A PRIORITY ARGUMENT IN DESCRIPTIVE SET THEORY

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Abstract. In 2009, Semmes announced that a function f on Baire space is decomposable into countably many Baire-one functions with G_δ domains if and only if the preimage of f under a F_σ set is $G_{\delta\sigma}$. In this short note, I will outline Semmes' proof with the emphasis on the use of a finite injury priority argument, but not a game-theoretic one, and try to clarify how his argument works.

1. Introduction

1.1. **Background.** In 2009, Semmes [4] announced a result extending the Jayne-Rogers theorem [2]. Since then, a number of experts tried to clarify and simplify Semmes' proof; however it seems that their attempt has eventually failed (see [1]). The previous expositions of Semmes' proof have laid emphasis on game-theoretic arguments. In this short note, we will take completely opposite approach: No determinacy argument has been used in Semmes' proof, and therefore, removing all the game-theoretic machineries makes the proof much clearer. As a consequence, the way of our exposition is slightly different from the original one, but all of the essential ideas are already contained in Semmes' original insightful proof.

1.2. **Notations.** $[\sigma]$ is the clopen set generated by $\sigma \in \omega^{<\omega}$. If a string σ is an initial segment of τ then we write $\sigma \sqsubseteq \tau$. If strings σ and τ are incomparable then we write $\sigma \perp \tau$. For a function $f\colon X \to Y$ and $A \subseteq X$, we use $f|_A$ to denote the restriction of f up to A. Let $\mathbf{\Gamma}$ and $\mathbf{\Lambda}$ be pointclasses. We write $f^{-1}\mathbf{\Gamma} \subseteq \mathbf{\Lambda}$ if the preimage of each $\mathbf{\Gamma}$ set under f is $\mathbf{\Lambda}$, that is,

$$A \in \mathbf{\Gamma} \implies f^{-1}[A] \in \mathbf{\Lambda} \text{ in } \mathrm{dom}(f).$$

For example, f is $\mathbf{\Sigma}^0_n$-measurable if and only if $f^{-1}\mathbf{\Sigma}^0_1 \subseteq \mathbf{\Sigma}^0_n$ holds.

If $\mathcal{F}$ is a class of functions, we also write $f \in \mathbf{dec}(\mathcal{F}/\mathbf{\Gamma})$ if f is decomposable into countably many $\mathcal{F}$-functions on $\mathbf{\Gamma}$ domains, that is, there is a countable $\mathbf{\Gamma}$ cover $(X_i)_{i\in\omega}$ of the domain of f such that $f|_{X_i} \in \mathcal{F}$ for each $i \in \omega$.

We also use $\mathbf{\Sigma}^0_n$ to denote the class of $\mathbf{\Sigma}^0_n$-measurable functions. For instance, the Jayne-Rogers theorem [2] can be stated as follows.

$$f^{-1}\mathbf{\Sigma}^0_2 \subseteq \mathbf{\Sigma}^0_2 \iff \mathbf{dec}(\mathbf{\Sigma}^0_1/\mathbf{\Pi}^0_1)$$

where f is a function from an analytic subset of a Polish space to a separable metrizable space.

It is easy to see the following (see Motto Ros [3] and Semmes [4, Lemma 4.3.1])

Observation 1. *The following equalities hold.*

$$\mathbf{dec}(\mathbf{\Sigma}^0_m/\mathbf{\Pi}^0_n) = \mathbf{dec}(\mathbf{\Sigma}^0_m/\mathbf{\Sigma}^0_{n+1}) = \mathbf{dec}(\mathbf{dec}(\mathbf{\Sigma}^0_m/\mathbf{\Pi}^0_n)/\mathbf{\Sigma}^0_{n+1}).$$

2. Proof

In his PhD thesis [4], Semmes showed that the following equivalence holds.

$$f^{-1}\mathbf{\Sigma}^0_2 \subseteq \mathbf{\Sigma}^0_3 \iff \mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Pi}^0_2).$$

The right-to-left implication is clear. Moreover, the condition $f^{-1}\mathbf{\Sigma}^0_2 \subseteq \mathbf{\Sigma}^0_3$ always implies that f is $\mathbf{\Sigma}^0_3$-measurable. Thus, to verify the above equivalence, it suffices to show the following:

Theorem 2 (Semmes [4])**.** *Suppose that $f : \omega^\omega \to \omega^\omega$ is $\mathbf{\Sigma}^0_3$-measurable. Then,*

$$f \notin \mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Pi}^0_2) \implies f^{-1}\mathbf{\Sigma}^0_2 \not\subseteq \mathbf{\Sigma}^0_3.$$

2.1. **Transfinite derivation process.** To prove Theorem 2, hereafter we fix a $\mathbf{\Sigma}^0_3$-measurable function $f : \mathbf{X} \to \mathbf{Y}$. Then, the preimage $f^{-1}[\sigma]$ of a clopen set is $\mathbf{\Sigma}^0_3$. Therefore, it can be written as a countable union of $\mathbf{\Pi}^0_2$ sets, say $f^{-1}[\sigma] = \bigcup_{s\in\omega} f^*_s[\sigma]$. Let $\mathbf{D}$ be a subset of $\mathbf{X}$, and put $h = f|_{\mathbf{D}}$. Then, define $h^*_s[\sigma] = f^*_s[\sigma] \cap \mathbf{D}$. In Section 2.1, we will present an essence of the argument of Semmes [4, Lemma 4.3.3].

Given $X \subseteq \mathbf{X}$, we define $[X; h]^\dagger_\sigma$ as follows:

$$[X; h]^\dagger_\sigma = X \setminus \bigcup\{J : h|_{\mathbf{D}\cap X\cap J\setminus h^{-1}[\sigma]} \in \mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Pi}^0_2)\},$$

where J ranges over open sets in $\mathbf{X}$. Moreover, given Y, we consider the following $[Y; h]^\star_{\sigma,s}$:

$$[Y; h]^\star_{\sigma,s} = \mathrm{cl}_Y(h^*_s[\sigma]),$$

where $\mathrm{cl}_Z A$ is the topological closure of a set $A \cap Z$ in a space Z. We call the above procedure a $\dagger_\sigma$-*derivation* (or a $\dagger$-derivation) and a $\star_{\sigma,s}$-*derivation* (or a $\star$-derivation), respectively. Clearly, $[X; h]^\dagger_\sigma$ and $[Y; h]^\star_{\sigma,s}$ are closed subsets of X and Y, respectively. In Semmes' thesis [4, Lemma 4.3.3], a $\dagger$-derivation and a $\star$-derivation are called a Ξ-operation and an Ω-operation, respectively.

We fix h, and simply write $X^\dagger_\sigma$ and $Y^\star_{\sigma,s}$ for $[X; h]^\dagger_\sigma$ and $[Y; h]^\star_{\sigma,s}$, respectively. We iterate these derivation procedures:

$$\begin{aligned} H^0_{\sigma,s} &= \mathbf{X}, \\ H^{\alpha+1}_{\sigma,s} &= ((H^\alpha_{\sigma,s})^\dagger_\sigma)^\star_{\sigma,s}, \\ H^\alpha_{\sigma,s} &= \bigcap_{\beta<\alpha} H^\beta_{\sigma,s} \text{ if } \alpha \text{ is a limit ordinal.} \end{aligned}$$

Note that there is a countable ordinal $\gamma(\sigma, s)$ such that $H^{\gamma(\sigma,s)+1}_{\sigma,s} = H^{\gamma(\sigma,s)}_{\sigma,s}$ since $(H^\alpha_{\sigma,s})_\alpha$ is a decreasing sequence of closed sets in $\mathbf{X}$. Clearly, $\gamma = \sup_{\sigma,s} \gamma(\sigma, s)$ is a countable ordinal since $\aleph_1$ is regular.

We divide the set $\mathbf{D}$ into three pieces. We first define the (σ, s)-*kernel* to be

$$K_{\sigma,s}h = H^{\gamma(\sigma,s)}_{\sigma,s}.$$

We say that a point $x \in X$ is *generic* if for every (σ, s), either $x \in h^{-1}[\sigma]$ or there exists α such that $x \in (H^\alpha_{\sigma,s})^\dagger_\sigma \setminus H^{\alpha+1}_{\sigma,s}$ (that is, x is removed by a $\star_{\sigma,s}$-derivation). Define $\mathbf{K} = \mathbf{D} \cap \bigcup_{\sigma,s} K_{\sigma,s}h$, $\mathbf{G}$ to be the set of all generic points $y \in \mathbf{D} \setminus \mathbf{K}$, and $\mathbf{A}$ to be the set of all other points in $\mathbf{D}$. Note that $x \in \mathbf{A}$ iff $x \notin h^{-1}[\sigma]$ holds, and x must be removed by a $\dagger$-derivation for some (σ, s).

Proposition 3.

(1) $\mathbf{A} \in \mathbf{\Delta}^0_3$, $\mathbf{G} \in \mathbf{\Delta}^0_3$, *and* $\mathbf{K} \in \mathbf{\Sigma}^0_2$, *in* $\mathbf{D}$.
(2) $h|_{\mathbf{G}}$ *is* $\mathbf{\Sigma}^0_2$*-measurable.*
(3) $h|_{\mathbf{A}} \in \mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Pi}^0_2)$.

To see this, we need the following characterization of the set $\mathbf{A}$.

Claim. Suppose $x \in \mathbf{D} \setminus \mathbf{K}$. Then, $x \in \mathbf{A}$ if and only if there are $(\sigma_0, s_0), (\sigma_1, s_1)$ and $\alpha_0, \alpha_1 < \gamma$ such that $[\sigma_0] \cap [\sigma_1] = \emptyset$ and $x \in H^{\alpha_i}_{\sigma_i,s_i} \setminus (H^{\alpha_i}_{\sigma_i,s_i})^\dagger_{\sigma_i}$ for every $i \in \{0, 1\}$.

Proof. The condition $x \notin \mathbf{K}$ means that for every (σ, s), there is $\alpha(\sigma, s)$ such that $x \in H^{\alpha(\sigma,s)}_{\sigma,s} \setminus H^{\alpha(\sigma,s)+1}_{\sigma,s}$. Now $x \in \mathbf{D}$ and thus $h(x)$ is defined.

If $x \in \mathbf{A}$ then $x \notin h^{-1}[\sigma_0]$ holds, and x must be removed by a $\dagger$-derivation, that is, $x \in H^{\alpha_0}_{\sigma_0,s_0} \setminus (H^{\alpha_0}_{\sigma_0,s_0})^\dagger_{\sigma_0}$ where $\alpha_0 = \alpha(\sigma_0, s_0)$. Since $h(x) \notin [\sigma_0]$ and since σ_0 is nonempty, there is σ_1

such that $[\sigma_0]\cap[\sigma_1]=\emptyset$ and $h(x)\in[\sigma_1]$. Let s_1 be such that $x\in h^*_{s_1}[\sigma_1]$. Note that for any Z, $x\in Z$ clearly implies $x\in\mathrm{cl}_Z(h^*_{s_1}[\sigma_1])$. Therefore, x is not removed by a $\star_{\sigma_1,s_1}$-derivation, and thus x must be removed by a $\dagger$-derivation. More precisely, for all α, $x\in(H^\alpha_{\sigma_1,s_1})^\dagger_{\sigma_1}$ implies $x\in H^{\alpha+1}_{\sigma_1,s_1}$; hence by putting $\alpha_1=\alpha(\sigma_1,s_1)$, we get $x\in H^{\alpha_1}_{\sigma_1,s_1}\setminus(H^{\alpha_1}_{\sigma_1,s_1})^\dagger_{\sigma_1}$ as desired.

We next verify the converse direction. Let x be a point in $\mathbf{D}\setminus\mathbf{K}$ satisfying the latter condition. Since $[\sigma_0]\cap[\sigma_1]=\emptyset$, we must have $x\notin h^{-1}[\sigma_i]$ for some $i<2$. Then, the pair (σ_i,s_i) witnesses that x is removed by a $\dagger_{\sigma_i}$-derivation. This implies that x is not generic. Hence, under our assumption that $x\notin\mathbf{K}$, we have $x\in\mathbf{A}$ as desired. □

Proof of Proposition 3. (1) By definition, clearly $\mathbf{K}\in\mathbf{\Sigma}^0_2$ in $\mathbf{D}$. Hence, by the above claim, $\mathbf{A}$ is $\mathbf{\Sigma}^0_2$ in $\mathbf{D}\setminus\mathbf{K}$, and thus $\mathbf{A}$ is the difference of two $\mathbf{\Sigma}^0_2$ sets in $\mathbf{D}$. Then, $\mathbf{G}$ is also contained in a finite level of the difference hierarchy over $\mathbf{\Sigma}^0_2$ in $\mathbf{D}$.

(2) Suppose that $x\in\mathbf{G}$. Then, x is generic, and $x\notin\mathbf{K}$. Given (σ,s), Let $\alpha(\sigma,s)$ witness $x\notin\mathbf{K}$ as in the previous claim. If $x\in h^{-1}[\sigma]$, there is s such that $x\in h^*_s[\sigma]$ by definition. Then, as mentioned in the previous claim, x is not removed by a $\star$-derivation, and thus removed by a $\dagger$-derivation: For all α, $x\in(H^\alpha_{\sigma,s})^\dagger_\sigma$ implies $x\in H^{\alpha+1}_{\sigma,s}$; hence $x\in H^{\alpha(\sigma,s)}_{\sigma,s}\setminus(H^{\alpha(\sigma,s)}_{\sigma,s})^\dagger_\sigma$. If $x\notin h^{-1}[\sigma]$, by our definition of genericity, x is always removed by a $\star$-derivation: For all s, there exists α such that $x\in(H^\alpha_{\sigma,s})^\dagger_\sigma\setminus H^{\alpha+1}_{\sigma,s}$, which means that $x\in(H^{\alpha(\sigma,s)}_{\sigma,s})^\dagger_\sigma\setminus H^{\alpha(\sigma,s)+1}_{\sigma,s}$. Consequently, whenever $x\in\mathbf{G}$, for any σ,

$$x\in h^{-1}[\sigma]\iff(\exists\alpha<\gamma)(\exists s\in\omega)\;x\in H^\alpha_{\sigma,s}\setminus(H^\alpha_{\sigma,s})^\dagger_\sigma.$$

The latter condition is clearly $\mathbf{\Sigma}^0_2$.

(3) Let $(\sigma_i,s_i,\alpha_i)_{i<2}$ be a witness of $x\in\mathbf{A}$ as in the previous claim. By our definition of the $\dagger$-derivation, for $Z=H^{\alpha_0}_{\sigma_0,s_0}\cap H^{\alpha_1}_{\sigma_1,s_1}$, there is a neighborhood J of x such that

$$h|_{\mathbf{D}\cap Z\cap J\setminus h^{-1}[\sigma_i]}\in\mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Pi^0_2}).$$

Clearly, $[\sigma_0]\cap[\sigma_1]=\emptyset$ implies $(J\setminus h^{-1}[\sigma_0])\cup(J\setminus h^{-1}[\sigma_1])=J$. Since $\mathbf{\Sigma}^0_3$-measurability of h and zero-dimensionality of $\mathbf{Y}$ implies that $h^{-1}[\sigma_i]$ is $\mathbf{\Delta}^0_3$ in $\mathbf{D}$, both $J\setminus h^{-1}[\sigma_0]$ and $J\setminus h^{-1}[\sigma_1]$ are $\mathbf{\Delta}^0_3$ in $\mathbf{D}$. Hence, $h|_{\mathbf{D}\cap Z\cap J}\in\mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Delta^0_3})$. Since there are only countably many candidates for such a witness $(\sigma_i,s_i,\alpha_i)_{i<2}$ (because $\alpha_i<\gamma$), $h|_{\mathbf{A}}$ is decomposable into countably many $\mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Delta^0_3})$-functions on closed domains $(H^{\alpha_0}_{\sigma_0,s_0}\cap H^{\alpha_1}_{\sigma_1,s_1}\cap J)_{\sigma_0,s_0,\alpha_0,\sigma_1,s_1,\alpha_1,J}$. Hence, by Observation 1, we conclude $h|_{\mathbf{A}}\in\mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Delta^0_3})$. □

2.2. **Chain of kernels.** As a consequence of Proposition 3, we obtain the following key lemma.

Lemma 4. *If $h\notin\mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Pi}^0_2)$, then $\mathbf{K}$ is nonempty.*

Proof. If $\mathbf{K}$ is empty, then $h=h|_{\mathbf{A}}\cup h|_{\mathbf{G}}$. By Proposition 3 (2) and (3), we have $h|_{\mathbf{A}},h|_{\mathbf{G}}\in\mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Pi}^0_2)$. Since $\mathbf{A}$ and $\mathbf{G}$ are $\mathbf{\Delta}^0_3$ by Proposition 3 (1), we also have $h\in\mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Pi}^0_2)$ by Observation 1. □

In particular, there is (σ,s) such that the (σ,s)-kernel $K_{\sigma,s}h$ is nonempty. It is easy to see that $K_{\sigma,s}h\subseteq\mathrm{cl}_{\mathbf{X}}\mathbf{D}$. Hereafter, if L is a closed set and p is a finite string, we write $p\in L$ if $L\cap[p]\neq\emptyset$.

2.2.1. *The $\star$-derivation.* The $\star$-derivation procedure for $h=f|_D$ ensures a *density* condition for $K_{\sigma,s}X$. We say that a triple (K,σ,s) of a nonempty closed set K, a finite string $\sigma\in K$, and a natural number s is a *bi-density triple* (w.r.t. f) if

(D1) $f^*_s[\sigma]$ is dense in K.
(D2) $f^*_t[\tau]$ is nowhere dense in K whenever $\sigma\perp\tau$ and $t\in\omega$.

Let $\mathbb{Q}$ be the set of all bi-density triples. The following proof is an analog of [4, Lemma 4.3.2].

Observation 5. *For any* $\mathbf{D} \subseteq \mathbf{X}$, σ *and* s, *we have* $(K_{\sigma,s}(f|_{\mathbf{D}}), \sigma, s) \in \mathbb{Q}$.

Proof. Put $h = f|_{\mathbf{D}}$. The $\star$-derivation procedure clearly ensures that $h^*_s[\sigma]$ is dense in $K_{\sigma,s}h$. Hence $f^*_s[\sigma]$ is also dense in $K_{\sigma,s}h$, that is, (D1) holds. Suppose for the sake of contradiction that the item (D2) fails. Then $f^*_t[\tau]$ is dense in $K_{\sigma,s}h \cap [\eta]$ for some $\eta \in \omega^{<\omega}$, and $f^*_s[\sigma]$ is also dense in $K_{\sigma,s}h \cap [\eta]$ by (D1). By definition, $f^*[\sigma]$ and $f^*[\tau]$ are $\mathbf{\Pi}^0_2$ in the Polish space $\mathbf{X}$. Thus, both are intersections of sequences of dense open sets in the closed set $K_{\sigma,s}h \cap [\eta]$. By the Baire category theorem, $f^*_s[\sigma]$ and $f^*_t[\tau]$ have an intersection. However, $\sigma \perp \tau$ implies $[\sigma] \cap [\tau] = \emptyset$ and thus we must have $f^*_s[\sigma] \cap f^*_t[\tau] \subseteq f^{-1}[\sigma] \cap f^{-1}[\tau] = \emptyset$. $\square$

2.2.2. *The* $\dagger$*-derivation.* The $\dagger$-derivation procedure ensures an *indecomposability* condition for $K_{\sigma,s}h$. We say that a triple $(L, p; V)$ of a nonempty closed set L, a finitary clopen set V, and a finite string $p \in L$ is an *indecomposability domain* (for f) such that if

$$(\forall q \in L)\ [q \sqsupseteq p \implies f|_{L \cap [q] \setminus f^{-1}[V]} \notin \mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Pi}^0_2)].$$

Let $\mathbb{L}_1$ be the set of all indecomposability domains. The $\dagger$-derivation procedure ensures the following.

Observation 6. *Assume that* $f|_{\mathbf{D}} \notin \mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Pi}^0_2)$, *and that* $\mathbf{D}$ *and* $f^{-1}[U]$ *have no intersection. Then, there exists* (σ, s) *such that* $(K_{\sigma,s}(f|_{\mathbf{D}}), \varepsilon; U \cup [\sigma]) \in \mathbb{L}_1$, *where* ε *denotes the empty string.*

Proof. Put $h = f|_{\mathbf{D}}$. By Lemma 4, there is (σ, s) such that $K_{\sigma,s}h$ is nonempty. By definition of a kernel $K = K_{\sigma,s}h$, we have $K^\dagger_\sigma = K$. Note that $\mathbf{D} \cap f^{-1}[U] = \emptyset$ implies that $\mathbf{D} \setminus f^{-1}[U \cup \sigma] = \mathbf{D} \setminus f^{-1}[\sigma] = \mathbf{D} \setminus h^{-1}[\sigma]$, and f and h agrees on this set. Therefore, by definition of the $\dagger$-derivation procedure, we have the following.

$$f|_{K \cap [q] \setminus f^{-1}[U \cup \sigma]} \supseteq f|_{\mathbf{D} \cap K \cap [q] \setminus f^{-1}[U \cup \sigma]} = h|_{\mathbf{D} \cap K \cap [q] \setminus h^{-1}[\sigma]} \notin \mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Pi}^0_2)$$

for any $q \in \omega^{<\omega}$. This means that $(K_{\sigma,s}h, \varepsilon; U \cup [\sigma])$ is an indecomposability domain. $\square$

The following states the basic properties of $\mathbb{L}_1$.

Observation 7.
(1) *If* $(L, p; V) \in \mathbb{L}_1$, *then for any* $V' \subseteq V$ *and* $q \in L$ *with* $q \sqsupseteq p$, *we have* $(L, q; V') \in \mathbb{L}_1$.
(2) *For any a pair* (J_0, J_1) *of disjoint finitary clopen sets, if* $(L, p; V) \in \mathbb{L}_1$ *then there are* $q \in L$ *with* $q \sqsupseteq p$ *and* $i < 2$ *such that* $(L, q; V \cup J_i) \in \mathbb{L}_1$.

Proof. (1) Obviously, if $f|_A \notin \mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Pi}^0_2)$ and $A \subseteq B$ then $f|_B \notin \mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Pi}^0_2)$. Now, note that $L \cap [q] \setminus f^{-1}[V] \subseteq L \cap [q] \setminus f^{-1}[V']$.

(2) Otherwise, $(L, p; V \cup J_0) \notin \mathbb{L}_1$ means that $f|_{L \cap [q] \setminus f^{-1}[V \cup J_0]} \in \mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Pi}^0_2)$ for some $q \in L$ with $q \sqsupseteq p$, and similarly, $(L, q; V \cup J_1) \notin \mathbb{L}_1$ means that $f|_{L \cap [r] \setminus f^{-1}[V \cup J_1]} \in \mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Pi}^0_2)$ for some $r \in L$ with $r \sqsupseteq q$. Since J_0 and J_1 are disjoint, we have

$$(L \cap [r]) \setminus f^{-1}[V] = (L \cap [r] \setminus f^{-1}[V \cup J_0]) \cup (L \cap [r] \setminus f^{-1}[V \cup J_1]).$$

By $\mathbf{\Sigma}^0_3$-measurability of f, $f^{-1}[V \cap J_i]$ is $\mathbf{\Delta}^0_3$, and therefore, again by Observation 1, we can see that $f|_{L \cap [r] \setminus f^{-1}[V]} \in \mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Pi}^0_2)$, and thus $(L, p; V) \notin \mathbb{L}_1$ since $p \sqsubseteq r \in L$. $\square$

Definition 8. We say that $((L_\ell)_{\ell \leq a}, p^a; V)$ is an indecomposability layer if

$$L_0 \supseteq L_1 \supseteq \cdots \supseteq L_{a-1} \supseteq L_a,$$
$$(L_a, p^a; V) \text{ is an indecomposability domain, i.e., in } \mathbb{L}_1,$$
$$(\forall q^a \sqsupseteq p^a)(\exists r^{a-1} \sqsupseteq q^a)\ ((L_\ell)_{\ell < a}, r^{a-1}; V) \text{ is an indecomposability layer,}$$

where q^a ranges over L_a and r^{a-1} ranges over L_{a-1}. Let $\mathbb{L}$ be the set of all indecomposability layers.

Observation 9. *If* $((L_\ell)_{\ell\leq a}, p^a; V) \in \mathbb{L}$, $q^a \in L_a$, *and* $q^a \sqsupseteq p^a$, *then* $((L_\ell)_{\ell\leq a}, q^a; V) \in \mathbb{L}$

Proof. By Observation 7 (2). □

According to Semmes' terminology, $((L_\ell)_{\ell\leq a}, p^a; V)$ is an indecomposability layer iff p^a is $(L_\ell)_{\ell\leq a}$-V^{c}-good. The following is a restatement of Semmes [4, Lemma 4.3.6] in our language, which generalizes Observation 7 (2).

Lemma 10. *Let* $(J_k)_{k<m}$ *be a collection of pairwise disjoint finitary clopen sets. If* $(\mathcal{L}, p^a; V) \in \mathbb{L}$, *then for all but* $|\mathcal{L}|$ *many indices* i, *we have* $(\mathcal{L}, q^a; V \cup J_i) \in \mathbb{L}$ *for some* $q^a \sqsupseteq p^a$.

Proof. First assume that $|\mathcal{L}| = 1$. In this case, $(\mathcal{L}, p; V) \in \mathbb{L}$ just means that $(\mathcal{L}, p; V) \in \mathbb{L}_1$. Thus, Observation 7 (2) clearly implies the assertion for $|\mathcal{L}| = 1$.

Consider the length $|\mathcal{L}| = a + 1$. In this proof, superscripts of variables x^a, y^{a-1}, etc. will indicate that x^a ranges over L_a, y^{a-1} ranges over L_{a-1}, etc. We put $S^i_a = \{q^a : (L_a, q^a; V \cup J_i) \in \mathbb{L}_1\}$ and $S^i_{<a} = \{q^{a-1} : ((L_\ell)_{\ell<a}, q^{a-1}; V \cup J_i) \in \mathbb{L}\}$. By Observations 7 (1) and 9, S^i_a and $S^i_{<a}$ are open in L_a. By induction, we assume the assertion for the length a, and fix $((L_\ell)_{\ell\leq a}, p^a; V) \in \mathbb{L}$. Since the third condition of $\mathbb{L}$ implies that, given $q^a \sqsupseteq p^a$, $((L_\ell)_{\ell<a}, r^{a-1}; V) \in \mathbb{L}$ for some $r^{a-1} \sqsupseteq q^a$, we get the following by the induction hypothesis:

(IH) Given $q^a \sqsupseteq p^a$, there is $r^{a-1} \sqsupseteq q^a$ such that for all but a many indices i, $u^{a-1} \in S^i_{<a}$ for some $u^{a-1} \sqsupseteq r^{a-1}$, that is, $S^i_{<a} \cap [r^{a-1}] \neq \emptyset$.

Note that (IH) implies that given $q^a \sqsupseteq p^a$, for all but a many indices i, $S^i_{<a} \cap [q^a] \neq \emptyset$.

Now we start to verify the assertion. By definition of $\mathbb{L}$, it suffices to show the following for all but $a + 1$ many indices $i < m$:

$$(\exists q^a \sqsupseteq p^a)\ q^a \in S^i_a \text{ and } (\forall r^a_0 \sqsupseteq q^a)(\exists r^{a-1}_1 \sqsupseteq r^a_0)\ r^{a-1}_1 \in S^i_{<a}. \tag{1}$$

Case 1. For any $i \leq m$, S^i_a is dense in $L_a \cap [p^a]$.

In this case, $\bigcap_{i<m} S^i_a$ is also dense in $L_a \cap [p^a]$ since the intersection of finitely many dense open sets is again dense. Let E be the set of all indices j such that the condition (1) fails. If $j \in E$, then since S^j_a is dense in $L_a \cap [p^a]$, the failure of (1) implies that

$$(\forall q^a_0 \sqsupseteq p^a)(\exists r^a_0 \sqsupseteq q^a_0)(\forall r^{a-1}_1 \sqsupseteq r^a_0)\ r^{a-1}_1 \notin S^j_{<a}.$$

Consider $\mathrm{ext}_{a-1} S^j_{<a} = \{q^{a-1} \in L_{a-1} : (\forall r^{a-1} \sqsupseteq q^{a-1})\ r^{a-1} \notin S^j_{<a}\}$, the exterior of $S^j_{<a}$ in L_{a-1}. Clearly, $\mathrm{ext}_{a-1} S^j_{<a}$ is open, and the above formula says that if $j \in E$, then $\mathrm{ext}_{a-1} S^j_{<a}$ is dense in $L_a \cap [p^a]$. Therefore, $\bigcap_{j\in E} \mathrm{ext}_{a-1} S^j_{<a}$ is also dense in $L_a \cap [p^a]$. In particular, there is $q^a_* \sqsupseteq p^a$ such that

$$(\forall j \in E)\ S^j_{<a} \cap [q^a_*] = \emptyset.$$

However, by (IH), for all but a many indices j, $S^j_{<a} \cap [q^a_*] \neq \emptyset$. Therefore, we have $|E| \leq a$.

Case 2. There is $i < m$ such that S^i_a is not dense in $L_a \cap [p^a]$.

In this case, $L_a \setminus S^i_a$ contains a nonempty open subset of $[p^a]$ in L_a, that is, there is $q^a \sqsupseteq p^a$ such that $q^a_0 \in L_a \setminus S^i_a$ for all $q^a_0 \sqsupseteq q^a$. By Observation 7 (2), if $j \neq i$, then for any $q^a_0 \sqsupseteq q^a$, there is $q^a_1 \sqsupseteq q^a_0$ such that $q^a_1 \in S^j_a$. In particular, $S^j_a \subseteq L_a$ is dense in $L_a \cap [q^a]$. Hence, $\bigcap_{i\neq j<m} S^j_a$ is dense in $L_a \cap [q^a]$.

Let E be the set of all indices j such that the condition (1) fails. If $j \in E$, $j \neq i$, then since S^j_a is dense in $L_a \cap [q^a]$, the failure of (1) implies that

$$(\forall q^a_0 \sqsupseteq q^a)(\exists r^a_0 \sqsupseteq q^a_0)(\forall r^{a-1}_1 \sqsupseteq r^a_0)\ r^{a-1}_1 \notin S^j_{<a}.$$

Thus, by the similar argument as before, we get $q^a_* \sqsupseteq p^a$ such that

$$(\forall j \in E \setminus \{i\})\ S^j_{<a} \cap [q^a_*] = \emptyset.$$

As before, (IH) implies $|E \setminus \{i\}| \leq a$. Hence, $|E| \leq a+1$. This concludes the proof. □

2.2.3. *Semmes conditions.* We now introduce a key notion, which we call a Semmes condition.

Definition 11. A tuple $((L_\ell, \sigma_\ell, s_\ell)_{\ell \leq a}, p^a, V)$ is called a *Semmes condition* if

$$\begin{gathered} ((L_\ell)_{\ell \leq a}, p^a, V) \text{ is an indecomposability layer, i.e., in } \mathbb{L}, \\ (\sigma_\ell)_{\ell \leq a} \text{ is pairwise incomparable, and } \sigma_\ell \in V, \\ (L_\ell, \sigma_\ell, s_\ell) \text{ is a bi-density witness, i.e., in } \mathbb{Q}, \text{ for all } \ell \leq a. \end{gathered}$$

Let $\mathbb{S}$ be the set of all Semmes conditions. We say that $((K_\ell, \sigma_\ell, s_\ell)_{\ell \leq a}, q^a, V') \in \mathbb{S}$ *extends* $((K_\ell, \sigma_\ell, s_\ell)_{\ell < a}; p^{a-1}, V) \in \mathbb{S}$ if

$$K_a \subseteq K_{a-1},\ V' \supsetneq V,\ \sigma_\ell \notin V, \text{ and } q^a \sqsupseteq p^{a-1}$$

We will need to ensure that a Semmes condition always has an extension. We utilize the indecomposability condition to construct an extension. The following lemma is buried in Semmes [4, p.37 in Theorem 4.3.7]

Lemma 12. *Let $\mathcal{K}$ and $\mathcal{L}_i$, $i < c$, be bi-density witnesses, and $(\mathcal{K}, p^{a-1}, V), (\mathcal{L}_i, p_i, V)$ are Semmes conditions. Then, there are $\mathcal{K}' \supseteq \mathcal{K}$, $q^a \sqsupseteq p^{a-1}$, $p'_i \sqsupseteq p_i$, and $V' \supseteq V$ such that $(\mathcal{K}', q^a, V')$ extends $(\mathcal{K}, p^{a-1}, V)$, and $(\mathcal{L}_i, p'_i, V')$ are still Semmes conditions.*

Proof. Let $\mathcal{K} = (\mathcal{K}_\ell)_{\ell < a}$ be given, where $\mathcal{K}_\ell$ is of the form $(K_\ell, \tau_\ell, t_\ell)$. We will construct K_a. We claim that for any z, there is a sequence $(K^{n+1}_a, q^a_{n+1}, \sigma_n)_{n \leq z}$ such that $(q^a_{n+1})_{n \leq z}$ is increasing,

$$\begin{gathered} K_{a-1} \supseteq K^0_a \supseteq K^1_a \supseteq \cdots \supseteq K^z_a \supseteq K^{z+1}_a, \\ (\sigma_n)_{n \leq z} \text{ is pairwise incomparable,} \\ (K^{n+1}_a, q^a_{n+1}; V \cup [\sigma_n]) \text{ is an idecomposable domain, i.e., in } \mathbb{L}_1, \text{ for any } n \leq z, \end{gathered}$$

We first define $K^0_a = K_{a-1}$, $q^a_0 = p^{a-1}$, and $U_0 = \emptyset$. Since $(\mathcal{K}, p^{a-1}, V) \in \mathbb{S}$, $(K^0_a, q^a_0; V \cup U_0)$ is an indecomposability domain. Inductively assume that $(K^n_a, q^a_n; V \cup U_n)$ is an indecomposability domain, that is, $f|_{L_n} \notin \mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Pi}^0_2)$, where $L_n = K^n_a \cap [q^a_n] \setminus f^{-1}[V \cup U_n]$. By applying Lemma 4 to $h = f|_{L_n}$, one obtains σ_n, s_n such that

$$\emptyset \neq K^{n+1}_a = K_{\sigma_n, s_n} h \subseteq \mathrm{cl}_{\mathbf{X}}(L_n) \subseteq K^n_a \cap [q^a_n].$$

Define $U_{n+1} = U_n \cup [\sigma_n]$. Since $\mathbf{D} = L_n$ and $f^{-1}[V \cup U_n]$ have no intersection, by Observation 6, $(K^{n+1}_a, \varepsilon; V \cup U_{n+1})$ is an indecomposability domain, and so is $(K^{n+1}_a, q^a_n; V \cup [\sigma_n])$ by Observation 7 (1). This verifies the third requirement of the claim whenever $q^a_n \sqsubseteq q^a_{n+1} \in K^a_{n+1}$. For the second requirement, as seen in the proof of Observation 5, $h^*_{s_n}[\sigma_n]$ is dense in K^{n+1}_a. In particular, $h^{-1}[\sigma_n] \supseteq h^*_{s_n}[\sigma_n]$ is nonempty. Since $h^{-1}[V \cup U_n]$ is empty, we have $\sigma_n \notin V \cup U_n$.

We now have two cases.

$$(\forall r^a \sqsupseteq q^a_n)(\exists u^{a-1} \sqsupseteq r^a)\ ((K_\ell)_{\ell < a}, u^{a-1}, V \cup [\sigma_n]) \in \mathbb{L},$$

where r^a ranges over K^{n+1}_a, and u^{a-1} ranges over K_{a-1}. In this case, by combining with the third condition of the previous claim, we get that $((K_\ell)_{\ell < a}{}^\frown K^{n+1}_a, q^a_n, V \cup [\sigma_n]) \in \mathbb{L}$. Then define $q^a_{n+1} = q^a_n$. Otherwise, we have

$$(\exists r^a \sqsupseteq q^a_n)(\forall u^{a-1} \sqsupseteq r^a)\ ((K_\ell)_{\ell < a}, u^{a-1}, V \cup [\sigma_n]) \notin \mathbb{L}.$$

In this case, we define $q^a_{n+1} = r^a$. In any case, $q^a_n \sqsubseteq q^a_{n+1} \in K^a_{n+1}$.

Let E be the set of all indices $n \leq z$ such that the second case applies. Recall that $((K_\ell)_{\ell < a}, q^a_{z+1}, V) \in \mathbb{L}$ and $(\sigma_n)_{n \leq z}$ is pairwise incomparable. Therefore, by Lemma 10, for all but a many indices $i \leq z$, we have $((K_\ell)_{\ell<a}, u^{a-1}, V \cup [\sigma_i]) \in \mathbb{L}$ for some $u^{a-1} \sqsupseteq q^a_{z+1}$. This means that $|E| \leq a$. Hence, for all but a many indices $n \leq z$, the first case applies, and we get that $((K_\ell)_{\ell<a}{}^\frown K^{n+1}_a, q^a_{n+1}, V \cup [\sigma_n]) \in \mathbb{L}$.

Put $z = a + b + 1$, where $b = \sum_{i<c} |\mathcal{L}_i|$. Then, $|z \setminus E| > b$. By Lemma 10, there is $n \in z \setminus E$ such that for all $i < c$, we have $(\mathcal{L}_i, p'_i, V \cup [\sigma_n]) \in \mathbb{S}$ for some $p'_i \sqsupseteq p_i$. Finally, put $K_a = K^{n+1}_a$, $q_a = q^a_{n+1}$ and $V' = V \cup [\sigma_n]$. By our choice of (σ_n, s_n) and by Observation 5, we have $\mathcal{K}_a := (K_a, \sigma_n, s_n) \in \mathbb{Q}$. Put $\mathcal{K}' = (\mathcal{K}_\ell)_{\ell \leq a}$. Then, we get $(\mathcal{K}', q^a, V') \in \mathbb{S}$, and it extends $(\mathcal{K}, p^{a-1}, V)$ as desired. □

Our construction makes an *injury* (in the sense of a priority argument), which may decreases the length of the Semmes condition. We say that $(\mathcal{K}', V', q^\ell) \in \mathbb{S}$ is a *shortening of* $(\mathcal{K}, V, p^a) \in \mathbb{S}$ if $\mathcal{K}'$ is an initial segment of $\mathcal{K}$, $V' = V$, and $q^\ell \sqsupseteq p^a$. The following corresponds to Semmes [4, Proposition 4.3.5].

Observation 13. *Every Semmes condiiton $(\mathcal{K}, p^a, V)$ has a shortening of length ℓ for any $\ell \leq |\mathcal{K}|$.*

Proof. Let a be the length of $\mathcal{K}$. Since $((K_j)_{j<a}, p^a, V) \in \mathbb{L}$, one can find a sequence $p^{a-1} \sqsupseteq p^{a-2} \sqsupseteq \ldots \sqsupseteq p^\ell$ such that $((K_j)_{j<a}, p^\ell, V) \in \mathbb{L}$. □

2.3. **Priority argument.** We are now ready to prove Theorem 2. Given a $\mathbf{\Sigma}^0_3$ set $U \subseteq 2^\omega$ we will construct a continuous function $\psi : \omega^\omega \to \omega^\omega$ and a set $V \subseteq \omega^{<\omega}$ of strings such that

$$x \in U \iff \psi(x) \in f^{-1}[V]$$

for every $x \in \omega^\omega$, where $[V]$ is the open set generated by V. Thus, this will ensure that $f^{-1}[V]$ is $\mathbf{\Sigma}^0_3$-complete for some set V of strings, which implies $f^{-1}\mathbf{\Sigma}^0_2 \not\subseteq \mathbf{\Sigma}^0_3$. We first describe $\mathbf{\Sigma}^0_3$ sets $f^{-1}[V]$ and U as follows:

$$\begin{aligned} x \in U &\iff (\exists a)(\forall b)(\exists c)\ S(x, a, b, c), \\ y \in f^{-1}[V] &\iff (\exists \sigma \in V)(\exists i)(\forall j)(\exists k)\ Q(y, \sigma, i, j, k). \end{aligned}$$

Here, $y \in f^*_i[\sigma]$ iff for all j, there exists k such that $Q(y, \sigma, i, j, k)$.

Requirements. The a-th requirements are given as follows:

$$\begin{aligned} \mathcal{N}^x_a : &\quad (\forall a' < a)(\exists b)(\forall c)\ S(x, a, b, c) \implies (\forall \sigma \in V_{s_a})(\forall i < a)(\exists j)(\forall k)\ Q(\psi(x), \sigma, i, j, k), \\ \mathcal{P}^x_a : &\quad (\forall b)(\exists c)\ S(x, a, b, c) \implies (\exists \sigma_a)(\exists i_a)(\forall j)(\exists k)\ Q(\psi(x), \sigma_a, i_a, j, k), \end{aligned}$$

where s_a is the first stage at which an a-th strategy acts along x (after the last initialization which may be caused by a higher-priority strategy).

Roughly speaking, every a-th strategy α_a believes that a is the least witness for $x \in U$. Then, the $\mathcal{P}$-action tries to keep $\psi(x) \in f^*_{i_a}[\sigma_a]$ and the $\mathcal{N}$-action forces $\psi(x) \notin \bigcup_{i<a} f^*_i[V_{s_a}]$. These requirements ensure that such ψ is a desired reduction as follows.

- If a is the smallest witness for $x \in U$, then the requirement $\mathcal{P}^x_a$ ensures that $\psi(x) \in f^*_{i_a}[\sigma_a]$. The a-th strategy will put σ_a into the set V in the construction, so by the requirement $\mathcal{P}^x_a$, we get that $x \in U$ implies $\psi(x) \in f^{-1}[V]$.
- If $x \notin U$, then for every a, the premise of the requirement $\mathcal{N}_a$ must be true, and then, the combination of requirements $\mathcal{N}^x_a$'s will eventually ensure that $\psi(x) \notin f^{-1}[V] = \bigcup_a f^{-1}[V_{s_a}]$.

Thus, it suffices to describe the strategy to satisfy the requirements $\mathcal{P}^x_a$ and $\mathcal{N}^x_a$.

Conditions. Fix a bijection $h : \omega^{<\omega} \to \omega$ such that $\alpha \sqsubseteq \beta$ implies $h(\alpha) \leq h(\beta)$. We simply write $\alpha \leq \beta$ if $h(\alpha) \leq h(\beta)$. At stage s, we will deal with the s-th binary string w.r.t. this order $\leq$. Given α, we use symbols $\alpha - 1$ and α^- to denote the immediate $\leq$-predecessor and the immediate $\sqsubseteq$-predecessor, respectively. At the α-th stage, we will construct $\mathcal{K}_\alpha$, $(p^\alpha_\beta)_{\beta\leq\alpha}$, and V_α satisfying the following condition.

- $(\mathcal{K}_\beta, p^\alpha_\beta, V_\alpha)$ is a Semmes condition for every $\alpha \in \omega^{<\omega}$ and $\beta \leq \alpha$.
- If $\beta \leq \alpha$, then $p^{\alpha-1}_\beta \sqsubseteq p^\alpha_\beta$ and $V_{\alpha-1} \subseteq V_\alpha$.
- $(\mathcal{K}_\alpha, p^\alpha_\alpha, V_\alpha)$ is either an extension or a shortening of $(\mathcal{K}_{\alpha^-}, p^{\alpha-1}_{\alpha^-}, V_{\alpha-1})$.

By the definition of a Semmes condition, $\mathcal{K}_\alpha$ is a sequence of bi-dense triples $(K_a, \sigma_a, i_a)_{a<\ell}$. Here, recall that (σ_a, i_a) witnesses the bi-dense property of K_a for every $a < \ell$:

$$\text{(D1)} \qquad (\forall q^a)(\exists y^a \sqsupset q^a)[\forall j \exists k Q(y^a, \sigma_a, i_a, j, k)],$$

$$\text{(D2)} \qquad (\forall i)(\forall q^a)(\exists r^a \sqsupseteq q^a)(\forall y^a \sqsupset r^a)[\exists j \forall k \neg Q(y^a, \tau, i, j, k)] \text{ if } \tau \perp \sigma_a,$$

where q^a, r^a and y^a range over K_a.

We now start to describe the proof of Theorem 2. For the reader who is familiar with priority arguments in computability theory, we first note that our proof is a *finite injury priority argument.*

Proof of Theorem 2. As mentioned before, we will construct Semmes conditions $(\mathcal{K}_\alpha, p^\alpha_\beta, V_\alpha)_{\beta\leq\alpha}$ at the α-th stage. If ℓ is the length of $\mathcal{K}_{\alpha^-}$, then every strategy $a \leq \ell$ is *eligible to act* at the α-th stage, that is, we deal with $\mathcal{P}_a$- and $\mathcal{N}_a$-strategies for any $a \leq \ell$. The state of the a-th strategy at the α-th stage s is written as $\mathtt{state}(a, \alpha)$. If $a < \ell$ then $\mathtt{state}(a, \alpha)$ takes a value in ω, and if $a = \ell$, then $\mathtt{state}(a, \alpha) = \mathtt{init} \notin \omega$.

At the first stage ε, where ε is the empty string, we first set $\mathtt{state}(a, \varepsilon) = \mathtt{init}$ for every $a \in \omega$. At the beginning of stage α we inductively assume that $\mathtt{state}(a, \beta)$ has already been defined for any $\beta < \alpha$. By our assumption, a Semmes condition $\mathbf{p} = (\mathcal{K}_{\alpha^-}, p^{\alpha-1}_{\alpha^-}, V_{\alpha-1})$ has also been constructed by the previous stage. Let ℓ be the length of $\mathcal{K}_{\alpha^-}$, that is, $\mathcal{K}_{\alpha^-}$ is of the form $(K_a, \sigma_a, i_a)_{a<\ell} \in \mathbb{Q}$ has been already constructed. At the a-th substage of stage α, we check whether the a-th strategy makes an action. Exactly one of the strategies acts at each stage.

Initial Action: If $\mathtt{state}(a, \alpha^-) = \mathtt{init}$, then $a = \ell$, so the ℓ-th strategy makes the following action.

(1) By Lemma 12, there are $\mathcal{K}_\ell = (K_\ell, \sigma_\ell, i_\ell)$, $p^\ell \sqsupseteq p^{\alpha-1}_\alpha$, $p^\alpha_\beta \sqsupseteq p^{\alpha-1}_\beta$, and $V_\alpha \supsetneq V_{\alpha-1}$ such that

$$((\mathcal{K}_a)_{a\leq\ell}, p^\ell, V_\alpha) \text{ extends } (\mathcal{K}_\alpha, p^{\alpha-1}_\alpha, V_{\alpha-1}),$$
$$(\mathcal{K}_\beta, p^\alpha_\beta, V_\alpha) \text{ is a Semmes conditions whenever } \beta \perp \alpha.$$

(2) To ensure $\mathcal{N}_\ell$, by the nowhere density condition (D2) for K_ℓ, since σ_ℓ is incomparable with any element in $V_{\alpha-1}$, there is a proper extension $q^\ell \sqsupset p^\ell$ such that

$$(\forall y^\ell \sqsupset q^\ell)(\forall \tau \in V_{\alpha-1})[\forall i < a \exists j \forall k \neg Q(y^\ell, \tau, i, j, k)].$$

Define $p^\alpha_\alpha = q^\ell$. By Observation 7 (1), $((\mathcal{K}_a)_{a\leq\ell}, p^\alpha_\alpha, V_\alpha)$ is still a Semmes condition, which extends the previous condition $\mathbf{p}$.

(3) Define $\mathtt{state}(\ell, \alpha) = 0$, and $\mathtt{state}(a, \alpha) = \mathtt{state}(a, \alpha^-)$ for $a \neq \ell$. Go to the next stage $\alpha + 1$.

The b-th Action: If $\mathtt{state}(a, \alpha) = b \in \omega$, then the a-strategy see if

$$(\forall b' \leq b)(\exists c \leq |\alpha|)\, S(\alpha, a, b', c).$$

If this does not hold, go to the next substage $a + 1$. If this condition is true, the a-th strategy acts as follows. By Observation 13, we get a length $a + 1$ shortening $((\mathcal{K}_b)_{b\leq a}, p^a, V_{\alpha-1}) \in \mathbb{S}$ of the previous condition $\mathbf{p}$. Now $\mathcal{K}_a$ is of the form $(K_a, \sigma_a, s_a) \in \mathbb{Q}$. By the density condition (D1) for K_a, there exists a proper extension $q^a \sqsupset p^a$ such that

$$(\forall j \leq b)(\exists k)\ Q(q^a, \sigma_a, i_a, j, k).$$

Define $p^\alpha_\alpha = q^a$. By Observation 7, $((\mathcal{K}_b)_{b\leq a}, p^\alpha_\alpha, V_{\alpha-1})$ is a Semmes condition, which is a shortening of the previous condition $\mathbf{p}$. Then, define $\mathcal{K}_\alpha = (\mathcal{K}_b)_{b\leq a}$, $V_\alpha = V_{\alpha-1}$, $\mathtt{state}(a,\alpha) = b+1$, $\mathtt{state}(i,\alpha) = \mathtt{state}(i,\alpha^-)$ for any $i < a$ and $\mathtt{state}(i,\alpha) = \mathtt{init}$ for any $i > a$. Go to the next stage $\alpha + 1$.

Outcomes: Put $V = \bigcup_{\alpha\in\omega^{<\omega}} V_\alpha$, and $\psi(x) = \bigcup_{s\in\omega} p^{x\restriction s}_{x\restriction s}$. Clearly, V is open since $(V_\alpha)_{\alpha\in\omega^{<\omega}}$ is a union of open open sets. Moreover, ψ is continuous since $(p^{x\restriction s}_{x\restriction s})_{s\in\omega}$ is increasing.

Lemma 14. *For any x, we have the following.*

$$x \notin U \iff \lim_{s\to\infty} \mathtt{state}(a, x \restriction s) \text{ converges for every } a.$$

Proof. ($\Rightarrow$) If $x \notin U$ then for any a there is b such that $(\exists c)S(x,a,b,c)$ fails. Then the b-th action never occur at any initial segment of x. Thus, we must have $\mathtt{state}(a, x\restriction s) \leq b$ for any s. If the state of a strategy does not change, then it does not injure any other strategy. Therefore, by induction, we can see $\lim_s \mathtt{state}(a, x \restriction s)$ converges for every a.

($\Leftarrow$) If $x \in U$ then there is a such that for every b, we have $(\exists c)S(x,a,b,c)$. Let a_0 be the smallest such a. Then it is easy to see that for any b, a_0 proceeds the b-th action at some stage s_b such that $\mathtt{state}(a_0, x \restriction s_b) = b$. In other words, $\lim_s \mathtt{state}(a_0, x\restriction s)$ diverges. □

One can also see that $\lim_{s\to\infty} \mathtt{state}(a, x \restriction s)$ converges for every a iff the a-th strategy acts at most finitely often for any a. We finally show the following.

Lemma 15. *For any x, we have the following.*

$$x \in U \iff \psi(x) \in f^{-1}[V].$$

Proof. ($\Leftarrow$) If $x \notin U$, then by Lemma 14, $\lim_s \mathtt{state}(a, x \restriction s)$ converges for every a. Then, for any a, there is a stage $\alpha_a \sqsubseteq x$ such that the strategy a proceeds the initial action at stage α_a and this action is never injured. Clearly, $(\alpha_a)_{a\in\omega}$ is strictly increasing. Then, by the initial action of a at stage α_a, for all $\tau \in V_{\alpha_a - 1}$ and $i < a$ we have $\exists j \forall k \neg Q(\psi(x), \tau, i, j, k)$ since $\psi(x)$ extends $p^{\alpha_a}_{\alpha_a}$. This means that $\psi(x) \notin f^*_i[V_{\alpha_a-1}]$ for any $i < a$. However, if $\psi(x) \in f^{-1}[V]$ then there are i and β such that $\psi(x) \in f^*_i[V_\beta]$. Let a be such that $i < a$ and $\beta < \alpha_a$. Then, $\psi(x) \in f^*_i[V_{\alpha_a-1}]$, a contradiction. Therefore, we get $\psi(x) \notin f^{-1}[V]$.

($\Rightarrow$) If $x \in U$, then by Lemma 14, there is a such that the a-th strategy acts infinitely often. Let a be the least such strategy. Then there is s such that a is never injured after $x \restriction s$. Assume that $\mathcal{K}_{x\restriction s}$ is of the form $(K^s_u, \sigma^s_u, i^s_u)_{u<\ell(s)}$, where we must have $a < \ell(s)$. Since a and hence any $u \leq a$ are never injured after $x \restriction s$, we have that $\sigma^s_u = \sigma^t_u$ and $i^s_u = i^t_u$ whenever $s \leq t$ and $u \leq a$. By the b-th action of a, we have $\forall j \leq b \exists k Q(\psi(x), \sigma^s_a, i^s_a, j, k)$ since $\psi(x)$ extends $p^{x\restriction s}_{x\restriction s}$. Since this holds for any b, we get $\forall j \exists k Q(\psi(x), \sigma^s_a, i^s_a, j, k)$. This means that $\psi(x) \in f^*_{i^s_a}[\sigma^s_a]$, and thus $\psi(x) \in f^*_{i^s_a}[V_{x\restriction s}] \subseteq f^{-1}[V]$ since $\sigma^s_a \in V_{x\restriction s}$. □

Lemma 15 shows that $f^{-1}[V]$ is $\mathbf{\Sigma}^0_3$-complete for some open set V. Hence, $f^{-1}[\omega^\omega \setminus V]$ is not $\mathbf{\Sigma}^0_3$ while $\omega^\omega \setminus V$ is $\mathbf{\Sigma}^0_2$. That is, $f^{-1}\mathbf{\Sigma}^0_2 \not\subseteq \mathbf{\Sigma}^0_3$. This concludes the proof. □

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