A Priority Argument in Descriptive Set Theory: The Proof of Semmes' Theorem

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1 Introduction

1.1 Notations

$[\sigma]$ is the clopen set generated by $\sigma \in \omega^{<\omega}$. If strings σ and τ are incomparable then we write $\sigma\perp\tau$. We write $f^{-1}\mathbf{\Sigma}^0_m \subseteq \mathbf{\Sigma}^0_n$ if the preimage of each $\mathbf{\Sigma}^0_m$ set under f is $\mathbf{\Sigma}^0_n$. Then, f is $\mathbf{\Sigma}^0_n$-measurable if and only if $f^{-1}\mathbf{\Sigma}^0_1 \subseteq \mathbf{\Sigma}^0_n$ holds. We also write $f \in \mathbf{dec}(\mathbf{\Sigma}^0_m/\mathbf{\Pi}^0_n)$ if f is decomposable into countably many $\mathbf{\Sigma}^0_m$-measurable functions on $\mathbf{\Pi}^0_n$ domains, that is, there is a countable $\mathbf{\Pi}^0_n$ partition $(X_i)_{i\in\omega}$ of the domain of f such that $f \upharpoonright X_i$ is $\mathbf{\Sigma}^0_m$-measurable for each $i \in \omega$.

2 Proof

Semmes [1] showed that the condition $f^{-1}\mathbf{\Sigma}^0_2 \subseteq \mathbf{\Sigma}^0_3$ implies $\mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Pi}^0_2)$. Clearly, the condition $f^{-1}\mathbf{\Sigma}^0_2 \subseteq \mathbf{\Sigma}^0_3$ always implies that f is $\mathbf{\Sigma}^0_3$-measurable. Thus, it suffices to show the following:

Theorem 1 (Semmes [1])**.** *Suppose that $f : \omega^\omega \to \omega^\omega$ is $\mathbf{\Sigma}^0_3$-measurable. Then,*

$$f \notin \mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Pi}^0_2) \implies f^{-1}\mathbf{\Sigma}^0_2 \not\subseteq \mathbf{\Sigma}^0_3.$$

If f is $\mathbf{\Sigma}^0_3$-measurable, then we can write the preimage $f^{-1}[\sigma]$ as $\bigcup_i f^{-1}[\sigma]_i$, where $f^{-1}[\sigma]_i$ is $\mathbf{\Pi}^0_2$ uniformly in σ and i. Given X, we define X'_σ as the set of all *irreducible points outside* σ, that is,

$$X'_\sigma = X \setminus \bigcup\{[\eta] : f|_{X\cap[\eta]\setminus f^{-1}[\sigma]} \in \mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Pi}^0_2)\}.$$

Moreover, given Y, we consider the following $Y^\star_{\sigma,i}$:

$$Y^\star_{\sigma,i} = \mathrm{cl}_Y(f^{-1}[\sigma]_i),$$

where $\mathrm{cl}_Z A$ is the topological closure of a set $A \cap Z$ in a space Z. Clearly, X'_σ and $Y^\star_{\sigma,i}$ are closed subsets of X and Y, respectively. We iterate this derivation procedure for any set $X \subseteq \omega^\omega$:

$$\begin{aligned} X^0_{\sigma,i} &= X, \\ X^{\alpha+1}_{\sigma,i} &= ((X^\alpha_{\sigma,i})'_\sigma)^\star_{\sigma,i}, \\ X^\alpha_{\sigma,i} &= \bigcap_{\beta<\alpha} X^\beta_{\sigma,i} \text{ if } \alpha \text{ is a limit ordinal.} \end{aligned}$$

Note that there is a countable ordinal $\gamma(\sigma,i)$ such that $X^{\gamma(\sigma,i)+1}_{\sigma,i} = X^{\gamma(\sigma,i)}_{\sigma,i}$ since $(X^\alpha_{\sigma,i})_\alpha$ is a decreasing sequence of closed sets. Clearly, $\gamma = \sup_{\sigma,i}\gamma(\sigma,i)$ is a countable ordinal since $\aleph_1$ is regular.

We divide the space X into three pieces. We first define the (σ,i)-*kernel* to be $K_{\sigma,i}X = X^\gamma$. We say that a point $x \in X$ is *generic* if for every (σ,i), either $x \in f^{-1}[\sigma]$ or there exists α such that $x \in (X^\alpha_{\sigma,i})'_\sigma \setminus X^{\alpha+1}_{\sigma,i}$. Define $C = \bigcup_{\sigma,i} K_{\sigma,i}X$, B to be the set of all generic points $y \notin C$, and A to be the set of all other points.

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Lemma 2. *Suppose $x \notin C$. Then, $x \in A$ if and only if there are $(\sigma_0, i_0), (\sigma_1, i_1)$ and $\alpha_0, \alpha_1 < \gamma$ such that $[\sigma_0] \cap [\sigma_1] = \emptyset$ and $x \in X^{\alpha_n}_{\sigma_n, i_n} \setminus (X^{\alpha_n}_{\sigma_n, i_n})'_{\sigma_n}$ for every $n \in \{0, 1\}$.*

Proof. The condition $x \notin C$ means that for every (σ, i), there is $\alpha(\sigma, i)$ such that $x \in X^{\alpha(\sigma,i)}_{\sigma,i} \setminus X^{\alpha(\sigma,i)+1}_{\sigma,i}$. Note also that x is not generic (in particular, $x \notin B$) if and only if there exists (σ_0, i_0) such that $x \notin f^{-1}[\sigma_0]$ and for all α, $x \in (X^{\alpha}_{\sigma_0,i_0})'_{\sigma_0}$ implies $x \in X^{\alpha+1}_{\sigma_0,i_0}$. Therefore, if $x \in A$ (i.e., $x \notin B \cup C$) and $\alpha_0 = \alpha(\sigma_0, i_0)$ then $x \notin f^{-1}[\sigma_0]$ and $x \in X^{\alpha_0}_{\sigma_0,i_0} \setminus (X^{\alpha_0}_{\sigma_0,i_0})'_{\sigma_0}$. Since $f(x) \notin [\sigma_0]$, by zero-dimensionality of ω^ω, there is σ_1 such that $[\sigma_0] \cap [\sigma_1] = \emptyset$ and $f(x) \in [\sigma_1]$. Now note that if $x \in f^{-1}[\sigma_1]$, then $x \in \mathrm{cl}_Z f^{-1}[\sigma_1]_{i_1}$ for any i_1 whenever $x \in Z$. Therefore, for all α, $x \in (X^{\alpha}_{\sigma_1,i_1})'_{\sigma_1}$ implies $x \in X^{\alpha+1}_{\sigma_1,i_1}$; and hence by putting $\alpha_1 = \alpha(\sigma_1, i_1)$, we get $x \in X^{\alpha_1}_{\sigma_1,i_1} \setminus (X^{\alpha_1}_{\sigma_1,i_1})'_{\sigma_1}$ as desired.

We next verify the converse direction. Let x be a point in X satisfying the latter condition. Since $[\sigma_0] \cap [\sigma_1] = \emptyset$, we must have $x \notin f^{-1}[\sigma_n]$ for some $n < 2$. Then, the pair (σ_n, i_n) witnesses that x is not generic since if $\alpha < \alpha_n$ then $x \in X^{\alpha_n+1}_{\sigma_n,i_n}$ and if $\alpha \geq \alpha_n$ then $x \notin (X^{\alpha_n}_{\sigma_n,i_n})'$. Hence, under our assumption that $x \notin C$, we have $x \in A$ as desired. □

Lemma 3. *$A \in \mathbf{\Delta}^0_3$, $B \in \mathbf{\Delta}^0_3$, and $C \in \mathbf{\Sigma}^0_2$.*

Proof. Clearly, $C \in \mathbf{\Sigma}^0_2$. Hence, by the previous lemma, A is the difference of two $\mathbf{\Sigma}^0_2$ sets. □

Lemma 4. *$f|_B$ is $\mathbf{\Sigma}^0_2$-measurable.*

Proof. Suppose that $x \in B$. Then, x is non-generic, and $x \notin C$. Let $\alpha(\sigma, i)$ witness $x \notin C$ as in the proof of Lemma 2. If $x \in f^{-1}[\sigma]$, as mentioned in the proof of Lemma 2, for all i and α, $x \in (X^{\alpha}_{\sigma,i})'_{\sigma}$ implies $x \in X^{\alpha+1}_{\sigma,i}$; hence $x \in X^{\alpha(\sigma,i)}_{\sigma,i} \setminus (X^{\alpha(\sigma,i)}_{\sigma,i})'_{\sigma}$. Therefore, by our definition of genericity, it is not hard to see that for any (σ, i), $x \in f^{-1}[\sigma]$ if and only if there exists $\alpha < \gamma$ such that $x \in X^{\alpha}_{\sigma,i} \setminus (X^{\alpha}_{\sigma,i})'_{\sigma}$. The latter condition is clearly $\mathbf{\Sigma}^0_2$. □

Lemma 5. *$f|_A \in \mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Pi}^0_2)$.*

Proof. Let $(\sigma_n, i_n, \alpha_n)_{n<2}$ be a witness of $x \in A$ as in Lemma 2. By our definition of the derivative, for $Z = X^{\alpha_0}_{\sigma_0,i_0} \cap X^{\alpha_1}_{\sigma_1,i_1}$, there is $\eta \in \omega^{<\omega}$ such that $x \in Z \cap [\eta]$ such that $f|_{Z \cap [\eta] \setminus f^{-1}[\sigma_i]} \in \mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Pi^0_2})$. Since zero-dimensionality of $\mathcal{Y}$ implies that $f^{-1}[\sigma_n]$ is $\mathbf{\Delta}^0_3$, and $[\sigma_0] \cap [\sigma_1] = \emptyset$ implies $([\eta] \setminus f^{-1}[\sigma_0]) \cup ([\eta] \setminus f^{-1}[\sigma_1]) = [\eta]$, we also have $f|_{Z \cap [\eta]} \in \mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Delta^0_3})$. Since there are only countably many candidates for such a witness $(\sigma_n, i_n, \alpha_n)_{n<2}$ (because $\alpha_n < \gamma$), $f|_A$ is decomposable into countably many $\mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Delta^0_3})$-functions on closed domains $(X^{\alpha_0}_{\sigma_0,i_0} \cap X^{\alpha_1}_{\sigma_1,i_1} \cap [\eta])_{\sigma_0,i_0,\alpha_0,\sigma_1,i_1,\alpha_1,\eta}$. □

Lemma 6 (Existence of a Nonempty Kernel)**.** *If $f \notin \mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Pi}^0_2)$, then C is nonempty, that is, there is (σ, i) such that the (σ, i)-kernel $K_{\sigma,i}X$ is nonempty.*

Proof. If C is empty, then $f = f|_A \cup f|_B$. By Lemmata 4 and 5 we have $f|_A, f|_B \in \mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Pi}^0_2)$. Since A and B are $\mathbf{\Delta}^0_3$ by Lemma 3, we also have $f \in \mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Pi}^0_2)$. □

The $'$-derivation procedure ensures that f is nowhere decomposable on $K_{\sigma,i}X$ outside $f^{-1}[\sigma]$, i.e., $f|_{K_{\sigma,i}X \cap [\eta] \setminus f^{-1}[\sigma]} \notin \mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Pi}^0_2)$ for all $\eta \in \omega^{<\omega}$. We will need to construct a decreasing chain of nonempty kernels. We utilize this nowhere-decomposability condition to extend such a chain, that is, the new kernel applied the previous kernel is still nonempty. The $\star$-derivation procedure ensures a density condition for $K_{\sigma,i}X$. Indeed, $K_{\sigma,i}X$ has the following "*both-density*" property.

1. $f^{-1}[\sigma]_i$ is dense in $K_{\sigma,i}X$.
2. $f^{-1}[\tau]_j$ is nowhere dense in $K_{\sigma,i}X$ whenever $\sigma \perp \tau$.

This is because the item (1) is ensured by the $\star$-derivation procedure, and if the item (2) fails, then $f^{-1}[\sigma]_i$ and $f^{-1}[\tau]_j$ are dense $\mathbf{\Pi}^0_2$ sets in $K_{\sigma,i} \cap [\eta]$ for some $\eta \in \omega^{<\omega}$, that is, intersections of sequences of dense open sets in $K_{\sigma_i} \cap [\eta]$. By the Baire category theorem, $f^{-1}[\sigma]_i$ and $f^{-1}[\tau]_j$ have an intersection. However, $f^{-1}[\sigma]_i \cap f^{-1}[\tau]_j$ must be empty since $\sigma \perp \tau$.

We assume that a chain $K_0 \supseteq K_1 \supseteq \cdots \supseteq K_{a-1}$ of kernels have been already constructed, where such a chain is associated with a single open set V_{a-1} such that f is nowhere decomposable on K_l outside $f^{-1}[V_{a-1}]$ for any $l < a$. We call such $(K_l)_{l<a}$ be a *good chain of kernels with an indecomposability-witness* V_{a-1}. Now we would like to make sure the (local) extendability of this chain by adding a new kernel K_a and by increasing an indecomposability-witness $V_a \supsetneq V_{a-1}$.

Lemma 7 (Extendability of a Good Chain)**.** *Suppose that $(K_l)_{l<a}$ be a good chain of kernels with an indecomposability-witness V. Then, there exist η and σ such that $V \cap [\sigma] = \emptyset$ such that f is nowhere decomposable on $K_l \cap [\eta]$ outside $f^{-1}[V \cup [\sigma]]$ for any $l \leq a$, where K_a is of the form $K_{\sigma,i}K$ for some closed set $K \subseteq K_{a-1}$ and $t \in \omega$. In particular, $(K_l \cap [\eta])_{l\leq a}$ is a good chain of kernels with an indecomposability-witness $V \cup [\sigma]$.*

Proof. Let C_0 and C_1 be a pair of pairwise disjoint clopen sets. If K_l contains a dense subset of reducible points outside $f^{-1}[V \cup C_n]$ for every $n < 2$, i.e., $f|_{[\eta]\cap K_l \setminus f^{-1}[V\cap C_n]} \in \mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Pi}^0_2)$, then by combining these two decomposable functions, we can see that f is somewhere decomposable on K_l outside $f^{-1}[V]$. By the similar argument, if f is nowhere decomposable on K_l outside $f^{-1}[V]$, given a collection $(C_k)_{k<m}$ of pairwise disjoint clopen sets, there can be at most one $k < m$ such that K_l contains a dense subset of reducible points outside $f^{-1}[V \cup C_n]$. Hence, if m is sufficiently big, there is $k < m$ such that f is nowhere decomposable on $K_l \cap [\eta]$ outside $f^{-1}[V \cup C_n]$ for any $l \leq a$.

By Lemma 6, we have an arbitrarily long chain $K_{a-1} \supseteq K_a^0 := K_{\sigma_0,i_0}K_{a-1} \supseteq K_a^1 := K_{\sigma_1,i_1}K_a^0 \supseteq \dots$ of kernels, where we apply Lemma 6 to $f|_{K_a^n \setminus f^{-1}[V\cup U_n]}$ to obtain K_a^{n+1}, where $U_n = \bigcup_{j\leq n}[\sigma_j]$. Then, there is n such that $K_a := K_a^n$ satisfies the desired condition. $\square$

We are now ready to prove Theorem 1. Given a $\mathbf{\Sigma}^0_3$ set $U \subseteq \omega^\omega$ we will construct a continuous function $\psi : \omega^\omega \to \omega^\omega$ and an set $V \subseteq \omega^{<\omega}$ of strings such that $x \in S$ iff $\psi(x) \in f^{-1}[V]$ for every $x \in \omega^\omega$, where $[V]$ is the open set generated by V. Thus, this will ensure that $f^{-1}[V]$ is $\mathbf{\Sigma}^0_3$-complete for some set V of strings, which implies $f^{-1}\mathbf{\Sigma}^0_2 \not\subseteq \mathbf{\Sigma}^0_3$. We first describe $\mathbf{\Sigma}^0_3$ sets $f^{-1}[V]$ and U as follows:

$$\begin{aligned} x \in U &\iff (\exists a)(\forall b)(\exists c)\ S(x,a,b,c), \\ y \in f^{-1}[V] &\iff (\exists \sigma \in V)(\exists i)(\forall j)(\exists k)\ Q(y,\sigma,i,j,k). \end{aligned}$$

The a-th requirements are given as follows:

$$\begin{aligned} \mathcal{N}_a : \quad & (\forall a' < a)(\exists b)(\forall c)\ S(x,a,b,c) \implies (\forall \sigma \in V_{s_a})(\forall i < a)(\exists j)(\forall k)\ Q(\psi(x),\sigma,i,j,k), \\ \mathcal{P}_a : \quad & (\forall b)(\exists c)\ S(x,a,b,c) \implies (\exists \sigma_a)(\exists i_a)(\forall j)(\exists k)\ Q(\psi(x),\sigma_a,i_a,j,k), \end{aligned}$$

where s_a is the first stage at which the a-th strategy acts (after the last initialization which may be caused by a higher-priority strategy). If a is the smallest witness for $y \in U$, then the requirement $\mathcal{P}_a$ ensures that $\psi(y) \in f^{-1}[\sigma_a]_{i_a}$. The a-th strategy will put σ_a into the set V in the construction, so by the requirement $\mathcal{P}_a$, we get that $y \in U$ implies $\psi(y) \in f^{-1}[V]$. If $y \notin U$, then for every a, the premise of the requirement $\mathcal{N}_a$ must be true, and then, the combination of requirements $\mathcal{N}_a$'s will eventually ensure that $\psi(y) \notin f^{-1}[V]$. Thus, it suffices to describe the strategy to satisfy the requirements $\mathcal{P}_a$ and $\mathcal{N}_a$.

At each stage s in our construction, we will have a chain $(K_a[s])_{a<l[s]}$ with a set V_s with witnesses $(\eta_a,\sigma_a,i_a)_{a<l[s]}$ by using Lemma 7. By the property of a kernel mentioned above, we have the following "*both-density*" property for every $a < l[s]$:

$$\begin{aligned} &\text{(D1)} \qquad (\forall \theta \succeq \eta_a[s])(\exists y \succ \theta)[\forall j \exists k Q(y,\sigma_a[s],i_a[s],j,k)], \\ &\text{(D2)} \qquad (\forall i)(\forall \theta)(\exists \rho \succeq \theta)(\forall y \succ \rho)[\exists j \forall k \neg Q(y,\tau,i,j,k)] \text{ if } \tau \perp \sigma_a[s], \end{aligned}$$

where θ, ρ ranges over $T_a[s]$ and y ranges over $K_a[s]$. The condition (D1) states that $f^{-1}[\sigma_a[s]]_{i_a[s]}$ is dense in $K_a[s]$, and the condition (D2) states that $f^{-1}[\tau]_i$ is nowhere dense in $K_a[s]$ whenever $\tau \perp \sigma_a[s]$.

Proof of Theorem 1. Suppose that $x \in \omega^\omega$ is given. The state of the a-th strategy at stage s is written as $\texttt{state}(a,s)$ which takes a value in $\omega \cup \{\texttt{init}\}$. We first set $\texttt{state}(a,0) = \texttt{init}$ for every $a \in \omega$. At stage s we inductively assume that $\texttt{state}(a,s)$ and V_s have already been defined. Let $l[s]$ be the length of the current good chain, that is, a good chain $(K_a[s])_{a<l[s]}$ of kernels with $[V_s]$ has been already constructed. Every strategy $a \leq l[s]$ is *eligible to act* at stage s.

Initial Action: Suppose that $\mathtt{state}(a, s) = \mathtt{init}$.

1. To ensure $\mathcal{N}_a$, by using (D2), choose an extension $\psi(x \upharpoonright s+1) \succ \psi(x \upharpoonright s)$ such that

$$(\forall y \succ \psi(x \upharpoonright s))(\forall \tau \in V_s)[\forall i < a \exists j \forall k \neg Q(y, \tau, i, j, k)].$$

2. Then the a-strategy defines (η_a, σ_a, i_a) satisfying the both-density condition (D1) and (D2), and enumerate σ_a into our set V_{s+1}.

3. Define $K_{a'}[s+1] = K_{a'}[s] \cap [\eta_a]$ for every $a' < a$ and $K_a := K^*_a \cap [\eta_a] \cap [\psi(y \upharpoonright s)]$. Define $\mathtt{state}(a, s+1) = 0$.

The b-th Action: Suppose that $\mathtt{state}(a, s) = b$. We say that the a-strategy *requires attention* at stage s if

$$(\forall b' \leq b)(\exists c \leq s)\ S(a, b', c).$$

If such a stage occurs, by using (D1), we may find $\psi(y \upharpoonright s) \in K_a$ extending $\psi(y \upharpoonright s-1)$ such that

$$(\forall j \leq b)(\exists k)\ Q(\psi(x \upharpoonright s), \sigma_a, i_a, j, k).$$

Then, the a-strategy makes a restraint $K_a[s+1] := K_a[s] \cap [\psi(y \upharpoonright s)]$, changes the state as $\mathtt{state}(a, s+1) = b+1$, and injures all lower-priority strategies $a' > a$ by destroying $K_{a'}$ and setting $\mathtt{state}(a') = \mathtt{init}$.

Outcomes: Put $V = \bigcup_s V_s$. First we can see that $x \notin S$ if and only if $\lim_s \mathtt{state}(a, s)$ converges for every a. If $x \notin S$ then for any a there is b such that $(\exists c)S(x, a, b, c)$. Thus, we must have $\mathtt{state}(a, s) \leq b$ for any stage s. If the state of a strategy does not change, then it does not injure any other strategy. Therefore, by induction, we can see $\lim_s \mathtt{state}(a, s)$ converges for every a. If $x \in S$ then there is a such that for every b, we have $(\exists c)S_{x,a,b,c}$. Let a_0 be the smallest such a. Then it is easy to see that for any b, a_0 requires attention at some stage s_b such that $\mathtt{state}(a_0, s_b) = b$. In other words, $\lim_s \mathtt{state}(a, s)$ does not converge.

Now we have two cases:

1. Suppose that $\lim_s \mathtt{state}(a, s)$ converges for every a. We claim that $\psi(x) \notin f^{-1}[V]$. Let $s_a \geq a$ be the first stage such that the strategy a is never injured after stage s_a. Then, by our construction, for all $\tau \in V_{s_a}$ and $i < a$ we have $\exists j \forall k \neg Q(\psi(x), \tau, j, k)$. However, if $z \in f^{-1}[V]$ then there is s such that $z \in f^{-1}[V_s]$. Therefore, we get $\psi(x) \notin f^{-1}[V]$.

2. Otherwise, there is a such that the a-strategy acts infinitely often. Then we claim that $\psi(x) \in f^{-1}[V]$. By our construction, we have $\forall j \exists k Q(\psi(x), \sigma_a, i_a, j, k)$.

□

References

[1] Semmes, B., A game for the Borel functions. *Ph.D. thesis, Universiteit van Amsterdam.* 2009.