HIGHER RANDOMNESS AND LIM-SUP FORCING WITHIN AND BEYOND HYPERARITHMETIC

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Abstract. We develop arboreal forcing in the context of hyperarithmetical randomness theory. In any transitive model of the Kripke-Platek set theory obtained as the companion of a Spector pointclass, we show that certain kinds of Σ_1-definable lim-sup tree creature forcings have the Σ_1-continuous reading of names, the Σ_1-fusion property, and so on. In this way, we show that the shape of Cichoń's diagram in hyperarithmetic theory is different from those in computability theory and set theory.

1. Summary

1.1. **Introduction.** In recent years, the theory of *hypercomputation* (it has been widely known as *generalized recursion theory* in the 20th century) have again received a lot of attention in computability theory and theoretical computer science. Of course, the theory of hypercomputation (in particular, recursion theory beyond hyperarithmetic, with many kinds of higher-type and ordinal computation models) was one of the central topics in computability theory in the mid to late 20th century, and an enormous number of mathematically deep researches on hypercomputation have been accomplished in parallel with developments of descriptive set theory, infinitary logic, and fine structure theory of Gödel's constructible universe (see also some textbooks [3, 16, 19, 38, 48] for developments of generalized recursion theory in the second half of the 20th century).

The randomness notion in the hyperarithmetical level was first introduced by Martin-Löf and developed by several researchers (see [21, 47, 52]) in the early days. After a blank period, very recently, algorithmic randomness researchers restarted to develop higher randomness theory based on modern notions and ideas from advanced study on algorithmic randomness ([4, 10, 11, 12, 31, 36, 37]; see also Nies [40, Chapter 9]). Technically, typical separation results in modern higher randomness theory are obtained by using standard arboreal forcing arguments such as Sacks forcing and clumpy Sacks forcing (infinite equal forcing). Indeed, some parts of modern algorithmic randomness theory have a strong connection to forcing theory, in particular, to the theory of cardinal characteristics (see Rupprecht [45, 46], Brendle et al. [6]). In modern set theory, these kinds of arboreal forcing, idealized forcing, and creature forcing have been deeply investigated in a sophisticatedly systematic manner (see Bartozyński-Judah [2], Rosłanowski-Shelah [44] and Zapletal [57] for instance). Our aim is to reflect the modern development of set theory in higher computability theory. In particular, we develop the general framework of *lim-sup tree forcing* in admissible sets generated by Spector pointclasses. We also propose a list of open problems.

2. Preliminaries

2.1. **Represented Spaces.** In this paper, most objects are coded by an element of ω^ω; e.g., a Borel set is coded by a Borel code. A *represented space* (see also [54]) is a set $\mathcal{X}$ equipped with a partial surjection $\rho_\mathcal{X} :\subseteq \omega^\omega \to \mathcal{X}$.[1] For instance, the pair $(\mathcal{B}, \mathrm{BC})$ of Borel sets $\mathcal{B}$ and Borel coding $\mathrm{BC} :\subseteq \omega^\omega \to \mathcal{B}$ forms a represented space.

Example 2.1. The following are examples of represented spaces (see also [29, 28]):

(1) (Hereditarily Countable Sets) The collection $\mathbf{H}_{\aleph_1}$ of all hereditarily countable sets is represented by $\rho_{\mathbf{HC}} :\subseteq \omega^\omega \to \mathbf{H}_{\aleph_1}$ such that

$$\rho_{\mathbf{HC}}(p) = X \iff (\omega, \{(n,m) \in \omega^2 : p(\langle n,m\rangle) = 1\}) \text{ is isomorphic to } (X, \in).$$

Here, $\langle \cdot, \cdot \rangle$ is a fixed effective bijection between ω^2 and ω.

(2) (Borel Coding) One can consider an ω^ω-representation of the class of all Borel sets (or all $\mathbf{\Sigma}^0_\alpha$ sets for a fixed rank $\alpha < \omega_1$) in a (recursively presented) Polish space $\mathcal{X}$. Formally, we introduce codings (representations) $\sigma^0_n : \omega^\omega \to \mathbf{\Sigma}^0_n(\mathcal{X})$ and $\pi^0_n : \omega^\omega \to \mathbf{\Pi}^0_n(\mathcal{X})$ in the following inductive way:

$$\sigma^0_1(p) = \bigcup_{m\in\omega} B_{p(m)}, \quad \pi^0_n(p) = \mathcal{X} \setminus \sigma^0_n(p), \quad \sigma^0_n(p) = \bigcup_{k\in\omega} \pi^0_n(p^{[k]}).$$

Here, $(B_n)_{n\in\omega}$ is a countable basis of $\mathcal{X}$, and the k-th section $p^{[k]}$ is defined by $p^{[k]}(m) = p(\langle k,m\rangle)$.

(3) Let $\mathcal{N}$ be the set of all null G_δ sets, and $\mathcal{M}$ be the set of all meager F_σ sets. Then, $\mathcal{N}$ and $\mathcal{M}$ are represented by the following representations:

$$\rho_\mathcal{N}(p) = \pi^0_2(p) = \bigcap_{n\in\omega} \sigma^0_1(p^{[n]}),$$
$$\mathrm{dom}(\rho_\mathcal{N}) = \{p : \lambda(\sigma^0_1(p^{[n]})) = 2^{-n} \text{ for all } n \in \omega\}.$$
$$\rho_\mathcal{M}(p) = \sigma^0_2(p) = \bigcup_{n\in\omega} \pi^0_1(p^{[n]}),$$
$$\mathrm{dom}(\rho_\mathcal{M}) = \{p : \pi^0_1(p^{[n]}) \text{ is nowhere dense for all } n \in \omega\}.$$

Here, λ is the Lebesgue measure on 2^ω.

Hereafter, for a pointclass Γ in a space $\mathcal{X}$ (that is, Γ is a collection of subsets of $\mathcal{X}$), we write $\check{\Gamma}$ for the dual of a pointclass Γ (that is, $\check{\Gamma} = \{A \subseteq \mathcal{X} : \mathcal{X} \setminus A \in \Gamma\}$), and Δ for the ambiguous pointclass $\Gamma \cap \check{\Gamma}$ as usual (see [39]). For a pointclass Γ in ω, we say that a point $x \in \omega^\omega$ is in Δ if $\{\langle m,n\rangle : x(m) = n\} \in \Delta$.

Definition 2.2 (see also [28])**.** Let $(\mathcal{X}, \rho_\mathcal{X})$ be a represented space, and Γ be a pointclass.

(1) We say that a point x in a represented space $(\mathcal{X}, \rho_\mathcal{X})$ is Δ-*computable* if x has a Δ-name, that is, there is $p \in \omega^\omega \cap \Delta$ such that $x = \rho_\mathcal{X}(p)$.

[1] To develop the framework of higher randomness theory, the author [28] used $\mathcal{O}\omega$-representations rather than ω^ω-representations since the paper [28] covers partial objects such as the Martin-Löf representation of Lebesgue null sets. If we wish to cover such a partial object in higher computability theory, we cannot replace $\mathcal{O}\omega$-representability with ω^ω-representability; however, in this article, we only work with total objects such as the Schnorr representation of Lebesgue null sets.

(2) A Δ-*computation* is a Γ-subset Φ of $\omega^\omega \times \omega^2$. We think of such Φ as a partial Γ-measurable function $\Phi :\subseteq \omega^\omega \to \omega^\omega$ ([28, 39]) by declaring that $\Phi(x) = p$ if the following holds:

$$p(m) = n \iff (x, m, n) \in \Phi.$$

(3) We say that $x \in \mathcal{X}$ *is* Δ-*reducible to* $y \in \omega^\omega$, or $y \in \omega^\omega$ Δ-*computes* $x \in \mathcal{X}$ (written as $x \leq_\Delta y$) if x has a $\Delta(y)$-name. In other words, there is a Δ-computation Φ such that $\Phi(y)$ is a name of x, that is, $\Phi(y) \in \rho_{\mathcal{X}}^{-1}\{x\}$.

Note that Δ-reducibility is equivalent to *relative* Δ-*definability* if $\mathcal{X} = 2^\omega$. Many natural reducibility notions in generalized recursion theory are obtained as effective measurable maps with respect to reasonable Σ-pointclasses (typically, Spector pointclasses). A *Spector pointclass* is a collection Γ of pointsets including Σ^0_1, closed under $\wedge$, $\vee$, $\exists^\omega$, and $\forall^\omega$, has the substitution property, ω-parametrized, and normed (see also Moschovakis [39] and Kechris [23, 24]). The notion of a Spector pointclass covers many kinds of generalized computations such as hyperarithmetical definability, (the 1- and 2-envelops of) finite type computability, and infinite-time-Turing-machine computability. For instance, the class of hyperarithmetical reductions is clearly equivalent to that of partial effective Π^1_1-measurable maps. Of course, there are many known natural computability-theoretic and descriptive set-theoretic pointclasses between Π^1_1 and Δ^1_2 (see [7, 8]), and hence, we have many natural reducibility notions between hyperarithmetical (Δ^1_1) reducibility and Δ^1_2-reducibility.

2.2. **Tukey Morphism and Muchnik Reduction.** A *Vojtáš triple* (see [5, 13]) is a multi-valued map $A : A_- \rightrightarrows A_+$ between represented spaces, that is, A_- and A_+ are represented via surjections $\rho_A^- : \widehat{A}_- \to A_-$ and $\rho_A^+ : \widehat{A}_+ \to A_+$ where $\widehat{A}_-$ and $\widehat{A}_+$ are subsets of ω^ω. The *dual* of A is a multi-valued map $A^\perp : A_+ \rightrightarrows A_-$ defined by $y \in A^\perp(x)$ if and only if $x \notin A(y)$. For Vojtáš triples A and B, a Δ-*Tukey morphism* from A to B is a pair of Γ-measurable maps $H : \widehat{A}_- \to \widehat{B}_-$ and $K : \widehat{B}_+ \to \widehat{A}_+$ such that

$$\text{if } G \text{ is a realizer of } B, \text{ then } K \circ G \circ H \text{ is a realizer of } A.$$

Here, a *realizer* of a multi-valued map $A : A_- \rightrightarrows A_+$ is a single valued map $\alpha : \widehat{A}_- \to \widehat{A}_+$ such that for a given name x of an instance of A, $\alpha(x)$ returns a name of an A-solution of the instance, that is, $\rho_A^+ \circ \alpha(x) \in A \circ \rho_A^-(x)$ for all $x \in \widehat{A}_-$. If a Δ-Tukey morphism exists from A to B, we write $A \leq_{\Delta\mathrm{sW}} B$ or simply $A \longrightarrow B$. Note that if $A \longrightarrow B$, then $||A|| \leq ||B||$, where $||C||$ is the associated cardinal of a Vojtáš triple $C : C_- \rightrightarrows C_+$ defined by

$$||C|| := \min\{\mathrm{card}(F) : (\forall x \in C_-)(\exists y \in F)\; y \in C(x)\}.$$

For a Vojtáš triple A, let $[A]_\Delta$ be the set of all instances $x \in A_-$ such that $A(x)$ has no Δ-computable solution, that is,

$$[A]_\Delta = \{x \in A_- : (\forall y \in A_+)\; y \leq_\Delta 0 \;\to\; y \notin A(x)]\}.$$

One can easily check that $[A^\perp]_\Delta$ is the set of all $y \in A_+$ such that y is an A-solution to all Δ-computable instances $x \in A_-$, that is,

$$[A^\perp]_\Delta = \{y \in A_+ : (\forall x \in A_-)\; x \leq_\Delta 0 \;\to\; y \in A(x)]\}.$$

For a represented space $\mathcal{X}$, the Δ-*degree spectrum* (see [30]) of $S \subseteq \mathcal{X}$ is the $\leq_\Delta$-upward closure of S, that is, $\{y \in 2^\omega : (\exists x \in S)\ x \leq_\Delta y\}$. The Δ^0_1-degree spectrum of $[A]_{\Delta^0_1}$ is also called the *Turing norm* of A (see Rupprecht [45, Definition IV.1]). The Turing norm is also studied by Brendle et al. [6]. We say that $P \subseteq \mathcal{X}$ is Δ-*Muchnik reducible to* $Q \subseteq \mathcal{Y}$ (written as $P \leq_{\Delta\mathrm{w}} Q$) if the Δ-degree spectrum of Q is included in that of P, that is,

$$(\forall y \in Q)(\forall q \in 2^\omega)\ [y \leq_\Delta q\ \rightarrow\ (\exists x \in P)\ x \leq_\Delta q].$$

Proposition 2.3. *Let A and B be any Vojtáš triples. If $A \leq_{\Delta\mathrm{sW}} B$ holds, then we have $[B]_\Delta \leq_{\Delta\mathrm{w}} [A]_\Delta$ and $[A^\perp]_\Delta \leq_{\Delta\mathrm{w}} [B^\perp]_\Delta$.*

Proof. Assume that $A \leq_{\Delta\mathrm{sW}} B$ via a Δ-Tukey morphism (H, K). Fix a real $q \in 2^\omega$. We first claim that for any $x \leq_\Delta q$, $x \in [A]_\Delta$ implies $H(x) \in [B]_\Delta$, that is, H is a Δ-Muchnik reduction witnessing $[B]_\Delta \leq_{\Delta\mathrm{w}} [A]_\Delta$. If $H(x) \notin [B]_\Delta$, then $B(H(x))$ has a Δ-computable solution y. Since (H, K) is a Δ-Tukey morphism, $K(y)$ is a solution to $A(x)$. Moreover, $K(y)$ is Δ-computable, since $K(y) \leq_\Delta y \leq_\Delta 0$. However, this contradicts our assumption that $A(x)$ is has no Δ-computable solution. We next claim that for any $y \leq_\Delta q$, $y \in [B^\perp]_\Delta$ implies $K(y) \in [A^\perp]_\Delta$, that is, K is a Δ-Muchnik reduction witnessing $[A^\perp]_\Delta \leq_{\Delta\mathrm{w}} [B^\perp]_\Delta$. To see this, let x be a Δ-computable instance of A. Then, $H(x)$ is an Δ-computable instance of B; therefore, y is a solution to $B(H(x))$ since $y \in [B^\perp]_\Delta$. Therefore, $K(y)$ is a solution to $A(x)$ since (H, K) is a Δ-Tukey morphism. Consequently, $K(y)$ is an A-solution to all Δ-computable instances. □

By $\mathcal{M}$ and $\mathcal{N}$ we denote σ-ideals consisting of meager sets and null sets endowed with the representations $\rho_\mathcal{M}$ and $\rho_\mathcal{N}$, respectively. We identify a Vojtáš triple with a triple (A_-, A_+, A), where we think of A as a relation $A \subseteq A_- \times A_+$ rather than a multi-valued map. Let $\mathcal{J}$ be a represented σ-ideal (e.g., $\mathcal{M}$ and $\mathcal{N}$). Consider the following Vojtáš triples (see also Rupprecht [45, Examples III.6 and 7]):

$$\begin{aligned} \mathsf{Add}\mathcal{J} &:= (\mathcal{J}, \mathcal{J}, \not\supseteq), & \mathsf{Cof}\mathcal{J} &:= \mathsf{Add}\mathcal{J}^\perp = (\mathcal{J}, \mathcal{J}, \subseteq), \\ \mathsf{Cov}\mathcal{J} &:= (2^\omega, \mathcal{J}, \in), & \mathsf{Non}\mathcal{J} &:= \mathsf{Cov}\mathcal{J}^\perp = (\mathcal{J}, 2^\omega, \not\ni), \\ \mathsf{B} &:= (\omega^\omega, \omega^\omega, \not\geq^*), & \mathsf{D} &:= \mathsf{B}^\perp = (\omega^\omega, \omega^\omega, \leq^*), \\ \mathsf{IE} &:= (\omega^\omega, \omega^\omega, =^\infty), & \mathsf{ED} &:= \mathsf{IE}^\perp = (\omega^\omega, \omega^\omega, \neg =^\infty), \\ \mathsf{IE}_h &:= ([h], [h], =^\infty), & \mathsf{ED}_h &:= \mathsf{IE}_h^\perp = ([h], [h], \neg =^\infty). \end{aligned}$$

Here, $f \leq^* g$ if and only if $f(n) \leq g(n)$ for almost all $n \in \omega$; $f =^\infty g$ if and only if $f(n) = g(n)$ for infinitely many $n \in \omega$, and $[h]$ is the represented subspace of ω^ω consisting of all h-bounded functions, that is, $[h] = \{g \in \omega^\omega : (\forall n \in \omega)\ g(n) < h(n)\}$ whose representation is induced from the identical representation of ω^ω.

Definition 2.4 (see also Brendle et al. [6])**.** Let $x \in 2^\omega$.

(1) x is Δ-$\mathcal{J}$-*engulfing* if there is $y \leq_\Delta x$ such that $y \in [\mathsf{Add}\mathcal{J}]_\Delta$, that is, x Δ-computes a $\mathcal{J}$-null set which covers all Δ-computable $\mathcal{J}$-null sets.
(2) x is *not* Δ-$\mathcal{J}$-*low* if there is $y \leq_\Delta x$ such that $y \in [\mathsf{Cof}\mathcal{J}]_\Delta$, that is, x Δ-computes a $\mathcal{J}$-null set which is not covered by any Δ-computable $\mathcal{J}$-null sets.
(3) x is *above-Δ-$\mathcal{J}$-quasigeneric* if there is $y \leq_\Delta x$ such that $y \in [\mathsf{Cov}\mathcal{J}]_\Delta$, that is, x Δ-computes a real which avoids all Δ-computable $\mathcal{J}$-null sets.

(4) x is *weakly Δ-$\mathcal{J}$-engulfing* if there is $y \leq_\Delta x$ such that $y \in [\mathsf{Non}\mathcal{J}]_\Delta$, that is, x Δ-computes a $\mathcal{J}$-null set which covers all Δ-computable reals.
(5) x is Δ-*dominant* if there is $y \leq_\Delta x$ such that $y \in [\mathsf{B}]_\Delta$, that is, x Δ-computes a function $f \in \omega^\omega$ which dominates all Δ-computable functions $g \in \omega^\omega$.
(6) x is Δ-*unbounded* if there is $y \leq_\Delta x$ such that $y \in [\mathsf{D}]_\Delta$, that is, x Δ-computes a function $f \in \omega^\omega$ which not dominated by any Δ-computable function $g \in \omega^\omega$.
(7) x is *Δ-h-eventually different* if there is $y \leq_\Delta x$ such that $y \in [\mathsf{IE}_h]_\Delta$, that is, x Δ-computes a h-bounded function $f \in \omega^\omega$ which eventually disagrees with any Δ-computable h-bounded function $g \in \omega^\omega$. We also say that x is *Δ-eventually different* if there is $y \leq_\Delta x$ such that $y \in [\mathsf{IE}]_\Delta$, that is, it is Δ-h-eventually different for all $h \in \omega^\omega \cap \Delta$.
(8) x is *Δ-h-infinitely often equal* if there is $y \leq_\Delta x$ such that $y \in [\mathsf{ED}_h]_\Delta$, that is, x Δ-computes a h-bounded function $f \in \omega^\omega$ which agrees infinitely often with any Δ-computable h-bounded function $g \in \omega^\omega$. We also say that x is *Δ-infinitely often equal* if there is $y \leq_\Delta x$ such that $y \in [\mathsf{ED}]_\Delta$, that is, it is Δ-h-infinitely often equal for all $h \in \omega^\omega \cap \Delta$.

2.3. **Background and Summary.** Over the past ten years, the notion of lowness properties was one of the central topics in algorithmic randomness theory [14, 40]. One of the important topics in algorithmic and higher randomness theory is the separation of various lowness notions in the sense of Muchnik reducibility. Known separation results of Cichoń's diagram in the context of Δ^0_1-Muchnik reducibility are summarized in [6]. For instance, consider the following Tukey morphisms:

$$\mathsf{Non}\mathcal{M} \longrightarrow \mathsf{Cof}\mathcal{M} \longrightarrow \mathsf{Cof}\mathcal{N}.$$

A standard forcing argument shows that there are no reversal definable morphisms. The same holds true for Δ^0_1-Muchnik reducibility. That is, the two facts that there is a real which is low for Kurtz randomness but not low for Schnorr randomness and that there is a real which is low for the pair of Schnorr randomness and Kurtz-randomness but not low for Kurtz randomness imply the following inequalities:

$$[\mathsf{Cof}\mathcal{N}]_{\Delta^0_1} <_{\Delta^0_1\mathrm{w}} [\mathsf{Cof}\mathcal{M}]_{\Delta^0_1} <_{\Delta^0_1\mathrm{w}} [\mathsf{Non}\mathcal{M}]_{\Delta^0_1}. \tag{1}$$

The main objects of higher randomness theory are randomness notions relative to the Spector pointclass Π^1_1 (and its companion model $L_{\omega_1^{\mathrm{CK}}}$). In such a theory, Chong et al. [10] used Sacks forcing to show that there are continuum many Δ^1_1-traceable reals (that is, reals that are low for Δ^1_1-randomness). Moreover, Kjos-Hanssen et al. [31] used forcing with clumpy trees to show that there are continuum many reals that are low for Δ^1_1-Kurtz randomness, but not low for Δ^1_1-randomness. The latter result implies that:

$$[\mathsf{Cof}\mathcal{N}]_{\Delta^1_1} <_{\Delta^1_1\mathrm{w}} [\mathsf{Cof}\mathcal{M}]_{\Delta^1_1} \leq_{\Delta^1_1\mathrm{w}} [\mathsf{Non}\mathcal{M}]_{\Delta^1_1}.$$

Later, we will see that the latter inequality is also proper, that is, we will show that there is a real which is low for the pair of Δ^1_1-randomness and Δ^1_1-Kurtz-randomness, but not low for Δ^1_1-Kurtz randomness (not low for Δ^1_1-generic as well).

We are now interested in the gap between Tukey morphism $A \longrightarrow B$ and Δ-Muchnik reducibility $[B]_\Delta \leq_{\Delta\mathrm{w}} [A]_\Delta$. One of the main technical differences between computability theory and set theory is the use of *time trick* (see [4]). This technique yields several strange phenomena in computability theory. For instance, consider the following Tukey morphisms:

$$\mathsf{Cov}\mathcal{M} \longrightarrow \mathsf{ED} \longrightarrow \mathsf{D},$$

whereas there are *no* reversal definable morphisms. Indeed, Laver forcing (see [2]) over Gödel's constructible universe L provides a model of ZFC satisfying $[\mathsf{D}]_L <_{L\mathrm{w}} [\mathsf{ED}]_L$ (see Proposition 3.6 (1a)), and Zapletal's uncountable dimensional forcing [58] over L provides a model of ZFC satisfying $[\mathsf{ED}]_L <_{L\mathrm{w}} [\mathsf{Cov}\mathcal{M}]_L$. However, it is known that if a real computes a function which is not dominated by any computable function, then it also computes a weakly 1-generic real. This implies that:

$$[\mathsf{Cov}\mathcal{M}]_{\Delta^0_1} \equiv_{\mathrm{w}} [\mathsf{ED}]_{\Delta^0_1} \equiv_{\mathrm{w}} [\mathsf{D}]_{\Delta^0_1}. \tag{2}$$

In Section 3.3, we will introduce a new Vojtáš triple $\mathsf{Cov}\mathcal{X}_{0.5}$, which is related to the notion of *coarse computability*. As observed by Andrews et al. [1], the above equalities (2) for instance implies that

$$[\mathsf{Cov}\mathcal{X}_{0.5}]_{\Delta^0_1} \leq_{\mathrm{w}} [\mathsf{D}]_{\Delta^0_1}, \tag{3}$$

whereas we will also see that there is *no* definable Tukey morphism $\mathsf{D} \longrightarrow \mathsf{Cov}\mathcal{X}_{0.5}$ (indeed, $[\mathsf{Cov}\mathcal{X}_{0.5}]_L \not\leq_{L\mathrm{w}} [\mathsf{D}]_L$). One may also observe the similar phenomenon for the following Tukey morphisms (see [2]):

$$\mathsf{Add}\mathcal{N} \longrightarrow \mathsf{Add}\mathcal{M} \longrightarrow \mathsf{B},$$

whereas there are *no* reversal definable morphisms. Indeed, $[\mathsf{B}]_L <_{L\mathrm{w}} [\mathsf{Add}\mathcal{M}]_L$ consistently holds (again in a Laver generic extension $L[G]$ of L; see [2]). However, Rupprecht [46] pointed out that if a real computes a function which dominates all computable functions, it also computes a computably null-engulfing real. This implies that:

$$[\mathsf{Add}\mathcal{N}]_{\Delta^0_1} \equiv_{\mathrm{w}} [\mathsf{Add}\mathcal{M}]_{\Delta^0_1} \equiv_{\mathrm{w}} [\mathsf{B}]_{\Delta^0_1}. \tag{4}$$

Thus, it is important to ask about the situation in higher randomness theory. We will show that the Δ^1_1-analog of (1) holds true as mentioned before; however, the Δ^1_1-analogues of (2) and (3) fail. Indeed, we show the main result for more general Spector pointclasses, e.g., $\Gamma \in \{\Sigma^0_{<\omega}, \Pi^1_1, \Game(D_{<\omega}\Sigma^0_1), {}_2\mathrm{env}(\mathrm{E}_n), \Game\Sigma^0_k\}$ where $k \neq 2$. Here $\Game$ is the game quantifier, $D_{<\omega}$ indicates the union of all finite ranks in the difference hierarchy, ${}_2\mathrm{env}$ is the 2-envelop (the collection of all sets of reals which are semi-computable in a given functional), and E_n is the n-th normal type-2 functional obtained by iterating Gandy's superjump operator.

Theorem 2.5. *Suppose that* $\Gamma \in \{\Sigma^0_{<\omega}, \Pi^1_1, \Game(D_{<\omega}\Sigma^0_1), {}_2\mathrm{env}(\mathrm{E}_n), \Game\Sigma^0_k\}$ *where* n *and* k *range over positive integers and* $k \neq 2$. *Then:*

$$[\mathsf{Cof}\mathcal{M}]_\Delta <_{\Delta\mathrm{w}} [\mathsf{Non}\mathcal{M}]_\Delta, \tag{5}$$

$$[\mathsf{D}]_\Delta <_{\Delta\mathrm{w}} [\mathsf{ED}]_\Delta, \tag{6}$$

$$[\mathsf{Cov}\mathcal{X}_{0.5}]_\Delta \not\leq_{\Delta\mathrm{w}} [\mathsf{D}]_\Delta. \tag{7}$$

From this result, we may read that infinitary computability theory is closer to set theory than to finitary computability theory. However, we have the following Δ^1_1-analog of (4):

Fact 2.6 (Monin)**.** *A function $x \in \omega^\omega$ dominates all Δ^1_1-functions if and only if $y \leq_T x$ for every hyperarithmetical real y. In particular, we have the following equivalences:*

$$[\mathsf{Add}\mathcal{N}]_{\Delta^1_1} \equiv_{\Delta^1_1 \mathrm{w}} [\mathsf{Add}\mathcal{M}]_{\Delta^1_1} \equiv_{\Delta^1_1 \mathrm{w}} [\mathsf{B}]_{\Delta^1_1}. \tag{8}$$

Furthermore, by the same argument as in Fact 2.6, one can also show that x dominates all arithmetically definable functions if and only if $\emptyset^{(n)} \leq_T x$ for every $n \in \omega$, and therefore, arithmetical dominance is also equivalent to being arithmetically meager engulfing (arithmetically null engulfing, resp.) Note that there is a Δ^1_1-dominant $x <_h \mathcal{O}$ (see Enderton-Putnam [15]) whereas x is arithmetically dominant if and only if $\emptyset^{(\omega)} \leq_a x$ since $\emptyset^{(\omega)}$ is a 2-least upper bound of $\{\emptyset^{(n)}\}_{n\in\omega}$ (see Enderton-Putnam [15]).

The above contrasting two results (Theorem 2.5 and Fact 2.6) can be explained in the context of forcing theory. The failure of (2) in the L-degrees can be witnessed by *lim-sup* tree forcing such as rational perfect set forcing $\mathbb{PT}$, whereas we need *lim-inf* tree forcing such as Laver forcing $\mathbb{LT}$ to show the failure of (4). The equivalences in (8) suggest that lim-inf tree forcing (in particular, Laver forcing $\mathbb{LT}$) does not work at $L_{\omega_1^{\mathrm{CK}}}$ (the companion model of Π^1_1). In this paper, we develop a general theory of lim-sup forcing over the companion model of a Spector pointclass to show Theorem 2.5.

3. Basic Properties

3.1. **Traceability.** To prove Theorem 2.5, we need a technical notion called traceability.

Definition 3.1 (see also [14, 28, 29, 40])**.** A *slalom* is a sequence $(T_n)_{n\in\omega}$ of finite subsets of ω such that $|T_n| \leq n$. It is a Γ-*slalom* if $\{(n,m) : m \in T_n\} \in \Gamma$. It is a Δ-*slalom* if the sequence of the canonical indices of T_n is in Δ, or equivalently, it is a Γ-slalom and $\lambda n.|T_n| \in \Delta$.

We say that a function $f \in \omega^\omega$ is *traced* (*infinitely often traced*, respectively) by a slalom $(T_n)_{n\in\omega}$ if $f(n) \in T_n$ for all $n \in \omega$ (for infinitely many $n \in \omega$, respectively). We also say that a function $f \in \omega^\omega$ is Δ-*often traced* by a slalom $(T_n)_{n\in\omega}$ if there is a Δ-function $h \in \omega^\omega$ such that for every $n \in \omega$, $f(k) \in T_k$ for some $k \in [h(n), h(n+1))$.

A real x is Γ-*traceable* (Δ-*traceable*, respectively) if every $f \leq_\Delta x$ is traced by a Γ-slalom (Δ-slalom, respectively). One can also define infinitely often (abbreviated as i.o.) Δ-traceability, Δ-often Δ-traceability, etc. in a straightforward manner (see also Kihara-Miyabe [28, 29]).

As pointed out by Kjos-Hanssen et al. [31, Lemma 4.5], (i.o.) Π^1_1-traceability is equivalent to (i.o.) Δ^1_1-traceability. Generally, if Γ is a Spector pointclass, then, by the Spector criterion, we have the equivalence of (i.o.) Γ-traceability and (i.o.) Δ-traceability. The notion of traceability characterizes (transitive-)additivity in set theory and lowness for randomness in computability theory (see [28]):

1. x is Δ-traceable if and only if $x \notin [\mathsf{Cof}\mathcal{N}]_\Delta$ (i.e., x is low for Δ-randomness).
2. x is Δ-often Δ-traceable if and only if $x \notin [\mathsf{Cof}\mathcal{M}]_\Delta$ (i.e., low for Δ-Kurtz randomness).
3. x is i.o. Δ-traceable if and only if $x \notin [\mathsf{Non}\mathcal{M}_\Delta]$ (i.e., low for the pair of Δ-randomness and Δ-Kurtz randomness).

We use the following traceability notion, which is related to the so-called *Laver property* of Laver forcing ([2, Definition 6.3.27]; see also Section 4.4).

Definition 3.2. Recall that a function $f \in \omega^\omega$ is Δ-bounded if there is $h \in \omega^\omega \cap \Delta$ such that $f(n) < h(n)$ for all $n \in \omega$. A real $x \in \omega^\omega$ is *Γ-Laver traceable* (*Δ-Laver traceable*) if for every Δ-bounded function $f \leq_\Delta x$, there is a Γ-slalom (a Δ-slalom) $(T_n)_{n\in\omega}$ such that $f(n) \in T_n$ for all $n \in \omega$.

Let λ^x_Γ be the supremum of all ordinals $\alpha \leq_\Delta x$. Then $\lambda_\Gamma := \lambda^\emptyset_\Gamma$ is the supremum of all Δ-ordinals. By modifying the argument in Kjos-Hanssen et al. [31, Lemma 4.5], one can easily show the following:

Lemma 3.3. *Let Γ be a Spector pointclass. Suppose that a real $x \in \omega^\omega$ satisfies $\lambda^x_\Gamma = \lambda_\Gamma$. Then, x is Γ-Laver traceable if and only if x is Δ-Laver traceable.* □

Note that most known traceability notions can be easily interpreted as properties concerning Kolmogorov complexity. A *machine* is a partial function $G :\subseteq 2^{<\omega} \to 2^{<\omega}$ with a prefix-free domain (here, we do not require a machine to be computable). A *Δ-machine* is a machine which has a Δ-graph (i.e., the graph is a Δ subset of $2^{<\omega} \times 2^{<\omega}$). The *Kolmogorov complexity* of a binary string $\sigma \in 2^{<\omega}$ with respect to a machine G is defined by $K_G(\sigma) = \min\{|\tau| : G(\tau) = \sigma\}$.

Definition 3.4 (see also [28])**.** A real x is *Δ-trivial* if for every Δ-machine H, there exists a Δ-machine G such that $K_G(x\restriction n) \leq K_H(n) + O(1)$ holds.

Note that this definition is a modification of *Schnorr triviality.* Note that Franklin-Stephan [17] found a traceability characterizatin of Schnorr triviality. Generally, a real x is Δ-trivial if and only if x is Δ-tt-traceable, that is, for every $h \in \omega^\omega \cap \Delta$, there is a Δ-slalom $(T_n)_{n\in\omega}$ such that $x\restriction h(n) \in T_n$ for all $n \in \omega$. The exactly same traceability notion is also used to characterize null-additivity in set theory. Therefore, Δ-triviality is equivalent to Δ-null-additivity (see [28]).

Proposition 3.5. *If $x \in \omega^\omega$ is Δ-Laver traceable, then every $y \leq_\Delta x$ with $y \in 2^\omega$ is Δ-trivial.* □

In our main theorem, we will construct a real $x \in \omega^\omega$ such that (i) x is not Δ-often Δ-traceable; in particular, x is not Δ-traceable; (ii) x is i.o. Δ-traceable; (iii) x is Δ-Laver traceable; hence, every $y \leq_\Delta x$ with $y \in 2^\omega$ is Δ-tt-traceable.

3.2. **Computable Tukey Morphism.** It is known that all implications in Cichoń's diagram are witnessed by computable Tukey morphisms (see also [41]). In this section, we will see some examples of computable Tukey morphisms outside Cichoń's diagram. A slalom $(T_n)_{n\in\omega}$ is *h-bounded* if $T_n \subseteq h(n)$ for all $n \in \omega$. Let $\mathcal{T}_h$ be the collection of all h-bounded slaloms, and $\mathsf{Non}\mathcal{T}_h$ be the Vojtáš triple $(\mathcal{T}_h, [h], \not\ni)$.

Proposition 3.6. *We have the following computable Tukey morphisms:*

1. *Let h be a function such that $h(n) > n$ for all n. Then, there is a computable Tukey morphism from $\mathsf{Non}\mathcal{T}_h$ to IE_h. In particular, we have the following:*
   - (a) *No Δ-Laver-traceable real Δ-computes an h-infinitely often equal real for any Δ-order h.*
   - (b) *Every weakly Δ-Laver-trace-engulfing real Δ-computes an h-eventually different real for any Δ-order h.*

(2) *(see also Rupprecht* [46, Proposition 16]*) There is a computable Tukey morphism from* $\mathsf{Cov}_{\mathcal{N}}$ *to* $\mathsf{IE}_{\lambda n.2^n}$*. In particular, we have the following:*
   (a) *If* x Δ*-computes a* $(\lambda n.2^n)$*-infinitely often equal real over* Δ*, then* x *is weakly null-engulfing over* Δ*.*
   (b) *Every* Δ*-Schnorr random real* Δ*-computes a* $(\lambda n.2^n)$*-eventually different real over* Δ*.*

Proof. (1) Let T_n be an h-bounded slalom with $|T_n| \leq n$. Then, we can choose some $f(n) < h(n)$ such that $f(n) \notin T_n$. Define $H(T_n : n \in \omega) = f$. If g is infinitely often equal to f, then it is easy to see that $K(g) := g$ is not traced by $(T_n)_{n\in\omega}$.

(2) Suppose that a real y is given. Let J_n be the interval $[\sum_{k=0}^{n-1} k, \sum_{k=0}^{n} k)$ of length n. Consider $H(y) := y \upharpoonright J_n$. If x is a $(\lambda n.2^n)$-infinite often equal real, then $x(n) = y \upharpoonright J_n$ for infinitely many n. Put $W_n = \bigcup_{m\geq n}[x(m)]$. Clearly, $K(x) := (W_n)_{n\in\omega}$ is a uniform sequence of open sets such that $\mu(W_n) = 2^{-n-1}$ for every n. Moreover, we have $y \in \bigcap_n W_n$.

The items (a) and (b) follow from the above results and Proposition 2.3. □

3.3. **Infinite Equality and Asymptotic Density.** We consider the asymptotic density version of infinite often equality. For a set $A \subseteq \omega$, the *lower density of* A is defined as follows:

$$\underline{\rho}(A) = \liminf_{n\to\infty} \frac{|\{k < n : k \in A\}|}{n}.$$

By using the asymptotic density, we consider the following "small" set for any $A \subseteq \omega$.

$$\langle A \rangle_r = \left\{B \subseteq \omega : \underline{\rho}(\{n \in \omega : A(n) = B(n)\}) \geq r\right\},$$

and then let $\mathcal{X}_r$ be the σ-ideal generated by $\{\langle A\rangle_r : A \subseteq \omega\}$. We then define the *coarse* Δ*-equality bounds* of $B \subseteq \omega$ as follows:

$$\mathfrak{g}_\Delta(B) = \sup\{r \in [0,1] : (\exists A \in \Delta)\ B \in \langle A\rangle_r\},$$
$$\mathfrak{G}_\Delta(x) = \inf\{\mathfrak{g}_\Delta(B) : B \leq_\Delta x\}.$$

The *upper density of* A, $\overline{\rho}(A)$, is defined in a similar manner. The *upper coarse* Δ*-equality bounds* $\overline{\mathfrak{g}}_\Delta$ and $\overline{\mathfrak{G}}_\Delta$ are also defined by the similar way. The above notions have already been studied in the computability theoretic context (see [1, 20]). In their terminology, $\mathfrak{g}_{\Delta^0_1}$ and $\mathfrak{G}_{\Delta^0_1}$ are denoted by γ and Γ, respectively.

Proposition 3.7.
(1) $\mathfrak{G}_\Delta(x) < r$ *if and only if there is* $y \leq_\Delta x$ *such that* $y \in [\mathsf{Cov}\mathcal{X}_r]_\Delta$*.*
(2) $\overline{\mathfrak{G}}_\Delta(x) \leq 1 - r$ *if and only if there is* $y \leq_\Delta x$ *such that* $y \in [\mathsf{Non}\mathcal{X}_r]_\Delta$*.*

Proof. The set $\langle A\rangle_r$ does not cover all computable reals if and only if there is a computable real B such that $\underline{\rho}(A = B) < r$. Consider the complement $C = \omega \setminus B$. Then,

$$\frac{|\{k < n : A(n) = C(n)\}|}{n} = 1 - \frac{|\{k < n : A(n) = B(n)\}|}{n}.$$

Since the infimum limit of the right-hand fraction is less than r, the supremum limit of the left-side value is greater than $1 - r$. In other words, $\overline{\mathfrak{g}}_\Delta(A) > 1 - r$. □

Next, we see the relationship between infinite often equality and lower asymptotic density.

Proposition 3.8. *We have the following computable Tukey morphisms:*

(1) *(see also Monin-Nies [37]) There is a computable Tukey morphism from* $\mathsf{Non}\mathcal{X}_{1/c}$ *to* $\mathsf{IE}_{\lambda n.2^{c^n}}$*. In particular, we have the following:*
   (a) *If x is $(\lambda n.2^{c^n})$-infinitely often equal over Δ, then $\mathfrak{G}_\Delta(x) \leq 1/c$. Hence, if x is $(\lambda n.2^{n^n})$-infinitely often equal over Δ, then $\mathfrak{G}_\Delta(x) = 0$.*
   (b) *If x is not $(\lambda n.2^{c^n})$-eventually different over Δ, then $\overline{\mathfrak{G}}_\Delta(x) \geq 1 - 1/c$. Hence, if x is not $(\lambda n.2^{n^n})$-eventually different over Δ, then $\overline{\mathfrak{G}}_\Delta(x) = 1$.*

(2) *(see also Andrews et al. [1]) There is a computable Tukey morphism from* $\mathsf{Cov}\mathcal{X}_{1/2}$ *to* $\mathsf{Cof}\mathcal{N}$*. In particular, we have the following:*
   (a) *If A is Δtraceable, then $\underline{\mathfrak{G}}_\Delta(A) \geq 1/2$.*
   (b) *If D is null-engulfing over Δ, then $\overline{\mathfrak{G}}_\Delta(D) \leq 1/2$.*

Proof. (1) Let B be any real, and J_n be the interval $[c^+_{n-1}, c^+_n)$, where $c^+_n = \sum_{k\leq n} c^k$. Define $H(B) := \lambda n.(\omega \setminus B) \upharpoonright J_n$. Note that $H(B)$ is 2^{c^n}-bounded. Let g be a function which is infinitely often equal to $H(B)$. Consider $K(g) := A = g(0)^\frown g(1)^\frown \dots$. For $n \in \omega$ such that $g(n) = H(B)(n)$, we have

$$\frac{\{k < c^+_n : A(k) = B(k)\}}{c^+_n} \leq \frac{c^+_n - c^n}{c^+_n} = \frac{c^n - 1}{c^{n+1} - 1}.$$

Clearly, the rightmost value converges to $1/c$ as n tends to infinity.

(2) We give a sketch of proof. Let A be a real, For $H(A) := \lambda n.(\omega \setminus A) \upharpoonright J_n$, let $(T_n)_{n\in\omega}$ be any slalom such that $(\omega \setminus A) \upharpoonright J_n \in T_n$ for almost all $n \in \omega$. By a probabilistic argument in [1, Proof of Theorem 1.10], we can construct $K(T_n : n \in \omega) := B$ such that $B \upharpoonright J_n$ is as close to T_n as possible. Let us consider $C = \{k \in \omega : B(k) \neq A(k)\}$. By [1, Lemmas 2.2], $\liminf_n \rho(C|J_n) \geq 1/2$, and indeed, $\underline{\rho}(C) \geq 1/2$. Then, $\overline{\rho}(\omega \setminus C) = 1 - \underline{\rho}(C) \leq 1/2$. Consequently, $\overline{\mathfrak{g}}_\Delta(B) \leq 1/2$. Therefore, B is a solution to the $\mathsf{Cov}\mathcal{X}_{1/2}$-instance generated from A by (the proof of) Proposition 3.7. □

4. Forcing Argument

4.1. **Models and Definability.** In this paper, we assume that any model $\mathbf{M}$ is the *companion* of a Spector pointclass (see Moschovakis [38, Theorem 9E.1]). Roughly speaking, the companion $\mathbf{M}$ of a Spector pointclass Γ is the set M_Δ of (Mostowski collapses of) all Δ-coded hereditarily countable sets (w.r.t. our representation of $\mathbf{H}_{\aleph_1}$; Example 2.1) with a certain relation R on M_Δ which for instance satisfies that a subset of ω is Γ if and only if it is Σ_1 on $\mathbf{M} = \langle M_\Delta, \in, R\rangle$. Indeed, a companion $\mathbf{M}$ is always of the form $\langle L_\alpha[R], \in, R\rangle$ where α is the supremum of M_Δ-ordinals (see also [16, Section 5.4]). Note that α is a countable R-admissible ordinal.

For a companion $\mathbf{M} = \langle L_\alpha[R], \in, R\rangle$ and a given real x, we consider the relativization $\mathbf{M}[x] := \langle L_\alpha[x, R], \in, R\rangle$. Note that $\mathbf{M}[x]$ may not be a companion of $\Gamma(x)$ even if $\mathbf{M}$ is a companion of Γ. For instance, if $\omega_1^x > \omega_1^{\mathrm{CK}}$ then $L_{\omega_1^{\mathrm{CK}}}[x]$ is not a companion of the pointclass $\Pi^1_1(x)$.

Definition 4.1. A partial function $\Phi :\subseteq \omega^\omega \to \omega^\omega$ is $\Sigma_1(\mathbf{M})$-*definable* if there is a $\Sigma_1(\mathbf{M})$-formula $\varphi(\sigma, x, \vec{v})$ with a parameter $\vec{v} \in \mathbf{M}$ such that for any $\sigma \in \omega^{<\omega}$ and $x \in \omega^\omega$

$$\Phi^{-1}[\sigma] = \mathrm{dom}(f) \cap \{x \in \omega^\omega : \mathbf{M}[x] \models \varphi(\sigma, x, \vec{v})\}.$$

A real $y \in \omega^\omega$ is $\mathbf{M}$-*reducible* to $x \in \omega^\omega$ (written as $y \leq_{\mathbf{M}} x$) if there is a partial $\Sigma_1(\mathbf{M})$-definable function $\Phi :\subseteq \omega^\omega \to \omega^\omega$ such that $x \in \mathrm{dom}(\Phi)$ and $\Phi(x) = y$.

Note that, even if $\mathbf{M}$ is the companion of a Spector pointclass Γ, the reducibillity notion $\leq_{\mathbf{M}}$ does not coincide with $\leq_{\Delta}$ (e.g., consider $\mathbf{M} = L_{\omega_1^{\mathrm{CK}}}$ and $\Gamma = \Pi^1_1$).

4.2. **Forcing and Fusion.** The notions of set-forcing (i.e., a forcing $\mathbb{P}$ with $\mathbb{P} \in \mathbf{M}$) in admissible recursion theory have been extensively studied in, for instance, Mathias [35], Sacks [48], and Zarach [59]. In particular, it is shown that any set-forcing preserves admissibility. However, most forcing notions which we are interested in are class-forcing notions in our companion model except for Cohen forcing. For instance, if $\mathbb{S}$ is the Sacks forcing notion (i.e., the set of all perfect subtrees of $2^{<\omega}$), we obviously have $\mathbb{S} \cap L_{\omega_1^{\mathrm{CK}}} \notin L_{\omega_1^{\mathrm{CK}}}$. To avoid some technical issues concerning class forcing, we will only consider a restricted class of forcing notions.

A *poset* is a triple $(\mathbb{P}, \leq_{\mathbb{P}}, \mathbf{1}_{\mathbb{P}})$ of a partial order $\leq_{\mathbb{P}}$ on a set $\mathbb{P}$ with a $\leq_{\mathbb{P}}$-greatest element $\mathbf{1}_{\mathbb{P}} \in \mathbb{P}$. In this paper, we always assume that a poset is *arboreal*, that is, $\mathbf{1}_{\mathbb{P}} := H$ is a perfect subtree of $\omega^{<\omega}$, each condition $p \in \mathbb{P}$ is a perfect subtree of $\mathbf{P}$, and $\leq_{\mathbb{P}}$ is the inclusion relation $\subseteq$. We also say that a poset is *strongly arboreal* if it is arboreal, and if $p \in \mathbb{P}$ and $\sigma \in p$ implies $p \upharpoonright \sigma \in \mathbb{P}$, where $p \upharpoonright \sigma = \{\tau \in p : \sigma \preceq \tau \text{ or } \tau \preceq \sigma\}$. For instance, Cohen forcing, random forcing, Hechler forcing, Mathias forcing, and Laver forcing are all strongly arboreal forcing notions (see [2, 57] for arboreal and idealized forcing). We define the relativization $\mathbb{P}^{\mathbf{M}} = \mathbb{P} \cap \mathbf{M}$. We say that $\mathbb{P}$ is a Σ_1*-forcing over* $\mathbf{M}$ if $\mathbb{P}^{\mathbf{M}}$, $\{(p,q) \in \mathbb{P}^{\mathbf{M}} : p \leq q\}$, and $\{(p,q) \in \mathbb{P}^{\mathbf{M}} : p \not\leq q\}$ are Σ_1-definable over $\mathbf{M}$ (see also Kechris [22]).

A set $U \subseteq \mathbb{P}$ is $\mathbb{P}$*-open* if $q \leq p \in U$ implies $q \in U$. A set $U \subseteq \mathbb{P}$ is $\mathbf{M}$-$\mathbb{P}$*-dense below* p if for every $q \in \mathbb{P}^{\mathbf{M}}$ with $q \leq p$ there is $r \in \mathbb{P}^{\mathbf{M}}$ such that $r \leq q$ and $r \in U$. A real $x \in [\mathbf{H}]$ is $\mathbb{P}$*-generic over* $\mathbf{M}$ if for every $\Sigma_n(\mathbf{M})$-definable $\mathbf{M}$-$\mathbb{P}$-dense $\mathbb{P}$-open set $D \subseteq \mathbb{P}$, there is $p \in \mathbb{P}^{\mathbf{M}}$ such that $x \in [p]$ and $p \in D$, where $[p]$ denotes the set of all infinite paths through a tree $p \subseteq H$.

We now introduce the forcing language $\mathcal{L}_{\mathbf{M}}$ (see also Sacks [48, Section 3.4]). Our model $\mathbf{M} = \langle L_\alpha[R], \in, R\rangle$ is the companion (Moschovakis [38, Section 9E]) of a Spector pointclass, and therefore it is *resolvable*, that is, there is a $\Delta_1(\mathbf{M})$ function $\tau : \alpha \to \mathbf{M}$ such that $\mathbf{M} = \bigcup_\xi \tau(\xi)$. Our forcing language $\mathcal{L}_{\mathbf{M}}$ consists of two relation symbols $\in$ and R, a constant symbol $\dot{r}_{gen}$ for a generic real, a constant symbol $\check{a}$ for every $a \in \mathbf{M}$, ranked variables $x^\xi, y^\xi, \dots$ (ranging over $\bigcup_{\zeta<\xi} \tau(\zeta)$) for $\xi < \alpha$ and unranked variables $x, y, \dots$ In particular, $\mathcal{L}_{\mathbf{M}}$ contains a name $\check{\omega}$ of ω since any companion $\mathbf{M}$ of a Spector pointclass is an ω-model. An $\mathcal{L}_{\mathbf{M}}$-formula is *ranked* if all of its variables are ranked. An $\mathcal{L}_{\mathbf{M}}$-formula is Σ_1 if it is of the form $\exists y \varphi(y, \check{\mathbf{v}})$ where φ has no quantification over unranked variables.

Fix an arboreal forcing notion $\mathbb{P}$. We define the (strong) forcing relation $\Vdash_{\mathbb{P}}$ by the following way (see also Sacks [48, Section 4.4.1]):

Suppose that $\varphi(\dot{r}_{gen}, \check{\mathbf{v}})$ is ranked.

(1) $p \Vdash_{\mathbb{P}} \varphi(\dot{r}_{gen}, \check{\mathbf{v}})$ if $[p] \subseteq \{x \in \omega^\omega : \mathbf{M}[x] \models \varphi(x, \mathbf{v})\}$.

Suppose that $\varphi(\dot{r}_{gen}, \check{\mathbf{v}})$ is unranked.

(2) $p \Vdash_{\mathbb{P}} \exists y^\xi\, \varphi(\dot{r}_{gen}, y^\xi, \check{\mathbf{v}})$ if $p \Vdash_{\mathbb{P}} \varphi(\dot{r}_{gen}, z, \check{\mathbf{v}})$ for some $z \in \bigcup_{\zeta<\xi} \tau(\zeta)$.
(3) $p \Vdash_{\mathbb{P}} \exists y\, \varphi(\dot{r}_{gen}, y, \check{\mathbf{v}})$ if $p \Vdash_{\mathbb{P}} \exists y^\xi\, \varphi(\dot{r}_{gen}, y^\xi, \check{\mathbf{v}})$ for some $\xi < \alpha$.
(4) $p \Vdash_{\mathbb{P}} \varphi \wedge \psi$ if $p \Vdash_{\mathbb{P}} \varphi$ and $p \Vdash_{\mathbb{P}} \psi$.
(5) $p \Vdash_{\mathbb{P}} \neg\varphi$ if $q \nVdash_{\mathbb{P}} \varphi$ for every $q \leq p$.

Note that for every $p \in \mathbb{P}$, the formula $\dot{r}_{gen} \in [\check{p}]$ is Δ_0 in $\mathbf{M}$ since $\Pi^1_1(u) \subseteq \Sigma_1(\mathbf{M})$ for all $u \in \mathbf{M}$. We abbreviate $\Vdash_{\mathbb{P}}$ as $\Vdash$ if $\mathbb{P}$ is clear from the context. As in Shinoda

[49, Lemma 4.3], one can easily show that the strong forcing relation for Σ_1-formulas on $\mathbf{M}$ is $\Sigma_1(\mathbf{M})$-definable (since a companion $\mathbf{M}$ is Σ_1-projectible to ω; see [16, 38]).

Lemma 4.2 (Definability; see Shinoda [49, Lemma 4.3])**.** *Suppose that M is a companion model, $\mathbb{P}$ is a Σ_1-forcing notion, and $\varphi(\dot{r}_{gen}, \check{\mathbf{v}})$ is a Σ_1 formula with parameters from M. Then, $p \Vdash \varphi(\dot{r}_{gen}, \check{\mathbf{v}})$ is Σ_1-definable over M uniformly in p, φ, and $\mathbf{v}$.* □

In set theory, the fusion argument is a key method to show various properties of forcing notions. For an arboreal forcing $\mathbb{P}$, we say that a collection $(q_i)_{i\in\omega} \subseteq \mathbb{P}$ is separated if $q_\eta \cap q_\lambda$ is finite (therefore, there is a finitely generated clopen (in the Baire topology) set $C \subseteq [H]$ such that $[q_\eta] \subseteq C$ and $C \cap [q_\lambda] = \emptyset$) whenever $\eta \neq \lambda$. A $\mathbb{P}$-open set U is Σ_1*-pre-regular* if U is closed under separated Σ_1-union, that is, the union of a separated $\Sigma_1(\mathbf{M})$-definable sequence $(q_i)_{i\in\omega} \subseteq U$ belongs to U.

Lemma 4.3. *Let φ be a $\Sigma_1(\mathbf{M})$-formula. Then $\{q : q \Vdash \varphi\}$ is Σ_1-pre-regular.*

Proof. If φ is ranked, by definition, it is clear that $\{q : q \Vdash \varphi\}$ is closed under arbitrary union. Assume that φ is an unranked formula of the form $\varphi \equiv \exists y\psi(y)$ such that ψ is ranked. Let q be the union of a $\Sigma_1(\mathbf{M})$-definable sequence $(q_i)_{i\in\omega}$ of forcing conditions such that $q_i \Vdash \varphi$ for each $i \in \omega$. Then, for any $i \in \omega$ there is $\xi(i) < \alpha$ such that $q_i \Vdash \exists y^{\xi(i)}\psi(y^{\xi(i)})$. By Lemma 4.2 and by admissibility of $\mathbf{M}$, the map $i \mapsto \xi(i)$ is $\Sigma_1(\mathbf{M})$, and therefore, there is an ordinal $\xi < \alpha$ such that $\xi(i) < \xi$ for all $i \in \omega$. Then $q \Vdash \exists y^\xi\psi(y^\xi)$ since ψ is ranked and so is $\exists y^\xi\psi(y^\xi)$. Consequently $q \Vdash \varphi$. □

Definition 4.4. Suppose that $\mathbb{P}$ is an arboreal Σ_1-forcing on $H \in \mathbf{M}$. We say that $\mathbb{P}$ satisfies the Σ_1*-fusion property over* $\mathbf{M}$ if for any $\Sigma_1(\mathbf{M})$-definable sequence $(D_n)_{n\in\omega}$ of Σ_1-pre-regular $\mathbf{M}$-$\mathbb{P}$-dense-below-p $\mathbb{P}$-open subsets of $\mathbb{P}$, their intersection $\mathbb{P}^{\mathbf{M}} \cap \bigcap_n D_n$ is nonempty below p.

By Lemmas 4.2 and 4.3, the Σ_1-fusion property enables us to use the following basic fusion argument: If $\varphi(n, x)$ is a $\Sigma_1(\mathbf{M})$-formula, and $q \Vdash \varphi(n, \dot{r}_{gen})$ is dense below p for any $n \in \omega$, then there is a condition $r \leq p$ that forces $\varphi(n, \dot{r}_{gen})$ for all $n \in \omega$ (consider $D_n = \{q : q \Vdash \varphi(n, \dot{r}_{gen})\}$). For a forcing $\mathbb{P}$ with the Σ_1-fusion property, one can easily check the following basic properties by using the standard argument.

Lemma 4.5 (see Sacks [48, Chapter IV])**.** *Suppose that $\mathbb{P}$ is a strongly arboreal Σ_1-forcing notion which satisfies the Σ_1-fusion property.*

1. *(Quasi-completeness) For any formula φ and condition $p \in \mathbb{P}$, there is a condition $q \leq p$ such that either $q \Vdash \varphi$ or $q \Vdash \neg\varphi$ holds.*
2. *(Forcing = Truth) For a $\mathbb{P}$-generic real z, $\mathbf{M}[z] \models \varphi(z)$ if and only if there exists a condition p such that $z \in [p]$ and $p \Vdash \varphi(\dot{r}_{gen})$.*
3. *If z is a $\mathbb{P}$-generic real, then, $\mathbf{M}[z]$ is admissible.*

As a consequence of Lemma 4.5 (3), such a forcing $\mathbb{P}$ satisfies the *"Borel reading of names"* in the following sense: For any $\mathbf{M}$-coded partial Π^1_1-measurable function $h :\subseteq [H] \to \omega^\omega$, if $z \in [H]$ is sufficiently $\mathbb{P}$-generic and $z \in \mathrm{dom}(h)$, then there is a partial $\Sigma_1(\mathbf{M})$-definable (in particular, Borel) function $g :\subseteq [H] \to \omega^\omega$ with $z \in \mathrm{dom}(g)$ such that $g(z) = h(z)$.

4.3. **Creature Forcing.** In this section, we deal with certain kinds of arboreal forcing notions induced by *norms* (see Rosłanowski-Shelah [44]). To simplify our argument, we only consider a very restricted class of lim-sup tree-creating creature forcings. However, most of our results can be easily generalized to a slightly larger class of lim-sup creature forcings (including several important finitary forgetful creature forcings such as Silver forcing).

Definition 4.6 (see also Rosłanowski-Shelah [44])**.** Let H be a subtree of $\omega^{<\omega}$. A *norm on* H is an ω-valued Borel function defined on all pairs (σ, A), where $\sigma \in H$ and A is a subset of $\{\sigma^\frown i \in H : i \in \omega\}$. Given a norm $\mathbf{nor}$ on H, we say that a tree $T \subseteq H$ is $\mathbf{nor}$*-perfect* if for any $n \in \omega$ and $\sigma \in T$, there is $\tau \in T$ extending σ such that $\mathbf{nor}(\tau, \mathrm{succ}_T(\tau)) > n$, where $\mathrm{succ}_T(\tau)$ is the set of all immediate successors of τ in T. Then, $\mathbb{Q}^{\mathrm{w}\infty}_{\mathbf{nor}}$ is the *forcing notion with* $\mathbf{nor}$*-perfect trees*, that is,

(1) A condition $p \in \mathbb{Q}^{\mathrm{w}\infty}_{\mathbf{nor}}$ is a $\mathbf{nor}$-perfect subtree of H.
(2) The order $p \leq q$ on $\mathbb{Q}^{\mathrm{w}\infty}_{\mathbf{nor}}$ is introduced by the inclusion $p \subseteq q$.

We say that $\mathbb{Q}$ is *a lim-sup tree forcing* if it is of the form $\mathbb{Q}^{\mathrm{w}\infty}_{\mathbf{nor}}$ where $\mathbf{nor}$ satisfies that $\mathbf{nor}(\sigma, A) = 0$ for any $\sigma \in H$ and $|A| \leq 1$ (this implies that every $\mathbf{nor}$-perfect tree is perfect), and moreover, that H is $\mathbf{nor}$-perfect. We also write $\mathbf{nor}_T(\sigma)$ for $\mathbf{nor}(\sigma, \mathrm{succ}_T(\sigma))$.

Let $\mathbb{Q}^{\mathrm{w}\infty+}_{\mathbf{nor}}$ be the suborder of $\mathbb{Q}^{\mathrm{w}\infty}_{\mathbf{nor}}$ defined as follows:

$$\mathbb{Q}^{\mathrm{w}\infty+}_{\mathbf{nor}} = \{p \in \mathbb{Q}^{\mathrm{w}\infty}_{\mathbf{nor}} : \limsup_{n\to\infty} \mathbf{nor}_p(x \restriction n) = \infty \text{ for all } x \in [p]\}.$$

Clearly, $\mathbb{Q}^{\mathrm{w}\infty+}_{\mathbf{nor}}$ is a dense suborder of $\mathbb{Q}^{\mathrm{w}\infty}_{\mathbf{nor}}$ (see the proof of Lemma 4.12). Therefore, we may assume that our lim-sup tree forcing notion is of the form $\mathbb{Q}^{\mathrm{w}\infty+}_{\mathbf{nor}}$.

Example 4.7. The following are examples of some lim-sup tree forcings.

(1) Let $H = 2^{<\omega}$, and define $\mathbf{nor}_T(\sigma) = |\{\tau \prec \sigma : \mathbf{nor}_T(\sigma) > 0\}| + 1$ if $|\mathrm{succ}_T(\sigma)| \geq 2$ (i.e., σ is branching), and $\mathbf{nor}_T(\sigma) = 0$ otherwise. Then, $\mathbb{Q}^{\mathrm{w}\infty}_{\mathbf{nor}}$ is *Sacks forcing.*
(2) Let $H = \prod_n 2^n$, and define $\mathbf{nor}_T(\sigma) = |\{\tau \prec \sigma : \mathbf{nor}_T(\sigma) > 0\}| + 1$ if $|\mathrm{succ}_T(\sigma)| = 2^{|\sigma|}$ (i.e., σ is full-branching), and $\mathbf{nor}_T(\sigma) = 0$ otherwise. Then, $\mathbb{Q}^{\mathrm{w}\infty}_{\mathbf{nor}}$ is *infinitely equal forcing.*
(3) Let $H = \omega^{<\omega}$, and define $\mathbf{nor}_T(\sigma) = |\{\tau \prec \sigma : \mathbf{nor}_T(\sigma) > 0\}| + 1$ if $|\mathrm{succ}_T(\sigma)| = \omega$ (i.e., σ has infinitely many immediate successors), and $\mathbf{nor}_T(\sigma) = 0$ otherwise. Then, $\mathbb{Q}^{\mathrm{w}\infty}_{\mathbf{nor}}$ is *rational perfect set forcing.*
(4) A limsup tree creature forcing with a nontrivial norm has been used by Goldstern-Shelah [18, Definition 2.6] and others.

If $\mathbb{Q}$ is a lim-sup tree forcing generated by $H, \mathbf{nor} \in \mathbf{M}$, we say that $\mathbb{Q}$ is a lim-sup tree forcing over $\mathbf{M}$.

Proposition 4.8. *Suppose that* $\mathbb{Q}$ *is a lim-sup tree forcing over* $\mathbf{M}$*. Then,* $\mathbb{Q}^{\mathbf{M}}$ *is a strongly arboreal* Σ_1*-forcing over* $\mathbf{M}$*.* □

For a tree p and $\sigma \in \omega^{<\omega}$, let $p(\sigma)$ be the σ-th branching node of p if such a node exists. In other words, $p(\emptyset)$ is the stem (the first branching node) of p, and $p(\sigma^\frown i)$ is the first branching node of p extending the i-th immediate successor of $p(\sigma)$ if such a node exists. Given σ, let $\tau_{p,n}(\sigma)$ be the lexicographically least $\tau \succeq \sigma$ such that $\mathbf{nor}_p(p(\tau)) \geq n$ if such a node exists; otherwise $\tau_{p,n}(\sigma)$ is undefined. Then, we define

$$B_n(p) := B_{n-1}(p) \cup \{\tau : (\exists \sigma \in n^n)\ \tau \preceq \tau_{p,n}(\sigma)\},$$
$$\overline{B}_n(p) := \{\tau : (\exists \sigma \in B_n(p))\ \tau \preceq p(\sigma)\}$$

where $B_{-1}(p) = \emptyset$ and n^n is the set of all strings σ of length n such that $\sigma(k) < n$ for all $k < n$. Roughly speaking, at stage n, for a given $\sigma \in n^n$, enumerate the first string τ extending σ whose norm is greater than or equal to n, and then define $\overline{B}_n(p)$ as the downward closure of such a collection. Note that $(\overline{B}_n(p))_{n\in\omega}$ is an increasing sequence of finite subtrees of p.

Definition 4.9. Let $\mathbb{Q}$ be a lim-sup tree forcing on H generated by **nor**. We define the order $\leq_n$ on $\mathbb{Q}$ as follows:

$$q \leq_n p \iff q \leq p \;\&\; (\forall \sigma \in \overline{B}_n(p))\ \mathrm{succ}_p(\sigma) = \mathrm{succ}_q(\sigma).$$

Note that $q \leq_0 p$ if $q \leq p$ and q has the same stem with p. It is easy to see that $q \leq_{n+1} p$ implies $q \leq_n p$. It is obvious that $\leq_n$ is $\Sigma_1(\mathbf{M})$ uniformly in $n \in \omega$ if $\mathbb{Q}$ is a lim-sup tree forcing over $\mathbf{M}$ by Proposition 4.8. The following is a key property of the ordering $\leq_n$.

Lemma 4.10 (Fusion)**.** *For every $(p_n)_{n\in\omega} \in \mathbb{Q}^\omega$ with $p_{n+1} \leq_n p_n$, then there exists $p \in \mathbb{Q}$ such that $p \leq_n p_n$ for all n.*

Proof. We define $p = \bigcap_s p_s$, and we will show that $p \in \mathbb{Q}$ and $p \leq_n p_n$ for all $n \in \omega$. To see this, we first note that if $\sigma \in p$, then $\sigma \in p_s$ for all $s \in \omega$; therefore, $\sigma \preceq p_0(\eta)$ for some $\eta \in m^m$ since p_0 is perfect. Fix any $n \geq m$. This implies that for any $s \in \omega$, $\sigma \preceq p_s(\eta_s)$ for some $\eta_s \in n^n$. By the definition of $\leq_n$, it is not hard to see that $\sigma \preceq p(\tau_{p_n,n}(\eta_n)) \in p$ and $\mathbf{nor}_p(p(\tau_{p_n,n}(\eta_n))) \geq n$. Consequently, p is **nor**-perfect. □

Here we discuss common properties of lim-sup tree forcing notions.

Definition 4.11. A Σ_1-forcing $\mathbb{P}$ on H satisfies the Σ_1*-continuous reading of names over* $\mathbf{M}$ (see also Zapletal [57]) if for every partial $\Sigma_1(M)$-definable function $\Phi :\subseteq [H] \to \omega^\omega$ such that if $z \in \mathrm{dom}(\Phi)$ for a $\mathbb{P}$-generic point z, then there is a $\Sigma_1(M)$-continuous function $f :\subseteq [H] \to \omega^\omega$ such that $z \in \mathrm{dom}(f)$ and $f(z) = \Phi(z)$.

Lemma 4.12. *Suppose that $\mathbb{Q}$ is a lim-sup forcing over* $\mathbf{M}$*. Then, we have the following:*

(1) *$\mathbb{P}$ satisfies the Σ_1-fusion property over* $\mathbf{M}$*.*
(2) *$\mathbb{P}$ satisfies the Σ_1-continuous reading of names over* $\mathbf{M}$*.*

Proof. (1) Let $(D_n)_{n\in\omega}$ be a $\Sigma_1(\mathbf{M})$-sequence of Σ_1-pre-regular $\mathbf{M}$-$\mathbb{P}$-dense $\mathbb{P}$-open set. We will construct a fusion sequence $(p_n)_{n\in\omega}$ such that $p_{n+1} \leq_n p_n$ for all $n \in \omega$. Put $p_0 = p$, and assume that p_n is given. Inductively we assume that $\tau_{p_n,n}(\sigma) = \sigma$ for any $\sigma \in n^n$, that is, $B_n(p_n) = \bigcup_{m\leq n} m^m$. We inductively define a sequence $(p_n^k)_{k\leq n+2}$ with $p_n^0 = p_n$. Given $k \leq n+1$, define $I_n^k := (n+1)^{k+1} \setminus \bigcup_{m\leq n} m^{\leq m}$. By $\mathbb{P}$-density of D_m for any m, for every $\sigma \in I_n^k$, one can find $q_\sigma \leq p_n^k \upharpoonright p_n^k(\sigma)$ such that $q_\sigma \in D_m$ for all $m \leq n$. Moreover, we may assume that the norm of the stem of q_σ is greater than the length of σ by replacing q_σ with $q_\sigma \upharpoonright q_\sigma(\tau_{q_\sigma,n+1}(\emptyset))$. Then, define p_n^{k+1} as the union of q_σ for $\sigma \in I_n^k$ and $p_n^k \upharpoonright p_n^k(\sigma)$ for σ such that $\sigma \not\succeq \tau$ for any $\tau \in I_n^k$. Finally, we define $p_{n+1} = p_n^{n+2}$. It is easy to see that $p_{n+1} \leq_n p_n$, and

$\tau_{p_{n+1},n+1}(\sigma) = \sigma$ for any $\sigma \in (n+1)^{n+1}$. Therefore, one can find a fusion $r \leq_n p_n$ for all $n \in \omega$ by Lemma 4.10 by an **M**-computable way.

For any $n \in \omega$ and $x \in \omega^\omega$, there are $m \geq n$ and $k \leq m+1$ such that $x \upharpoonright k+1 \in I^k_m$. We define $\sigma(x) = x \upharpoonright k+1$ for minimal such k. Then, clearly there is a computable enumeration $(\sigma_i)_{i\in\omega}$ of the range of σ, and by minimality of $\sigma(x)$, $\{\sigma_i : i \in \omega\} \subseteq \omega^{<\omega}$ forms an antichain. If $r(\sigma_i)$ is defined, then q_{σ_i} in our construction belongs to D_n. Moreover, it is easy to see that $r(\sigma_i) \leq q_{\sigma_i}$, and therefore, $r(\sigma_i) \in D_n$ since D_n is $\mathbb{P}$-open. Then r is the Σ_1-separated union of $(r \upharpoonright r(\sigma_i))_{i\in\omega}$ and $r(\sigma_i) \in D_n$ for all $i \in \omega$. Since D_n is Σ_1-pre-regular, we have $r \in D_n$. Consequently, $r \in \bigcap_n D_n$.

(2) Let Φ be a $\Sigma_1(\mathbf{M})$-definable function. Suppose that $p \Vdash \dot{r}_{gen} \in \mathrm{dom}(\check{\Phi})$, that is, $p \Vdash (\forall n)(\exists k)\ \check{\Phi}(\dot{r}_{gen})(n) \downarrow= k$. As in the above argument, for any $\sigma \in I^k_n$, one can find $q_\sigma \leq p^k_n \upharpoonright p^k_n(\sigma)$ and $v_\sigma \in \omega^n$ such that $q_\sigma \Vdash \check{\Phi}(\dot{r}_{gen}) \upharpoonright n = v_\sigma$. We can find such q_σ and v_σ by a **M**-computable way. Define $h(\sigma) = v_\sigma$, and $p_{n+1} = p^{n+2}_n$. As before, we get a fusion $r \leq_n p_n$ for all $n \in \omega$ by Lemma 4.10. We define $\hat{h}(x) = \bigcup_{\sigma \prec x} h(\sigma)$ for all $x \in [r]$. Clearly, $\hat{h}$ is a continuous function defined on $[r]$. If z is an infinite path through r, then for any n there is $\sigma_n \prec z$ and $\sigma_n \in I^k_m$ for some $m \geq n$ and k. Hence, $r \upharpoonright \sigma_n \Vdash \check{\Phi}(\dot{r}_{gen}) \upharpoonright k = h(\sigma_n)$. Therefore, by Lemma 4.5 (2), if z is a $\mathbb{Q}$-generic point in $[r]$, then $\Phi(z) = \hat{h}(z)$ as desired. □

We say that a real $z \in \omega^\omega$ is *weakly meager engulfing over* **M** (see also Definition 2.4) if z **M**-computes a meager set that covers all **M**-coded reals.

Lemma 4.13. *Suppose that $\mathbb{Q}$ is a lim-sup tree forcing over* **M**. *Then, every $\mathbb{Q}$-generic real z over* **M** *is not weakly meager engulfing over* **M**.

Proof. For any real $x \in 2^\omega$ and order $h \in \omega^\omega$, let $E(h,x)$ denote the meager set defined by

$$E(h,x) = \{y \in 2^\omega : (\forall^\infty k)(\exists n \in [h(k), h(k+1)))\ y(n) \neq x(n)\}.$$

Note that every meager set in 2^ω is covered by a set of the form $E(h,x)$ (see [2, Theorem 2.2.4]). Assume that there are two M-computations Φ, ψ such that $E(\Phi(z), \psi(z))$ is meager. Then, there is a condition $p \in \mathbb{P}$ such that

$$p \Vdash \check{\Phi}(\dot{r}_{gen}) \in \omega^\omega \ \wedge\ \check{\psi}(\dot{r}_{gen}) \in 2^\omega$$

By continuous reading of names (Lemma 4.12), there is $q \leq p$ such that for every n and $\sigma \in n^{\leq n}$, $q(\sigma)$ decides the values $\check{\Phi}(\dot{r}_{gen}) \upharpoonright n$ and $\check{\psi}(\dot{r}_{gen}) \upharpoonright n$. In other words, we have monotone functions $h : q \to \omega^{<\omega}$ and $x : q \to 2^{<\omega}$ in M such that for every n and $\sigma \in n^{\leq n}$,

$$q \upharpoonright q(\sigma) \Vdash \check{\Phi}(\dot{r}_{gen}) \upharpoonright n = h(q(\sigma)) \upharpoonright n \ \wedge\ \check{\psi}(\dot{r}_{gen}) \upharpoonright n = x(q(\sigma)) \upharpoonright n.$$

We define $h(y) = \bigcup_n h(y \upharpoonright n)$ and $x(y) = \bigcup_n x(y \upharpoonright n)$ for $y \in 2^\omega$. We say that r is *q-rational* if r is the leftmost path through q which extends $q(\sigma)$ for some $\sigma \in \omega^{<\omega}$. Note that the set $\mathbb{Q}_q$ of all q-rationals can be coded in **M**. Hence, $E_q = \bigcup_{y \in \mathbb{Q}_q} E(h(y), x(y))$ is an **M**-coded meager set. Therefore, there is an **M**-coded real $c \notin E_q$.

Let z be a $\mathbb{Q}$-generic real over **M**. It suffices to show $c \notin E(\Phi(z), \psi(z))$. We construct a decreasing fusion sequence $q = q_0 \geq_0 q_1 \geq_1 \dots$ such that each $[q_n]$ is a clopen subset of $[q]$; therefore, $\mathbb{Q}_q$ is dense in $[q_n]$ for any $n \in \omega$. Suppose that q_n is already constructed. Let S_n be the set of all successors of a level n node in q_n. For every $\tau \in S_n$, we choose a q-rational v_τ extending τ in $[q_n]$. Then $c \notin E(h(v_\tau), x(v_\tau))$ since $E(h(v_\tau), x(v_\tau)) \subseteq E_q$. In other words,

$$(\exists^\infty k)\ x(v_\tau)\upharpoonright[h(v_\tau)(k), h(v_\tau)(k+1)) = c\upharpoonright[h(v_\tau)(k), h(v_\tau)(k+1)).$$

Thus, one can find a sufficiently long initial segment $\eta_\tau \prec v_\tau$ such that

$$(\exists k > n)\ x(\eta_\tau)\upharpoonright[h(\eta_\tau)(k), h(\eta_\tau)(k+1)) = c\upharpoonright[h(\eta_\tau)(k), h(\eta_\tau)(k+1)).$$

Since $\limsup_{\eta \prec v_\tau} \mathbf{nor}(\eta) = \infty$, there must exist η^*_τ such that $\eta_\tau \preceq \eta^*_\tau \prec v_\tau$ and $\mathbf{nor}(\eta^*_\tau) > n$. Then, we define

$$q_{n+1} = \bigcup_{\tau \in S_n} q_n \upharpoonright \eta^*_\tau.$$

Clearly, q_{n+1} is a clopen subset of q_n. Let r be the fused condition obtained from the sequence $\langle q_n \rangle_{n\in\omega}$. Then,

$$r \Vdash \exists^\infty k\ \check{x}(\dot{r}_{gen})\upharpoonright[\check{h}(\dot{r}_{gen})(k), \check{h}(\dot{r}_{gen})(k+1)) = \check{c}\upharpoonright[\check{h}(\dot{r}_{gen})(k), \check{h}(\dot{r}_{gen})(k+1)).$$

Hence, this implies $c \notin E(\Phi(z), \psi(z))$. □

Corollary 4.14. *Let $\mathbb{Q}$ be a lim-sup tree forcing over $L_{\omega_1^{\mathrm{CK}}}$. If z is $\mathbb{Q}$-generic over $L_{\omega_1^{\mathrm{CK}}}$, then z is i.o. Δ^1_1-traceable.*

Proof. Since $\mathbb{Q}$ is a lim-sup tree forcing over $L_{\omega_1^{\mathrm{CK}}}$, $\mathbb{Q}$ satisfies the $\Sigma_1(L_{\omega_1^{\mathrm{CK}}})$-fusion property by Lemma 4.12. Therefore, $\omega_1^{\mathrm{CK},z} = \omega_1^{\mathrm{CK}}$ by Lemma 4.5. By Lemma 4.4, z is not weakly meager engulfing over $L_{\omega_1^{\mathrm{CK}}}$. Therefore, z is not Δ^1_1-weakly meager engulfing since $\omega_1^{\mathrm{CK},z} = \omega_1^{\mathrm{CK}}$. Hence, z is i.o. Δ^1_1-traceable as mentioned in Section 3.1. □

4.4. **Examples of Lim-Sup Forcings.** Here, we see properties of some examples of lim-sup forcings. A notion $\mathbb{P}$ of forcing has the Σ_1*-Laver property over* $\mathbf{M}$ if for every $\mathbf{M}$-computation Φ and for every bound $f \in \mathbf{M} \cap \omega^\omega$, if

$$\Vdash_{\mathbb{P}} (\forall n \in \check{\omega})\ \check{\Phi}(\dot{r}_{gen})(n) \downarrow < \check{f}(n),$$

then there exists a $\Sigma_1(\mathbf{M})$-slalom $\{T_n\}_{n\in\omega}$ with $|T_n| \leq n$ such that

$$\Vdash_{\mathbb{P}} (\forall n \in \check{\omega})\ \check{\Phi}(\dot{r}_{gen})(n) \in \check{T}_n.$$

The items (1) and (2) in the following lemma can be proved by using the straightforward modification of the standard argument (see Bartoszyński-Judah [2]).

Lemma 4.15. *Let $\mathbb{PT}$ be the rational perfect set forcing (Example 4.7 (3)).*

1. *Every $\mathbb{PT}$-generic real over $\mathbf{M}$ is $\mathbf{M}$-unbounded.*
2. *The forcing $\mathbb{PT}$ satisfies the Σ_1-Laver property; therefore, every $\mathbb{PT}$-generic real over $\mathbf{M}$ is $\Sigma_1(\mathbf{M})$-Laver traceable.*
3. *If z is $\mathbb{PT}$-generic real over $\mathbf{M}$, then we have $\mathfrak{G}_{\mathbf{M}}(z) = 1/2$.*

Proof. (2) Suppose that $f \in \omega^\omega \cap \mathbf{M}$ is given, and assume that $p \Vdash \check{\Phi}(\dot{r}_{gen})(n) < f(n)$. By continuous reading of names (Lemma 4.12), there is $q \leq p$ such that for every n and $\sigma \in n^n$, $q(\sigma)$ decides the values $\check{\Phi}(\dot{r}_{gen})\upharpoonright n$. In other words, we have a monotone function $h : q \to \omega^{<\omega}$ in M such that for every n and $\sigma \in n^n$,

$$q\upharpoonright q(\sigma) \Vdash \check{\Phi}(\dot{r}_{gen})\upharpoonright n = h(q(\sigma))\upharpoonright n.$$

Moreover, without loss of generality, we may assume that $h(\tau)(n) < f(n)$ for all $\tau \in q$ and $n \in \omega$, and every branching node of q is infinitely branching. We will define a decreasing sequence $(q_n)_{n\in\omega} \subseteq \mathbb{PT}$ such that $q_0 = q$ and $q_{n+1} \leq_n q_n$. Given q_n and $\sigma \in n^{\leq n}$, since $q_n(\sigma)$ is infinitely branching, by the pigeonhole principle, there is $k_\sigma < f(n)$, there are infinitely many j such that $h(q_n(\sigma^\frown j))(n) = k$. Let J_σ be all such j's, and define $S = \{\sigma^\frown j : \sigma \in n^{\leq n},\ j \in J_\sigma \text{ and } \sigma^\frown j \notin n^{\leq n}\}$, and $T_n = \{k_\sigma : \sigma \in n^{\leq n}\}$. Then, define $q_{n+1} = \bigcup_{\sigma^\frown j\in S} q_n(\sigma^\frown j)$. It is not hard to see that $q_n \geq_n q_{n+1} \in \mathbb{PT}$. Let r be a fusion such that $r \leq_n q_n$ for all $n \in \omega$. One can check that $h(z)(n) \in T_n$ for all $z \in [r]$ and $n \in \omega$.

(3) We give a sketch of the proof. We follow the argument of Andrews et al. [1] which shows that computably traceability of x implies $\mathfrak{G}_{\Delta^0_1}(x) \geq 1/2$. Let z be a $\mathbb{PT}$-generic real over $\mathbf{M}$. By the Σ_1-Laver property, for every $A \leq_{\mathbf{M}} z$, there is a $\mathbf{M}$-coded slalom $(T_n)_{n\in\omega}$ such that $A\upharpoonright J_n \in T_n$ for almost all n. Thus, as in Andrews et al. [1], we can find a set $B \subseteq \omega$ in $\mathbf{M}$ such that $\underline{\rho}_{\mathbf{M}}(\{n : A(n) = B(n)\}) \geq 1/2$. Thus, $\mathfrak{G}_{\mathbf{M}}(z) \geq 1/2$. Conversely, $\mathfrak{G}_{\mathbf{M}}(z) \leq 1/2$ since $z \notin \mathbf{M}$ (see [1]). Hence, $\mathfrak{G}_{\mathbf{M}}(z) = 1/2$. □

As in the argument in Corollary 4.14, we can see the following:

Corollary 4.16. *Let z be a $\mathbb{PT}$-generic real over $L_{\omega_1^{\mathrm{CK}}}$. Then, z is Δ^1_1-unbounded, Δ^1_1-Laver traceable (hence, not Δ^1_1-infinitely often equal), and $\mathfrak{G}_{\Delta^1_1}(z) = 1/2$.*

Proof. We only show that z is not Δ^1_1-infinitely often equal. As in Corollary 4.14, $\omega_1^{\mathrm{CK},z} = \omega_1^{\mathrm{CK}}$ by Lemmata 4.12 and 4.5. By Lemma 4.15, z is $\Sigma_1(L_{\omega_1^{\mathrm{CK}}})$-Laver traceable. Therefore, z is Π^1_1-Laver traceable since $\omega_1^{\mathrm{CK},z} = \omega_1^{\mathrm{CK}}$. By Lemma 3.3, z is Δ^1_1-Laver traceable. Consequently, by Proposition 3.6 (1a), z is not Δ^1_1-infinitely often equal. □

We are now ready to prove our main theorem for $\Gamma = \Pi^1_1$.

Proof of Theorem 2.5 for $\Gamma = \Pi^1_1$. Let z be a $\mathbb{PT}$-generic real over $L_{\omega_1^{\mathrm{CK}}}$. By Corollary 4.16, $z \in [\mathsf{D}]_\Delta$ (i.e., z is Δ^1_1-bounded), but $z \notin [\mathsf{ED}]_\Delta$ (i.e., z is not Δ-infinitely often equal) and $z \notin [\mathsf{Cov}\mathcal{X}_{0.5}]_\Delta$ (since $\mathfrak{G}_{\Delta^1_1}(z) = 1/2$). Therefore, we have $[\mathsf{ED}]_{\Delta^1_1} \not\leq_{\Delta^1_1\mathrm{w}} [\mathsf{D}]_{\Delta^1_1}$ and $[\mathsf{Cov}\mathcal{X}_{0.5}]_{\Delta^1_1} \not\leq_{\Delta^1_1\mathrm{w}} [\mathsf{D}]_{\Delta^1_1}$. Moreover, since there is a computable Tukey morphism $\mathsf{D} \longrightarrow \mathsf{Cof}\mathcal{M}$, we have $[\mathsf{D}]_{\Delta^1_1} \subseteq [\mathsf{Cof}\mathcal{M}]_{\Delta^1_1}$. Therefore $z \in [\mathsf{Cof}\mathcal{M}]_{\Delta^1_1}$ (i.e., z is not low for Δ^1_1-generic). Finally, by Lemma and by $\omega_1^{\mathrm{CK},z} = \omega_1^{\mathrm{CK}}$, z is not weakly Δ^1_1-meager engulfing, that is, $z \notin [\mathsf{Non}\mathcal{M}]_{\Delta^1_1}$. Consequently, we have $[\mathsf{Non}\mathcal{M}]_{\Delta^1_1} \not\leq_{\Delta^1_1} [\mathsf{Cof}\mathcal{M}]_{\Delta^1_1}$. □

The infinite equal forcing $\mathbb{EE}$ (Example 4.7 (4)) is useful to show the properness of the Muchnik reduction $[\mathsf{Non}\mathcal{N}]_L <_{L\mathrm{w}} [\mathsf{Cof}\mathcal{M}]_L$ in Cichoń's diagram. As in the usual argument (see Bartzyński-Judah [2, Section 7.4.C], Kjos-Hanssen et al. [31]), one can easily see that Every $\mathbb{EE}$-generic real z over $\mathbf{M}$ is an $(\lambda n.2^n)$-infinitely often equal real over $\mathbf{M}$ (in particular, z is weakly null-engulfing over $\mathbf{M}$ by Proposition 3.6), and every $\mathbb{EE}$-generic is $\mathbf{M}$-often $\mathbf{M}$-traceable. Hence, every $\mathbb{EE}$-generic real over $L_{\omega_1^{\mathrm{CK}}}$ is Δ^1_1-weakly null-engulfing, and Δ^1_1-often Δ^1_1-traceable (i.e., low for Δ^1_1-Kurtz randomness).

4.5. **Arithmetical Definability.** One can also ask whether the similar results also hold for the arithmetical degrees. Of course, we cannot use the forcing relation introduced in Section 4 since the universe $L_{\omega+1}$ of arithmetically coded sets is not admissible, and the definition of $\Vdash_{\mathbb{Q}}$ is Π^1_1. To overcome this difficulty, we consider the following local Cohen forcing argument: If $\mathbb{Q}$ is lim-sup tree forcing over $\mathbf{M} = L_\alpha[R]$, then the forcing relation $\Vdash_{\mathbb{Q}}$ for $\Delta_0(\mathbf{M})$-formulas can be characterized by using *local Cohen forcing*. Indeed, by using the usual fusion argument, we can show that the following two conditions are equivalent for any Δ_0 formula φ (see also Lemma 4.27):

$$(\exists q \leq p)\ q \Vdash_{\mathbb{Q}} \varphi(\dot{r}_{gen}, \vec{v}), \tag{9}$$

$$(\exists r \leq p)\ [r \in \mathbb{Q}\ \&\ \mathbf{1} \Vdash_{\mathbb{C}(r)} \varphi(\dot{r}_{gen}, \vec{v})]. \tag{10}$$

Here $\mathbb{C}(r)$ is the local Cohen forcing inside the tree $r \subseteq H$, that is, each forcing condition is a string in r, and $p \leq q$ if and only if p extends q. The implication from (9) to (10) is obvious by the definition of the forcing relation $\Vdash_{\mathbb{Q}}$ for Δ_0 formulas. For the converse direction, assume $\mathbf{1} \Vdash_{\mathbb{C}(r)} \varphi(\dot{r}_{gen}, \vec{v})$, and find a sufficiently large rank $\gamma \in \mathbf{M}$ such that $L_\gamma[R]$ is provident (see Mathias [35]) which contains r, φ and $\vec{v}$. By fusion argument inside r (see also Lemmas 4.27 for the detail), we can find a $\mathbb{Q}$-condition $q \leq r$ such that every element of $[q]$ is $\mathbb{C}(r)$-generic over $L_{\gamma+1}[R]$. Then, every $x \in [q]$ satisfies $\varphi(x, \vec{v})$ since x is an r-local Cohen real, and $\mathbf{1} \Vdash_{\mathbb{C}(r)} \varphi(\dot{r}_{gen}, \vec{v})$. Hence, $[q] \subseteq \{x \in \omega^\omega : \varphi(x, \mathbf{v})\}$, and $q \Vdash_{\mathbb{Q}} \varphi(\dot{r}_{gen}, \vec{v})$ by the definition.

It is well-known that the Cohen forcing relation $\Vdash_{\mathbb{C}(r)}$ for Σ^0_n formulas is Σ^0_n-definable relative to r. Thus, by using the condition (10) instead of the condition (9), we have the similar results for arithmetical degrees: Every arithmetically $\mathbb{PT}$-generic real z is arithmetically unbounded, arithmetically i.o. traceable, arithmetically Laver traceable (hence, not arithmetically infinitely often equal), and the arithmetical $\mathfrak{G}$-value of z is $1/2$.

4.6. **Infinite Time Register Machine Computability.** The next Spector pointclasses above Π^1_1 are obtained as the Nikodym hierarchy of Selivanovskij's $\mathcal{C}$-sets (that is, the hierarchy obtained by iterating Suslin's $\mathcal{A}$-operation). This Nikodym hierarchy has a logical representation by using the game quantifier $\Game(D_\alpha\Sigma^0_1)$ for the α-th level $D_\alpha\Sigma^0_1$ of the (lightface) Hausdorff difference hierarchy starting from (c.e.) open sets (see Burgess [7], and Tanaka [53]), where recall that $\Game\Sigma^0_1 = \Pi^1_1$ and $\Game\Pi^0_1 = \Sigma^1_1$. Of course, the companions of these pointclasses are an initial segment of admissible sets $L_{\omega_1^{\rm CK}}, L_{\omega_2^{\rm CK}}, L_{\omega_3^{\rm CK}}, \dots$. The first ω admissible ordinals and their limit also naturally occur in the context of *infinite time register machines*.

At the beginning of the 1960s, computability theorists (proof theorists, and set theorists) started to develop ordinal-length computations. Earlier developments (the 1960s–'80s) on computability on ordinals have been summarized in several classical textbooks such as Sacks [48] (for α-recursion theory) and Hinman [19, Chapter 8] (for ordinal recursion theory). *Infinite time Turing machines* (ITTMs) and *infinite time register machines* (ITRMs) are special kinds of computation models on ordinals. An ITTM/ITRM is designed as the same as a usual TM/RM, but an ITTM/ITRM-computation is allowed to run for ordinal steps, while the memory storage of a machine is limited to ω as a usual TM/RM (see Koepke [32], Koepke-Miller [33] and Carl-Schlicht [9]).

The theory of ITTMs has been found to be very interesting because it involves a new kind of large countable ordinals (see Welch [56]) which has not been discovered in the '80s. The ordinals associated with ITRMs are relatively small as described below. We use the symbol $\omega^{\rm CK}_\alpha$ (and $\omega^{{\rm CK},z}_\alpha$) to denote the α-th ordinal which is admissible or the limit of admissible ordinals (relative to z). Note that $\omega^{\rm CK}_\omega$ itself is not admissible, and clearly, it is much smaller than the first recursively inaccessible ordinal $\omega_1^{\mathrm{E}_1}$ (Indeed, $\omega_1^{\mathrm{E}_1}$ is the least ordinal α such that $\alpha = \omega_\alpha^{\mathrm{CK}}$).

Fact 4.17 (see Koepke [32])**.** *A real x is ITRM-computable relative to a real y if and only if $x \in L_{\omega_\omega^{\mathrm{CK},y}}[y]$.*

Now we show our main result for ITRM computability by combining a local Cohen forcing argument and a lim-sup tree forcing argument.

Fact 4.18 (Folklore)**.** *Suppose that z is a $\mathbb{C}(r)$-generic real over $L_{\omega_n^{\mathrm{CK},r}+1}[r]$. Then, $\omega_n^{\mathrm{CK},z\oplus r} = \omega_n^{\mathrm{CK},r}$.* □

One can again use local Cohen forcing argument to show the following.

Lemma 4.19. *Suppose that $\mathbb{Q}$ is a lim-sup tree forcing with an ITRM-computable norm. There exists a real z such that*

(1) *z is $\mathbb{Q}$-generic over $L_{\omega_n^{\mathrm{CK}}}$ for infinitely many $n \in \omega$.*
(2) *$\omega_\omega^{\mathrm{CK}} = \omega_\omega^{\mathrm{CK},z}$.*

Proof. We start from the empty condition $p_0 = \mathbf{1}_{\mathbb{Q}}$. Suppose that a condition $p_n \in \mathbb{Q} \cap L_{\omega_\omega^{\mathrm{CK}}}$ is given. Choose $k(n) \in \omega$ such that $p_n \in L_{\omega_{k(n)}^{\mathrm{CK}}}$. In the next admissible rank (i.e., in $L_{\omega_{k(n)+1}^{\mathrm{CK}}}$), we construct a sequence $p_n = q_n^0 \geq_0 q_n^1 \geq_1 q_n^2 \geq_2 \dots$ such that q_n^e decides the e-th sentence over $L_{\omega_{k(n)}^{\mathrm{CK}}}$. By fusion argument, one can find a **nor**-perfect tree $q_n \in L_{\omega_{k(n)+1}^{\mathrm{CK}}}$ below p_n such that for any $x \in [q_n]$, x is $\mathbb{Q}$-generic over $L^{\mathrm{CK}}_{\omega_{k(n)}}$.

Then, we can next find a **nor**-perfect tree $p_{n+1} \subseteq q_n$ in $L_{\omega_{k(n)+n+1}^{\mathrm{CK}}}$ such that for any $x \in [p_{n+1}]$, x is $\mathbb{C}(q_n)$-generic over $L_{\omega_{k(n)+n}^{\mathrm{CK}}}$, where $\mathbb{C}(q_n)$ is "local" Cohen forcing inside the closed subspace $[q_n] \subseteq \omega^\omega$. We may assume that the length of the stem of p_{n+1} is greater than n. Since $q_n \in L_{\omega_{k(n)}^{\mathrm{CK}}}$, for every $1 \leq i < n$, $L_{\omega_{k(n)+i}^{\mathrm{CK}}}$ is admissible in any $x \in [p_{n+1}]$ by Fact 4.18. Then, for such $x \in [p_{n+1}]$, there are at least n many admissible-in-x ordinals below $L_{\omega_\omega^{\mathrm{CK}}}$. Therefore, if $z \in \bigcap_n [p_n]$, then there are ω many admissible-in-z ordinals below $L_{\omega_\omega^{\mathrm{CK}}}$. This implies $\omega_\omega^{\mathrm{CK},z} = \omega_\omega^{\mathrm{CK},z}$. □

Corollary 4.20. *There is a real z such that z is ITRM-unbounded, i.o. ITRM-traceable, ITRM-Laver traceable (hence, not ITRM-infinitely often equal), and that $\mathfrak{G}_{\mathrm{ITRM}}(z) = 1/2$.*

Proof. For $\mathbb{Q} = \mathbb{PT}$, suppose that z is a real as in Lemma 4.19. We only show that z is ITRM-Laver-traceable. If $g \in \omega^\omega$ is ITRM-computable in z, then $g \in L_{\omega_\omega^{\mathrm{CK},z}}[z] = L_{\omega_\omega^{\mathrm{CK}}}[z]$ by Lemma 4.19. In particular, there is n such that $g \in L_{\omega_n^{\mathrm{CK}}}[z]$. Suppose that g is ITRM-bounded, say, there is $h \in L_{\omega_k^{\mathrm{CK}}}$ such that $g(n) < h(n)$ for all $n \in \omega$ So, we can choose some $m \geq n, k$ such that z is $\mathbb{PT}$-generic over $L_{\omega_m^{\mathrm{CK}}}$. Then, by Lemma 4.15, there is a $\Sigma_1(L_{\omega_m^{\mathrm{CK}}})$-slalom $(T_n)_{n\in\omega}$ such that $g \in T_n$ for all $n \in \omega$. Consequently, every ITRM-bounded function $g \leq_{\mathrm{ITRM}} z$ is traced by a ITRM-computable slalom $(T_n)_{n\in\omega}$, that is, z is ITRM-Laver traceable. □

4.7. **Recursively Hyper-Inaccessibles.** Our results in Section 4 can be applied to Kleene's higher type computability theory, where recall that the 2-envelope of any normal type 2 functional forms a Spector pointclass (see Kechris [24]). We first consider a hierarchy of Spector pointclasses generated by a normal type 3 functional sJ called *Gandy's superjump operator* (see Hinman [19] for instance), where $\mathrm{sJ}(F) : \omega \times \omega^\omega \to 2$ is defined as follows for any type 2 functional F:

$$\mathrm{sJ}(F)(e,x) = \begin{cases} 1 & \text{if } \Phi_e^F(x)\downarrow, \\ 0 & \text{if } \Phi_e^F(x)\uparrow \end{cases}$$

Here, Φ_e^F is the e-th computation relative to the functional F in the sense of Kleene's finite type computability. Define E_0 to be the Turing jump operator, and define E_{n+1} to be the superjump $\mathrm{sJ}(\mathrm{E}_n)$ of E_n. Note that E_1 is essentially equivalent to the hyperjump operator (hence, a transfinite iteration of hyperjumps is computable in E_1). We can introduce the E_n-reducibility $\leq_{\mathrm{E}_n}$ via the Spector pointclass ${}_2\mathrm{env}(\mathrm{E}_n)$, where ${}_2\mathrm{env}(\mathrm{E}_n)$ denotes the 2-envelop of the normal type 2 functional E_n, that is, the collection of all sets A of reals such that A is semi-computable in E_n. Clearly, $\leq_{\mathrm{E}_0}$ is equivalent to the hyperarithmetical reducibility $\leq_{\mathrm{h}}$. The structure of the E_n-degrees has been studied by Shinoda [49, 50].

Shinoda [49] showed that for every $n \in \omega$, the Sacks forcing over $L_{\omega_1^{\mathrm{E}_n}}$ preserves the least n-recursively inaccessible ordinal $\omega_1^{\mathrm{E}_n}$ under the n-th normal type 2 hypercomputation E_n, by using the Σ_1-fusion property and local Cohen forcing. By the same argument, we show that any sufficiently generic with respect to a lim-sup tree forcing also preserves $\omega_1^{\mathrm{E}_n}$. First note that if z is a (local) Cohen-generic real over $L_{\omega_1^{\mathrm{E}_n}}$, then, $\omega_1^{\mathrm{E}_n,z} = \omega_1^{\mathrm{E}_n}$ (Shinoda [49, Theorem 3.9]). The following two lemmas are modifications of Shinoda [49, Lemma 4.6] and Shinoda [49, Theorem 4.7], respectively.

Lemma 4.21. *Suppose that κ is n-recursively inaccessible, and p is a* **nor***-perfect tree in L_κ. Then, there exists a* **nor***-perfect tree $p^* \subseteq p$ in L_{κ^+} such that for any $x \in [p^*]$, x is p-Cohen over L_κ and κ is n-recursively-in-x inaccessible, where κ^+ is the first admissible ordinal larger than κ.* □

Lemma 4.22. *Suppose that $\mathbb{Q}$ is a lim-sup tree forcing. For every $n \in \omega$, there exists a $\mathbb{Q}$-generic real x over $L_{\omega_1^{\mathrm{E}_n}}$ such that $\omega_1^{\mathrm{E}_n,x} = \omega_1^{\mathrm{E}_n}$.* □

As a consequence of the above lemmas combined with Sections 4.3 and 4.4, we can see the following:

Corollary 4.23. *For every $n \in \omega$, there is a real z such that z is E_n-unbounded, i.o. E_n-traceable, E_n-Laver traceable (hence, not E_n-infinitely often equal), and $\mathfrak{G}_{\mathrm{E}_n}(z) = 1/2$.* □

4.8. **Reflecting Spector Pointclasses.** In this subsection, we deal with the following kind of pointclasses indroduced by Kechris [25, Definition 1.11]. A pointclass Γ is *nice* if Γ contains Δ^0_1 and closed under computable substitutions, and $\vee, \wedge, \exists^\omega$, Γ is ω-parametrized and scaled, and Γ contains Π^0_1 unless $\Gamma = \Sigma^0_1$. For instance, the pointclass Σ^0_n for each $n \geq 1$ is nice. A Spector pointclass Γ is $\Game$-*generated* if there is a nice pointclass Γ' such that $\Gamma = \Game\Gamma'$. Moreover, a Spector pointclass Γ is *reflecting* if for any oracle α and $\Gamma(\alpha)$ set $R \subseteq 2^\omega$, we have

$$\exists A \in \Gamma(\alpha)\ R(A) \ \Rightarrow\ \exists B \in \Delta(\alpha)\ R(B).$$

Kechris [25] pointed out that $\Game\Gamma'$ is reflecting for every nice pointclass $\Gamma' \supseteq \Pi^0_2$. Basically, we follow the argument in Kechris [22]. Let $A \subseteq \omega^\omega$. We say that p *forces*

A (written as $p \Vdash \dot{r}_{gen} \in A$ or $p \Vdash A$) if

$$(\forall p_0 \leq p)(\exists p_1 \leq p_0)(\forall p_2 \leq p_1)(\exists p_3 \leq p_2) \ldots \lim_n p_n \in A.$$

Here, $\lim p_n$ is the unique element contained in $\bigcap_n [p_n]$ if $\bigcap_n [p_n]$ is a singleton.

(1) $p \Vdash A$ and $A \subseteq B$ implies $p \Vdash B$.
(2) $p \Vdash A$ and $q \leq p$ implies $q \Vdash A$.

We say that $A \subseteq \omega^\omega$ is $\mathbb{P}$-*measurable* (or it has *the Baire property with respect to* $\mathbb{P}$) if there is a $\mathbb{P}$-open set $U \subseteq \omega^\omega$ (i.e., a set of the form $\bigcup_{p \in W}[p]$ for some $W \subseteq \mathbb{P}$) such that $\mathbf{1} \Vdash A = U$ (i.e., $\mathbf{1} \Vdash \dot{r}_{gen} \notin A \triangle U$).

Proposition 4.24 (Kechris [22])**.** *Suppose that A is $\mathbb{P}$-measurable.*

(1) *(Quasi-completeness) $p \Vdash A$ or there exists $q \leq p$ such that $q \Vdash \neg A$.*
(2) *(Forcing = Truth) For all sufficiently $\mathbb{P}$-generic reals $r \in \omega^\omega$, $r \in A$ if and only if there exists $p \in \mathbb{P}$ such that $r \in [p]$ and $p \Vdash A$.*

Proposition 4.25 (Kechris [22])**.** *Suppose that A and $\{A_i\}_{i\in\omega}$ are $\mathbb{P}$-measurable.*

(1) *$p \Vdash \neg A$ if and only if $q \nVdash A$ for every $q \leq p$.*
(2) *$p \Vdash \bigcap_i A_i$ if and only if $p \Vdash A_i$ for all $i \in \omega$.*
(3) *$p \Vdash \bigcup_i A_i$ if and only if for every $q \leq p$, there are $r \leq q$ and $i \in \omega$ such that $r \Vdash A_i$.*

The problem is to estimate the complexity of $p \Vdash A$. Most pointclasses Γ which we will consider in this section satisfy the following condition:

(11) $\Gamma = \Game\Gamma'$ is a $\Game$-generated reflecting Spector pointclass and $\sigma\mathbf{\Gamma}'$-determinacy holds,

where $\sigma\mathbf{\Gamma}'$ is the smallest σ-algebra including $\mathbf{\Gamma}'$. The Borel determinacy implies that the Spector pointclass $\Game\Sigma^0_n$ for each $n \geq 3$ satisfies (11). We use the notation $\mathbb{P}$ for a Γ-definable strongly arboreal forcing, and $\mathbb{Q}$ for the following

(12) $\mathbb{Q}$ is a lim-sup tree forcing with Δ-trees and Δ-norms.

Note that such a forcing $\mathbb{Q}$ must be Γ-definable forcing.

Lemma 4.26 (see also Kechris [22, Theorem 5.2.2])**.** *Suppose that Γ is a Spector pointclass satisfying (11), and $\mathbb{P}$ is a Γ-forcing. Then, every $\mathbf{\Gamma}$ set is $\mathbb{P}$-measurable.*

Sketch of Proof. As in Kechris [22, Theorem 5.2.2], it suffices to show that the Banach-Mazur game (with respect to the topology induced by the forcing $\mathbb{P}$) with the winning set $A \cap F$ is determined for any $A \in \mathbf{\Gamma}$ and a closed set F. Note that the game on forcing conditions can be identified with a game on ω by using the oracle $D = \{\langle i,j\rangle \in \omega^2 : \pi(i), \pi(j) \in \mathbb{P} \ \&\ \pi(i) \leq \pi(j)\}$, where $\pi :\subseteq \omega \twoheadrightarrow \Delta$ is a partial surjection. Since $A \in \Game\Gamma'$, by absorbing the Γ' game for A into the Banach-Mazur game, the whole game can also be viewed as a $\mathbf{\Gamma}'$ game on ω with parameter D. Hence, we have the desired conclusion by $\mathbf{\Gamma}'$-determinacy. □

Now, we see that the forcing relation of any lim-sup tree forcings $\mathbb{Q}$ is characterized in the context of local Cohen forcing.

Lemma 4.27 (see also Kechris [25, Theorem1.14]). *Suppose that* Γ *is a Spector pointclass satisfying (11), and* $\mathbb{Q}$ *is a forcing satisfying (12). Then, the following are equivalent:*

$$(\exists q \leq p)\ q \Vdash_{\mathbb{Q}} (\dot{r}_{gen}, n) \in A, \tag{13}$$

$$(\exists r \leq p)\ [r \in \mathbb{Q}\ \&\ \mathbf{1} \Vdash_{\mathbb{C}(r)} (\dot{r}_{gen}, n) \in A] \tag{14}$$

Sketch of Proof. We follow the argument in Kechris [25, Theorem 1.14]. We first see that (13) implies (14). Let $D = \{\langle i, j\rangle \in \omega^2 : \pi(i), \pi(j) \in \mathbb{Q}\ \&\ \pi(i) \leq \pi(j)\}$. As before, there is a $\Gamma'(D)$-game G such that q forces $(\dot{r}_{gen}, n) \in A$ for some $q \leq p$ if and only if $\exists$-player wins the game G. Then, there is a $\Delta(D)$ winning strategy τ for $\exists$-player in the game associated with $q \Vdash_{\mathbb{M}} (\dot{r}_{gen}, n) \in A$ for some $q \leq p$.

We construct an $\omega^{<\omega}$-indexed set $\{r_\sigma\}_{\sigma\in\omega^{<\omega}}$ of **nor**-perfect subtrees of q according to the strategy τ. Let $r_{\langle\rangle} = q$, and suppose that we have already constructed a **nor**-perfect tree $r_\sigma \in \mathbb{Q}$. Let ρ_σ be the first node of the **nor**-perfect tree r_σ such that $\mathbf{nor}(\rho_\sigma, \mathrm{succ}_{r_\sigma}(\rho_\sigma)) > |\sigma|$. Then, $\forall$-player defines $r^*_{\sigma^\frown\langle n\rangle}$ to be $r_\sigma \upharpoonright \rho_\sigma{}^\frown\langle s_n\rangle$, where s_n is the n-th immediate successor node of ρ_σ in the tree r_σ. Then, the winning strategy τ for $\exists$-player returns a **nor**-perfect tree $r_{\sigma^\frown\langle n\rangle}$ from $r^*(\sigma^\frown\langle n\rangle)$. Clearly, $\{\rho_\sigma : \sigma \in \omega^{<\omega}\}$ generates a $\Delta(D)$ **nor**-perfect tree $r \leq p$ such that $[r] \subseteq \{x : (x, n) \in A\}$, since τ is a $\Delta(D)$-winning strategy.

Now, we proceeds local Cohen forcing $\mathbb{C}(r)$ inside **nor**-perfect trees r. Clearly, the empty condition forces $(\dot{r}_{gen}, n) \in A$ inside r. Then,

$$(\exists r \in \Delta(D))\ [r \leq p\ \&\ r \in \mathbb{Q}\ \&\ \mathbf{1} \Vdash_{\mathbb{C}(r)} (\dot{r}_{gen}, n) \in A]$$

is represented by a Γ formula with a parameter n and D, since $\mathbb{C}(r)$ is set-forcing. Since Γ is reflecting, we can replace D with a Δ set. Consequently, the condition (13) implies the condition (14).

Conversely, if (14) holds, by using a Δ winning strategy associated with $\Vdash_{\mathbb{C}(r)}$ $(\dot{r}_{gen}, n) \in A$, we may find a Δ **nor**-perfect tree q inside the **nor**-perfect tree r such that $[q] \subseteq \{x : (x, n) \in A\}$ by the similar argument as above. Clearly, q forces $(\dot{r}_{gen}, n) \in A$. □

As a consequence, for every Γ set A,

$$\{(p, n) \in \omega^\omega \times \omega : (\exists q \leq p)\ q \Vdash_{\mathbb{Q}} (\dot{r}_{gen}, n) \in A\}$$

is in Γ, since by the definability of (local) Cohen forcing. Therefore, by quasi-completeness,

$$\{(i, n) \in \omega^2 : p_i \nVdash_{\mathbb{P}} (\dot{r}_{gen}, n) \notin A\}$$

is in Γ, since this is equivalent to the consequence of the previous lemma provided every $\mathbf{\Gamma}$ set is $\mathbb{P}$-measurable.

Lemma 4.28 (Kechris [22, Theorem 4.4.1]). *Let* Γ *be a Spector pointclass, and* $\mathbb{P}$ *be a* Γ*-forcing notion. Moreover, suppose that every* $\mathbf{\Gamma}$ *subset of* ω^ω *is* $\mathbb{P}$*-measurable. For every* Γ *set* $A \subseteq \omega^\omega \times \omega$*, if*

$$\{(i, n) \in \omega^2 : p_i \nVdash_{\mathbb{P}} (\dot{r}_{gen}, n) \notin A\}$$

is in Γ*, then for all sufficiently* $\mathbb{P}$*-generic* x*, we have* $\lambda^x_\Gamma = \lambda_\Gamma$. □

Lemma 4.29 (see also Kechris [25, Lemma 1.14]). *Suppose that* Γ *is a Spector pointclass satisfying (11), and* $\mathbb{Q}$ *is a forcing satisfying (12). Then, every sufficiently* $\mathbb{Q}$*-generic real* z *satisfies* $\lambda^z_\Gamma = \lambda_\Gamma$.

By combining the results from Section 4 and Lemma 4.29 (with Kechris [25, Lemma 1.5]), we have the following:

Lemma 4.30. *Suppose that Γ is a Spector pointclass satisfying (11), and $\mathbb{Q}$ is a forcing satisfying (12). Then,*

(1) *$\mathbb{Q}$ has the Γ-continuous reading of names.*
(2) *$\mathbb{Q}$ adds no weakly Γ-meager engulfing real.*

Lemma 4.31. *Suppose that Γ is a Spector pointclass satisfying (11), and $\mathbb{PT}(\Delta)$ is the rational perfect set forcing with Δ-trees. Then,*

(1) *$\mathbb{PT}(\Delta)$ adds a Δ-unbounded real.*
(2) *$\mathbb{PT}(\Delta)$ has the Γ-Laver property.*
(3) *For any $\mathbb{PT}(\Delta)$-generic real z, we have $\mathfrak{G}_\Delta(z) = 1/2$.*

Corollary 4.32. *Suppose that Γ is a Spector pointclass satisfying (11), and $\mathbb{PT}(\Delta)$ is the rational perfect set forcing with Δ-trees. Then, for every $\mathbb{PT}(\Delta)$-generic real z, z is Δ-unbounded, i.o. Δ-traceable, Δ-Laver traceable (hence, not Δ-infinitely often equal), and $\mathfrak{G}_\Delta(z) = 1/2$.* □

5. Open Questions

In this section, we propose several open problems. First we focus on the separation problem of $[\mathsf{Cov}\mathcal{N}]_\Delta$ and $[\mathsf{Non}\mathcal{M}]_\Delta$, that is, the problem asking the existence of a real which is not i.o. Δ-traceable, and Δ-computes no Δ-random real. Since i.o. Δ-tt-traceability is equivalent to Hausdorff h-nullness for all Δ-gauge functions h (see [28]), there must be an extremely long hierarchy of Hausdorff measures (i.e., a hierarchy based on growing rates of Kolmogorov complexity) between $[\mathsf{Cov}\mathcal{N}]_\Delta$ and $[\mathsf{Non}\mathcal{M}]_\Delta$. However, as shown in Lemma 4.4, any real added by a lim-sup tree forcing is captured by a ground model set which is Hausdorff h-null for all gauge functions h in the ground model. Therefore, to separate these two notions, we need to develop a new technique other than lim-sup tree forcing.

Problem 5.1. *Does there exist a forcing over a companion model $\mathbf{M}$ (e.g., $\mathbf{M} = L_{\omega_1^{\mathrm{CK}}}$) which satisfies the Σ_1-fusion property, but adds a weakly meager engulfing real?*

The second problem is whether the lim-sup forcing argument is generally applicable to all Spector pointclasses. At least, as observed in this article, we know the lim-sup forcing argument works at pointclasses $\Game\Sigma^0_1$, $\Game(D_{<\omega}\Sigma^0_1)$, and $\Game\Sigma^0_n$ for $n \geq 3$. However, our work does not cover various important pointclasses; e.g., any pointclasses in the $\mathcal{R}$-hierarchy (the hierarchy between $\Game\Sigma^0_2$ and $\Game\Delta^0_3$; see [8]). Note that the ordinal for $\Game\Sigma^0_2$ (or equivalently $\mathrm{Ind}(\Sigma^1_1)$) is characterized as the least Σ^1_1-reflecting ordinal by Richter-Aczel [43]. More generally, the ordinal for $\Game(D_\alpha\Sigma^0_2)$ (where $D_\alpha\Sigma^0_2$ is the α-th level of the difference hierarchy starting from the pointclass Σ^0_2) is characterized by Lubersky [34] in the context of reflecting ordinal. Beyond the $\mathcal{R}$-hierarchy, Welch [55] showed that Sacks forcing is available at the level of infinite-time Turing machine (ITTM) computability. It is natural to ask whether any lim-sup forcing works at these natural pointclasses. For instance:

Problem 5.2. *Let λ^x be the supremum of infinite-time Turing machine writable ordinals relative to x (see [56]). Suppose that $\mathbb{Q}$ is a lim-sup tree forcing over L_λ. Then, for every $\mathbb{Q}$-generic real z over L_λ, do we have $\lambda^z = \lambda$?*

The problem can be reduced to the problem asking whether one can define such ordinals (Σ^1_1-reflecting ordinals, n-gap reflecting ordinals, writable ordinals, etc.) in the context of admissibility. For instance, recursively inaccessible ordinals are defined as the admissible limits of admissible ordinals; therefore, preservation of admissible ordinals implies preservation of recursively inaccessible ordinals, too (indeed, this is the essence in the proofs of results in Section 4.7). Hiroshi Fujita taught the author that, shortly after he obtained master degree under Juichi Shinoda in 1989, Shinoda had given him a similar problem.

Next, we devote our attention to lim-inf forcing. To carry out the fusion argument with Laver trees, the standard method requires us to decide the truth-value a given Σ_1-sentence (to define a ranking function). Of course, this is impossible if our model is the companion of a Spector pointclass. Therefore, we conjecture that Laver forcing is unavailable over the companion of any Spector pointclass. At least, as mentioned in Section 2.3, we know that Laver forcing is not available at the level of arithmetical and hyperarithmetical degrees because (hyper)arithmetical dominant computes all (hyper)arithmetical reals. We may ask whether unavailability of Laver forcing over any companion model can be witnessed in this way, that is,

(∗) if a function $f \in \omega^\omega$ dominates all Δ-functions, then does there exist $x \leq_\Delta f$ such that x computes all Δ-reals?

As a first step, we may consider the following special case:

Problem 5.3. *If a function $f \in \omega^\omega$ dominates all $\mathcal{O}$-hyperarithmetic functions, then does f compute all $\mathcal{O}$-hyperarithmetical reals?*

The fourth problem is about infinite often equality. The reason of the emphasis on the notion of infinite often equality in this article lies in previous works on forcing theory. It is known that two iterations of any forcing adding an infinitely often equal real always add a Cohen real (see [2, Lemma 2.4.8]). Fremlin's so-called "half-a-Cohen-real problem" asks whether there is a forcing adding an infinitely often equal real over the ground model without adding a Cohen real. Zapletal [58] recently solved the problem by showing that forcing with nonzero-dimensional subcompacta of a Henderson compactum (a hereditarily infinite dimensional compactum) adds an infinitely often equal real, but no Cohen real (Zapletal also pointed out that if X is uncountable dimensional compactum, forcing with uncountable-dimensional subcompacta of X satisfies Fremlin's condition). It is very unusual that infinite dimensionality is applied in set theory and mathematical logic in such an essential way (except for set-theoretic topology). Therefore, the role of infinite dimensionality in set theory and logic is yet to be understood.

As another recent use of infinite dimensional topology in mathematical logic, Kihara-Pauly [30] observed that a non-total degree (in the sense of computability theory) can be thought of as an effective genericity with respect to countable-dimensional subsets of Hilbert cube. Based on this observation, the authors [30] found a connection between computability theory and infinite dimensional topology, and used the connection to solve some problems in descriptive set theory and Banach space theory. We are interested in whether there is a hidden connection between these two recent discoveries [30, 58]. To develop an understanding of the relationship between these works, it seems reasonable to study the effective version of Zapletal's forcing. This is the reason why we consider the following problem as an important one:

Problem 5.4. *Does there exist a Δ^1_1-infinitely often equal real $x \in 2^\omega$ such that no $y \leq_h x$ is Δ^1_1-generic?*

Zapletal's proof heavily uses the result of R. Pol and Zakrzewski [42, Theorem 5.1] to show that Zapletal's uncountable-dimensional forcing has the 1-1 or constant property (Spector-Sacks' minimality condition). However, it seems that the proof of Pol-Zakrzewski [42, Theorem 5.1] requires a stronger separation axiom than KP.

Additional Comments on Problem 5.3. An anonymous referee answered Problem 5.3 negatively, while Day-Greenberg-Turetsky (in a private communication) answered the "*correct*" version of Problem 5.3 affirmatively.

On the one hand, an anonymous referee suggested to use Solovay's characterization [51] that *computably encodable sets* are exactly hyperarithmetical sets. Given a reducibility notion $\leq_r$, we say that $A \in \omega^\omega$ is $\leq_r$*-encodable* if for every $X \in [\omega]^\omega$ there is $Y \in [X]^\omega$ such that $A \leq_r Y$. We also say that $A \in \omega^\omega$ admits a $\leq_r$*-modulus* if there is $f \in \omega^\omega$ such that, for any $g \in \omega^\omega$ dominating f, $A \leq_r g$ holds. Obviously, if A admits a $\leq_r$-modulus, then it is $\leq_r$-encodable.

Fact 5.5 (see Solovay [51])**.** *Let A be a real.*

1. *A is computably encodable if and only if A admits a computable modulus if and only if A is hyperarithmetical.*
2. *If A is arithmetically encodable then $A \in L_{\omega_1^{\mathrm{E}_1}}$ (that is, $A \leq_{\mathrm{E}_1} \emptyset$).*
3. *A is hyperarithmetically encodable if and only if A admits a hyperarithmetical modulus if and only if $A \in L_{\sigma^1_1}$ where σ^1_1 is the least Σ^1_1-reflecting ordinal (that is, $A \leq_\Delta \emptyset$ for $\Gamma = \Game\Sigma^0_2$ as mentioned above).*

This refutes Problem 5.3. More generally, Fact 5.5 implies the following:

Corollary 5.6. *Let Γ be a countable pointclass.*

1. *If $\Delta \supsetneq \Delta^1_1$ then there are a Δ-dominant f and a Δ real x such that x is not computable in f.*
2. *If Δ contains a real which is not E_1-computable, then there are a Δ-dominant f and a Δ real x such that x is not arithmetical in f.*
3. *If Δ contains a real which is not $\Game\Sigma^0_2$-computable, then there are a Δ-dominant f and a Δ real x such that x is not hyperarithmetical in f.*

On the other hand, Day-Greenberg-Turetsky pointed out that if f dominates all $\mathcal{O}$-computable functions, all $\mathcal{O}$-computable reals are *arithmetical* in f. Indeed, their argument shows that if f dominates all $\mathcal{O}^n$-computable functions, all $\mathcal{O}^n$-computable reals are arithmetical in f. In particular, if a function f dominates all ITRM-computable functions, then all ITRM-computable reals are arithmetical in f. As in Fact 2.6, this shows that

$$[\mathsf{Add}\mathcal{N}]_{\mathrm{ITRM}} \equiv_{\mathrm{ITRMw}} [\mathsf{Add}\mathcal{M}]_{\mathrm{ITRM}} \equiv_{\mathrm{ITRMw}} [\mathsf{B}]_{\mathrm{ITRM}}.$$

Fact 5.5 (2) leads us to the conjecture that Day-Greenberg-Turetsky's observation can be extended to E_1-computability. We also note that the least n-recursively inaccessible ordinal $\omega_1^{\mathrm{E}_n}$ is much smaller than the least Σ^1_1-reflecting ordinal σ^1_1, while the ITTM ordinal λ is far larger than σ^1_1. Therefore, for instance, one can naturally ask the following:

Problem 5.7.

(1) *If a function $f \in \omega^\omega$ dominates all* E_1*-computable functions, then are all* E_1*-computable reals arithmetical in* f*?*
(2) *For* $n > 1$*, if a function* $f \in \omega^\omega$ *dominates all* E_n*-computable functions, then are all* E_n*-computable reals hyperarithmetical in* f*?*
(3) *If a function* $f \in \omega^\omega$ *dominates all* $\Game\Sigma^0_2$*-computable functions, then are all* $\Game\Sigma^0_2$*-computable reals hyperarithmetical in* f*?*
(4) *If a function* $f \in \omega^\omega$ *dominates all ITTM-computable functions, then are all ITTM-computable reals ITRM-computable in* f*?*

Finally, we point out the limit of our approach. We write $\Gamma' \ll \Gamma$ if there is a Δ-enumeration of all total $\Delta'(x)$-functions on ω for any Δ-real x. Our strategy to show the Δ-analog of the property (8) is to find a pointclass $\Gamma' \ll \Gamma$ (which has a good closure property) such that, if f is a Δ-dominant, then every Δ-real is Δ'-reducible to f. By using the notion from Q-theory, Kechris [26] proved an analog of Solovay's characterization of encodable sets. For the basic terminology in Q-theory, we refer the reader to Kechris-Martin-Solovay [27].

Fact 5.8 (Kechris [26])**.** *Assume Projective Determinacy. For* $n > 0$*, the* Q_{2n+1}*-encodable reals are exactly* Q_{2n+1}*-reals.*

This implies that, if $\Gamma \supsetneq Q_{2n+1}$, some Δ-real is not Q_{2n+1}-reducible to some Δ-dominant. Therefore, for instance, if $\Gamma \supsetneq Q_3$ is close to Q_3 (that is, $\Gamma' \ll \Gamma$ implies $\Gamma' \subseteq Q_3$), this fact limits us to use the above mentioned strategy.

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