1^{-1}[V]$. Clearly, $A, B \in \Pi_\alpha$. Moreover,

$$f(X) \leq p \implies U \circ \tau_0(X) = 0 \iff X \in A \implies f(X) < r$$
$$f(X) \geq q \implies V \circ \tau_1(X) = 1 \iff X \in B \implies f(X) > s.$$

This implies that $\{f \leq p\} \subseteq A$, $\{f \geq q\} \subseteq B$, and $A \cap B = \emptyset$. By the separation property of Π_α (Fact 1), there is a Δ_α set $C \subseteq \omega^\omega$ such that $C \subseteq A$ and $\omega^\omega \setminus C \subseteq B$. Then,

$$\{f \leq p\} \subseteq C, \text{ and } \{f \geq q\} \subseteq \omega^\omega \setminus C.$$

This shows that $\mathrm{Lev}_f \upharpoonright \langle p,q\rangle \leq_w C$, and therefore, $\mathrm{Lev}_f \upharpoonright \langle p,q\rangle$ is $\Delta^{\diamondsuit}_{\alpha}$, which verifies our claim. Consequently, $\mathrm{Lev}_f = \bigoplus_{p,q} \mathrm{Lev}_f \upharpoonright \langle p,q\rangle$ is also $\Delta^{\diamondsuit}_{\alpha}$, and in particular, Lev_f cannot be $\Gamma_{\mathbf{d}}$-complete. □

This concludes the proof of Theorem 3.3.

Proof (Theorem 1.2). For any Wadge degree $\mathbf{d}$ of a subset of ω^ω, $\mathbf{d}$ is the Wadge degree of a Σ_α- or Π_α-complete set for some $\alpha < \Theta$, or a Δ_α set for some $\alpha < \Theta$ whose cofinality is countable by Fact 2. As seen in the last paragraph in Section 2.2, every selfdual Wadge degree splits into two m-Wadge degrees (which are linearly ordered), and nonselfdual Wadge degrees remains the same. By Theorem 3.3 and Lemma 3.4 (5), we conclude that the structure of $\mathbf{m}$-degrees of real-valued functions on 2^ω looks like Figure 2 as desired. □

4. The Bourgain rank

In this section, we will generalize Day-Downey-Westrick's result on the Bourgain rank α to an arbitrary Polish domain. Pequignot [22] extended the notion of Wadge reducibility to subsets of second countable spaces based on the theory of admissible representation. An *admissible representation* of a topological space $\mathcal{X}$ is a partial continuous surjection $\delta :\subseteq \omega^\omega \twoheadrightarrow \mathcal{X}$ which has the following universal property: For any partial continuous function $f :\subseteq \omega^\omega \to \mathcal{X}$, there is a partial continuous function $\tau :\subseteq \omega^\omega \to \omega^\omega$ such that $f = \delta \circ \tau$. An admissible representation is *open* if it is an open map, and *total* if its domain is ω^ω. The following result seems folklore, but we present the proof for the sake of completeness.

Lemma 4.1. *Every Polish space has a total open admissible representation.*

Proof. If $\mathcal{X}$ is Polish, as in Kechris [14, Exercise 7.14], one can construct a Suslin scheme $\mathcal{U} = (U_s)_{s\in\omega^{<\omega}}$ on $\mathcal{X}$ whose associate map is a total open continuous surjection $\delta^{\mathcal{U}}_{\rm Sus}\colon \omega^\omega \twoheadrightarrow \mathcal{X}$, where $\delta^{\mathcal{U}}_{\rm Sus}(p)$ is defined as a unique element in $\bigcap_n U_{p\upharpoonright n}$. Here, this particular Suslin scheme satisfies $U_\emptyset = \mathcal{X}$ and $U_s = \bigcup_n U_{s^\frown n}$. Although it is not hard to check that $\delta^{\mathcal{U}}_{\rm Sus}$ is admissible, we here give the proof for the sake of completeness.

To see that $\delta^{\mathcal{U}}_{\rm Sus}$ is admissible, recall that the "neighborhood filter" representation is always admissible, that is, for any countable basis $\mathcal{B} = (B_e)_{e\in\omega}$ of $\mathcal{X}$ with $B_0 = \mathcal{X}$, consider $\mathcal{N}^{\mathcal{B}}_x = \{e \in \omega : x \in B_e\}$, and then define $\delta^{\mathcal{B}}_{\rm nbhd}(p) = x$ if the range of p is equal to $\mathcal{N}^{\mathcal{B}}_x$. It is well-known that $\delta^{\mathcal{B}}_{\rm nbhd}$ is an admissible representation of $\mathcal{X}$ (cf. [9, Theorem 48]).

It suffices to find a partial continuous function $\tau :\subseteq \omega^\omega \to \omega^\omega$ such that $\delta^{\mathcal{B}}_{\rm nbhd} = \delta^{\mathcal{U}}_{\rm Sus} \circ \tau$. Given $p \in \omega^\omega$, wait for the first $t \in \omega$ such that $B_{p(t)} \subseteq U_{\langle n\rangle}$ for some $n \in \omega$. If there are such t and n, define $\tau(p)(0) = n$. Assume that $s = \tau(p) \upharpoonright \ell$ has already been defined. Then, wait for the first $t \in \omega$ such that $B_{p(t)} \subseteq U_{s^\frown n}$ for some $n \in \omega$. If there are such t and n, define $\tau(p)(\ell) = n$. Continue this procedure. Clearly τ is continuous. If $\delta^{\mathcal{U}}_{\rm nbhd}(p) = x \in U_s = \bigcup_n U_{s^\frown n}$, one can find t such that $x \in B_{p(t)} \subseteq U_{s^\frown n}$ for some n since $\mathcal{N}^{\mathcal{B}}_x$ is a local basis of $\mathcal{X}$ at x. Therefore, if $p \in {\rm dom}(\delta^{\mathcal{U}}_{\rm nbhd})$, one can inductively show that $\tau(p) \in \omega^\omega$, and it is easy to check that $\delta^{\mathcal{U}}_{\rm nbhd} = \delta^{\mathcal{U}}_{\rm Sus}(\tau(p))$. □

Observation 4.2. *Let $\mathcal{X}$ be a Polish space. For any function $f\colon \mathcal{X} \to \mathbb{R}$, the* **m***-degree of $f \circ \delta\colon \omega^\omega \to \mathbb{R}$ is independent of the choice of a total admissible representation δ.*

Proof. Let δ and γ be total admissible representations of $\mathcal{X}$. By admissibility, $\gamma = \delta\circ\theta$ for some continuous function θ. Thus, ${\rm Lev}_{f\circ\gamma}(\langle p, q\rangle^\frown X) = {\rm Lev}_f(\langle p, q\rangle^\frown \gamma(X)) = {\rm Lev}_f(\langle p, q\rangle^\frown \delta \circ \theta(X)) = {\rm Lev}_{f\circ\delta}(\langle p, q\rangle^\frown \theta(X))$. Hence, ${\rm Lev}_{f\circ\gamma} \leq_{mw} {\rm Lev}_{f\circ\delta}$. □

Hence, given a Polish space $\mathcal{X}$, by Lemma 4.1, one can fix a total open continuous admissible representation δ of $\mathcal{X}$. Then, the **m**-degree of $f\colon \mathcal{X} \to \mathbb{R}$ is defined as the **m**-degree of $f \circ \delta\colon \omega^\omega \to \mathbb{R}$.

Observation 4.3. *The* **m***-degrees of real-valued functions on Polish spaces form a semi-well-order of length Θ.*

4.1. Baire one functions.

We now consider the Cantor-Bendixson-type derivation procedure for Baire-one functions, which is originally introduced by Bourgain [5] in Banach space theory, and extensively studied in Kechris-Louveau [13].

4.1.1. *Bourgain rank.* Let $\mathcal{X}$ be a Polish space, and consider a function $f\colon \mathcal{X} \to \mathbb{R}$. Then, given $P \subseteq \mathcal{X}$, the $(f; p, q)$-derivative of P is defined as follows.

$$D_{f,p,q}P = P \setminus \bigcup \{J : J \subseteq \{f < q\} \text{ or } J \subseteq \{f > p\}\}$$

where J ranges over open sets in $\mathcal{X}$. Then iterate this procedure: $D^0_{f,p,q}P = P$, $D^{\nu+1}_{f,p,q}P = D_{f,p,q}(D^\nu_{f,p,q}P)$ for $\nu < \omega_1$, and $D^\lambda_{f,p,q}P = \bigcap_{\nu<\lambda} D^\nu_{f,p,q}P$ for a limit ordinal $\lambda < \omega_1$. Let $|x|_{f,p,q}$ be the least α such that $x \notin D^\alpha_{f,p,q}\mathcal{X}$. Let $\alpha(f, p, q)$ be the least α such that $D^\alpha_{f,p,q}\mathcal{X} = \emptyset$. Note that $\alpha(f, p, q) = \sup_{x\in\mathcal{X}} |x|_{f,p,q}$. Then the *Bourgain rank of* f is defined by $\alpha(f) = \sup_{p<q} \alpha(f, p, q)$.

By definition, $|x|_{f,p,q}$ is always a successor ordinal. If the domain is not compact, the rank $\alpha(f, p, q)$ can be a limit ordinal. In this case, there is no x such that $|x|_{f,p,q} = \alpha(f, p, q)$. Hence, if the domain is not compact, $\alpha(f) = \alpha(f, p, q)$ can happen even if $\alpha(f)$ is limit.

4.1.2. *Sided-conditions.* The following is a key notion for characterizing the Bourgain rank.

Definition 4.4 (Day-Downey-Westrick [8])**.** Let $f\colon \mathcal{X} \to \mathbb{R}$ be a Baire-one function.

(1) f is *two-sided* if there are rationals $p < q$ such that $\alpha(f, p, q) = \alpha(f)$ and that for any $\nu < \alpha(f)$, $f(x) \leq p < q \leq f(y)$ for some $x, y \in D^\nu_{f,p,q}\mathcal{X}$.
(2) If f is not two-sided, it is called *one-sided*.
(3) f is *left-sided* if for any rationals $p < q$ with $\alpha(f, p, q) = \alpha(f)$, there is $\nu < \alpha(f)$ such that $D^\nu_{f,p,q}\mathcal{X} \subseteq \{f < p\}$.
(4) f is *right-sided* if for any rationals $p < q$ with $\alpha(f, p, q) = \alpha(f)$, there is $\nu < \alpha(f)$ such that $D^\nu_{f,p,q}\mathcal{X} \subseteq \{f > q\}$.
(5) f is *irreducible* if there are rationals $p < q$ such that $\alpha(f) = \alpha(f, p, q)$. Otherwise, f is called *reducible*.

Observation 4.5. *Let $f\colon \mathcal{X} \to \mathbb{R}$ be a Baire-one function.*

(1) *If $\alpha(f)$ is a successor ordinal, then f is irreducible.*
(2) *If $\alpha(f)$ is a limit ordinal, and f is irreducible, then f is two-sided.*
(3) *f is both left- and right-sided if and only if f is reducible.*

Proof. (1) Since $\alpha(f) = \sup_{p<q} \alpha(f; p, q)$, if $\alpha(f)$, then there must exist $p < q$ such that $\alpha(f) = \alpha(f; p, q)$.

(2) Let $p < q$ be rationals such that $\alpha(f, p, q) = \alpha(f)$. For any $\nu < \alpha(f)$ if $\nu < \xi < \alpha(f)$, then there exists x such that $|x|_{f,p,q} = \xi$. Therefore, for any open neighborhood J of x, there are $y, z \in J \cap D^\nu_{f,p,q}\mathcal{X}$ such that $f(y) \leq p < q \leq f(z)$.

(3) By definition, if there is no $p < q$ such that $\alpha(f) = \alpha(f, p, q)$, then f is one-sided, and moreover, left- and right-sided. For the converse, let f be both left- and right-sided. Suppose for the sake of contradiction that f is irreducible. Since f is one-sided and irreducible, by (2), $\alpha(f)$ must be a successor ordinal, $\alpha(f) = \nu + 1$ say. Then $D^\nu_{f,p,q}\mathcal{X} \neq \emptyset$. Since f is both left- and right-sided, we also have $D^\nu_{f,p,q}\mathcal{X} \subseteq \{p < f < q\}$. Therefore, $x \in D^\nu_{f,p,q}\mathcal{X}$ implies $p < f(x) < q$, and then let r be such that $f(x) \leq r < q$. Then, we have $\nu + 1 = \alpha(f) = \alpha(f, p, q) \leq \alpha(f, r, q)$ and

$$x \in D^\nu_{f,p,q}\mathcal{X} \subseteq D^\nu_{f,r,q}\mathcal{X} \not\subseteq \{f > r\}.$$

Therefore, f is not right-sided. $\square$

Define the *type of* f as follows: If f is two-sided, define $\mathrm{type}(f) = \mathtt{t}$. If f is one-sided, but neither left- nor right-sided, define $\mathrm{type}(f) = \mathtt{o}$. If f is left-sided, but not right-sided, define $\mathrm{type}(f) = \mathtt{l}$. If f is right-sided, but not left-sided, define $\mathrm{type}(f) = \mathtt{r}$. If f is both left- and right-sided (that is, f is reducible by Observation 4.5 (3)), define $\mathrm{type}(f) = \mathtt{f}$. We define the order on these types as follows:

$$\mathtt{f} < \mathtt{l}, \mathtt{r} < \mathtt{o} < \mathtt{t}.$$

4.1.3. *Open continuous surjection.*

Lemma 4.6. *Let $\delta\colon \mathcal{Z} \to \mathcal{X}$ be an open continuous surjection, and $f\colon \mathcal{X} \to \mathbb{R}$ be a Baire-one function. Then, $|x|_{f\circ\delta,p,q} = |\delta(x)|_{f,p,q}$. In particular, $\alpha(f) = \alpha(f \circ \delta)$ and $\mathrm{type}(f) = \mathrm{type}(f \circ \delta)$.*

Proof. We claim that $\delta^{-1}[D_{f,p,q}P] = D_{f\circ\delta,p,q}\delta^{-1}[P]$ for any $P \subseteq \mathcal{X}$. If $z \in \delta^{-1}[D_{f,p,q}P]$, then there is an open neighborhood J of $\delta(z)$ in P such that $f(x) \le p$ and $f(y) \ge q$ for some $x, y \in J \cap P$. Then, $\delta^{-1}[J]$ is an open neighborhood of z in $\delta^{-1}[P]$, and since δ is surjective, there are $u, v \in \delta^{-1}[J] \cap \delta^{-1}[P]$ such that $\delta(u) = x$ and $\delta(v) = y$. Since $f \circ \delta(u) \le p$ and $f \circ \delta(v) \ge q$, we have $z \in D_{f\circ\delta,p,q}\delta^{-1}[P]$. Conversely, if $z \in D_{f\circ\delta,p,q}\delta^{-1}[P]$, then there is an open neighborhood J of z in $\delta^{-1}[P]$ such that $f \circ \delta(x) \le p$ and $f \circ \delta(y) \ge q$ for some $x, y \in J \cap \delta^{-1}[P]$. Since δ is an open map, $\delta[J]$ is an open neighborhood of $\delta(z)$, and we also have $\delta(x), \delta(y) \in \delta[J] \cap P$. Consequently, $\delta(z) \in D_{f,p,q}P$, and thus $z \in \delta^{-1}[D_{f,p,q}P]$. This verifies the claim.

We will inductively show that $\delta^{-1}[D^{\xi}_{f,p,q}\mathcal{X}] = D^{\xi}_{f\circ\delta,p,q}\mathcal{Z}$ for any ξ. It is obvious for a limit step. For a successor ordinal $\xi + 1$,

$$D^{\xi+1}_{f\circ\delta,p,q}\mathcal{Z} = D_{f\circ\delta,p,q}\delta^{-1}[D^{\xi}_{f,p,q}\mathcal{X}] = \delta^{-1}[D_{f,p,q}D^{\xi}_{f,p,q}\mathcal{X}] = \delta^{-1}[D^{\xi+1}_{f,p,q}\mathcal{X}].$$

The first equality follows from the induction hypothesis, and the second equality follows from the previous claim with $P = D^{\xi}_{f,p,q}\mathcal{X}$. Consequently, $x \in D^{\xi}_{f\circ\delta,p,q}\mathcal{Z}$ iff $\delta(x) \in D^{\xi}_{f,p,q}\mathcal{X}$, and hence $|x|_{f\circ\delta,p,q} = |\delta(x)|_{f,p,q}$ as desired. □

4.1.4. *The Day-Downey-Westrick Theorem.*

Theorem 4.7. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Polish spaces and let $f\colon \mathcal{X} \to \mathbb{R}$ and $g\colon \mathcal{Y} \to \mathbb{R}$ be Baire-one functions. Then, $f \le_{\mathbf{m}} g$ if and only if either $\alpha(f) < \alpha(g)$ holds or both $\alpha(f) = \alpha(g)$ and $\mathrm{type}(f) \le \mathrm{type}(g)$ hold.*

Proof. By Observation 4.2 and Lemma 4.6, one can assume that $\mathcal{X} = \mathcal{Y} = \omega^\omega$ by considering $f \circ \delta_{\mathcal{X}}$ and $g \circ \delta_{\mathcal{Y}}$ instead of f and y, where $\delta_{\mathcal{X}}$ and $\delta_{\mathcal{Y}}$ are total open admissible representations of $\mathcal{X}$ and $\mathcal{Y}$ ensured by Lemma 4.1, respectively.

For the left-to-right direction, it is not hard to check that a straightforward modification of the argument in Day-Downey-Westrick [8] gives us the desired condition.

For the right-to-left direction, assume that either $\alpha(f) < \alpha(g)$ holds or both $\alpha(f) = \alpha(g)$ and $\mathrm{type}(f) \le \mathrm{type}(g)$ hold. Given $p < q$ we need to find $r < s$ satisfying the following property: For any x, there is y such that

$$|x|_{f,p,q} \le |y|_{g,r,s}, \text{ and } \mathrm{Lev}_f(\langle p, q\rangle^\frown x) \le_{\mathbf{2}_\perp} \mathrm{Lev}_g(\langle r, s\rangle^\frown y). \tag{1}$$

If $\alpha(f; p, q) < \alpha(g)$, then there are $r < s$ such that $\alpha(f; p, q) < \alpha(g; r, s)$. Choose such $r < s$. In particular, if either $\alpha(f) < \alpha(g)$ or $\mathrm{type}(f) = \mathtt{f}$, then we get (1). Now assume $\alpha(f) = \alpha(g)$ and $\mathrm{type}(f) \ne \mathtt{f}$. In this case, there are $p < q$ such that $\alpha(f; p, q) = \alpha(f) = \alpha(g)$.

The rest of the proof is similar to the argument as in Day-Downey-Westrick [8]. □

Note that Lemma 4.6 only requires δ to be an open continuous surjection (that is, δ is not necessarily admissible). Hence, Theorem 4.7 implies that the **m**-degree of a Baire-one function $f \circ \delta$ is independent of the choice of an open continuous surjection δ.

One of the most important conclusions of Theorems 1.2 and 4.7 is that one can characterize the Bourgain rank in terms of the descriptive complexity as follows:

Corollary 4.8. *Let $\delta\colon \omega^\omega \to \mathcal{X}$ be an open continuous surjection, and $f\colon \mathcal{X} \to \mathbb{R}$ be a Baire-one function. Then, for any $\xi < \omega_1$,*

(1) $\mathrm{Lev}_{f\circ\delta}$ *is either $\Sigma^\diamondsuit_\xi$- or $\Pi^\diamondsuit_\xi$-complete if and only if $\alpha(f) = \xi + 1$ and either f is left- or right-sided.*
(2) $\mathrm{Lev}_{f\circ\delta}$ *is $\Delta^\diamondsuit_\xi$-complete and m-σ-ji if and only if $\alpha(f) = \xi$ and f is two-sided.*
(3) $\mathrm{Lev}_{f\circ\delta}$ *is $\Delta^\diamondsuit_\xi$-complete and m-σ-jr if and only if $\alpha(f) = \xi$ and either* $\mathrm{type}(f) = \mathtt{o}$ *(if ξ is successor) or* $\mathrm{type}(f) = \mathtt{f}$ *(if ξ is limit).*

Proof. By Lemma 4.6, we can assume that $\mathcal{X} = \omega^\omega$ and $\delta = \mathrm{id}$. By induction. If $\alpha(f)$ is successor, then $\mathrm{type}(f) \in \{\mathtt{l}, \mathtt{r}, \mathtt{o}, \mathtt{t}\}$ by Observation 4.5 (1). If $\alpha(f)$ is limit, then $\mathrm{type}(f) \in \{\mathtt{f}, \mathtt{t}\}$ by Observation 4.5 (2) and (3). Moreover, it is easy to see that every such type is realized by some function whenever $\alpha(f) > 1$. One can see that if $\alpha(f) = 1$ then $\mathrm{type}(f) \in \{\mathtt{o}, \mathtt{t}\}$, that is, either f is constant or f is nonconstant and continuous.

Let ξ be a limit ordinal. If $\xi = 0$, put $\mathtt{c} = \mathtt{o}$; otherwise put $\mathtt{c} = \mathtt{f}$. By Theorem 4.7, one can easily see that for any $n \in \omega$,

$$\begin{aligned}
\mathrm{rank}_{\mathbf{m}}(f) = \xi + 3n &\iff \alpha(f) = 1 + \xi + n \text{ and } \mathrm{type}(f) = \mathtt{c}\\
\mathrm{rank}_{\mathbf{m}}(f) = \xi + 3n + 1 &\iff \alpha(f) = 1 + \xi + n \text{ and } \mathrm{type}(f) = \mathtt{t}\\
\mathrm{rank}_{\mathbf{m}}(f) = \xi + 3n + 2 &\iff \alpha(f) = 1 + \xi + n + 1 \text{ and } \mathrm{type}(f) \in \{\mathtt{l}, \mathtt{r}\},
\end{aligned}$$

where $\mathrm{rank}_{\mathbf{m}}(f)$ denotes the **m**-rank of f. The proof of Theorems 1.2 and Theorem 3.3 shows that

$$\begin{aligned}
\mathrm{rank}_{\mathbf{m}}(f) = \xi + 3n &\iff \mathrm{Lev}_f \text{ is } \Delta^\diamondsuit_{1+\xi+n}\text{-complete and } m\text{-jr}\\
\mathrm{rank}_{\mathbf{m}}(f) = \xi + 3n + 1 &\iff \mathrm{Lev}_f \text{ is } \Delta^\diamondsuit_{1+\xi+n}\text{-complete and } m\text{-ji}\\
\mathrm{rank}_{\mathbf{m}}(f) = \xi + 3n + 2 &\iff \mathrm{Lev}_f \text{ is } \Sigma^\diamondsuit_{1+\xi+n}\text{- or } \Pi^\diamondsuit_{1+\xi+n}\text{-complete.}
\end{aligned}$$

This concludes the proof. □

4.1.5. *Separation rank.* We characterize the Bourgain rank in the context of the Hausdorff difference hierarchy by modifying the α_1-rank introduced by Elekes-Kiss-Vidnyánszky [11]. Recall from Example 1.4 that, for $\Gamma \in \{\Sigma, \Pi, \Delta\}$ and $\xi < \omega_1$, the class Γ_ξ corresponds to the ξ^{th} level of the Hausdorff difference hierarchy. For a Baire-one function $f\colon \mathcal{X} \to \mathbb{R}$ and rationals $p < q$, let $\alpha_1^{\mathrm{sep}}(f; p, q)$ be the least ordinal α such that a Δ_α set separates $\{f \le p\}$ from $\{f \ge q\}$. Then, we define

$$\alpha_1^{\mathrm{sep}}(f) = \sup_{p<q} \alpha_1^{\mathrm{sep}}(f; p, q).$$

By using a variant of the Vaught transform, one can characterize α-rank in the context of m-Wadge reducibility:

Lemma 4.9. *Let $\delta\colon \omega^\omega \to \mathcal{X}$ be a total open admissible representation of a Polish space $\mathcal{X}$, and $f\colon \mathcal{X} \to \mathbb{R}$ be a Baire-one function. Then, $\alpha_1^{\rm sep}(f) \leq \xi$ iff $\mathrm{Lev}_{f\circ\delta} \leq_{mw} E_\xi$, where E_ξ is an m-σ-ji Δ_ξ-complete subset of ω^ω.*

Proof. Given $\xi < \omega_1$, assume that a Δ_ξ set D separates $\{f \leq p\}$ from $\{f \geq q\}$. Then, $\delta^{-1}[D]$ is Δ_ξ, and clearly separates $\{f\circ\delta \leq p\}$ from $\{f\circ\delta \geq q\}$. Conversely, assume that a Δ_ξ set D separates $\{f\circ\delta \leq p\}$ from $\{f\circ\delta \geq q\}$. Then, consider the following:

$$\delta^*[D] = \big\{x \in \mathcal{X} : D \cap \delta^{-1}\{x\} \text{ is not meager in } \delta^{-1}\{x\}\big\}.$$

Then, $\delta^*[D]$ is Δ_ξ (see de Brecht [9, Theorem 68]), and separates $\{f \leq p\}$ from $\{f \geq q\}$: If $x \in \{f \leq p\}$, i.e., $f(x) \leq p$, then, for any $z \in \delta^{-1}\{x\}$ we have $f\circ\delta(z) \leq p$, so $z \in D$. This implies $x \in \delta^*[D]$. Similarly, if $x \in \{f \geq q\}$, i.e., $f(x) \geq q$, then, for any $z \in \delta^{-1}\{x\}$ we have $f\circ\delta(z) \geq q$, so $z \notin D$. In this case, $D \cap \delta^{-1}\{x\}$ is empty, and therefore, $x \notin \delta^*[D]$.

Now, it is clear that a Δ_ξ set separates $\{f\circ\delta \leq p\}$ and $\{f\circ\delta \geq q\}$ for any p, q if and only if $\mathrm{Lev}_{f\circ\delta}$ is $\mathbf{2}_\perp$-Wadge reducible to a Δ_ξ-complete set, and it is equivalent to saying that Lev_f is $\mathbf{2}_\perp$-m-Wadge reducible to an m-σ-ji Δ_ξ-complete set. □

Proposition 4.10. $\alpha(f) = \alpha_1^{\rm sep}(f)$.

Proof. Let $\delta\colon \omega^\omega \to \mathcal{X}$ be an open continuous surjection. By Corollary 4.8, if ξ is successer, $\xi = \eta + 1$ say, $\alpha(f) = \xi$ iff $\mathrm{Lev}_{f\circ\delta}$ is Γ-complete for some $\Gamma \in \{\Sigma_\eta, \Pi_\eta, \Delta^{jr}_{\eta+1}, \Delta_{\eta+1}\}$. If ξ is limit, $\alpha(f) = \xi$ iff $\mathrm{Lev}_{f\circ\delta}$ is either Δ^{jr}_ξ- or Δ_ξ-complete. In any case, $\alpha(f) \leq \xi$ iff $\mathrm{Lev}_{f\circ\delta} \leq_{mw} E_\xi$. By Lemma 4.9, the latter is equivalent to saying that $\alpha_1^{\rm sep}(f) \leq \xi$. Consequently, $\alpha(f) \leq \xi$ iff $\alpha_1^{\rm sep}(f) \leq \xi$. □

4.2. **Higher Baire ranks.** For a function $\theta : \mathcal{X} \to \mathcal{Y}$ we write $\theta^{-1}\mathbf{\Sigma}^0_\xi \subseteq \mathbf{\Sigma}^0_\xi$ if the preimage of a $\mathbf{\Sigma}^0_\xi$ set under θ is $\mathbf{\Sigma}^0_\xi$. A function satisfying the condition $\theta^{-1}\mathbf{\Sigma}^0_{1+\xi} \subseteq \mathbf{\Sigma}^0_{1+\xi}$ is also known as a *ξ-th level Borel function.* There are several works on analyzing subsets of ω^ω using ξ-th level Borel functions instead of a continuous function (see Motto Ros [20, 21]). For $\mathcal{Q}$-valued functions $\mathcal{A}, \mathcal{B} : \omega^\omega \to \mathcal{Q}$, we say that $\mathcal{A}$ is $\mathbf{\Delta}^0_\xi$-Wadge reducible to $\mathcal{B}$ if there is a function θ with $\theta^{-1}\mathbf{\Sigma}^0_\xi \subseteq \mathbf{\Sigma}^0_\xi$ such that $\mathcal{A}(X) \leq_\mathcal{Q} \mathcal{B}\circ\theta(X)$ for any $X \in \omega^\omega$. This notion has also been studied by Kihara-Selivanov [17].

Then we introduce the $\mathbf{\Delta}^0_\xi$-version of many-one reducibility as follows: We say that $f\colon \omega^\omega \to \mathcal{Q}$ is *$\mathbf{\Delta}^0_\xi$-many-one reducible to* $g\colon \omega^\omega \to \mathcal{Q}$ (written $f \leq^{\Delta^0_\xi}_{\mathbf{m}} g$) if for any $p < q$ there exist $r < s$ such that $(\{f \leq p\}, \{f \geq q\})$ is $\mathbf{\Delta}^0_\xi$-reducible to $(\{g \leq r\}, \{g \geq s\})$. As before, if the domains of f and g are Polish spaces $\mathcal{X}$ and $\mathcal{Y}$ with admissible representations δ_X and δ_Y, respectively, then define $f \leq^{\Delta^0_\xi}_{\mathbf{m}} g$ if $(\{f\circ\delta_X \leq p\}, \{f\circ\delta_X \geq q\})$ is $\mathbf{\Delta}^0_\xi$-reducible to $(\{g\circ\delta_Y \leq r\}, \{g\circ\delta_Y \geq s\})$.

It is not hard to describe the first ω_1 ranks in the $\mathbf{\Delta}^0_\xi$-**m**-degrees: Andretta-Martin [3, Theorems 23 and 24] showed that the first ω_1 initial segment of the Borel-Wadge degrees is equal to the difference hierarchy over $\mathbf{\Pi}^1_1$. Then Motto Ros [20, Section 4.6] pointed out that a straightforward modification of their argument shows an analogous result for the $\mathbf{\Delta}^0_\xi$-Wadge degrees, that is, the first ω_1 initial segment of the $\mathbf{\Delta}^0_\xi$-Wadge degrees is equal to the difference hierarchy over $\mathbf{\Sigma}^0_\xi$. To be precise, let η-$\mathbf{\Delta}^0_\xi$ be the delta-class in the η^{th} level of the difference hierarchy over

$\mathbf{\Sigma}^0_\xi$. Then, η-$\mathbf{\Delta}^0_\xi$ is exactly the η^{th} selfdual pointclass in the $\mathbf{\Delta}^0_\xi$-Wadge degrees. For more detainls on $\mathbf{\Delta}^0_\xi$-degrees for $\mathcal{Q}$-valued functions, see also [17].

Consequently, the $\mathbf{\Delta}^0_\xi$-**m**-degrees of Baire class ξ functions are closely related to the separation rank introduced by Elekes-Kiss-Vidnyańszky [11, Definition 4.2]: As in the definition of $\alpha_1^{\rm sep}$, for a Baire class ξ function $f\colon \mathcal{X}\to\mathbb{R}$, we define $\alpha_\xi^{\rm sep}(f)$ as the first rank in the difference hierarchy over $\mathbf{\Sigma}^0_\xi$ at which we can separate all upper and lower level sets of f. Note that if a η-$\mathbf{\Delta}^0_\xi$ set D separates $\{f\leq p\}$ from $\{f\geq q\}$, then $\delta^{-1}[D]$ is η-$\mathbf{\Delta}^0_\xi$, and separates $\{f\circ\delta\leq p\}$ from $\{f\circ\delta\geq p\}$. Conversely, if a η-$\mathbf{\Delta}^0_\xi$ set D separates $\{f\circ\delta\leq p\}$ from $\{f\circ\delta\geq p\}$, then its Vaught-like transform $\delta^*[D]$ is η-$\mathbf{\Delta}^0_\xi$ and separates $\{f\leq p\}$ from $\{f\geq q\}$ (as in the proof of Lemma 4.9). Hence, the $\mathbf{\Delta}^0_\xi$-**m**-rank is essentially the same as the separation rank $\alpha_\xi^{\rm sep}$.

Elekes-Kiss-Vidnyańszky [11, Section 5] related this separation rank to the Baire-ξ version of the Bourgain rank. Given a Baire-ξ function $f:(X,\tau)\to\mathbb{R}$, let $T_{f,\xi}$ be the set of all topologies τ' on X refining τ such that $\tau'\subseteq\mathbf{\Sigma}^0_\xi(\tau)$ and f is Baire-one w.r.t. τ':

$$T_{f,\xi}=\{\tau'\supseteq\tau\text{ Polish}:\tau'\subseteq\mathbf{\Sigma}^0_\xi(\tau)\text{ and }f\in\mathcal{B}_1(\tau')\}.$$

Then we define $\alpha^*_\xi(f)$ as the least possible α-rank of the Baire-one function f with respect to some topology $\tau'\in T_{f,\xi}$. For instance, Elekes-Kiss-Vidnyańszky [11, Theorem 5.17] showed the inequality $\alpha^*_\xi(f)\leq\alpha^{\rm sep}_\xi(f)\leq 2\alpha^*_\xi(f)$.

Following the above idea, we consider the "change of topology" version of many-one reducibility $\leq_{\mathbf{m}}$: For Baire class ξ functions $f\colon X\to\mathbb{R}$ and $g\colon Y\to\mathbb{R}$, we write $f\leq_{\mathbf{m}}^{{\rm top}(\xi)} g$ if for any topology $\tau_g\in T_{g,\xi}$, there exists a topology $\tau_f\in T_{f,\xi}$ such that $(f,\tau_f)\leq_{\mathbf{m}}(g,\tau_g)$; that is, for any $p<q$ there are $r<s$ such that $(\{f\leq p\},\{f\geq q\})$ is Wadge reducible to $(\{g\leq r\},\{g\geq s\})$ via some continuous function $\theta\colon(X,\tau_f)\to(Y,\tau_g)$. First, it is easy to see the following:

Lemma 4.11. *If $f\leq_{\mathbf{m}}^{\Delta^0_\xi} g$ then $f\leq_{\mathbf{m}}^{{\rm top}(\xi)} g$.*

Proof. If $f\leq_m^{\mathbf{\Delta}^0_\xi} g$, then there are $\mathbf{\Delta}^0_\xi$-function θ witnessing that $(\{f\leq p\},\{f\geq q\})$ is $\mathbf{\Delta}^0_\xi$-reducible to $(\{g\leq r\},\{g\geq s\})$. Let $\tau_g\in T_{g,\xi}$ be a Polish topology. Since τ_g has a countable basis, and contained in $\mathbf{\Sigma}^0_\alpha(\tau)$, by Kuratowski's theorem (see Kechris [14, Theorem 22.18]), there is a Polish topology $\tau_f\in T_{f,\xi}$ extending τ_g such that $\theta\colon(2^\omega,\tau_f)\to(2^\omega,\tau_g)$ is continuous. See also Elekes et al. [11, Proposition 5.2] and Motto Ros [21, Lemma 5.2]. □

In particular, the $\mathbf{\Delta}^0_\xi$-**m**-rank of a function is greater than or equal to its $\leq_{\mathbf{m}}^{{\rm top}(\xi)}$-rank. Now, by the Day-Downey-Westrick Theorem on non-compact Polish spaces (Theorem 4.7), the relation $(f,\tau_f)\leq_{\mathbf{m}}(g,\tau_g)$ is characterized by Bourgain rank α and some sided-condition. Recall

$$\alpha^*_\xi(f)=\min\{\alpha(f,\tau_f):\tau_f\in T_{f,\xi}\}\leq\alpha^{\rm sep}_\xi(f)$$

as mentioned above. Conversely, take a topology $\tau_f\in T_{f,\xi}$ such that $\alpha(f,\tau_f)=\alpha^*_\xi(f)$. Then, by Proposition 4.10, we have $\alpha(f,\tau_f)=\alpha^{\rm sep}_1(f,\tau_f)$. In the last paragraph of the proof of [11, Theorem 5.17], Elekes et al. has shown that $\alpha^{\rm sep}_\xi(f)\leq\alpha^{\rm sep}_1(f,\tau_f)$. Hence, $\alpha^{\rm sep}_\xi(f)=\alpha^*_\xi(f)$.

As mentioned above, the $\mathbf{\Delta}^0_\xi$-$\mathbf{m}$-rank of a Baire-ξ function is essentially the same as $\alpha^{\rm sep}_\xi(f)$. Consequently, we finally obtain the exact relationship between the $\mathbf{\Delta}^0_\xi$-$\mathbf{m}$-degrees of Baire class ξ functions and their ξ-Bourgain ranks α^*_ξ.

5. T-degrees and the Martin ordering

In this section, we will see a strong connection between the structure of Day-Downey-Westrick's **T**-degrees of real-valued functions [8] and the *Martin ordering on the uniform Turing degree invariant functions.* Then, by combining with Becker's result [4], we will see that the **T**-degrees of real-valued functions form a well-order of type Θ.

5.1. **Reducibility notions.** There are a number of works on Wadge-like classifications of functions on ω^ω. For instance, Carroy [7] adopted *continuous strong Weihrauch reducibility* as a tool to provide a reasonable classification of functions on ω^ω, where for $f,g\colon\omega^\omega\to\omega^\omega$, we say that f is continuously strongly Weihrauch reducible to g (written as $f\leq^c_{sW} g$) if there are continuous functions $\Phi,\Psi\colon\omega^\omega\to\omega^\omega$ such that

$$f=\Phi\circ g\circ\Psi.$$

In general, one can define continuous strong Weihrauch reducibility for other spaces (although the definition is not directly used in this article): If X, Y, Z and W are Polish spaces, by Lemma 4.1, they have total admissible representations δ_X, δ_Y, δ_Z and δ_W, respectively. Then, we say that $f\colon X\to Y$ is continuous strong Weihrauch reducible to $g\colon Z\to W$ if there are continuous functions $\Phi,\Psi\colon\omega^\omega\to\omega^\omega$ such that, if $\delta_X(p)=x$, $\delta_Z(\Psi(p))=z$, and $\delta_W(q)=g(z)$ then $\delta_Y(\Phi(q))=f(x)$. For the details, see also Brattka-Gherardi-Pauly [6].

Subsequently, Day-Downey-Westrick [8] adopted *parallelized continuous strong (p.c.s.) Weihrauch reducibility* as a formalization of topological "Turing reducibility" for real-valued functions. Given a function $h\colon\mathcal{X}\to\mathcal{Y}$, define the *parallelization of* h as the following function $\widehat{h}\colon\mathcal{X}^\omega\to\mathcal{Y}^\omega$:

$$\widehat{h}(\langle x_n:n\in\omega\rangle)=\langle h(x_n):n\in\omega\rangle.$$

A function $h\colon\mathcal{X}\to\mathcal{Y}$ is *parallelizable* if $\widehat{h}$ is continuously strongly Weihrauch reducible to h. We use $\leq^{\widehat{c}}_{sW}$ to denote the p.c.s. Weihrauch reducibility, that is, $f\leq^{\widehat{c}}_{sW} g$ iff $f\leq^c_{sW}\widehat{g}$. In this article, $\mathcal{X}$ and $\mathcal{Y}$ are 2^ω, ω^ω or $\mathbb{R}$. For $f,g\colon\omega^\omega\to\mathbb{R}$, we use only the following property: One can deduce $f\leq^{\widehat{c}}_{sW} g$ whenever there are continuous functions $\Phi:\subseteq\mathbb{R}^\omega\to\mathbb{R}$ and $\Psi\colon\omega\times\omega^\omega\to\omega^\omega$ such that

$$(\forall X\in\omega^\omega)\ f(X)=\Phi\left(\langle g(\Psi(i,X)):i\in\omega\rangle\right).$$

We connect the reducibility notion $\leq^{\widehat{c}}_{sW}$ with the *uniform Martin conjecture* [27, 25]. To explain this, we need to introduce several notions from computability theory. For $X,Y\in 2^\omega$, we say that Y is Turing reducible to X (written $Y\leq_T X$) if there is a partial computable function $\Phi:\subseteq 2^\omega\to 2^\omega$ such that $\Phi(X)=Y$. We write $X\equiv_T Y$ if $X\leq_T Y$ and $Y\leq_T X$. We fix an effective enumeration of all partial computable functions. If Φ in the definition of Turing reducibility is given as the e-th partial computable function, then we say that $Y\leq_T X$ via e.

As a kind of extension of the question that asks a natural solution to Post's problem, in 1960s, Martin conjectured that natural (in a certain sense) Turing degrees are well-ordered, and the successor rank is given by the Turing jump. Normally we require naturalness to be relativizable and Turing degree invariant (see [19]), but we here also require the uniformity on Turing degree invariance. This version is known as the uniform Martin conjecture, which has been solved by [27, 25]. To be precise, the uniform Martin conjecture deals with the following types of functions:

Definition 5.1 ([27, 25, 4])**.** A function $f\colon 2^\omega \to 2^\omega$ is *uniformly Turing degree invariant* (UI) if there is a function $u\colon \omega^2 \to \omega^2$ such that

$$X \equiv_T Y \text{ via } (d,e) \implies f(X) \equiv_T f(Y) \text{ via } u(d,e).$$

A function $f\colon 2^\omega \to 2^\omega$ is *uniformly Turing order preserving* (UOP) if there is a function $u\colon \omega \to \omega$ such that

$$X \leq_T Y \text{ via } e \implies f(X) \leq_T f(Y) \text{ via } u(e).$$

For functions $f, g\colon 2^\omega \to 2^\omega$, we say that f *is Martin reducible to* g (or f *is Turing reducible to* g *a.e.*; written $f \leq_T^\triangledown g$) if

$$(\exists C \in 2^\omega)(\forall X \geq_T C)\ f(X) \leq_T g(X).$$

It is clear that every UOP function is UI. The converse also holds up to the Turing equivalence a.e.

Fact 6 (Becker [4])**.** *Every UI function is Turing equivalent to a UOP function a.e.*

However, this reducibility notion $\leq_T^\triangledown$ is badly behaved for constant functions. In this article, we use the following variant of $\leq_T^\triangledown$:

Definition 5.2. For functions $f, g\colon 2^\omega \to 2^\omega$, we write $f \leq_{\mathbf{T}}^\triangledown g$ if

$$(\exists C \in 2^\omega)(\forall X \geq_T C)\ f(X) \leq_T g(X) \oplus C.$$

A function $f\colon 2^\omega \to 2^\omega$ is *increasing a.e.* if there is $C \in 2^\omega$ such that $f(X) \geq_T X$ for all $X \geq_T C$. A function $f\colon 2^\omega \to 2^\omega$ is *constant a.e.* if there is $C \in 2^\omega$ such that $f(X) \equiv_T C$ for all $X \geq_T C$.

Fact 7 (Slaman-Steel [25])**.** *For a UI function $f\colon 2^\omega \to 2^\omega$, either f is constant a.e. or f is increasing a.e.*

Observation 5.3. *Let $f, g\colon 2^\omega \to 2^\omega$ be UI functions. If g is not constant a.e., then*

$$f \leq_T^\triangledown g \iff f \leq_{\mathbf{T}}^\triangledown g.$$

Proof. Assume that $f \leq_{\mathbf{T}}^\triangledown g$ via C. By Fact 7, g is increasing a.e. Therefore, there is $D \geq_T C$ such that $g(X) \geq_T X$ for all $X \geq_T D$. For such X, $f(X) \leq_T g(X) \oplus C \leq_T g(X)$. Hence, $f \leq_T^\triangledown g$. □

The $\leq_{\mathbf{T}}^\triangledown$-degrees of UOP functions forms a well-order of height Θ, and the successor rank is given by the Turing jump (cf. Steel [27] and Becker [4]).

By UOP we denote the collection of UOP functions, and by $\mathcal{F}$ we denote the collection of real-valued functions on ω^ω. In this section, we will show the following.

Theorem 5.4. *The identity map induces an isomorphism between quotients of* $(\mathrm{UOP}, \leq_{\mathbf{T}}^\triangledown)$ *and* $(\mathcal{F}, \leq_{sW}^{\widehat{c}})$.

As a corollary, by Observation 5.3, the identity map induces an isomorphism between the Martin ordering on the UOP operators which is not constant a.e. and the parallel continuous strong Weihrauch degrees of real-valued non-constant functions. Theorem 5.4 also implies the following.

Theorem 5.5. *The p.c.s. Weihrauch degrees of real-valued functions on ω^ω form a well-order of type Θ. Moreover, if $g\colon 2^\omega \to 2^\omega$ has nonzero p.c.s. Weihrauch rank α, then $\widehat{g}'$ has p.c.s. Weihrauch rank $\alpha+1$, where $h'(x)$ is defined as the Turing jump of $h(x)$.*

In particular, for any parallelizable function g (that is, $g \equiv^c_{sW} \widehat{g}$), if g has continuous strong Weihrauch rank α, then g' has continuous strong Weihrauch rank $\alpha+1$. Note that UOP operators are always parallelizable. What this concludes is that, surprisingly, continuous strong Weihrauch reducibility $\leq^c_{sW}$ gives us a very natural well-ordering of functions beyond UOP functions.

5.2. **Injectivity.** We first show that the identity map gives an embedding of (UOP, $\leq^\nabla_T$) into $(\mathcal{F}, \leq^{\widehat{c}}_{sW})$. We will use the following notion. A *uniformly pointed perfect tree* (u.p.p. tree) is a perfect tree $T \subseteq 2^{<\omega}$ such that $T \leq_T T[X]$ via some index e independent of $X \in 2^\omega$, where we often think of a perfect tree T as a continuous embedding $T[\cdot]\colon 2^\omega \to 2^\omega$, that is, $T[X]$ is the X-th infinite path through T.

Fact 8 (Martin; see [18])**.** *For any countable partition $(P_i)_{i\in\omega}$ of 2^ω, there is $i \in \omega$ such that P_i includes the set of all infinite paths through a u.p.p. tree.*

Lemma 5.6. *Assume that $f, g\colon 2^\omega \to 2^\omega$ are UOP functions. Then,*

$$f \leq^\nabla_{\mathbf{T}} g \iff f \leq^c_{sW} g \iff f \leq^c_{sW} \widehat{g}.$$

Proof. Assume that $f \leq^\nabla_{\mathbf{T}} g$ via C. By Fact 8, there are a u.p.p. tree T and an index e such that for any X, $f(T[X]) \leq_T g(T[X]) \oplus C$ via Φ_e. Note that $\Phi^C_e\colon Z \mapsto \Phi_e(Z \oplus C)$ is continuous. Assume that f is UOP via u. For an index d witnessing $X \leq_T T[X]$, we have $f(X) \leq_T f(T[X])$ via $\Phi_{u(d)}$. Then, we have

$$f(X) = \Phi_{u(d)}(f(T[X])) = \Phi_{u(d)}(\Phi_e(g(T[X]) \oplus C)) = \Phi_{u(d)}(\Phi^C_e(g(T[X]))).$$

This implies that $f = \Phi_{u(d)} \circ \Phi^C_e \circ g \circ T$, and thus, $f \leq^c_{sW} g$ as desired.

Conversely, assume that $f \leq^c_{sW} \widehat{g}$. Then, there are continuous functions Φ, Ψ such that $f(X) = \Phi(\langle g(\Psi(i,X))\rangle_i)$ for all X. Let C be an oracle such that Φ and Ψ are C-computable. If $X \geq_T C$, then $\Psi(i,X)$ is X-computable uniformly in i, that is, there is a computable function p such that $\Psi(i,X) \leq_T X$ via $\Phi_{p(i)}$. Let u witness that g is UOP. Then, if $X \geq_T C$, then we have $g(\Psi(i,X)) \leq_T g(X)$ via $\Phi_{u\circ p(i)}$. Therefore, $\bigoplus_i g(\Psi(i,X)) \leq_T g(X) \oplus u$. Then, for any $X \geq_T C$,

$$f(X) = \Phi\left(\langle g(\Psi(i,X))\rangle_i\right) \leq_T g(X) \oplus u \oplus C.$$

Consequently, we get that $f \leq^\nabla_{\mathbf{T}} g$. □

5.3. **Surjectivity.** To prove Theorem 5.4, it remains to show that every function $f\colon \omega^\omega \to \mathbb{R}$ is $\equiv^{\widehat{c}}_{sW}$-equivalent to a UOP function. Clearly, every constant function is UOP, and any two constant functions are $\equiv^\nabla_{\mathbf{T}}$-equivalent, and $\equiv^{\widehat{c}}_{sW}$-equivalent. We hereafter assume that $f\colon \omega^\omega \to \mathbb{R}$ is not constant.

5.3.1. *Continuous functions.* By Theorem 3.3, the Wadge degrees of $\mathbf{2}_\perp$-valued functions of the form Lev_f can be identified with the Wadge degrees of subsets of ω^ω. For **m**-degrees, recall that each selfdual degree splits into two degrees, but this spliting happens only for **m**-degrees. Actually, one can see that parallel continuous strong Weihrauch reducibility for non-constant functions is coarser than Wadge reducibility as follows.

Lemma 5.7. *Assume that $g\colon \omega^\omega \to \mathbb{R}$ is not constant. For any $f\colon \omega^\omega \to \mathbb{R}$, if $\mathrm{Lev}_f \leq_w \mathrm{Lev}_g$, then $f \leq^c_{sW} \widehat{g}$.*

To show Lemma 5.7, we need the following sublemma. We write $f \leq^c_W g$ if $f \leq^c_{sW} (\mathrm{id}, g)$, where given functions f, h, define $(f,h)\colon x \mapsto (f(x), h(x))$.

Lemma 5.8. *Let $f, g\colon \omega^\omega \to \mathbb{R}$ be functions. Assume that g is not constant. Then, $f \leq^c_{sW} \widehat{g}$ if and only if $f \leq^c_W \widehat{g}$.*

Proof. It suffices to show that $(\mathrm{id}, \widehat{g}) \leq^c_{sW} \widehat{g}$. Since g is not constant, there are $Y_0, Y_1 \in \omega^\omega$ such that $g(Y_0) \neq g(Y_1)$. Let $U_0, U_1 \subseteq \mathbb{R}$ be disjoint rational open intervals such that $g(Y_i) \subseteq U_i$ for each $i < 2$. Given $X \in 2^\omega$, define $\rho(i, X) = Y_{X(i)}$. Then $\rho\colon \omega \times 2^\omega \to \omega^\omega$ is continuous. We also define $\tau(\bigoplus_i Z_i)(n) = i$ if $Z_n \in U_i$. Obviously, $\tau :\subseteq \mathbb{R} \to 2^\omega$ is partial continuous. Then,

$$X = \tau\left(\bigoplus_i g(Y_{X(i)})\right) = \tau\left(\bigoplus_i g \circ \rho(i, X)\right).$$

Thus, the pair $\langle \rho, \tau \rangle$ witnesses that $\mathrm{id} \leq^c_{sW} \widehat{g}$. Consequently, $(\mathrm{id}, \widehat{g}) \leq^c_{sW} (\widehat{g}, \widehat{g}) \equiv_{sW} \widehat{g}$. □

Note that the outer reduction τ in the proof of Lemma 5.8 is clearly computable. Later we will use this observation to show Theorem 5.5.

Proof (Lemma 5.7). Assume that $\mathrm{Lev}_f \leq_w \mathrm{Lev}_g$. Then, there are continuous functions $r, s\colon \mathbb{Q}^2 \times \omega^\omega \to \mathbb{Q}$ and $\psi\colon \mathbb{Q}^2 \times \omega^\omega \to \omega^\omega$ such that for any $p < q$ and $X \in \omega^\omega$, we have $r(p,q,X) < s(p,q,X)$ and

$$f(X) \leq p \implies g(\psi(p,q,X)) \leq r(p,q,X),$$
$$f(X) \geq q \implies g(\psi(p,q,X)) \geq s(p,q,X).$$

Here, $[\mathbb{Q}]^2$ is identified with ω, so endowed with the discrete topology. Then, given $X \in \omega^\omega$ and an $[\mathbb{Q}^2]$-indexed sequence $y = (y_{pq})_{p<q}$ of reals, consider $L_{X,y} = \{p \in \mathbb{Q} : y_{pq} > r(p,q,X)\}$ and $R_{X,y} = \{q \in \mathbb{Q} : y_{pq} < s(p,q,X)\}$. If $\sup L_{X,y} = \inf R_{X,y}$ then define $\Phi(X,y) = \sup L_{X,y}$. If there are rationals $p < q$ such that $p, q \notin L_{X,y} \cup R_{X,y}$ then $s(p,q,X) \leq y_{pq} \leq r(p,q,X)$, which is impossible. Thus, if $L_{X,y}$ and $R_{X,y}$ are disjoint, then $\Phi(X,y)$ is defined. One can easily check that the partial function $\Phi :\subseteq \omega^\omega \times \mathbb{R}^\omega \to \mathbb{R}$ is continuous. If $y_{pq} = g(\psi(p,q,X))$ then, by the above property, we have $L_{X,y} \subseteq \{p \in \mathbb{Q} : f(X) > p\}$ and $R_{X,y} \subseteq \{q \in \mathbb{Q} : f(X) < q\}$, so we get $\Phi(X,y) = \sup L_{X,y} = \inf R_{X,y} = f(X)$. Consequently, $\langle \psi, \Phi \rangle$ witnesses that $f \leq^c_W \widehat{g}$, and thus $f \leq^c_{sW} \widehat{g}$ by Lemma 5.8. □

By Lemma 5.7, the $\equiv^{\widehat{c}}_{sW}$-degrees of continuous functions consist only of two degrees, that is, if f and g are continuous, but not constant, then $f \equiv^{\widehat{c}}_{sW} g$. In particular, $f \equiv^{\widehat{c}}_{sW} \mathrm{id}$, where note that the identity map id is clearly a UOP function. Hence, it remains to consider the case that f is discontinuous.

5.3.2. *Nonselfdual functions.* Assume that Lev_f is nonselfdual. As in Becker [4], we first assign a UOP function to each nonselfdual Wadge degree. Following Becker [4, Definiton 2.2], we say that a pointclass Γ is *reasonable* if Γ is ω-parametrized, contains all computable sets, and has the substitution property. We do not explain the details of this definition, since only the facts described below are used, rather than the definition itself.

For a reasonable pointclass Γ, a Γ-*indexing* is a Γ set $U \subseteq \omega^2 \times 2^\omega$ such that for any Γ set $V \subseteq \omega \times 2^\omega$ such that there exists e such that, for any $n \in \omega$ and $X \in 2^\omega$, $U(e,n,X)$ if and only if $V(n,X)$. In particular, a Γ-indexing is Γ-complete. As in [4, Definiton 2.4], given a Γ-indexing U, we define $J^U_\Gamma : 2^\omega \to 2^\omega$ as follows:

$$J^U_\Gamma(X) = \{\langle m,n\rangle : U(m,n,X)\}.$$

Fact 9 (Becker [4, Lemmas 2.5 and 2.6])**.** *For any reasonable pointclass Γ and its indexing U, J^U_Γ is a UOP function which is increasing a.e. Moreover, the $\equiv^\nabla_T$-degree of J^U_Γ is independent of the choice of U.*

By Fact 9 and Lemma 5.6, there is no harm if we use the symbol J_Γ to denote J^U_Γ. As in Section 2.1, given a pointclass Γ, we define $\Gamma^\diamondsuit = \{\mathcal{A} : \omega^\omega \to \mathbf{2}_\perp \mid (\exists S \in \Gamma)\ \mathcal{A} \leq_w S\}$.

Lemma 5.9. *For any reasonable pointclass Γ and function $f\colon \omega^\omega \to \mathbb{R}$, if Lev_f is $\Gamma^\diamondsuit$-complete, then $f \equiv^{\widehat{c}}_{sW} J_\Gamma$.*

Proof. Let U be a Γ-indexing, which is, in particular, Γ-complete. Since $\mathrm{Lev}_f \in \Gamma^\diamondsuit$, there is a continuous function θ such that $\mathrm{Lev}_f(\langle p,q\rangle^\frown X) \leq_{\mathbf{2}_\perp} U \circ \theta(p,q,X)$. Then $\theta(p,q,X)$ is of the form $(\tau_{pq}(X), \psi_{pq}(X))$, where $\tau_{pq}(X) \in \omega^2$ and $\psi_{pq}(X) \in \omega^\omega$. Thus, $\mathrm{Lev}_f(\langle p,q\rangle^\frown X) \leq_{\mathbf{2}_\perp} J^U_\Gamma(\psi_{pq}(X))(\tau_{pq}(X))$. Then, given $X \in \omega^\omega$ and a $[\mathbb{Q}]^2$-sequence $(y_{pq})_{p<q}$ of elements in 2^ω, as in the proof of Lemma 5.7, consider $L_{X,y} = \{p \in \mathbb{Q} : y_{pq}(\tau_{pq}(X)) = 0\}$ and $R_{X,y} = \{q \in \mathbb{Q} : y_{pq}(\tau_{pq}(X)) = 1\}$. If $\sup L_{X,y} = \inf R_{X,y}$ then define $\Phi(X,y) = \sup L_{X,y}$. Then the partial function $\Phi :\subseteq \omega^\omega \times (2^\omega)^\omega \to \mathbb{R}$ is continuous. If $y_{pq} = J^U_\Gamma(\psi_{pq}(X))$ then, as in the proof of Lemma 5.7, we get $\Phi(X,y) = \sup L_{X,y} = \inf R_{X,y} = f(X)$. Thus, $\langle \psi, \Phi\rangle$ witnesses that $f \leq^c_W \widehat{J_\Gamma}$. Consequently, $f \leq^c_{sW} \widehat{J_\Gamma}$ by Lemma 5.8 since J_Γ is not constant by Fact 9.

Conversely, since $U \in \Gamma$ and Lev_f is $\Gamma^\diamondsuit$-complete, there are continuous functions $r_{mn}, s_{mn}, \psi_{mn}$ such that $U(m,n,X) = \mathrm{Lev}_f(\langle r_{mn}(X), s_{mn}(X)\rangle^\frown \psi_{mn}(X))$. Thus, $f(\psi_{mn}(X)) < s_{mn}(X)$ then $J_\Gamma(X)(m,n) = 0$; and $f(\psi_{mn}(X)) > r_{mn}(X)$ then $J_\Gamma(X)(m,n) = 1$. Now, by the similar argument as above, one can show that $J_\Gamma \leq^c_{sW} \widehat{f}$. □

Becker [4, Lemma 3.4] showed that for every nonzero ordinal $\alpha < \Theta$, there are reasonable pointclasses Σ and Π such that $\mathbf{\Sigma} = \Sigma_\alpha$ and $\mathbf{\Pi} = \Pi_\alpha$. Consequently, if Lev_f is nonselfdual, then there is a UOP function g (i.e., $g = J_\Sigma$ or $g = J_\Pi$) such that $f \equiv^{\widehat{c}}_{sW} g$.

5.3.3. *Selfdual functions.* It remains to consider the case that Lev_f is selfdual, and f is discontinuous. By discontinuity of f, we have $\mathrm{Lev}_f \not\in \Delta^\diamondsuit_1$. Generally, the following lemma states that we do not need to deal with a selfdual Wadge degree of successor rank.

Lemma 5.10. *Assume $\alpha > 0$. For any $f, g\colon \omega^\omega \to \mathbb{R}$, if $\mathrm{Lev}_f \in \Delta^\diamondsuit_{\alpha+1}$ and if Lev_g is $\Sigma^\diamondsuit_\alpha$-complete, then $f \leq^c_{sW} \widehat{g}$.*

Proof. Given $g\colon \omega^\omega \to \mathbb{R}$, define $-g$ by $(-g)(X) = -g(X)$. Note that Lev_{-g} is $\Pi^\diamondsuit_\alpha$-complete whenever Lev_g is $\Sigma^\diamondsuit_\alpha$-complete. Define h by $h(0X) = g(X)$ and $h(1X) = -g(X)$. Then, Lev_h is $\Delta^\diamondsuit_{\alpha+1}$-complete, and therefore, if $\mathrm{Lev}_f \in \Delta^\diamondsuit_{\alpha+1}$, then $\mathrm{Lev}_f \leq_w \mathrm{Lev}_h$. Thus, by Lemma 5.7, we have $f \leq^c_{sW} \widehat{h}$. Therefore, it suffices to show that $h \leq^c_{sW} \widehat{g}$. Since $\alpha > 0$, g cannot be constant, that is, there are Z_0, Z_1 such that $g(Z_0) \neq g(Z_1)$. Let $U_0, U_1 \subseteq \mathbb{R}$ be disjoint open sets such that $g(Z_i) \subseteq U_i$ for each $i < 2$. Define $\Psi(0iX) = Z_i$, and $\Psi(1iX) = X$. Then define $\Phi(z \oplus y) = y$ if $z \in U_0$, and $\Phi(z \oplus y) = -y$ if we find that $z \in U_1$. Obviously, $\Psi :\subseteq \omega^\omega \to \omega^\omega$ and $\Phi :\subseteq \mathbb{R}^2 \to \mathbb{R}$ are continuous. Then, we get

$$h(iX) = \Phi(g(\Psi(0iX)) \oplus g(\Psi(1iX))).$$

Consequently, $h \leq^c_{sW} \widehat{g}$ as desired. □

If α is a limit ordinal of countable cofinality, there is a sequence $\beta_n < \alpha$ such that $\alpha = \sup_n \beta_n$. Let J_{β_n} be a UOP function corresponding to the reasonable pointclass Σ_{β_n}. Then, as in Becker [4], define J_α as follows:

$$J_\alpha(X) = \bigoplus_{n \in \omega} J_{\beta_n}(X).$$

Becker [4] showed that J_α is a UOP function on a u.p.p. tree, that is, there is a u.p.p tree T such that $J^*_\alpha := J_\alpha \circ T$ is UOP. Note that T can be identified with the homeomorphism $T[\cdot] \colon 2^\omega \simeq [T]$, where $[T]$ is the set of infinite paths through T.

Lemma 5.11. *Let α be a limit ordinal of countable cofinality. For any $f \colon \omega^\omega \to \mathbb{R}$, if Lev_f is $\Delta^\diamondsuit_\alpha$-complete, then $f \equiv^{\widehat{c}}_{sW} J^*_\alpha$.*

Proof. Note that f can be written as a join of functions f_{β_n} such that Lev_{f_n} is $\Sigma^\diamondsuit_{\beta_n}$-complete. By Lemma 5.9, $f_n \equiv^{\widehat{c}}_{sW} J_{\beta_n}$. Hence, $f \equiv^{\widehat{c}}_{sW} J_\alpha$. Then the assertion holds since $J_\alpha \equiv^c_{sW} J^*_\alpha$ via T and T^{-1}. □

Proof (Theorem 5.4). In Section 5.2, we have already seen that the identity map is an embedding from the quotient of $(\mathrm{UOP}, \leq^\nabla_{\mathbf{T}})$ into that of $(\mathcal{F}, \leq^{\widehat{c}}_{sW})$. It suffices to show that it is surjective; that is, for any function $f \colon \omega^\omega \to \mathbb{R}$ there exists a UOP function J such that $f \equiv^{\widehat{c}}_{sW} J$. For any $f \colon \omega^\omega \to \mathbb{R}$, there exists α such that Lev_f is Γ_α-complete, where $\Gamma \in \{\Sigma, \Pi, \Delta\}$. If $\Gamma \in \{\Sigma, \Pi\}$, then by (the paragraph below) Lemma 5.9, $f \equiv^{\widehat{c}}_{sW} J$ for some UOP function J. If $\Gamma = \Delta$ then by Fact 2 the cofinality of α is countable. Thus, by Lemma 5.10 if α is successor, and by Lemma 5.11 if α is limit, there exists a UOP function J such that $f \equiv^{\widehat{c}}_{sW} J$. This completes the proof. □

We finally show Theorem 5.5 saying that the jump of the parallelization always gives the successor rank.

Proof (Theorem 5.5). As mentioned before, the $\leq^\nabla_{\mathbf{T}}$-degrees of UOP functions form a well-order of height Θ, and therefore, by Theorem 5.4, so are the p.c.s. Weihrauch degrees of real-valued functions.

We claim that for non-constant functions $f, g \colon 2^\omega \to 2^\omega$, if $f \leq^{\widehat{c}}_{sW} g$ then $f' \leq^c_{sW} \widehat{g}'$. First note that the Turing jump $X \mapsto X'$ is UOP, so let u be a witness of UOP-ness of the Turing jump, that is, if $X \leq_T Y$ via e then $X' \leq_T Y'$ via $u(e)$. If $f \leq^{\widehat{c}}_{sW} g$ then there are continuous functions h and k such that $f = k \circ \widehat{g} \circ h$. Put $B_X = \widehat{g}(h(X))$ for any $X \in 2^\omega$. As k is continuous, there is an oracle C such that k is C-computable. Hence, $f(X) \leq_T B_X \oplus C$ via some index e independent of X. As in the proof of Lemma 5.8, since g is not constant, one can find $\langle Z^C_n : n \in \omega \rangle$ such that $\widehat{g}(\langle Z^C_n : n \in \omega \rangle)$ computes C. Assume that $h(X) = \langle h_n(X) : n \in \omega \rangle$. Then define $r_{2n}(X) = h_n(X)$, $r_{2n+1}(X) = Z^C_n$, and $r(X) = \langle r_n(X) : n \in \omega \rangle$. Since $B_X \oplus C \leq_T \widehat{g}(r(X))$ via some index d (independent of X), $(B_X \oplus C)' \leq_T \widehat{g}(r(X))'$

via $u(d)$. Moreover, since $f(X) \leq_T B_X \oplus C$ via e, we have $f(X)' \leq_T (B_X \oplus C)'$ via $u(e)$. Hence, there is an index c independent of X such that $f(X)' \leq_T \widehat{g}(r(X))'$ via c. In other words, the pair (r, Φ_c) witnesses $f' \leq^c_{sW} \widehat{g}'$.

Let f be a non-constant function of p.c.s. Weihrauch rank α. By Theorem 5.4, there is a UOP function g such that $f \equiv^{\widehat{c}}_{sW} g$, and the $\leq^{\nabla}_{\mathbf{T}}$-rank of g is also α. It is easy to see that $\widehat{g}$ and $\widehat{g}'$ are also UOP, and clearly $\widehat{g} \equiv^{\widehat{c}}_{sW} g$. Hence, by Theorem 5.4, the $\leq^{\nabla}_{\mathbf{T}}$-rank of $\widehat{g}$ is still α. Then, by Steel's theorem [27], the $\leq^{\nabla}_{\mathbf{T}}$-rank of $\widehat{g}'$ is $\alpha + 1$, and again by Theorem 5.4, so is the p.c.s. Weihrauch rank. By the above claim, we have $\widehat{f}' \equiv^{\widehat{c}}_{sW} \widehat{g}'$. Consequently, the p.c.s. Weihrauch rank of $\widehat{f}'$ is $\alpha + 1$ □

The claim in the above proof also shows that if f is non-constant and parallelizable, so is f': By parallelizability of f, we have $\widehat{f} \leq^c_{sW} f$, and by the above claim, we also have $(\widehat{f})' \leq^c_{sW} f'$. For $A_n = f(X_n)$, as the Turing jump is UOP, $\bigoplus_{n\in\omega} A'_n$ is computable in $(\bigoplus_{n\in\omega} A_n)'$ in a uniform manner; hence $\widehat{(f')} \leq^c_{sW} (\widehat{f})'$. Therefore, f' is parallelizable.

As a consequence, if f is non-constant and parallelizable, then one can obtain an ω_1-sequence of p.c.s. Weihrauch successor ranks of f only by iterating the jump $g \mapsto g'$.

Acknowledgements. The author was partially supported by JSPS KAKENHI Grant 19K03602, 21H03392, the JSPS Core-to-Core Program (A. Advanced Research Networks), and the JSPS-RFBR Bilateral Joint Research Project JPJSBP120204809. The author would like to thank Adam Day, Rod Downey, Antonio Montalbán, and Linda Brown Westrick for valuable discussions.

References

[1] Alessandro Andretta. The SLO principle and the Wadge hierarchy. In *Foundations of the formal sciences V*, volume 11 of *Stud. Log. (Lond.)*, pages 1–38. Coll. Publ., London, 2007.
[2] Alessandro Andretta and Alain Louveau. Wadge degrees and pointclasses. Introduction to Part III. In *Wadge degrees and projective ordinals. The Cabal Seminar. Volume II*, volume 37 of *Lect. Notes Log.*, pages 3–23. Assoc. Symbol. Logic, La Jolla, CA, 2012.
[3] Alessandro Andretta and Donald A. Martin. Borel-Wadge degrees. *Fund. Math.*, 177(2):175–192, 2003.
[4] Howard Becker. A characterization of jump operators. *J. Symbolic Logic*, 53(3):708–728, 1988.
[5] J. Bourgain. On convergent sequences of continuous functions. *Bull. Soc. Math. Belg. Sér. B*, 32(2):235–249, 1980.
[6] Vasco Brattka, Guido Gherardi, and Arno Pauly. Weihrauch complexity in computable analysis. arXiv 1707.03202, 2017.
[7] Raphaël Carroy. A quasi-order on continuous functions. *J. Symbolic Logic*, 78(2):633–648, 2013.
[8] Adam Day, Rod Downey, and Linda Brown Westrick. Three topological reducibilities for discontinuous functions. submitted, available at arXiv:1906.07600.
[9] Matthew de Brecht. Quasi-Polish spaces. *Ann. Pure Appl. Logic*, 164(3):356–381, 2013.
[10] J. Duparc. The Steel hierarchy of ordinal valued Borel mappings. *J. Symbolic Logic*, 68(1):187–234, 2003.
[11] Márton Elekes, Viktor Kiss, and Zoltán Vidnyánszky. Ranks on the Baire class ξ functions. *Trans. Amer. Math. Soc.*, 368(11):8111–8143, 2016.
[12] Daisuke Ikegami, Philipp Schlicht, and Hisao Tanaka. Borel subsets of the real line and continuous reducibility. *Fund. Math.*, 244(3):209–241, 2019.
[13] A. S. Kechris and A. Louveau. A classification of Baire class 1 functions. *Trans. Amer. Math. Soc.*, 318(1):209–236, 1990.

[14] Alexander S. Kechris. *Classical descriptive set theory*, volume 156 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[15] Takayuki Kihara and Antonio Montalbán. The uniform Martin's conjecture for many-one degrees. *Trans. Amer. Math. Soc.*, 370(12):9025–9044, 2018.

[16] Takayuki Kihara and Antonio Montalbán. On the structure of the Wadge degrees of bqo-valued Borel functions. *Trans. Amer. Math. Soc.*, 371(11):7885–7923, 2019.

[17] Takayuki Kihara and Victor Selivanov. Wadge-like degrees of borel bqo-valued functions. submitted, available at arXiv:1909.10835.

[18] Andrew Marks, Theodore Slaman, and John Steel. Martin's conjecture, arithmetic equivalence, and countable Borel equivalence relations. In *Ordinal Definability and Recursion Theory: The Cabal Seminar, Volume III*, pages 493–519. Cambridge University Press, 2016.

[19] Antonio Montalbán. Martin's conjecture: a classification of the naturally occurring Turing degrees. *Notices Amer. Math. Soc.*, 66(8):1209–1215, 2019.

[20] Luca Motto Ros. *General reducibilities for sets of reals*. PhD thesis, Polytechnic of Turin, 2007.

[21] Luca Motto Ros. Borel-amenable reducibilities for sets of reals. *J. Symbolic Logic*, 74(1):27–49, 2009.

[22] Yann Pequignot. A Wadge hierarchy for second countable spaces. *Arch. Math. Logic*, 54(5-6):659–683, 2015.

[23] Philipp Schlicht. Continuous reducibility and dimension of metric spaces. *Arch. Math. Logic*, 57(3-4):329–359, 2018.

[24] Victor L. Selivanov. Extending Wadge theory to k-partitions. In Jarkko Kari, Florin Manea, and Ion Petre, editors, *Unveiling Dynamics and Complexity - 13th Conference on Computability in Europe, CiE 2017, Turku, Finland, June 12-16, 2017, Proceedings*, volume 10307 of *Lecture Notes in Computer Science*, pages 387–399. Springer, 2017.

[25] Theodore A. Slaman and John R. Steel. Definable functions on degrees. In *Cabal Seminar 81–85*, volume 1333 of *Lecture Notes in Math.*, pages 37–55. Springer, Berlin, 1988.

[26] John R. Steel. Determinateness and the separation property. *J. Symbolic Logic*, 46(1):41–44, 1981.

[27] John R. Steel. A classification of jump operators. *J. Symbolic Logic*, 47(2):347–358, 1982.

[28] Robert Van Wesep. Wadge degrees and descriptive set theory. In *Cabal Seminar 76–77 (Proc. Caltech-UCLA Logic Sem., 1976–77)*, volume 689 of *Lecture Notes in Math.*, pages 151–170. Springer, Berlin, 1978.

[29] Robert A. Van Wesep. Separation principles and the axiom of determinateness. *J. Symbolic Logic*, 43(1):77–81, 1978.

[30] William Wilfred Wadge. *Reducibility and Determinateness on the Baire Space*. ProQuest LLC, Ann Arbor, MI, 1983. Thesis (Ph.D.)–University of California, Berkeley.

[31] W. Hugh Woodin. Supercompact cardinals, sets of reals, and weakly homogeneous trees. *Proc. Nat. Acad. Sci. U.S.A.*, 85(18):6587–6591, 1988.

(Takayuki Kihara) Graduate School of Informatics, Nagoya University, Japan
E-mail address: kihara@i.nagoya-u.ac.jp