de Groot-like duality for represented spaces

Takayuki Kihara
Department of Mathematical Informatics
Nagoya University, Japan
kihara@i.nagoya-u.ac.jp

Arno Pauly
School of Mathematics & Computer Science
Swansea University, Swansea, United Kingdom
Arno.Pauly@cl.cam.ac.uk

We explore de Groot duality in the setting of represented spaces. The de Groot dual of a space is the space of closures of its singletons, with the representation inherited from the hyperspace of closed subsets. This yields an elegant duality, in particular between Hausdorff spaces and compact T_1-spaces. As some applications of the concept, we also study the Weihrauch complexity of the unique closed choice on the second Kleene-Kreisel space, and the point degree spectrum of the dual of Baire space.

Contents

1 Introduction

In this article, through the theory of represented spaces, higher type computability, synthetic topology, synthetic domain theory, and so on, we give a unified treatment of the studies of Π^0_1 *singletons* in classical computability theory [20, Definition XII.2.13] and *de Groot duality* in general topology [8, Section 9.1.2]. The former notion has been associated with *implicit definability* in classical logic [20, Definition XII.2.13]; hence, this unified treatment gives de Groot duality a new interpretation: the duality of "explicit" and "implicit". Conversely, the pure topological aspect of the latter also provides a renewed understanding of Π^0_1 singletons. By exploring these notions, in this article, we see an elegant duality between Hausdorff spaces and compact T_1-spaces.

We primarily have in mind the category of represented spaces and computable functions (see Example 2.10 and 4.2), but we also work with a variety of similar categories, including some full subcategories of the category of topological spaces and continuous maps. For every space $\mathbf{X}$ under consideration in this article, we assume that one can construct the hyperspace $\mathcal{A}(\mathbf{X})$ of all closed subsets of $\mathbf{X}$. For example, for any represented space $\mathbf{X}$, we obtain the represented space $\mathcal{A}(\mathbf{X})$ of closed subsets by identifying a set with the characteristic function of its complement into Sierpiński space. The exact same construction is possible for exponentiable topological spaces (Section 2.1).

The de Groot dual of a T_1-space $\mathbf{X}$ is introduced as the restriction of $\mathcal{A}(\mathbf{X})$ to the set of all singletons. If a non-T_1 space is also under consideration, it is the space of the closures of all singletons. For example, the following definition is given for represented spaces.

Definition 1.1. For a represented space $\mathbf{X}$, let $\mathbf{X}^\mathsf{d}$ denote the space $\{\overline{\{x\}} \mid x \in \mathbf{X}\} \subseteq \mathcal{A}(\mathbf{X})$. We call $\mathbf{X}^\mathsf{d}$ the *de Groot dual* of $\mathbf{X}$.

For a more precise description of the de Groot dual, see Section 3. For a T_1-space $\mathbf{X}$, if there is no risk of confusion, a point $\{x\}$ in the de Groot dual $\mathbf{X}^\mathsf{d}$ is simply written as x, so we often consider $\mathbf{X}$ and $\mathbf{X}^\mathsf{d}$ have the same underlying set.

One of our main theorems states that, with respect to T_1-spaces, the dual exhibits very elegant properties, and in particular becomes a duality between Hausdorffness and compactness. The following theorem lays out how the duality works.

Theorem 1.2. Let $\mathbf{X}$ be computably admissible and T_1, and let $\mathbf{X}$ and $\mathbf{X}^\mathsf{d}$ each contain two computable points. Then:

1. $\mathrm{id}\colon \mathbf{X} \to \mathbf{X}^\mathsf{dd}$ is computable.
2. $\mathbf{X}^\mathsf{d} \cong \mathbf{X}^\mathsf{ddd}$.
3. The following are equivalent:
   (a) $\mathbf{X}$ is computably Hausdorff.
   (b) $\mathbf{X}$ is computably Hausdorff and $\mathbf{X} \cong \mathbf{X}^\mathsf{dd}$.
   (c) $\mathbf{X}^\mathsf{dd}$ is computably Hausdorff.

   (d) $\mathbf{X}^{\mathsf{d}}$ is computably compact.
   (e) $\mathrm{id}\colon \mathbf{X} \to \mathbf{X}^{\mathsf{d}}$ is computable.
   (f) $\mathrm{id}\colon \mathbf{X}^{\mathsf{dd}} \to \mathbf{X}^{\mathsf{d}}$ is computable.

4. The following are equivalent:
   (a) $\mathbf{X}$ is computably compact.
   (b) $\mathbf{X}^{\mathsf{dd}}$ is computably compact.
   (c) $\mathbf{X}^{\mathsf{d}}$ is computably Hausdorff.
   (d) $\mathrm{id}\colon \mathbf{X}^{\mathsf{d}} \to \mathbf{X}$ is computable.
   (e) $\mathrm{id}\colon \mathbf{X}^{\mathsf{d}} \to \mathbf{X}^{\mathsf{dd}}$ is computable.

5. The following are equivalent:
   (a) $\mathbf{X}$ is computably compact and computably Hausdorff.
   (b) $\mathbf{X} \cong \mathbf{X}^{\mathsf{d}}$.

The notion of computability in the assertion here disappears later, so the reader who is not familiar with computability theory will also be able to understand the whole picture. All unexplained terms will be introduced in later sections. For now, it is sufficient to understand somewhat that de Groot dual interchanges Hausdorff and compact. We here emphasize that this “compact vs. Hausdorff” duality is not previously known in general topology. The reason will be explained in Section 5.4.

This article is an extended version of the conference paper [17]. While in [17] the authors focused only on the category of represented spaces and computable functions (as can be seen from Theorem 1.2), this article extracts the essence of the previous proofs and extends main theorems to more general categories. We also give complete proofs for some parts of the theorems that were omitted in [17].

Note that the authors’ previous proof of Theorem 1.2 uses arguments specific to represented spaces, so it is not at all clear whether Theorem 1.2 can be generalized to other settings (such as the category of exponentiable topological spaces and continuous maps). The notion of representation is of the essence, and its topological meaning must be understood. The key idea is to consider a represented space as a subquotient of Baire space. In this article, we show that Theorem 1.2 holds for any cartesian closed category of “topological subquotient spaces” (see Section 4 for the precise meaning).

In Section 2, we explain some background on general topology, and introduce (some fragments of) synthetic topology and synthetic domain theory. In Section 3, we introduce the notion of de Groot dual, and show that $\mathbf{X}^{\mathsf{ddd}} \simeq \mathbf{X}^{\mathsf{d}}$ holds in a fairly general setting (almost only cartesian closedness is used!). In Section 4, we generalize the notion of represented space, based on our analysis of the conditions under which Theorem 1.2 holds. In other words, it is a description of our setting, a cartesian closed category of “topological subquotient spaces.” In Section 5, we prove an analogue of Theorem 1.2 in this general setting. Section 6 discusses some specific examples and counterexamples. In Section 7, the focus shifts to pure computability theory, and we begin by explaining the true motivation for our work. Following this true motivation, we present some new results on Weihrauch degrees of unique closed choice, and the point degree spectrum of the space of singletons.

2 Preliminaries

In Section 2.1, we introduce some background information on exponentiability in the context of general topology. In Section 2.2, we describe our setup: It is a cartesian closed (concrete) category with finite limit, where an equalizer is described as a restriction. In Section 2.3, we extract the essence of the Sierpiński space and give proofs of the necessary lemmata.

2.1 Topology

For topological spaces X and Y, it is an important question whether there is a good topology on the set $C(X, Y)$ of continuous functions from X to Y. Of particular importance is what is called an exponential topology (see e.g. [6]).

Definition 2.1. An *exponential* topology on $C(X, Y)$ is one in which continuity of $f\colon Z \times X \to Y$ and that of its currying $\lambda f\colon Z \to C(X, Y)$ are equivalent, where $(\lambda f)(z)\colon x \mapsto f(z, x)$.

A topological space X is *exponentiable* if there exists an exponential topology on $C(X, Y)$ for any topological space Y. If $C(X, Y)$ is endowed with such a topology, we simply write Y^X.

We denote the *Sierpiński space* by $\mathbb{S}$, which consists of a closed point $\bot$ and an open point $\top$; that is, the open sets in $\mathbb{S}$ are $\emptyset$, $\{\top\}$ and $\{\top, \bot\}$. The space $\mathbb{S}$ is non-Hausdorff (indeed, non-T_1), but is by no means indispensable for treating various notions of topological space in a functional way. One can easily see the following for any topological space X:

1. A subset A of X is open iff its characteristic map $\chi_A\colon X \to \mathbb{S}$ is continuous.
2. X is Hausdorff iff $\neq\colon X \times X \to \mathbb{S}$ is continuous.
3. X is T_1 iff, for any $x \in X$, $\neq_x\colon X \to \mathbb{S}$ is continuous, where $\neq_x(y)$ is the truth value of $x \neq y$.

Here, $\top$ is interpreted as "true" and $\bot$ as "false." Roughly speaking, the first property (1) means that the Sierpiński space is an "open subobject classifier"; see also Section 2.3. Identifying a subset with its characteristic map, if X is exponentiable, the hyperspace $\mathcal{O}(X)$ of all open subsets of X can be defined by the exponential $\mathbb{S}^X$. In a similar way, the hyperspace $\mathcal{A}(X)$ of all closed sets can also be defined. Thereafter, by abuse of notation, we often write $A(x)$ for $\chi_A(x)$.

Observation 2.2. Let X and Y be exponentiable spaces. Then $f\colon X \to Y$ is continuous iff $f^{-1}\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ is continuous.

Proof. For the backward direction, if $f^{-1}\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ is well-defined, then f must be continuous. To show the forward direction, observe that $f^{-1} = \lambda U.U \circ f$. This is continuous because it is the currying of the composition of $\mathrm{id} \times f\colon (U, x) \mapsto (U, f(x))$ and $\mathsf{eval}\colon (U, f(x)) \mapsto U(f(x))$. □

There is also a nontrivial characterization of compactness:

Fact 2.3 ([5, 6]). Let X be an exponentiable space. Then, a subset A of X is compact iff $\forall_A\colon \mathcal{O}(X) \to \mathbb{S}$ is continuous, where $\forall_A(U)$ is the truth value of "$A \subseteq U$."

This fact follows from the result that the exponential topology on $\mathcal{O}(X)$ coincides with the Scott topology and the observation that A is compact iff $\{U \in \mathcal{O}(X) : A \subseteq U\}$ is Scott open [5,6].

If X is exponentable, the neighborhood filter $\eta_X(x) = \{U \in \mathcal{O}(X) : x \in U\}$ of a point $x \in X$ is just the evaluation map $\mathsf{eval}_x = \lambda U.U(x) \colon \mathcal{O}(X) \to \mathbb{S}$, which is continuous. If, moreover, $\mathcal{O}(X)$ is exponentable, $\eta_X \colon X \to \mathcal{O}\mathcal{O}(X)$ is the currying $\lambda x.\lambda U.U(x)$ of the evaluation map, which is also continuous. A topological space X is T_0 if two different points have different neighborhood filters; that is, η_X is injective. Under the exponentiability assumption, one can recover any point $x \in X$ from its neighborhood filter $\eta_X(x)$ in the following sense:

Observation 2.4. If $\mathcal{O}(X)$ is exponentable, then X is T_0 iff $\eta_X \colon X \to \mathcal{O}\mathcal{O}(X)$ has a partial continuous left inverse.

Proof. As mentioned above, X is T_0 iff there exists a function $\eta_X^- \colon \subseteq \mathcal{O}\mathcal{O}(X) \to X$ such that $\eta_X^-(\mathsf{eval}_x) = x$. It suffices to show that such an η_X^- must be continuous; that is, if $U \subseteq X$ is open, then $(\eta_X^-)^{-1}[U] \subseteq \mathcal{O}\mathcal{O}(X)$ is also open in the domain of η_X^-. One can identify $(\eta_X^-)^{-1}[U]$ with $U \circ \eta_X^- \colon \subseteq \mathcal{O}\mathcal{O}(X) \to \mathbb{S}$. Consider the evaluation map $\mathsf{eval} \colon \mathcal{O}\mathcal{O}(X) \times \mathcal{O}(X) \to \mathbb{S}$. We have $\mathsf{eval}(\mathsf{eval}_x, U) = U(x) = U \circ \eta_X^-(\mathsf{eval}_x)$. Thus, $U \circ \eta_X^-$ is the restriction of the continuous map $\lambda e.\mathsf{eval}(e, U) \colon \mathcal{O}\mathcal{O}(X) \to \mathbb{S}$ to the domain of η_X^-. □

A complete characterization of exponentiability is already known:

Fact 2.5 (see [6])**.** A topological space is exponentiable if and only if it is core compact.

Note that an exponential Y^X of an exponentiable space X is not necessarily exponentiable. In particular, the category of exponentiable spaces and continuous functions is not cartesian closed. The idea to solve this problem is to “represent” a space by an exponentiable space, which leads us to the notion of a *core-compactly generated* space. An important observation for our purposes is that this notion can be characterized as follows:

Fact 2.6 ([7])**.** A topological space is core-compactly generated iff it is a quotient of an exponentiable space.

This notion complements several important objects that are left out of a notable convenient category of topological spaces, the category of compactly generated spaces: Any compactly generated space is core-compactly generated. A Hausdorff space is compactly generated iff core-compactly generated. But for us, non-Hausdorff spaces, $\mathbb{S}, \mathcal{O}(X), \mathcal{O}\mathcal{O}(X)$, etc., are an absolute necessity.

Fact 2.7 ([7])**.** The category of core-compactly generated spaces and continuous functions is cartesian closed.

However, core-compact spaces have poor behavior on subspaces, so it remains open whether our results are applicable to this category. In technical terms, the category of core-compactly generated spaces is the coreflective hull of the category of exponentiable spaces. There is also the notion of a hereditarily coreflexive hull, which is better behaved with respect to subspaces, but this is not what we are looking for. The condition we need is that a generator behaves well with respect to subspaces; see Section 4.2.

2.2 Category

In the previous article [17], the authors have dealt with the category of represented sets and computable functions. This article also attempts to deal with some subcategories of the category of core-compactly generated spaces and continuous maps. Another category worth considering is that of equilogical spaces and equivariant maps [2].

First, all of these categories consist of structured sets and structure-preserving functions. Namely, concrete categories. The most important common feature is that they are all cartesian closed, and the forgetful functor to **Set** preserves a lot of structure.

C1. Every object is a (structured) set, and every morphism is a (structure-preserving) function.

C2. It has a finite product:
   (a) $X_1 \times \cdots \times X_n$ is the set $\{\langle x_1, \dots, x_n \rangle : x_i \in X_i \text{ for each } i \leq n\}$.
   (b) Any projection $\pi_i \colon X_1 \times \cdots \times X_n \to X_i$ is a morphism.
   (c) If $f_i \colon T \to X_i$ is a morphism for each $i < n$, then so is $t \mapsto \langle f_0(t), f_1(t) \dots, f_{n-1}(t) \rangle \colon T \to X_1 \times X_2 \times \cdots \times X_n$.

C3. It is cartesian closed:
   (a) Y^X is a set of (not necessarily all) functions from X to Y.
   (b) $f \colon Z \times X \to Y$ is a morphism if and only if its currying $\lambda f \colon Z \to Y^X$ is a morphism, where $(\lambda f)(z)(x) = f(z, x)$.
   (c) The evaluation map $\mathsf{eval} \colon Y^X \times X \to Y$ is a morphism, where $\mathsf{eval}(f, x) = f(x)$.

Example 2.8. The category of core-compactly generated spaces and continuous functions satisfies (C1)–(C3).

Example 2.9 (see [28])**.** A qcb$_0$ space is a T_0 topological space which is the quotient of a second countable space. The category of qcb$_0$ spaces and continuous functions satisfies (C1)–(C3).

Example 2.10 (see e.g. [22, 28])**.** A *represented space* is a set X equipped with a partial surjection $\delta_X \colon \subseteq \mathbb{N}^{\mathbb{N}} \to X$. If $\delta_X(p) = x$ then we say that p is a name of x. A point $x \in \mathbf{X}$ is computable if it has a computable name. A function $f \colon \mathbf{X} \to \mathbf{Y}$ is computable if there exists a computable function which, given a name of $x \in \mathbf{X}$, returns a name of $f(x) \in \mathbf{Y}$. The category **Rep** of represented spaces and computable functions satisfies (C1)–(C3).

The Baire space $\mathbb{N}^{\mathbb{N}}$ in the definition of the representation above can be replaced with any partial combinatory algebra [21]. The correspondence between the category of represented spaces and that of partial equivalence relations is also well known.

Example 2.11 ([2])**.** An equilogical space is a T_0 topological space X equipped with an equivalence relation $\equiv_X$. For equilogical spaces $\mathbf{X}$ and $\mathbf{Y}$, an equivariant map from $\mathbf{X}$ to $\mathbf{Y}$ is the equivalence class of a continuous map $f \colon X \to Y$ preserving the equivalence relation; that is, $x \equiv_X x'$ implies $f(x) \equiv_Y f(x')$. Moreover, $f \equiv_{X \to Y} g$ if $x \equiv_X x'$ implies $f(x) \equiv_Y g(x')$.

Another notion we need is that of "substructure" or "subspace"; that is, if $\mathbf{X} = (X, \tau)$ is a structured set and $A \subseteq X$ then one may often consider the substructure $\mathbf{X} \upharpoonright A = (A, \tau \upharpoonright A)$.

S1. For any object $\mathbf{X} = (X, \tau)$, and any subset $A \subseteq X$, there is an object $\mathbf{X} \upharpoonright A$ of $\mathbf{X}$ whose underlying set is A.

S2. If $f \colon \mathbf{X} \to \mathbf{Y}$ is a morphism then the restriction $f|_A \colon \mathbf{X} \upharpoonright A \to \mathbf{Y} \upharpoonright B$ is also a morphism whenever $A \subseteq X$, $B \subseteq Y$ and $f[A] \subseteq B$.

Here, since f is just a (structure-preserving) function, the notion of restriction $f|_A$ is already defined as the restriction of f to the domain A. In technical terms, (S2) merely states that the restriction always gives a regular subobject (an equalizer). The other conditions required for $\mathbf{X} \upharpoonright A$ will be described later.

Example 2.12. For a represented space $\mathbf{X}$ and a subset $A \subseteq X$, the space $\mathbf{X} \upharpoonright A$ is the set A represented by the restriction of δ_X to $\delta_X^{-1}[A]$.

Example 2.13. For a qcb$_0$ space $\mathbf{X}$ and a subset $A \subseteq X$, the space $\mathbf{X} \upharpoonright A$ is the set A endowed with the sequentialization of the subspace topology.

Definition 2.14. A morphism $f\colon \mathbf{X} \upharpoonright A \to \mathbf{Y}$ for some $A \subseteq X$ is called a *partial morphism from* $\mathbf{X}$ *to* $\mathbf{Y}$, and written $f\colon \subseteq \mathbf{X} \to \mathbf{Y}$.

As we work in a concrete category, one can take an element $x \in X$ of an object $\mathbf{X}$, but note that it cannot necessarily be written as a morphism $\mathbf{1} \to \mathbf{X}$ from the terminal object. A (unique value of) a morphism $\mathbf{1} \to \mathbf{X}$ is called a *point* or sometimes a computable point. For example, a non-computable element $x \in \mathbf{X}$ in a represented space $\mathbf{X}$ is an element, but not a point in this sense.

Remark (Technical comments)**.** For the very same reason, the underlying set of an exponential object $\mathbf{Y}^{\mathbf{X}}$ is not necessarily the set of all morphisms from $\mathbf{X}$ to $\mathbf{Y}$. For example, in **Rep**, $\mathbf{Y}^{\mathbf{X}}$ may contain non-computable continuously realizable functions. Nevertheless, most arguments work even if one thinks of each $f \in \mathbf{Y}^{\mathbf{X}}$ as a morphism from $\mathbf{X}$ to $\mathbf{Y}$. Formally, for any f, consider $\mathbf{o}_f := \mathbf{Y}^{\mathbf{X}} \upharpoonright \{f\}$. Then the restriction of the identity function, $\mathbf{o}_f \to \mathbf{Y}^{\mathbf{X}}$, is a morphism by (S2), so consider its uncurrying $\mathbf{o}_f \times \mathbf{X} \to \mathbf{Y}$ as if it were $f\colon \mathbf{X} \to \mathbf{Y}$ itself. We often do this implicitly. One may think of $\mathbf{o}_f$ as an "oracle" that computes f. The reader who does not want to be bothered by such complications should consider only those categories where the underlying set of $\mathbf{Y}^{\mathbf{X}}$ coincides with the set of all morphisms from $\mathbf{X}$ to $\mathbf{Y}$.

2.3 Dominance

2.3.1 Synthetic topology

As we have already seen, the Sierpiński space $\mathbb{S}$ is essential when dealing with topological notions. In order to extract which properties of $\mathbb{S}$ are essential, let us consider a slight abstraction of the Sierpiński space. From the topological point of view, the Sierpiński space is an "open subobject classifier." Similarly, from the viewpoint of computability theory, Sierpiński space is the space of Σ^0_1 truth values. A similar idea can be used to consider the space of Σ^0_n truth values, etc.

Definition 2.15 ([19, 25])**.** A *predominance* is an object $\mathbb{S}$ with a distinguished point $\top \in \mathbb{S}$ such that, for any object $\mathbf{X}$ and any subset $A \subseteq X$, there are at most one morphism $\chi_A\colon \mathbf{X} \to \mathbb{S}$ such that $x \in A$ iff $\chi_A(x) = \top$ for any $x \in X$. Such a χ_A is called a characteristic morphism of A. A subset $A \subseteq X$ is *effectively* $\mathbb{S}$*-open* if its characteristic morphism $\chi_A\colon X \to \mathbb{S}$ exists.

In other words, the notion of predominance is a generalization of subobject classifier, where only some subobjects may be classified. As our category is cartesian closed, one can also deal with $\mathcal{O}(X) := \mathbb{S}^X$, $\mathcal{O}\mathcal{O}(X)$, and so on. An element $A \in \mathcal{O}(X)$ is called $\mathbb{S}$*-open*. Then, $A \subseteq X$ is (effectively) $\mathbb{S}$-closed if it is the complement of an (effectively) $\mathbb{S}$-open set. We may also define the object $\mathcal{A}(X)$ for $\mathbb{S}$-closed subsets, where any argument on closed sets is always reduced to an argument on open sets via the bijection $\lambda A.X \setminus A\colon \mathcal{O}(X) \simeq \mathcal{A}(X)$.

Example 2.16. The topological Sierpiński space $\mathbb{S}$ introduced in Section 2.1 with $\top \in \mathbb{S}$ is a predominance in any full subcategory of **Top** having $\mathbb{S}$ as an object. The (effectively) $\mathbb{S}$-open sets are exactly the open sets.

In the category **Rep**, we also use $\mathbb{S}$ to denote the represented Sierpiński space, which consists of a point $\bot$ (whose name is $000\dots$) and a point $\top$ (whose names are other sequences). This is also a predominance in **Rep**.

Indeed, in our concrete setting, a predominance is always two-valued; that is, the underlying set of $\mathbb{S}$ can be assumed to be either $\{\top\}$ or $\{\top, \bot\}$.

Proposition 2.17. An object $\mathbb{S}$ is a predominance iff $\mathbb{S}$ consists only of at most two elements, one of which is a point.

Proof. The backward direction is obvious since a morphism is a (structure-preserving) function in our category. Conversely, suppose that $\mathbb{S}$ has at least two elements a, b that differ from $\top$. Then (the restriction of) each projection, $\mathbb{S} \times \mathbb{S} \upharpoonright \{\langle a, b\rangle\} \to \mathbb{S}$, is a morphism by (S2). As a and b differ from $\top$, it is a characteristic morphism of the empty set $\emptyset \subseteq \{\langle a, b\rangle\}$. This means that $\emptyset \subseteq \{\langle a, b\rangle\}$ has two characteristic morphisms $x \mapsto a$ and $x \mapsto b$. $\square$

Remark. Proposition 2.17 does not mean that there are only a few predominances. This is because there are a huge number of "binary truth values." Computability theorists and (descriptive) set theorists should instantly come up with a myriad of predominances: Σ^0_n truth values, Π^1_1 truth values, F_σ truth values, Borel truth values, projective truth values, and so on.

Some topological notions can be relativized to a given predominance.

Definition 2.18 (see also [5, 22])**.** Let $\mathbf{X}$ be an object in a cartesian closed concrete category with a predominance $\mathbb{S}$.

1. $\mathbf{X}$ is $\mathbb{S}$-*Hausdorff* if $\neq\colon \mathbf{X} \times \mathbf{X} \to \mathbb{S}$ is a morphism.
2. $\mathbf{X}$ is $\mathbb{S}$-T_1 if every singleton is $\mathbb{S}$-closed; that is, $\{x\} \in \mathcal{A}(\mathbf{X})$ for any $x \in X$.
3. $\mathbf{X}$ is $\mathbb{S}$-*compact* if $\forall_X\colon \mathcal{O}(\mathbf{X}) \to \mathbb{S}$ is a morphism.
4. $\mathbf{X}$ is $\mathbb{S}$-T_0 if $\eta_X\colon \mathbf{X} \to \mathcal{O}\mathcal{O}(\mathbf{X})$ is a monomorphism.
5. $\mathbf{X}$ is $\mathbb{S}$-*admissible* if $\eta_X\colon \mathbf{X} \to \mathcal{O}\mathcal{O}(\mathbf{X})$ has a partial left inverse which is a morphism.

The item (3) reflects Fact 2.3. It is easy to see that X is $\mathbb{S}$-compact iff $\texttt{IsEmpty}\colon \mathcal{A}(\mathbf{X}) \to \mathbb{S}$ is a morphism, where $\texttt{IsEmpty}(A)$ is the truth value of "$A = \emptyset$." Note that both of the conditions (4) and (5) correspond to T_0-ness as seen in Observation 2.4. An object satisfying (4) is also called an $\mathbb{S}$-space, and an object satisfying (5) is called an extensional $\mathbb{S}$-space, e.g. in [12]. Some notions essentially equivalent to the predominance versions of Hausdorff-ness and compactness have been studied, for example, in [24].

Let us introduce an analogous notion of relative or induced topology.

Definition 2.19 (Relative topology)**.** An $\mathbb{S}$-*subspace* of an object $\mathbf{X}$ is an object $\mathbf{A}$ with $A \subseteq X$ such that $U \subseteq A$ is $\mathbb{S}$-open iff there exists an $\mathbb{S}$-open set $\widehat{U} \subseteq X$ such that $U = \widehat{U} \cap A$.

Observation 2.20. If $\mathbf{A}$ is an $\mathbb{S}$-subspace of $\mathbf{X}$, then every morphism $f\colon \mathbf{A} \to \mathbb{S}$ can be extended to a total morphism $\widehat{f}\colon \mathbf{X} \to \mathbb{S}$.

2.3.2 Admissibility

This subsection is intended to illustrate the idea of the notion of admissibility. This is used, for example, to link represented spaces with topological spaces (Corollary 4.8), but is not used in the body of the proof, so it can be skipped.

Based on Observation 2.2, one can introduce the notion of continuity relative to a given predominance:

Definition 2.21. A function $f\colon \mathbf{X} \to \mathbf{Y}$ is $\mathbb{S}$-*continuous* if $f^{-1}\colon \mathcal{O}(\mathbf{Y}) \to \mathcal{O}(\mathbf{X})$ is a morphism.

Observe that every morphism is $\mathbb{S}$-continuous since $f^{-1} = \lambda U.U \circ f$. The converse does not hold in general. The notion of continuity and morphism can be linked via the neighborhood filter $\eta_X \colon \mathbf{X} \to \mathcal{O}\mathcal{O}(\mathbf{X})$, where recall $\eta_X(x) = \mathsf{eval}_x = \lambda U.U(x)$. To see this, let us define $\mathbf{X}^{\mathsf{top}} = \mathcal{O}\mathcal{O}(\mathbf{X}) \restriction \eta_X[X]$. In [17], this notion has been called the admissibilification of $\mathbf{X}$. Note that $\mathbf{X}$ is $\mathbb{S}$-admissible (Definition 2.18) iff $\mathbf{X} \simeq \mathbf{X}^{\mathsf{top}}$ via η_X.

Proposition 2.22. $f \colon \mathbf{X} \to \mathbf{Y}$ is $\mathbb{S}$-continuous iff $\eta_X \circ f \colon \mathbf{X} \to \mathbf{Y}^{\mathsf{top}}$ is a morphism.

Proof. First observe $\eta_X(f(x)) = \mathsf{eval}_{f(x)} = \lambda U.U(f(x)) \colon \mathcal{O}(\mathbf{X}) \to \mathbb{S}$. Therefore, $\eta_X \circ f = \lambda x.\lambda U.U(f(x)) \colon \mathbf{X} \to \mathcal{O}\mathcal{O}(\mathbf{Y})$. One can easily see that this is a morphism iff $f^{-1} = \lambda U.U \circ f \colon \mathcal{O}(\mathbf{Y}) \to \mathcal{O}(\mathbf{X})$ is a morphism. The latter means that f is $\mathbb{S}$-continuous. □

As a consequence, if $\mathbf{Y}$ is $\mathbb{S}$-admissible, under the identification of $\mathbf{Y}$ and $\mathbf{Y}^{\mathsf{top}}$, a function $f \colon \mathbf{X} \to \mathbf{Y}$ is $\mathbb{S}$-continuous iff it is a morphism. Let us take a look at what kinds of objects are $\mathbb{S}$-admissible. The following results have already been shown in [22, Section 9] in the category of represented spaces, and similar ideas are applicable to our setting.

Proposition 2.23. Any predominance $\mathbb{S}$ is $\mathbb{S}$-admissible. If $\mathbf{Y}$ is $\mathbb{S}$-admissible, then so is any exponential $\mathbf{Y}^{\mathbf{X}}$.

Proof. Indeed, $\eta_{\mathbb{S}} \colon \mathbb{S} \to \mathcal{O}\mathcal{O}(\mathbb{S})$ has a left-inverse $\lambda U.U(\mathrm{id}_{\mathbb{S}}) \colon \mathcal{O}\mathcal{O}(\mathbb{S}) \to \mathbb{S}$, where note that $\eta_{\mathbb{S}}(t)(\mathrm{id}_{\mathbb{S}}) = \mathrm{id}_{\mathbb{S}}(t) = t$. For the latter claim, we first describe the process $(\mathsf{eval}_f, x) \mapsto \mathsf{eval}_{f(x)}$. This is given by restricting the morphism $H := \lambda\alpha.\lambda x.\lambda U.\alpha(\lambda g.U(g(x))) \colon \mathcal{O}\mathcal{O}(\mathbf{Y}^{\mathbf{X}}) \to \mathbf{X} \to \mathcal{O}(\mathbf{Y}) \to \mathbb{S}$. This is because $H(\mathsf{eval}_f)(x)(U) = \mathsf{eval}_f(\lambda g.U(g(x))) = U(f(x)) = \mathsf{eval}_{f(x)}(U)$. This yields a function of type $(\mathbf{Y}^{\mathbf{X}})^{\mathsf{top}} \times \mathbf{X} \to \mathbf{Y}^{\mathsf{top}}$, which is a morphism by (S2). By $\mathbb{S}$-admissibility of $\mathbf{Y}$, we have a morphism $\mathsf{eval}_{f(x)} \mapsto f(x) \colon \mathbf{Y}^{\mathsf{top}} \to \mathbf{Y}$. The rest is done through composition and currying. □

Proposition 2.24. If $\mathbf{X}$ is $\mathbb{S}$-admissible, so is any restriction $\mathbf{X} \restriction A$.

Proof. Put $\mathbf{A} = \mathbf{X} \restriction A$, and fix an inclusion morphism $i \colon \mathbf{A} \hookrightarrow \mathbf{X}$. Let us first describe the process $\eta_A(x) \mapsto \eta_X(x)$. This is given by the morphism $H := \lambda\alpha.\lambda U.\alpha(U \circ i) \colon \mathcal{O}\mathcal{O}(\mathbf{A}) \to \mathcal{O}\mathcal{O}(\mathbf{X})$. This is because $H(\eta_A(x))(U) = \mathsf{eval}_x(U \circ i) = U \circ i(x) = U(x) = \eta_X(U)$. Let $\eta_X^- \colon \mathcal{O}\mathcal{O}(\mathbf{X}) \to \mathbf{X}$ be a partial left inverse of η_X. Note that the image of $\eta_X^- \circ H|_{\mathbf{A}^{\mathsf{top}}}$ is included in A. Thus, by the regularity requirement (S2), $\eta_X^- \circ H|_{\mathbf{A}^{\mathsf{top}}}$ can be viewed as a morphism $\mathbf{A}^{\mathsf{top}} \to \mathbf{A}$. Therefore, $\mathbf{A}$ is $\mathbb{S}$-admissible. □

Corollary 2.25. $\mathbf{X}^{\mathsf{top}}$ is $\mathbb{S}$-admissible, i.e., $(\mathbf{X}^{\mathsf{top}})^{\mathsf{top}} \simeq \mathbf{X}^{\mathsf{top}}$.

Proof. By Proposition 2.23, $\mathcal{O}\mathcal{O}(\mathbf{X})$ is always $\mathbb{S}$-admissible. As $\mathbf{X}^{\mathsf{top}}$ is a restriction of $\mathcal{O}\mathcal{O}(\mathbf{X})$, by Proposition 2.24, $\mathbf{X}^{\mathsf{top}}$ is $\mathbb{S}$-admissible. □

One can also prove that this process preserves various topological properties. For example:

Observation 2.26. If $\mathbf{X}$ is computably Hausdorff, so is $\mathbf{X}^{\mathsf{top}}$.

Proof. Currying $\neq \colon \mathbf{X} \times \mathbf{X} \to \mathbb{S}$ yields a morphism $\lambda x.X \setminus \{x\} \colon \mathbf{X} \to \mathcal{O}(\mathbf{X})$. Thus, $\langle \eta_X(x), y\rangle \mapsto (x \neq y) \colon \mathbf{X}^{\mathsf{top}} \times \mathbf{X} \to \mathbb{S}$ is a morphism since this is $\langle \eta_X(x), y\rangle \mapsto \eta_X(x)(X \setminus \{y\})$. This shows that $\eta_X(x) \mapsto X \setminus \{x\} \colon \mathbf{X}^{\mathsf{top}} \to \mathcal{O}(\mathbf{X})$ is a morphism. Thus, by the same argument as above, we see that $\neq \colon \mathbf{X}^{\mathsf{top}} \times \mathbf{X}^{\mathsf{top}} \to \mathbb{S}$ is a morphism. □

For further background on admissibility, see [22, 28].

2.3.3 Lift

Although it would be difficult to apply the results of this article to a predominance other than the Sierpiński space, we nevertheless deal with a predominance because it is deeply connected to the notion of lift, which plays an essential role in our proof. The following are basic "lifting" properties of a predominance (see [12] and also [19, Chapter 4]):

1. A predominance $\mathbb{S}$ always yields an $\mathbb{S}$-partial map classifier $\mathbf{Y}_\perp$ whose underlying set is $Y_\perp = Y \cup \{\perp_Y\}$; that is, any partial morphism $f\colon \subseteq \mathbf{X} \to \mathbf{Y}$ with an effective $\mathbb{S}$-open domain can be extended to a total morphism $f^\perp\colon \mathbf{X} \to \mathbf{Y}_\perp$ such that if $x \in \mathrm{dom}(f)$ then $f^\perp(x) = f(x)$; otherwise $f^\perp(x) = \perp_Y$.
2. It also yields a lifting functor; in particular, for any morphism $f\colon \mathbf{X} \to \mathbf{Y}$ we have a morphism $f_\perp\colon \mathbf{X}_\perp \to \mathbf{Y}_\perp$ such that if $x \in X$ then $f_\perp(x) = f(x)$; otherwise $f_\perp(x) = \perp_Y$.

Often $\perp_Y$ is abbreviated to $\perp$ when there is no confusion. For readers who do not want to be bothered with abstractions, it is sufficient to consider only standard examples in Example 2.16. It is quite obvious that the above lifting properties hold for these standard examples.

Definition 2.27 ([19,25])**.** A predominance $\mathbb{S}$ is a *dominance* if a retraction $\mu\colon \mathbb{S}_\perp \to \mathbb{S}$ exists; that is, μ is a morphism such that $\mu(x) = x$ for $x \in \mathbb{S}$ and $\mu(\perp_\mathbb{S}) = \perp$.

Lemma 2.28. If $\mathbb{S}$ is a dominance, then a retraction $\mathcal{O}(\mathbf{X})_\perp \to \mathcal{O}(\mathbf{X})$ exists.

Proof. As A is effectively $\mathbb{S}$-open in $\mathbf{A}_\perp$, one can see that $A \times B$ is also effectively $\mathbb{S}$-open in $\mathbf{A}_\perp \times \mathbf{B}$, since $\chi_A \circ \pi_0\colon \mathbf{A}_\perp \times \mathbf{B} \to \mathbb{S}$ is a characteristic morphism of $A \times B \subseteq A_\perp \times B$. Thus, $\mathrm{id}\colon \mathbf{A}\times\mathbf{B} \to \mathbf{A}\times\mathbf{B}$ is extended to $\mathrm{id}^\perp\colon \mathbf{A}_\perp\times\mathbf{B} \to (\mathbf{A}\times\mathbf{B})_\perp$. Now, consider $\mathbf{A} = \mathbb{S}^\mathbf{X}$ and $\mathbf{B} = \mathbf{X}$. Compose $\mathrm{id}^\perp$ and the lift $\mathsf{eval}_\perp\colon (\mathbb{S}^\mathbf{X}\times\mathbf{X})_\perp \to \mathbb{S}_\perp$ of $\mathsf{eval}\colon \mathbb{S}^\mathbf{X}\times\mathbf{X} \to \mathbb{S}$. As $\mathbb{S}$ is a dominance, we have an arrow $\mu\colon \mathbb{S}_\perp \to \mathbb{S}$, so we obtain the morphism $e := \mu \circ \mathsf{eval}_\perp \circ \mathrm{id}^\perp\colon (\mathbb{S}^\mathbf{X})_\perp \times \mathbf{X} \to \mathbb{S}$. By currying, we get $\tilde{e}\colon (\mathbb{S}^X)_\perp \to \mathbb{S}^X$. Then we have $\tilde{e}(\perp) = \lambda x.\perp$, and $f \neq \perp$ implies $\tilde{e}(f) = f$. □

Given a morphism $\delta\colon D \to \mathcal{O}(\mathbf{X})$, by Lemma 2.28, we get a lift $\delta'\colon D_\perp \to \mathcal{O}(\mathbf{X})_\perp \to \mathcal{O}(\mathbf{X})$; that is, if $p \neq \perp$ then $\delta(p) = p$; and if $p = \perp$ then $\delta(p) = \emptyset$, where note that $\lambda x.\perp$ is identified with the empty set $\emptyset$.

In the standard topological setting, $\perp$ is the bottom element in the specialization order on $\mathbf{X}_\perp$. We will show that the same fact holds in our general setting.

Definition 2.29. For an object $\mathbf{X}$, given $x, y \in X$, we write $x \leq_\mathbf{X} y$ if $x \in U$ implies $y \in U$ for any $\mathbb{S}$-open set $U \subseteq X$. This preorder $\leq$ is called the $\mathbb{S}$-*specialization preorder* (or the intrinsic preorder [12]) on X.

Observation 2.30. For any morphism $f\colon \mathbf{X} \to \mathbf{Y}$, $x \leq_\mathbf{X} y$ implies $f(x) \leq_\mathbf{Y} f(y)$.

Proof. $f(x) \in U$ implies $x \in f^{-1}[U]$, which implies $y \in f^{-1}[U]$ by the assumption $x \leq_\mathbf{X} y$; hence $f(y) \in U$. □

For the behavior of Sierpiński spaces in **Top** or **Rep**, one can easily see the following properties:

- D1. $\mathbb{S}$ is a lattice with the top element $\top$ and the bottom element $\perp$ under the $\mathbb{S}$-specialization order.
- D2. $\vee, \wedge\colon \mathbb{S}^2 \to \mathbb{S}$ are morphisms.

The standard assumptions on synthetic domain theory guarantees these properties (see e.g. [12]). Under the requirement (D2), the union and intersection operations can be treated as morphisms:

Observation 2.31. For any object $\mathbf{X}$, the operations $\cup, \cap \colon \mathcal{O}(\mathbf{X})^2 \to \mathcal{O}(\mathbf{X})$ are morphisms.

Proof. Note that $\cup = \lambda\langle U, V\rangle.\lambda x.U(x) \vee V(x)$ and $\cap = \lambda\langle U, V\rangle.\lambda x.U(x) \wedge V(x)$. $\square$

Under the requirements (D1) and (D2), the specialization order on the open sets coincides with the inclusion relation:

Lemma 2.32. For any $\mathbb{S}$-open sets $U, V \in \mathcal{O}(\mathbf{X})$, $U \leq_{\mathcal{O}(\mathbf{X})} V$ iff $U \subseteq V$.

Proof. Assume $U \leq_{\mathcal{O}(\mathbf{X})} V$. We show that $U(x) = \top$ implies $V(x) = \top$. As $\lambda x.\lambda U.U(x) \colon \mathbf{X} \to \mathcal{O}\mathcal{O}(\mathbf{X})$ is a morphism, we have $\alpha := \lambda U.U(x) \in \mathcal{O}\mathcal{O}(\mathbf{X})$. By our assumption $U \leq_{\mathcal{O}(\mathbf{X})} V$, $\alpha(U) = \top$ implies $\alpha(V) = \top$. This means that $x \in U$ implies $x \in V$.

Assume $U \subseteq V$. Consider $H := \lambda\langle s, t\rangle.\lambda u.s \vee (t \wedge u) \colon \mathbb{S}^2 \to \mathbb{S}^{\mathbb{S}}$. Then $H(U(x), V(x)) = \lambda u.U(x) \vee (V(x) \wedge u)$, so $H(U(x), V(x))(\bot) = U(x)$ and $H(U(x), V(x))(\top) = U(x) \vee V(x) = V(x)$ since $U \subseteq V$ implies $U(x) \leq_{\mathbb{S}} V(x)$. Then we get $G := \lambda t.\lambda x.H(U(x), V(x))(t) \colon \mathbb{S} \to \mathcal{O}(\mathbf{X})$. Note that $G(\bot) = U$ and $G(\top) = V$. For any $\mathbb{S}$-open $\alpha \colon \mathcal{O}(\mathbf{X}) \to \mathbb{S}$, by Observation 2.30, $\alpha \circ G \colon \mathbb{S} \to \mathbb{S}$ must be order preserving, which means that $\alpha(U) \leq_{\mathbb{S}} \alpha(V)$. Thus, $\alpha(U) = \top$ implies $\alpha(V) = \top$. $\square$

One can see that $\bot \in X_\bot$ is actually the bottom in the specialization order; that is, for any $\mathbb{S}$-open set $U \subseteq X_\bot$, if $\bot \in U$ then $U = X_\bot$.

Lemma 2.33. For any object $\mathbf{X}$, the bottom $\bot$ is the least element in the $\mathbb{S}$-specialization order on $\mathbf{X}_\bot$. More generally, for any object $\mathbf{Y}$, we have $(\bot, y) \leq_{\mathbf{X}_\bot \times \mathbf{Y}} (x, y)$ for any $x \in X_\bot$ and $y \in Y$.

Proof. Let $U \colon \mathbf{X}_\bot \times \mathbf{Y} \to \mathbb{S}$ be a morphism. The restriction $\bar{\pi}$ of the projection $\pi \colon \mathbb{S} \times \mathbf{X} \to \mathbf{X}$ to $\{\top\} \times X$ is a morphism by (S2). As $\{\top\} \times X \subseteq \mathbb{S} \times X$ is effectively $\mathbb{S}$-open, $\bar{\pi}$ is extended to a morphism $\bar{\pi}^\bot \colon \mathbb{S} \times \mathbf{X} \to \mathbf{X}_\bot$. For $V := \lambda\langle x, y\rangle.\lambda t.U(\bar{\pi}^\bot(t, x), y) \colon \mathbf{X} \times \mathbf{Y} \to \mathcal{O}(\mathbb{S})$, we see $V(x, y)(\top) = U(\pi(\top, x), y) = U(x, y)$ and $V(x, y)(\bot) = U(\bot, y)$. As $V(x, y) \in \mathcal{O}(\mathbb{S})$, $V(x, y)$ is $\mathbb{S}$-open in $\mathbb{S}$. As $\bot$ is the bottom element w.r.t. $\leq_{\mathbb{S}}$, $\bot \in V(x, y)$ implies $t \in V(x, y)$ for any $t \in \mathbb{S}$. If $U(\bot, y) = \top$, then $\bot \in V(x, y)$, so $\top \in V(x, y)$, which means that $U(x, y) = V(x, y)(\top) = \top$. $\square$

Note that this means that, for any morphism $f \colon \mathbf{X}_\bot \times \mathbf{Y} \to \mathbb{S}$, if $f(\bot, y) = \top$ then we must have $f(x, y) = \top$ for any $x \in X$.

3 De Groot dual

3.1 De Groot duality for T_1 spaces

Our main object of study in this article is the notion of de Groot dual. We begin with an explanation of de Groot dual in general topology. If (X, τ) is a T_1 space, the (topological) *de Groot dual* is the set X endowed with the cocompact topology τ^{d}; that is, $U \subseteq X$ is τ^{d}-open iff $X \setminus U$ is compact. For further background on topological de Groot dual, see [11] and [8, Section 9.1.2].

This notion can also be derived from the compact-open topology of function spaces. Let $C_{\mathsf{co}}(X,Y)$ be the space of continuous maps from X to Y endowed with the compact-open topology. We write $\mathcal{A}_{\mathsf{co}}(X)$ for the hyperspace of closed sets obtained by identifying $f \in C_{\mathsf{co}}(X,\mathbb{S})$ with a closed set $\{x \in X : f(x) = \bot\}$.

Observation 3.1. The de Groot dual of a T_1 topological space X is homeomorphic to the subspace of $\mathcal{A}_{\mathsf{co}}(X)$ consisting of all singletons in X.

Proof. As $\chi_U[K] \subseteq \{\top\}$ iff $K \subseteq U$, the topology on $\mathcal{O}_{\mathsf{co}}(X) := C_{\mathsf{co}}(X,\mathbb{S})$ is generated from $\{U \in \mathcal{O}_{\mathsf{co}}(X) : K \subseteq U\}$, where K is compact. As $K \subseteq X \setminus \{x\}$ iff $x \notin K$, the topology on co-singletons in $\mathcal{O}_{\mathsf{co}}(\mathbf{X})$ is generated from $\{X \setminus \{x\} : x \notin K\}$, where K is compact. The map $x \mapsto X \setminus \{x\}$ gives a homeomorphism between the de Groot dual of $\mathbf{X}$ and the subspace of $\mathcal{O}_{\mathsf{co}}(\mathbf{X})$ consisting of co-singletons. The latter is clearly homeomorphic to the subspace of $\mathcal{A}_{\mathsf{co}}(\mathbf{X})$ consisting of singletons. $\square$

Recall that $\mathcal{A}(\mathbf{X})$ is the hyperspace of all closed sets in $\mathbf{X}$ equipped with the exponential topology; that is, it is the space obtained via the indentification $\mathcal{A}(\mathbf{X}) \simeq \mathbb{S}^{\mathbf{X}}$.

Corollary 3.2. If a topological space $\mathbf{X}$ is exponentiable, locally compact and T_1, then its de Groot dual is homeomophic to the subspace of $\mathcal{A}(\mathbf{X})$ consisting of all singletons in $\mathbf{X}$.

Proof. If $\mathbf{X}$ is exponentiable, locally compact, then the exponential topology on $C(\mathbf{X},\mathbb{S})$ coincides with the compact-open topology; see [5, Section 8.5]. $\square$

Based on this observation, our main idea is to define the de Groot dual of $\mathbf{X}$ as the restriction of the exponential $\mathcal{A}(\mathbf{X})$ to the set of all singletons. Of course, $\mathcal{A}(\mathbf{X})$ and $\mathcal{A}_{\mathsf{co}}(\mathbf{X})$ do not necessarily coincide, so it may be possible that this definition may not match the topological de Groot dual. But the difference, if any, between $\mathcal{A}(\mathbf{X})$ and $\mathcal{A}_{\mathsf{co}}(\mathbf{X})$ is marginal. In later sections, we will discuss the relationship between the two definitions in a setting that also includes non-T_1 spaces.

Now, let us consider a cartesian closed concrete category $\mathcal{C}$ equipped with a predominance $\mathbb{S}$ and a restriction $\upharpoonright$. Then we define the de Groot dual of a $\mathcal{C}$-object as follows:

Definition 3.3. The *de Groot dual* of an $\mathbb{S}$-T_1 object $\mathbf{X}$ is defined as $\mathcal{A}(\mathbf{X}) \upharpoonright \{\{x\} : x \in X\}$.

This definition is inspired by the notion of Π^0_1 singleton in computability theory. See Section 7.1 for details.

Example 3.4. Computable points in $(\mathbb{N}^{\mathbb{N}})^{\mathsf{d}}$ are exactly Π^0_1 singletons in $\mathbb{N}^{\mathbb{N}}$.

By Propositions 2.23 and 2.24, the de Groot dual $\mathbf{X}^{\mathsf{d}}$ is always $\mathbb{S}$-admissible. The conditions on a category that make Theorem 1.2 on T_1 de Groot duals valid are a bit complicated, so before giving details, we prove a theorem that holds under looser conditions.

3.2 De Groot duality for non-T_1 spaces

In the standard topological setting, one can define the de Groot dual of a non-T_1-space [8, Section 9.1.2]: An open set in the de Groot dual X^{d} is the complement of a saturated compact set in X, where a set $A \subseteq X$ is *saturated* if it is an intersection of open sets.

One can also introduce the de Groot dual as a subspace of the hyperspace $\mathcal{A}(X)$ of closed sets, following the form of Definition 3.3. However, in a non-T_1-space, a singleton is no longer a closed set, so it is necessary to take its closure. This process allows us to extend Observation 3.5 to non-T_1 spaces.

Observation 3.5. The de Groot dual of a T_0 topological space X is homeomorphic to the subspace of $\mathcal{A}_{\mathsf{co}}(X)$ consisting of all closures of singletons in X.

Proof. As X is T_0, $x \mapsto \overline{\{x\}}$ is bijective. A closed set $A \subseteq X$, if it does not intersect with $S \subseteq X$, does not intersect with the saturation of S either. Note also that the saturation of a compact set K is also compact since an open cover of K always covers its saturation. Thus, as in Observation 3.5, one can easily see that $x \mapsto \overline{\{x\}}$ is indeed a homeomorphism. □

It is useful to introduce various topological notions in terms of specialization order. Note that $x \leq_X y$ iff $\overline{\{x\}} \subseteq \overline{\{y\}}$. Thus, the closure $\overline{\{x\}}$ of a singleton corresponds to the downward closure $\downarrow x := \{y \in X : y \leq_X x\}$, while the saturation of a set $A \subseteq X$ is the upward closure $\uparrow A$.

Based on this observation, one may attempt to introduce the de Groot dual as the restriction of $\mathcal{A}(X)$ to the closure of points. A predominance yields a specialization order, so it seems possible to treat the closure of a point as $\downarrow x$. However, in too general a setting, $\downarrow x$ may not be a closed set. Therefore, when dealing with de Groot duals for non-T_1 spaces, we will have to add the following ad hoc requirement.

D3. For any $x \in \mathbf{X}$, $\downarrow x$ is $\mathbb{S}$-closed, i.e., $\downarrow x \in \mathcal{A}(\mathbf{X})$.

Natural examples such as the category of represented spaces and computable functions (Example 2.10) and the category of core-compactly generated spaces and continuous maps (Example 2.8) – indeed, any full subcategory of **Top** – obviously satisfies (D3) with respect to the standard Sierpiński space $\mathbb{S}$ (Example 2.16), so it is best to keep those categories in mind.

Definition 3.6. The *de Groot dual* of an object $\mathbf{X}$ is defined as $\mathbf{X}^{\mathsf{d}} = \mathcal{A}(\mathbf{X}) \restriction \{\downarrow x : x \in X\}$.

If $\mathbf{X}$ is $\mathbb{S}$-T_0, then the map $x \mapsto \downarrow x$ is bijective, so one can think of an underlying set of $\mathbf{X}^{\mathsf{d}}$ as X by identifying $\downarrow x$ with x. Hereafter, we assume that an object $\mathbf{X}$ is always $\mathbb{S}$-T_0.

Even in natural examples, it is not obvious at first glance how much difference there is between $\mathcal{A}(\mathbf{X})$ and $\mathcal{A}_{\mathsf{co}}(\mathbf{X})$, and thus it is not clear how our definition connects to the topological de Groot dual. In the case of represented spaces, we know a few things:

Example 3.7. By Schröder [27, Proposition 4.2.5 (4)], the standard exponential representation is admissible with respect to the compact open topology on $C(X, Y)$ whenever X is sequential. In particular, the standard representation of the hyperspace of closed sets is admissible with respect to $\mathcal{A}_{\mathsf{co}}(X)$. By restricting its domain, we get an admissible representation of the de Groot dual $\mathbf{X}^{\mathsf{d}} \simeq \{\overline{\{x\}} : x \in X\} \subseteq \mathcal{A}_{\mathsf{co}}(X)$. By Schröder [27, Proposition 2.4.18 (4)], this implies that the de Groot dual built as a represented space is homeomorphic to the sequentialization of the topological de Groot dual.

It remains open as to whether the sequentialization is necessary.

Problem 3.8. For any qcb$_0$ space $\mathbf{X}$, is $\mathbf{X}^{\mathsf{d}}$ built in the category of qcb$_0$ spaces homeomorphic to the topological de Groot dual of $\mathbf{X}$?

To analyze this in a general setting, let us introduce an object $\mathcal{K}(\mathbf{X})$ consisting of saturated compact subsets. A subset $A \subseteq X$ is $\mathbb{S}$-compact iff $\forall_A : \mathcal{O}(\mathbf{X}) \to \mathbb{S}$ is a morphism, where $\forall_A(U)$ is the truth value of "$A \subseteq U$." Therefore, it seems natural to identify a compact subset of X with an element of $\mathcal{O}\mathcal{O}(\mathbf{X})$. However, note that for $\mathbb{S}$-compact subsets $A, B \subseteq X$, A and B have the same $\mathbb{S}$-saturation iff $\forall_A = \forall_B$. Here, the $\mathbb{S}$-saturation of $A \subseteq X$ is defined as the upward closure $\uparrow A$ w.r.t. the $\mathbb{S}$-specialization order on X. Therefore, it is reasonable to regard $\forall_A$ as the name of the saturation of A. By restricting $\mathcal{O}\mathcal{O}(\mathbf{X})$ to the elements of the form $\forall_A$, we can introduce objects consisting of saturated compacts.

Definition 3.9. The object $\mathcal{K}(\mathbf{X})$ is defined as the restriction of $\mathcal{OO}(\mathbf{X})$ to the set of all $\forall_A$ such that $A \subseteq X$ is $\mathbb{S}$-saturated and $\mathbb{S}$-compact.

The following already shows that there is a genuine connection: The de Groot dual as we defined it here is in fact the "least" representation such that saturated compacts of the original space translate to complements of open subsets.

Proposition 3.10. $\mathrm{id}\colon \mathcal{K}(\mathbf{X}) \to \mathcal{A}(\mathbf{X}^{\mathsf{d}})$ is well-defined and a morphism. Moreover, if $\mathbf{X}'$ is an object with the same underlying set X such that $\mathrm{id}\colon \mathcal{K}(\mathbf{X}) \to \mathcal{A}(\mathbf{X}')$ is well-defined and a morphism, then $\mathrm{id}\colon \mathbf{X}' \to \mathbf{X}^{\mathsf{d}}$ is a morphism.

Proof. For the first claim: Given $A \in \mathcal{K}(\mathbf{X})$ and $\downarrow x \in \mathbf{X}^{\mathsf{d}}$ we can confirm $x \notin A$ by detecting that $A \subseteq X \setminus \downarrow x$, i.e., $\forall_A(X \setminus \downarrow x) = \top$. It is obvious that the condition suffices. To see that it is necessary, assume $y \in A \cap \downarrow x$. Note that $y \in \downarrow x$ iff $x \in \uparrow y$. Since A is saturated, then $y \in A$ implies $x \in \uparrow y \subseteq \uparrow A = A$.

For the second claim, we start with $x \in \mathbf{X}'$. Given any $y \in \mathbf{X}$, since $y \in U$ iff $\uparrow y \subseteq U$ for any open set U, the evaluation map $\in\colon \mathbf{X} \times \mathcal{O}(\mathbf{X}) \to \mathbb{S}$ yields a name of the saturation of its singleton $\uparrow y \in \mathcal{K}(\mathbf{X})$. By assumption, this lets us get $\uparrow y \in \mathcal{A}(\mathbf{X}')$. By composing the evaluation map $\notin\colon \mathbf{X}' \times \mathcal{A}(\mathbf{X}') \to \mathbb{S}$, we now get a morphism $y \mapsto (x \notin \uparrow y) = (y \notin \downarrow x)\colon \mathbf{X} \to \mathbb{S}$, which realizes $\downarrow x \in \mathbf{X}^{\mathsf{d}}$. $\square$

Even in a full subcategory $\mathcal{C}$ of **Top**, the reason why Proposition 3.10 does not already tell us that our definition of $\mathbf{X}^{\mathsf{d}}$ yields the de Groot dual is two-fold: First, we do not merely require that complements of saturated compacts are open, but we require that they are open in a uniform way. Second, it only shows that $\mathbf{X}^{\mathsf{d}}$ is minimal amongst the $\mathcal{C}$-objects for this property – we would need to separately ensure that the topological de Groot dual of a $\mathcal{C}$-object is indeed a $\mathcal{C}$-object once more.

At any rate, it will not cause any major problems if we consider our de Groot dual to be the "best approximation" of the topological de Groot dual within a given category $\mathcal{C}$.

3.3 Iterated dual

For the topological de Groot dual, Kovár has shown that taking iterated duals will yield at most four distinct topological spaces [18]. In our setting, based on the cartesian closedness requirement for our category, we observe that the iterated dual will only yield at most three distinct represented spaces, with an argument that is similar to but simpler than the one by Kovár.

Theorem 3.11. $\mathbf{X}^{\mathsf{ddd}} \simeq \mathbf{X}^{\mathsf{d}}$ for any object $\mathbf{X}$.

To prove this, we need some preparations. Topologically, the complement of $\downarrow x = \overline{\{x\}}$ is the exterior $\mathsf{ext}(\{x\})$, so it is reasonable to consider this as an intuitionistic negation. That is, we may write $\neg x = X \setminus \downarrow x$ and use it instead of $\downarrow x$.

Definition 3.12. The de Groot dual of an object $\mathbf{X}$ is defined as $\mathbf{X}^{\mathsf{d}} = \mathcal{O}(\mathbf{X}) \restriction \{\neg x : x \in \mathbf{X}\}$.

The two de Groot duals (Definitions 3.6 and 3.12) are isomorphic, so we can use either one, but we may gain a new perspective by viewing the de Groot dual as a kind of negation. For example, we can first derive the following "contrapositive rule" $A \to B \equiv \neg B \to \neg A$ for the specialization order:

Proposition 3.13. For any $x, y \in \mathbf{X}$, $x \leq_{\mathbf{X}} y$ iff $\neg y \leq_{\mathbf{X}^{\mathsf{d}}} \neg x$.

Proof. By definition, $x \leq_{\mathbf{X}} y$ iff $\neg y \subseteq \neg x$. All that remains is to show that $\neg y \leq_{\mathbf{X}^{\mathsf{d}}} \neg x$ if and only if $\neg y \subseteq \neg x$. For this, note that our proof of Lemma 2.32 can be applied to any restriction of $\mathcal{O}(\mathbf{X})$ by (S2). Let us describe this carefully to illustrate the use of (S2).

For the forward direction, we restrict $\lambda U.\lambda z.U(z) \colon \mathcal{O}(\mathbf{X}) \to \mathbb{S}^{\mathbf{X}}$ to get a morphism of type $\mathbf{X}^{\mathsf{d}} \to \mathbb{S}^{\mathbf{X}}$ by (S2). By uncurrying and currying, we get a morphism $\lambda z.\lambda U.U(x) \colon \mathbf{X} \to \mathcal{O}(\mathbf{X}^{\mathsf{d}})$, so we have $\alpha := \lambda U.U(z) \in \mathcal{O}(\mathbf{X}^{\mathsf{d}})$. If $\neg y \leq_{\mathbf{X}^{\mathsf{d}}} \neg x$, then $\alpha(\neg y) = \top$ implies $\alpha(\neg x) = \top$. This means that $z \in \neg y$ implies $z \in \neg x$.

For the backward direction, assume $U \subseteq V$. Consider $H \colon \mathbb{S}^2 \to \mathbb{S}^{\mathbb{S}}$ in Lemma 2.32. Then $H(U(z), V(z))(\bot) = U(z)$ and $H(U(z), V(z))(\top) = V(z)$. If $U, V \in \mathbf{X}^{\mathsf{d}}$ then we have $\lambda z.H(U(z), V(z))(t) \in \{U, V\} \subseteq \mathbf{X}^{\mathsf{d}}$ for any $t \in \mathbb{S}$. Thus, the range of the morphism $G := \lambda t.\lambda z.H(U(z), V(z))(t) \colon \mathbb{S} \to \mathcal{O}(\mathbf{X})$ is included in $\mathbf{X}^{\mathsf{d}}$, so the regularity requirement (S2) enables us to think of the codomain of G as $\mathbf{X}^{\mathsf{d}}$. For any $\mathbb{S}$-open $\alpha \colon \mathbf{X}^{\mathsf{d}} \to \mathbb{S}$, by Observation 2.30, $\alpha \circ G \colon \mathbb{S} \to \mathbb{S}$ must be order preserving, which means $\alpha(U) \leq_{\mathbb{S}} \alpha(V)$. Thus, $\alpha(U) = \top$ implies $\alpha(V) = \top$. This means $U \leq_{\mathbf{X}^{\mathsf{d}}} V$. Finally, consider $U = \neg y$ and $V = \neg x$. □

Observe that an object is $\mathbb{S}$-T_1 iff its specialization order has no nontrivial chain. The forward direction is the same as in the standard topological argument. For the backward direction, the latter condition implies ${\downarrow} x = \{x\}$, which is $\mathbb{S}$-closed by (D3). Combining Proposition 3.13 with this observation, we conclude that $\mathbf{X}$ is T_1 iff $\mathbf{X}^{\mathsf{d}}$ is T_1. Thus, the sequences of iterated duals of $\mathbb{S}$-T_1 and non-$\mathbb{S}$-T_1 objects never intersect.

Using Proposition 3.13, we can also derive the double negation rule $A \to \neg\neg A$:

Corollary 3.14. $\neg\neg \colon \mathbf{X} \to \mathbf{X}^{\mathsf{dd}}$ is a morphism.

Proof. By Proposition 3.13, $\neg\neg x(\neg y) = \top$ iff $\neg y \not\leq_{\mathbf{X}^{\mathsf{d}}} \neg x$ iff $x \not\leq_{\mathbf{X}} y$ iff $\neg y(x) = \top$. This means $\neg\neg x = \lambda U.U(x)$. Therefore, $\neg\neg = \lambda x.\lambda U.U(x)$. □

Observation 3.15. For any morphism $f \colon \mathbf{X} \to \mathbf{Y}$, if $f \colon (X, \leq_{\mathbf{X}}) \to (Y, \leq_{\mathbf{Y}})$ is an order isomorphism, then $f^{-1} \colon \mathbf{Y}^{\mathsf{d}} \to \mathbf{X}^{\mathsf{d}}$ is well-defined and a morphism.

Proof. For any $y \in Y$, by surjectivity, we have some $x \in X$ such that $f(x) = y$. To see that f^{-1} is well-defined, it suffices to check that $f(x) = y$ implies $f^{-1}[\neg y] = \neg x$. Since f is an order isomorphism, $x' \leq_{\mathbf{X}} x$ if and only if $f(x') \leq_{\mathbf{Y}} f(x) = y$. This implies $f^{-1}[{\downarrow} y] = {\downarrow} x$, which also means $f^{-1}[\neg y] = \neg x$. Thus, $f^{-1} \colon \mathbf{Y}^{\mathsf{d}} \to \mathbf{X}^{\mathsf{d}}$ is well-defined. This can be viewed as a restriction of $f^{-1} = \lambda U.U \circ f \colon \mathcal{O}(\mathbf{Y}) \to \mathcal{O}(\mathbf{X})$, so it is a morphism by the regularity requirement (S2). □

For de Groot duality, $X^{\mathsf{dd}} \cong X$ does not necessarily hold. This may be viewed as the failure of double negation elimination $\neg\neg A \to A$ as in the intuitionistic logic. In contrast, we can derive the triple negation rule $\neg\neg\neg A \to \neg A$. This is also the same as in the intuitionistic logic.

Corollary 3.16. $(\neg\neg)^{-1} \colon \mathbf{X}^{\mathsf{ddd}} \to \mathbf{X}^{\mathsf{d}}$ is a morphism.

Proof. By Corollary 3.14, $\neg\neg \colon \mathbf{X} \to \mathbf{X}^{\mathsf{dd}}$ is a morphism. Moreover, it is an order isomorphism since $\mathbf{X}$ and $\mathbf{X}^{\mathsf{dd}}$ have the same specialization order by Proposition 3.13. Hence, we just need to apply Observation 3.15 to $\neg\neg \colon \mathbf{X} \to \mathbf{X}^{\mathsf{dd}}$. □

Now, Theorem 3.11 follows from Corollaries 3.14 and 3.16. Consequently, the iteration sequence of the de Groot dual of any object terminates in at most three steps.

Example 3.17. In the category of core-compactly generated spaces and continuous maps, we always have $\mathbf{X}^{\mathsf{ddd}} \simeq \mathbf{X}^{\mathsf{d}}$.

This contrasts with the existence of a topological space whose iterated dual sequence does not terminate in at three steps [18]. We will see later an example showing that $\mathbf{X}, \mathbf{X}^{\mathsf{d}}$ and $\mathbf{X}^{\mathsf{dd}}$ can indeed be three non-isomorphic represented spaces (Section 6).

4 Generalized represented space

4.1 Quotient space

Theorem 3.11 (i.e., $\mathbf{X}^{\mathsf{ddd}} \simeq \mathbf{X}^{\mathsf{d}}$) has been proved under the rather loose assumption of a finitely complete cartesian closed concrete category with a predominance (more precisely, (C1)–(C3), (S1)–(S2) and (D1)–(D3)). Unfortunately, however, the requirements necessary to show Theorem 1.2 are not this loose.

Now our declaration is as follows: To deal with topological notions, our category $\mathcal{C}$ is a cartesian closed concrete category (C1)–(C3) equipped with a predominance $\mathbb{S}$. Moreover, the category $\mathcal{C}$ consists of *(sub-)quotient spaces* (w.r.t. the $\mathbb{S}$-topology); that is, every $\mathcal{C}$-object is the $\mathbb{S}$-topological (sub-)quotient of an object in some well-behaved full subcategory $\mathcal{B}$ of $\mathcal{C}$ (which is not necessarily cartesian closed). In detail, first, the "$\mathbb{S}$-topological quotient" formally means the following:

Definition 4.1. Let $\mathcal{B}$ be a full subcategory of $\mathcal{C}$. A *quotient representation* of a $\mathcal{C}$-object $\mathbf{X}$ is a surjection $\delta\colon D \to X$ for some $\mathcal{B}$-object $\mathbf{D}$ such that $A \subseteq X$ is $\mathbb{S}$-open in $\mathbf{X}$ if and only if $\delta^{-1}[A] \subseteq D$ is $\mathbb{S}$-open in $\mathbf{D}$.

In other words, $f\colon \mathbf{X} \to \mathbb{S}$ is a morphism iff there exists a morphism $F\colon \mathbf{D} \to \mathbb{S}$ such that $F = f \circ \delta$. If $\delta(p) = x$ then we often say that p is a *δ-name* (or simply a name) of x.

Example 4.2 (Represented space)**.** Let $\mathcal{B}$ be a concrete category. A *$\mathcal{B}$-represented space* is a set X equipped with a surjection $\delta_X\colon D_X \to X$ from some $\mathcal{B}$-object D_X. A function $f\colon (X, \delta_X) \to (Y, \delta_Y)$ is *$\mathcal{B}$-realizable* if there exists a $\mathcal{B}$-morphism $F\colon D_X \to D_Y$ which, given a δ_X-name of $x \in X$, returns a δ_Y-name of $f(x) \in Y$. By $\mathbf{Rep}_{\mathcal{B}}$ we denote the category of $\mathcal{B}$-represented spaces and $\mathcal{B}$-realizable functions. One can represent each $\mathcal{B}$-object by the identity function, so one can think of $\mathcal{B}$ as a full subcategory of $\mathbf{Rep}_{\mathcal{B}}$.

Example 4.3. If $\mathcal{B}$ is the category of subsets of $\mathbb{N}^{\mathbb{N}}$ and computable functions, then $\mathbf{Rep}_{\mathcal{B}}$ is the category of represented spaces and computable functions.

If $\mathcal{B}$ is the category of T_0 spaces and continuous maps, then $\mathbf{Rep}_{\mathcal{B}}$ is equivalent to the category of equilogical spaces and equivariant maps; see also [1].

Observation 4.4. Assume that $\mathbf{Rep}_{\mathcal{B}}$ is equipped with a predominance $\mathbb{S}$. Then every $\mathcal{B}$-representation $\delta_X\colon D_X \to X$ is indeed a quotient representation $(D_X, \mathrm{id}) \to (X, \delta_X)$.

Proof. Assume that $f\colon \mathbf{X} \to \mathbb{S}$ is a $\mathbf{Rep}_{\mathcal{B}}$-morphism, which is equivalent to the existence of a $\mathcal{B}$-morphism $F\colon D_X \to D_{\mathbb{S}}$ such that $f \circ \delta_X = \delta_{\mathbb{S}} \circ F$. Then one can see that $\delta_{\mathbb{S}} \circ F\colon (D_X, \mathrm{id}) \to \mathbb{S}$ is a $\mathbf{Rep}_{\mathcal{B}}$-morphism (realized by F). Conversely, assume that there exists a $\mathbf{Rep}_{\mathcal{B}}$-morphism $g\colon (D_X, \mathrm{id}) \to \mathbb{S}$ such that $f \circ \delta_X = g$. Then there exists a $\mathcal{B}$-morphism $G\colon D_X \to D_{\mathbb{S}}$ such that $g = \delta_{\mathbb{S}} \circ G$. Thus, we have $f \circ \delta_X = \delta_{\mathbb{S}} \circ G$, which means that f is a $\mathbf{Rep}_{\mathcal{B}}$-morphism. $\square$

Hereafter, for a $\mathcal{B}$-object D, the represented space (D, id) is sometimes simply written as D. Next, we present more topological examples of quotient representations.

Example 4.5 (Quotient space)**.** Let $\mathcal{B}$ be a concrete category with a predominance $\mathbb{S}$. For a $\mathcal{B}$-represented spaces (X, δ_X), we say that $A \subseteq X$ is δ_X-open if $\delta_X^{-1}[A]$ is $\mathbb{S}$-open. This may be viewed as a generalization of the Ershov topology in the theory of numberings.

For $\mathcal{B}$-represented spaces $\mathbf{X}$ and $\mathbf{Y}$, a function $f\colon X \to Y$ is $\mathbb{S}$-*continuous* if, for any δ_Y-open set $U \subseteq Y$, the preimage $f^{-1}[U] \subseteq X$ is δ_X-open. By $\mathbf{Top}_{\mathcal{B}}$ we denote the category of $\mathcal{B}$-represented spaces and $\mathbb{S}$-continuous functions.

Example 4.6. If $\mathcal{B}$ is the category of second-countable T_0-spaces and continuous maps, where the equipped predominance is the topological Sierpiński space, then $\mathbf{Top}_{\mathcal{B}}$ is exactly the category of qcb_0 spaces and continuous maps.

If $\mathcal{B}$ is the category of exponentiable spaces and continuous maps, where the equipped predominance is the topological Sierpiński space, then $\mathbf{Top}_{\mathcal{B}}$ is exactly the category of core-compactly generated spaces and continuous maps (Fact 2.6).

Observation 4.7. Every $\mathcal{B}$-representation $\delta_X\colon D_X \to X$ is indeed a quotient representation $(D_X, \mathrm{id}) \to (X, \delta_X)$ in $\mathbf{Top}_{\mathcal{B}}$.

Proof. Recall that every predominance is at most two-valued by Proposition 2.17. Combining with (D1), one can see that $\{\top\}$ is the unique nontrivial open set in $\mathbb{S}$. Thus, $f\colon \mathbf{X} \to \mathbb{S}$ is $\mathbb{S}$-continuous iff $f^{-1}\{\top\} \subseteq X$ is δ_X-open iff $(f \circ \delta_X)^{-1}[\top] \subseteq D_X$ is $\mathbb{S}$-open iff $f \circ \delta_X\colon D_X \to \mathbb{S}$ is a $\mathcal{B}$-morphism iff $f \circ \delta_X\colon (D_X, \mathrm{id}) \to \mathbb{S}$ is $\mathbb{S}$-continuous. $\square$

Note that every realizable function between $\mathcal{B}$-represented spaces is $\mathbb{S}$-continuous. Observation 4.7 also means that $f\colon \mathbf{X} \to \mathbb{S}$ is realizable iff it is $\mathbb{S}$-continuous. As in Observation 2.2, if $\mathcal{C} \in \{\mathbf{Rep}_{\mathcal{B}}, \mathbf{Top}_{\mathcal{B}}\}$ is cartesian closed, one can see that $f\colon \mathbf{X} \to \mathbf{Y}$ is $\mathbb{S}$-continuous iff $f^{-1}\colon \mathcal{O}(\mathbf{Y}) \to \mathcal{O}(\mathbf{X})$ is a $\mathcal{C}$-morphism.

A more precise relationship between the categories $\mathbf{Rep}_{\mathcal{B}}$ and $\mathbf{Top}_{\mathcal{B}}$ can be given via the neighborhood filter $\eta_X\colon \mathbf{X} \to \mathcal{O}\mathcal{O}(\mathbf{X})$. By Proposition 2.22, if $\mathbf{Rep}_{\mathcal{B}}$ is cartesian closed and $\mathbf{Y}$ is $\mathbb{S}$-admissible, then $f\colon \mathbf{X} \to \mathbf{Y}$ is $\mathbb{S}$-continuous iff it is realizable. Even if $\mathbf{Y}$ is not $\mathbb{S}$-admissible, under the assumption that $\mathbf{Rep}_{\mathcal{B}}$ and $\mathbf{Top}_{\mathcal{B}}$ are both cartesian closed, $\eta_X \circ f\colon \mathbf{X} \to \mathbf{Y}^{\mathsf{top}}$ is $\mathbb{S}$-continuous iff it is realizable. Consequently, an $\mathbb{S}$-continuous map can always be treated as a realizable map in any category consisting of T_0 quotient spaces within the scope of this article.

Corollary 4.8. Assume that every object in $\mathbf{Top}_{\mathcal{B}}$ is $\mathbb{S}$-T_0. Then, $\mathbf{Top}_{\mathcal{B}}$ is equivalent to the full subcategory of $\mathbf{Rep}_{\mathcal{B}}$ consisting of all $\mathbb{S}$-admissible objects.

Proof. Restrict $\mathbf{Rep}_{\mathcal{B}}$ to objects of the form $\mathbf{X}^{\mathsf{top}}$. By Proposition 2.25, $\mathbf{X}^{\mathsf{top}}$ is $\mathbb{S}$-admissible, so by Proposition 2.22, every $\mathbb{S}$-continuous map $\mathbf{X} \to \mathbf{Y}$ gives rise to a realizable map $\mathbf{X}^{\mathsf{top}} \to \mathbf{Y}^{\mathsf{top}}$. $\square$

4.2 Requirements for $\mathcal{B}$

So far we have analyzed the idea of representing a category $\mathcal{C}$ by its subcategory $\mathcal{B}$. Of course, there is no merit in representing $\mathcal{C}$ using $\mathcal{B}$ unless $\mathcal{B}$ behaves better than $\mathcal{C}$ in some sense. As of now, we have no specific requirements for $\mathcal{B}$, so let us add a few objects to $\mathcal{B}$ to make it behave better. We define a full subcategory $\mathcal{B}^*$ of $\mathcal{C}$ by the following inductive clause:

1. Every $\mathcal{B}$-object is a $\mathcal{B}^*$-object.

2. If D and E are $\mathcal{B}^*$-objects then $D_\perp$ and $D \times E$ are $\mathcal{B}^*$-object.

Here, as in Section 2.3.3, where there is a predominance $\mathbb{S}$, there is always $D_\perp$. To describe the requirement for $\mathcal{B}^*$, recall that our category has the notion of restriction $\mathbf{X} \upharpoonright A$ of an object $\mathbf{X}$ to a subset $A \subseteq X$. Our key assumption is that this subcategory $\mathcal{B}^*$ behaves very well with respect to the restriction, which is essential to our proof.

S3. For any subset A of a $\mathcal{B}^*$-object D, the restriction $D \upharpoonright A$ is an $\mathbb{S}$-subspace of D.

Here, recall Definition 2.19 for the notion of $\mathbb{S}$-subspace.

Example 4.9. The category of represented spaces and computable functions, the category of qcb$_0$ spaces and continuous maps, and the category of equilogical spaces and equivariant maps are all cartesian closed categories satisfying (S3). Here, $\mathcal{B}$ and $\mathbb{S}$ are those chosen in Examples 4.3 and 4.5, and for a $\mathcal{B}^*$-object $\mathbf{D}$, the restriction $\mathbf{D} \upharpoonright A$ is defined as the (topological) subspace of $\mathbf{D}$ whose underlying set is A.

Requirement (S3) may seem a bit demanding, but one can always obtain a new category $\mathcal{B}^+$ by adding $\mathbb{S}$-subspaces of $\mathcal{B}^*$ to $\mathcal{B}$ if necessary, and by redefining the restriction $\upharpoonright$ on $\mathcal{B}^+$, one can always assume (S3). For example, $\mathbf{Rep}_{\mathcal{B}^+}$ or $\mathbf{Top}_{\mathcal{B}^+}$ automatically fulfills (S3).

Declaration: Our target category $\mathcal{C}$ is $\mathbf{Rep}_{\mathcal{B}}$ or $\mathbf{Top}_{\mathcal{B}}$ equipped with a dominance $\mathbb{S}$ and a restriction $\upharpoonright$. Here, we require (C1)–(C3) for $\mathcal{C}$, (C1)–(C2) for $\mathcal{B}$, (D1)–(D2) for $\mathbb{S}$, and (S1)–(S3) for $\upharpoonright$.

The reader may think that there are too many requirements to understand. But most of them are obvious and few are essential. There are only two nontrivial requirements, the subspace requirement (S3) for $(\mathcal{B}, \upharpoonright)$ and the cartesian closedness requirement (C3) for $\mathcal{C}$, so in a practical situation, it would be safe to assume that the other requirements do not exist (as they are trivially met by natural examples). Furthermore, the former (S3) is made obvious by considering $\mathbf{Rep}_{\mathcal{B}^+}$ or $\mathbf{Top}_{\mathcal{B}^+}$. In a nutshell, $\mathcal{C}$ is a cartesian closed concrete category consisting of $\mathcal{B}$-subquotient spaces w.r.t. the $\mathbb{S}$-topology.

4.3 Basic properties

In practice, we usually define a restriction of an object as a restriction of its representation, as in Example 2.12. That is, our quotient representations always satisfy the following property:

Definition 4.10. A *hereditarily quotient representation* $\delta\colon \mathbf{D} \to \mathbf{X}$ is a quotient representation of a $\mathcal{C}$-object $\mathbf{X}$ such that, if $A \subseteq X$ then $\delta|^A$ is a quotient representation of $\mathbf{X} \upharpoonright A$, where $\delta|^A\colon \mathbf{D} \upharpoonright \delta^{-1}[A] \to \mathbf{X} \upharpoonright A$ is the restriction of δ to $\delta^{-1}[A]$.

As far as standard examples are concerned, any representation also behaves well with respect to the product.

Observation 4.11. If $\mathcal{B}$ has binary product, then so is $\mathbf{Rep}_{\mathcal{B}}$. Indeed, $(X \times Y, \delta_X \times \delta_Y) \simeq (X, \delta_X) \times (Y, \delta_Y)$ in $\mathbf{Rep}_{\mathcal{B}}$.

In the case of $\mathbf{Top}_{\mathcal{B}}$, it does not do as well, but it behaves well enough for our purposes. To explain this, let us write $\mathbf{X} \times_{\mathcal{B}} \mathbf{Y}$ for $(X \times Y, \delta_X \times \delta_Y)$. The above observation says that $(\mathbf{X} \times \mathbf{Y}) \simeq (\mathbf{X} \times_{\mathcal{B}} \mathbf{Y})$ in $\mathbf{Rep}_{\mathcal{B}}$.

Proposition 4.12. If $\mathbf{X}$ and $\mathbf{Y}$ are $\mathbb{S}$-T_0, then $(\mathbf{X} \times \mathbf{Y}) \simeq (\mathbf{X}^{\mathsf{top}} \times_{\mathcal{B}} \mathbf{Y}^{\mathsf{top}})$ in $\mathbf{Top}_{\mathcal{B}}$.

Proof. Let $\mathbf{X}$ and $\mathbf{Y}$ be $\mathcal{B}$-represented spaces, and let $\mathbf{X}\times\mathbf{Y}$ be their binary product in $\mathbf{Top}_{\mathcal{B}}$. By Proposition 2.22 in $\mathbf{Rep}_{\mathcal{B}}$, the projection $\mathbf{X}^{\mathsf{top}} \times_{\mathcal{B}} \mathbf{Y}^{\mathsf{top}} \to \mathbf{X}^{\mathsf{top}}$ yields an $\mathbb{S}$-continuous function $\mathbf{X}^{\mathsf{top}} \times_{\mathcal{B}} \mathbf{Y}^{\mathsf{top}} \to \mathbf{X}$, where such a function is well-defined since $\mathbf{X}$ is $\mathbb{S}$-T_0. By universality of the binary product, we get an $\mathbb{S}$-continuous function $\mathbf{X}^{\mathsf{top}} \times_{\mathcal{B}} \mathbf{Y}^{\mathsf{top}} \to \mathbf{X}\times\mathbf{Y}$, which can be viewed as the identity map. Conversely, consider the projection $\pi_0 \colon \mathbf{X}\times\mathbf{Y} \to \mathbf{X}$, which is $\mathbb{S}$-continuous; hence $\pi_0 \colon \mathbf{X}\times\mathbf{Y} \to \mathbf{X}^{\mathsf{top}}$ is realizable by Proposition 2.22 in $\mathbf{Rep}_{\mathcal{B}}$. By universality of the binary product, we get a realizable function $\mathbf{X}\times\mathbf{Y} \to \mathbf{X}^{\mathsf{top}} \times_{\mathcal{B}} \mathbf{Y}^{\mathsf{top}}$, which can also be viewed as the identity map. □

Note that the second half of the proof does not use the T_0 property. Thus, $\eta_X \times \eta_Y \colon \mathbf{X}\times\mathbf{Y} \to \mathbf{X}^{\mathsf{top}} \times_{\mathcal{B}} \mathbf{Y}^{\mathsf{top}}$ is always a $\mathbf{Top}_{\mathcal{B}}$-morphism. This completes the list of properties that we will use in later proofs.

4.4 Abstraction

Next, as an option, we suggest the possibility of handling other categories than $\mathbf{Rep}_{\mathcal{B}^+}$ and $\mathbf{Top}_{\mathcal{B}^+}$. Those who are not interested in abstraction may skip this part, but our real purpose is not so much to generalize as to extract in advance which properties of $\mathbf{Rep}_{\mathcal{B}^+}$ and $\mathbf{Top}_{\mathcal{B}^+}$ are essential for the proofs. Based on the above observations, we make the following requirements for our categories $\mathcal{B}$ and $\mathcal{C}$.

- Q1. Every $\mathcal{C}$-object $\mathbf{X}$ is equipped with a hereditarily quotient representation δ_X.
- Q2. $\mathcal{B}$ satisfies (C1)–(C2), and for any $\mathcal{C}$-object $\mathbf{X}$ and $\mathbf{Y}$, $\delta_X \times \delta_Y$ is a hereditarily quotient representation of some $\mathcal{C}$-object $\mathbf{X} \times_{\mathcal{B}} \mathbf{Y}$ whose underlying set is $X \times Y$.
- Q3. $\eta_X \times \eta_Y \colon \mathbf{X}\times\mathbf{Y} \to \mathbf{X}^{\mathsf{top}} \times_{\mathcal{B}} \mathbf{Y}^{\mathsf{top}}$ is a morphism.

Of course, (Q2) and (Q3) correspond to Proposition 4.12. Note that both $\mathbf{Rep}_{\mathcal{B}}$ and $\mathbf{Top}_{\mathcal{B}}$ fulfill the requirements (Q1)–(Q3). The item (Q3) ensures that, if $\mathbf{X}$ and $\mathbf{Y}$ are $\mathbb{S}$-admissible, any morphism $\mathbf{X} \times_{\mathcal{B}} \mathbf{Y} \to \mathbf{Z}$ gives rise to a morphism $\mathbf{X}\times\mathbf{Y} \to \mathbf{Z}$.

The above are all the requirements that $(\mathcal{C}, \mathcal{B}, \mathbb{S}, \upharpoonright)$ must satisfy. Importantly, our target is divisible into two layers, the “extensional” level $\mathcal{C}$ and the “intentional” level $\mathcal{B}$.

1. $\mathbb{S}$ is a dominance that satisfies a few mild requirements (included in the standard requirements in synthetic domain theory).
2. $\mathcal{B}$ is a concrete category that fulfills a few mild requirements (that most interesting examples satisfy) and the non-mild essential requirements (S3) and (Q3).
3. $\mathcal{C}$ is a concrete category consisting of $\mathcal{B}$-quotient spaces, which fulfills some mild requirements and the non-mild essential requirement (C3) of being cartesian closed.

In most interesting cases, $\mathcal{C}$ is either $\mathbf{Rep}_{\mathcal{B}}$ or $\mathbf{Top}_{\mathcal{B}}$. In this case, the requirement (Q3) is automatically fulfilled. When considering $\mathbf{Rep}_{\mathcal{B}^+}$ or $\mathbf{Top}_{\mathcal{B}^+}$ the requirement (S3) is also automatically fulfilled, so the only nontrivial requirement is cartesian closedness (C3) for $\mathcal{C}$. This is our setup.

5 Duality for T_1 represented spaces

Under the requirements described in Section 4.2 (or more generally, in Section 4.4), we finally prove an analogue of Theorem 1.2.

Theorem 5.1. Let $\mathbf{X}$ be $\mathbb{S}$-admissible and $\mathbb{S}$-T_1, and let $\mathbf{X}$ and $\mathbf{X}^{\mathsf{d}}$ each contain two points. Then:

1. $\mathrm{id}\colon \mathbf{X} \to \mathbf{X}^{\mathsf{dd}}$ is a morphism.
2. $\mathbf{X}^{\mathsf{d}} \cong \mathbf{X}^{\mathsf{ddd}}$.
3. The following are equivalent:
   (a) $\mathbf{X}$ is $\mathbb{S}$-Hausdorff.
   (b) $\mathbf{X}$ is $\mathbb{S}$-Hausdorff and $\mathbf{X} \cong \mathbf{X}^{\mathsf{dd}}$.
   (c) $\mathbf{X}^{\mathsf{dd}}$ is $\mathbb{S}$-Hausdorff.
   (d) $\mathbf{X}^{\mathsf{d}}$ is $\mathbb{S}$-compact.
   (e) $\mathrm{id}\colon \mathbf{X} \to \mathbf{X}^{\mathsf{d}}$ is a morphism.
   (f) $\mathrm{id}\colon \mathbf{X}^{\mathsf{dd}} \to \mathbf{X}^{\mathsf{d}}$ is a morphism.
4. The following are equivalent:
   (a) $\mathbf{X}$ is $\mathbb{S}$-compact.
   (b) $\mathbf{X}^{\mathsf{dd}}$ is $\mathbb{S}$-compact.
   (c) $\mathbf{X}^{\mathsf{d}}$ is $\mathbb{S}$-Hausdorff.
   (d) $\mathrm{id}\colon \mathbf{X}^{\mathsf{d}} \to \mathbf{X}$ is a morphism.
   (e) $\mathrm{id}\colon \mathbf{X}^{\mathsf{d}} \to \mathbf{X}^{\mathsf{dd}}$ is a morphism.
5. The following are equivalent:
   (a) $\mathbf{X}$ is $\mathbb{S}$-compact and $\mathbb{S}$-Hausdorff.
   (b) $\mathbf{X} \cong \mathbf{X}^{\mathsf{d}}$.

The proofs of its claims are spread throughout Subsection 5.1 below. The requirement for the space and or its dual to contain one or two (computable) points are used only for a few of the implications. We do not know whether these requirements are needed, but having some computable points seems like a sufficiently innocent restriction.

While the situation of Hausdorffness and compactness are mostly symmetrical in our main theorem, there is a notable absence: For computably compact $\mathbf{X}$ we cannot conclude that $\mathbf{X} \cong \mathbf{X}^{\mathsf{dd}}$. An example for this is exhibited in Section 6.

Henceforth, to clarify the comparison with the discussion in the category of represented spaces and computable functions [17], even in our general setting, we refer to a morphism as a "computable function" and to $\mathbb{S}$-T for each topological notion T as "computably T." For example, an $\mathbb{S}$-Hausdorff object is called a computably Hausdorff space.

5.1 Proofs of the basics.

We proceed to prove the various components of Theorem 5.1. Most of the proofs are presented by crystal-clear arguments based on higher type computability. These seem to fit well with synthetic topology [5], with the exception of the proofs of Propositions 5.6 and 5.13 and Lemma 5.11. The essence of proving these three exceptions is the requirement (S3) introduced in Section 4.

Throughout this section, the space $\mathbf{X}$ is assumed to be $\mathbb{S}$-T_1 without this being necessarily stated explicitly. If there is no risk of confusion, a point $\{x\}$ in the de Groot dual $\mathbf{X}^{\mathsf{d}}$ is simply written as x.

Observation 5.2. The map $\neq : \mathbf{X} \times \mathbf{X}^{\mathsf{d}} \to \mathbb{S}$ is computable.

Proof. As $\mathcal{A}(\mathbf{X}) \simeq \mathbb{S}^{\mathbf{X}}$, the non-membership relation $\notin : \mathbf{X} \times \mathcal{A}(\mathbf{X}) \to \mathbb{S}$ is exactly the evaluation map, so it is computable. For $x, y \in X$, note that $x \notin \{y\}$ iff $x \neq y$. Thus, the non-membership relation $\notin$ restricted to $\mathbf{X} \times \mathbf{X}^{\mathsf{d}}$ is exactly the non-equality relation $\neq$ via the identification of $\{y\}$ with $y \in \mathbf{X}^{\mathsf{d}}$. Therefore, $\neq : \mathbf{X} \times \mathbf{X}^{\mathsf{d}} \to \mathbb{S}$ is computable. □

Corollary 5.3.

1. $\mathbf{X}^{\mathsf{d}}$ is T_1 (and thus $\mathbf{X}^{\mathsf{dd}}$ is well-defined).
2. $\mathrm{id} : \mathbf{X} \to \mathbf{X}^{\mathsf{dd}}$ is computable.

Proof. For (1), currying the function $\neq$ in Observation 5.2 yields the function $x \mapsto X \setminus \{x\} : \mathbf{X} \to \mathcal{O}(\mathbf{X}^{\mathsf{d}})$. In particular, $X \setminus \{x\}$ is open in $\mathbf{X}^{\mathsf{d}}$, which means that $\mathbf{X}^{\mathsf{d}}$ is T_1. For (2), as currying preserves computability, the above function is computable, and an $\mathcal{O}(\mathbf{X}^{\mathsf{d}})$-name of $X \setminus \{x\}$ is exactly an $\mathbf{X}^{\mathsf{dd}}$-name of x. □

The following is essentially just a rephrasing of the definition of being computably Hausdorff:

Observation 5.4. $\mathrm{id} : \mathbf{X} \to \mathbf{X}^{\mathsf{d}}$ is computable iff $\mathbf{X}$ is computably Hausdorff.

Proof. As in Corollary 5.3, one can see that $\mathrm{id} : \mathbf{X} \to \mathbf{X}^{\mathsf{d}}$ is computable iff $\neq : X \times X \to \mathbb{S}$ is computable, which means that $\mathbf{X}$ is computably Hausdorff. □

5.2 The connection to unique closed choice.

More or less by the definition of admissibility, we find that $\mathrm{id} : \mathbf{X}^{\mathsf{d}} \to \mathbf{X}$ is computable for a computably compact computably admissible space:

Observation 5.5.

1. If $\mathbf{X}$ is computably compact and computably admissible, then $\mathrm{id} : \mathbf{X}^{\mathsf{d}} \to \mathbf{X}$ is computable.
2. If $\mathbf{X}$ is computably compact, then $\mathbf{X}^{\mathsf{d}}$ is computably Hausdorff.

Proof. (1) A name of a given $x \in \mathbf{X}^{\mathsf{d}}$ is also a name of $X \setminus \{x\} \in \mathcal{O}(\mathbf{X})$. By computable compactness and Observation 2.31, $F := \lambda V.\lambda U.\forall_A(V \cup U) \in \mathcal{O}(\mathbf{X}) \to \mathcal{O}(\mathbf{X}) \to \mathbb{S}$ is computable. Given $U \in \mathcal{O}(\mathbf{X})$, note that $(X \setminus \{x\}) \cup U = X$ iff $x \in U$. Thus, $F(X \setminus \{x\}) = \eta_X(x)$. By computable admissibility, this yields an $\mathbf{X}$-name of x.

(2) Names of $x, y \in \mathbf{X}^{\mathsf{d}}$ are also names of $X \setminus \{x\}, X \setminus \{y\} \in \mathcal{O}(\mathbf{X})$. By computable compactness and Observation 2.31, $G := \lambda\langle V, U\rangle.\forall_A(V \cup U) \in \mathcal{O}(\mathbf{X}) \times \mathcal{O}(\mathbf{X}) \to \mathbb{S}$ is computable. Note that $(X \setminus \{x\}) \cup (X \setminus \{y\}) = X$ iff $x \neq y$; that is, $G(X \setminus \{x\}) \cup (X \setminus \{y\})$ is the truth value of "$x \neq y$." This shows that $\neq : \mathbf{X}^{\mathsf{d}} \times \mathbf{X}^{\mathsf{d}} \to \mathbb{S}$ is computable. Consequently, $\mathbf{X}^{\mathsf{d}}$ is computably Hausdorff. □

Before proving the next Proposition, let us point out that the map $\mathrm{id} : \mathbf{X}^{\mathsf{d}} \to \mathbf{X}$ is just another perspective on the principle of *unique closed choice*, studied in [3]. To be more precise, as $\mathbf{X}^{\mathsf{d}}$ is a restriction of $\mathcal{A}(\mathbf{X})$, one may think of $\mathrm{id} : \mathbf{X}^{\mathsf{d}} \to \mathbf{X}$ as a partial morphism $\mathrm{UC}_{\mathbf{X}} : \subseteq \mathcal{A}(\mathbf{X}) \to \mathbf{X}$, whose domain is the set of all closed singletons, and $\mathrm{UC}_{\mathbf{X}}(\{a\}) = a$ for any $a \in X$.

Proposition 5.6. Let $\mathbf{X}$ contain a computable point. If $\mathrm{id} : \mathbf{X}^{\mathsf{d}} \to \mathbf{X}$ is computable, then $\mathbf{X}$ is computably compact.

Proof. To prove that $\mathbf{X}$ is computably compact, it suffices to show that $\texttt{IsEmpty}\colon \mathcal{A}(\mathbf{X}) \to \mathbb{S}$ is computable. Fix a quotient representation $\delta\colon D \to \mathcal{A}(\mathbf{X})$. As in the comment immediately after Lemma 2.28, consider the lift $\delta'\colon D_\perp \to \mathcal{A}(\mathbf{X})$. By Observation 2.31, the function $\delta'' = \cap \circ (\delta \times \delta')\colon D \times D_\perp \to \mathcal{A}(\mathbf{X})$ is computable. Here, if $q = \perp$ then $\delta''(p,q) = \delta(p)$; otherwise $\delta''(p,q) = \delta(p) \cap \delta(q)$. As mentioned above, one may think of $\mathrm{id}\colon \mathbf{X}^{\mathsf{d}} \to \mathbf{X}$ as a partial morphism $\mathrm{UC}_{\mathbf{X}}\colon\subseteq \mathcal{A}(\mathbf{X}) \to \mathbf{X}$. Then the partial function $f := \lambda\langle A,B\rangle.A(\mathrm{UC}_{\mathbf{X}}(A \cup B))\colon\subseteq A(\mathbf{X})^2 \to \mathbb{S}$ is computable. By Observation 2.20 together with (S3), $f \circ (\delta'' \times \delta'')\colon\subseteq (D \times D_\perp)^2 \to \mathbb{S}$ has a total extension $F\colon (D \times D_\perp)^2 \to \mathbb{S}$ since $(D \times D_\perp)^2$ is a $\mathcal{B}^*$-object.

We claim that, for any $\mathtt{p},\mathtt{q} \in D$, if $\mathtt{p}$ is a δ-name of a nonempty closed set P, then we must have $F(\langle \mathtt{p},\perp\rangle,\langle \mathtt{q},\perp\rangle) = \perp$. To see this, let x be an element of P. Note that $\{x\}$ is a closed set as X is T_1. Let $\dot{\mathtt{x}}$ and $\mathtt{o}$ be δ-names of the closed sets $\{x\}$ and $\emptyset$, respectively; so we now have $\mathtt{p},\dot{\mathtt{x}},\mathtt{o} \in D$ such that $\delta(\mathtt{p}) = P$, $\delta(\dot{\mathtt{x}}) = \{x\}$ and $\delta(\mathtt{o}) = \emptyset$. Then we have $\delta''(\langle \mathtt{p},\dot{\mathtt{x}}\rangle) = P \cap \{x\} = \{x\}$ and $\delta''(\langle \mathtt{q},\mathtt{o}\rangle) = \emptyset$. Observe that $\mathrm{UC}_{\mathbf{X}}(\{x\} \cup \emptyset) = x$ and $x \in \{x\}$, so we have $f(\{x\},\emptyset) = \perp$. Hence, we get $F(\langle \mathtt{p},\dot{\mathtt{x}}\rangle,\langle \mathtt{q},\mathtt{o}\rangle) = \perp$. By Lemma 2.33, we obtain $F(\langle \mathtt{p},\perp\rangle,\langle \mathtt{q},\perp\rangle) = \perp$. This verifies the claim.

Next, let us consider the case where $\mathtt{p}$ is a δ-name of the empty set. By our assumption, we are given a name $\mathtt{z}$ of a computable point $z \in \mathbf{X}^{\mathsf{d}}$. As $\mathbf{X}^{\mathsf{d}}$ is a restriction of $\mathcal{A}(\mathbf{X})$, by (Q1), one may ensure that the name $\mathtt{z}$ is also a name of the closed set $\{z\} \in \mathcal{A}(\mathbf{X})$. Observe that $\mathrm{UC}_{\mathbf{X}}(\emptyset \cup \{z\}) = z$ and $z \notin \emptyset$, so we have $f(\emptyset,\{z\}) = \top$. Hence, we get $F(\langle \mathtt{p},\perp\rangle,\langle \mathtt{z},\perp\rangle) = \top$. Combined with the above claim, for any $\mathtt{p} \in D$, we now conclude that $\mathtt{p}$ is a δ-name of the empty set iff $F(\langle \mathtt{p},\perp\rangle,\langle \mathtt{z},\perp\rangle) = \top$. This means that for the computable function $E := \lambda\mathtt{p}.F(\langle \mathtt{p},\perp\rangle,\langle \mathtt{z},\perp\rangle)\colon D \to \mathbb{S}$ we have $E(\mathtt{p}) = \texttt{IsEmpty}(\delta(\mathtt{p}))$. As δ is a quotient representation, this shows that $\texttt{IsEmpty}\colon \mathcal{A}(X) \to \mathbb{S}$ is computable. □

Corollary 5.7. Let $\mathbf{X}$ contain a computable point. Then the following are equivalent:

1. $\mathrm{id}\colon \mathbf{X} \to \mathbf{X}^{\mathsf{d}}$ and $\mathrm{id}\colon \mathbf{X}^{\mathsf{d}} \to \mathbf{X}$ are both computable.
2. $\mathbf{X}$ is computably admissible, computably compact and computably Hausdorff.

Proof. The direction from (2) to (1) follows from Observations 5.4 and 5.5 (1). For the direction from (1) to (2), Observation 5.4 and Proposition 5.6 show that $\mathbf{X}$ is computably compact and computably Hausdorff. It remains to show that $\mathbf{X}$ is computably admissible. Let $\mathsf{eval}_x\colon \mathcal{O}(\mathbf{X}) \to \mathbb{S}$ defined by $\mathsf{eval}_x(U) = (x \in U)$ be given. As $y \mapsto X \setminus \{y\}\colon \mathbf{X}^{\mathsf{d}} \to \mathcal{O}(\mathbf{X})$ is a computable embedding, the function $y \mapsto \mathsf{eval}_x(X \setminus \{y\})\colon \mathbf{X}^{d} \to \mathbb{S}$ is also computable. Note that $\mathsf{eval}_x(X \setminus \{y\}) = \top$ iff $x \neq y$, so this yields a name of $X \setminus \{x\} \in \mathcal{O}(\mathbf{X}^{\mathsf{d}})$, which is exactly an $\mathbf{X}^{\mathsf{dd}}$-name of x. This shows that $\mathsf{eval}_x \mapsto x\colon\subseteq \mathcal{O}\mathcal{O}(\mathbf{X}) \to \mathbf{X}^{\mathsf{dd}}$ is always computable. By using our assumption (1) twice, we see that $\mathrm{id}\colon \mathbf{X}^{\mathsf{dd}} \to \mathbf{X}$ is computable, so we conclude that $\mathbf{X}$ is computably admissible. □

Remark (Computability theory)**.** Observation 5.5 (1) generalizes the classical observation that a Π^0_1 singleton in Cantor space is computable [20, Exercise XII.2.15 (c)] (whose uniform version is given in [3, Corollary 6.4] in the context of a unique closed choice). Proposition 5.6 gives a topological interpretation of the classical observation that a Π^0_1 singleton in Baire space (which is non-compact) is not necessarily computable [20, Exercise XII.2.15 (d)].

5.3 More on Hausdorffness.

Proposition 5.8. If $\mathbf{X}$ is computably Hausdorff, then $\mathrm{id}\colon \mathbf{X}^{\mathsf{dd}} \to \mathbf{X}^{\mathsf{d}}$ is computable.

Proof. By Observation 5.2, $\neq\colon \mathbf{X}^{\mathsf{d}} \times \mathbf{X}^{\mathsf{dd}} \to \mathbb{S}$ is computable. Since $\mathbf{X}$ is computably Hausdorff, by Observation 5.4, $\mathrm{id}\colon \mathbf{X} \to \mathbf{X}^{\mathsf{d}}$ is computable, so $\neq\colon \mathbf{X} \times \mathbf{X}^{\mathsf{dd}} \to \mathbb{S}$ is computable. By currying, the function $x \mapsto X \setminus \{x\}\colon \mathbf{X}^{\mathsf{dd}} \to \mathcal{O}(\mathbf{X})$ is computable, which means that $\mathrm{id}\colon \mathbf{X}^{\mathsf{dd}} \to \mathbf{X}^{\mathsf{d}}$ is computable. □

Corollary 5.9. Let $\mathbf{X}$ be computably Hausdorff and contain a computable point. Then $\mathbf{X}^{\mathsf{d}}$ is computably compact.

Proof. By Proposition 5.8 and Proposition 5.6 (applied to $\mathbf{X}^{\mathsf{d}}$ rather than $\mathbf{X}$). Note that by Corollary 5.3 (2), we obtain a computable point in $\mathbf{X}^{\mathsf{dd}}$ from the one we have in $\mathbf{X}$. □

Corollary 5.10. If $\mathbf{X}^{\mathsf{d}}$ is computably compact, then $\mathbf{X}$ is computably Hausdorff.

Proof. By Observation 5.5 (2), if $\mathbf{X}^{\mathsf{d}}$ is computably compact, then $\mathbf{X}^{\mathsf{dd}}$ is computably Hausdorff. By Corollary 5.3, $\mathrm{id}\colon \mathbf{X} \to \mathbf{X}^{\mathsf{dd}}$ is computable, so $\mathbf{X}$ admits a computable injection into a computable Hausdorff space, and is thus itself computably Hausdorff. □

If objects $\mathbf{X}$ and $\mathbf{Y}$ have the same underlying set, then define $\mathbf{X} \wedge \mathbf{Y}$ as the restriction of $\mathbf{X} \times \mathbf{Y}$ to the diagonal $\{(x, x) : x \in X\}$.

Lemma 5.11. Let $\mathbf{X}^{\mathsf{d}}$ contain two computable points and let $\mathbf{X}$ be computably admissible. Then $\mathrm{id}\colon (\mathbf{X}^{\mathsf{d}} \wedge \mathbf{X}^{\mathsf{dd}}) \to \mathbf{X}$ is computable.

In order to verify the assertion, we also consider $\mathbf{X} \wedge_{\mathcal{B}} \mathbf{Y}$, which is defined as the restriction of $\mathbf{X} \times_{\mathcal{B}} \mathbf{Y}$ to the diagonal. By (S2), (Q3), and admissibility of $\mathbf{X}$, note that $\mathrm{id}\colon (\mathbf{X}^{\mathsf{d}} \wedge \mathbf{X}^{\mathsf{dd}}) \to (\mathbf{X}^{\mathsf{d}} \wedge_{\mathcal{B}} \mathbf{X}^{\mathsf{dd}})$ is computable.

Proof. As our space $\mathbf{X}$ is computably admissible, it suffices to show that $x \mapsto \eta_{\mathbf{X}}(x)\colon \mathbf{X}^{\mathsf{d}} \wedge \mathbf{X}^{\mathsf{dd}} \to \mathcal{O}\mathcal{O}(\mathbf{X})$ is computable. Fix quotient representations $\delta\colon D \to \mathcal{A}(X)$ and $\rho\colon E \to X^{\mathsf{dd}}$. As in the comment immediately after Lemma 2.28, consider the lift $\delta'\colon D_\perp \to \mathcal{A}(\mathbf{X})$. Moreover, with a slight modification of the construction in Proposition 5.6, one may construct $\delta'''\colon D^3 \times D_\perp \to \mathcal{A}(\mathbf{X})^3$ by setting $\delta'''(p, q, r, s) = \langle \delta''(p, s), \delta''(q, s), \delta''(r, s)\rangle$. That is, if $\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{s} \in D$ are δ-names of A, B, C, S, respectively, we have $\delta'''(\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{s}) = \langle A \cap S, B \cap S, C \cap S\rangle$, and $\delta'''(\mathtt{a}, \mathtt{b}, \mathtt{c}, \perp) = \langle A, B, C\rangle$.

By Observation 5.2, the non-equality $\neq\colon \mathbf{X}^{\mathsf{dd}} \times \mathbf{X}^{\mathsf{d}} \to \mathbb{S}$ is computable. As $\mathbf{X}^{\mathsf{d}}$ is a restriction of $\mathcal{A}(\mathbf{X})$, one can think of $\neq$ as a partial morphism $\nu\colon \subseteq \mathbf{X}^{\mathsf{dd}} \times \mathcal{A}(\mathbf{X}) \to \mathbb{S}$. Here, $\nu(x, \{y\}) = \top$ iff $x \neq y$. Let us now think the computable function $h\colon \mathcal{A}(\mathbf{X})^3 \to \mathcal{A}(\mathbf{X})$ defined by $h(A, B, C) = A \cup (B \cap C)$; then $g := \lambda\langle x, A, B, C\rangle.\nu(x, h(A, B, C))\colon \subseteq \mathbf{X}^{\mathsf{dd}} \times \mathcal{A}(\mathbf{X})^3 \to \mathbb{S}$ is computable. By Observation 2.20 together with (S3), $g \circ (\rho \times \delta''')\colon \subseteq E \times D^3 \times D_\perp \to \mathbb{S}$ is extended to $G\colon E \times D^3 \times D_\perp \to \mathbb{S}$, since $E \times D^3 \times D_\perp$ is a $\mathcal{B}^*$-object.

Observe that if $h(A, B, C)$ is a singleton, then $x \in h(A, B, C)$ implies $g(x, A, B, C) = \perp$. Even if $h(A, B, C)$ is not a singleton, let us show that the same property holds at the intentional level. To be more precise, for any ρ-name $\tilde{\mathtt{x}} \in E$ of $x \in \mathbf{X}^{\mathsf{dd}}$ and δ-names $\mathtt{a}, \mathtt{b}, \mathtt{c} \in D$ of A, B, C, respectively, we claim that $x \in h(A, B, C)$ implies $G(\tilde{\mathtt{x}}, \mathtt{a}, \mathtt{b}, \mathtt{c}, \perp) = \perp$.

This is because, as $\mathbf{X}$ is T_1, there exists a δ-name $\dot{\mathtt{x}} \in D$ of $\{x\} \in \mathcal{A}(\mathbf{X})$. Then we have $\delta'''(\mathtt{a}, \mathtt{b}, \mathtt{c}, \dot{\mathtt{x}}) = \langle A \cap \{x\}, B \cap \{x\}, C \cap \{x\}\rangle$, and note that $x \in h(A, B, C)$ implies $h(A \cap \{x\}, B \cap$

$\{x\}, C \cap \{x\}) = \{x\}$. By definition, we have $\nu(x, \{x\}) = \bot$, so we get $G(\tilde{\mathtt{x}}, \mathtt{a}, \mathtt{b}, \mathtt{c}, \dot{\mathtt{x}}) = \bot$. By Lemma 2.33, we must have $G(\tilde{\mathtt{x}}, \mathtt{a}, \mathtt{b}, \mathtt{c}, \bot) = \bot$. This verifies the claim.

Next, given $x, y, z \in \mathbf{X}^{\mathsf{d}}$ and $U \in \mathcal{O}(\mathbf{X})$ one can construct the following closed sets $P_y, P_z \in \mathcal{A}(\mathbf{X})$:

$$P_y = \begin{cases} \{y\} & \text{if } x \in U \\ \{x, y\} & \text{if } x \notin U \end{cases} \qquad\qquad P_z = \begin{cases} \{z\} & \text{if } x \in U \\ \{x, z\} & \text{if } x \notin U \end{cases}$$

Indeed, we have $P_y = h(\{y\}, \{x\}, X \setminus U)$ and $P_z = h(\{z\}, \{x\}, X \setminus U)$. Now we fix two computable points $y, z \in \mathbf{X}^{\mathsf{d}}$.

The next step is to find a procedure to track the function $\lambda\langle x, x, U\rangle.U(x)\colon (\mathbf{X}^{\mathsf{d}} \wedge \mathbf{X}^{\mathsf{dd}}) \times \mathcal{O}(\mathbf{X}) \to \mathbb{S}$. Let $\langle \dot{\mathtt{x}}, \tilde{\mathtt{x}}\rangle \in D \times E$ be a name of $\langle x, x\rangle \in \mathbf{X}^{\mathsf{d}} \wedge \mathbf{X}^{\mathsf{dd}}$, and $\bar{\mathtt{u}} \in D$ be a name of $X \setminus U \in \mathcal{A}(\mathbf{X})$. For each $t \in \{y, z\}$, note that $h(\{t\}, \{x\}, X \setminus U) = P_t$. As seen above, if $x \in U$ then $P_t = \{t\}$ and if $x \notin U$ then $P_t = \{x, t\}$. Now, let $\dot{\mathtt{y}}, \dot{\mathtt{z}} \in D$ be names of $y, z \in \mathbf{X}^{\mathsf{d}}$, respectively. If $x \notin U$ then $x \in h(\{t\}, \{x\}, X \setminus U)$, so by the above claim, $G(\tilde{\mathtt{x}}, \dot{\mathtt{t}}, \dot{\mathtt{x}}, \bar{\mathtt{u}}, \bot) = \bot$ must hold for each $\dot{\mathtt{t}} \in \{\dot{\mathtt{y}}, \dot{\mathtt{z}}\}$. If $x \in U$ then P_t is a singleton, so $\nu(x, P_t) = \top$ iff $t \neq x$. As either $y \neq x$ or $z \neq x$ holds, we have $\nu(x, P_t) = \top$ for some $t \in \{y, z\}$, so $g(x, \{t\}, \{x\}, X \setminus U) = \top$. Hence, we get $G(\tilde{\mathtt{x}}, \dot{\mathtt{t}}, \dot{\mathtt{x}}, \bar{\mathtt{u}}, \bot) = \top$ for some $\dot{\mathtt{t}} \in \{\dot{\mathtt{y}}, \dot{\mathtt{z}}\}$.

Thus, let us consider the computable function $H := \lambda\langle \dot{\mathtt{x}}, \tilde{\mathtt{x}}, \bar{\mathtt{u}}\rangle.G(\tilde{\mathtt{x}}, \dot{\mathtt{y}}, \dot{\mathtt{x}}, \bar{\mathtt{u}}, \bot) \vee G(\tilde{\mathtt{x}}, \dot{\mathtt{z}}, \dot{\mathtt{x}}, \bar{\mathtt{u}}, \bot)$. The output of H is determined by the truth value of $x \in U$, so the function H tracks the function $\lambda\langle x, x, U\rangle.U(x)\colon (\mathbf{X}^{\mathsf{d}} \wedge_{\mathcal{B}} \mathbf{X}^{\mathsf{dd}}) \times_{\mathcal{B}} \mathcal{O}(\mathbf{X}) \to \mathbb{S}$. Let $R \subseteq D \times E$ be the set of all names of elements of $\mathbf{X}^{\mathsf{d}} \wedge \mathbf{X}^{\mathsf{dd}}$. Then the restriction of $\delta \times \rho$ to R gives a quotient representation of $\mathbf{X}^{\mathsf{d}} \wedge_{\mathcal{B}} \mathbf{X}^{\mathsf{dd}}$. Hence, computability of $H\colon R \times D \to \mathbb{S}$ implies computability of $\lambda\langle x, x, U\rangle.U(x)\colon (\mathbf{X}^{\mathsf{d}} \wedge_{\mathcal{B}} \mathbf{X}^{\mathsf{dd}}) \times_{\mathcal{B}} \mathcal{O}(\mathbf{X}) \to \mathbb{S}$. By the argument as above, this gives rise to $(\mathbf{X}^{\mathsf{d}} \wedge \mathbf{X}^{\mathsf{dd}}) \times \mathcal{O}(\mathbf{X}) \to \mathbb{S}$, where note that $\mathcal{O}(\mathbf{X})$ is also admissible by Proposition 2.23. Its currying $(\mathbf{X}^{\mathsf{d}} \wedge \mathbf{X}^{\mathsf{dd}}) \to \mathcal{O}\mathcal{O}(\mathbf{X})$ gives the desired function. $\square$

Corollary 5.12. Let $\mathbf{X}$ be computably Hausdorff, computably admissible and contain two computable points. Then $\mathbf{X} \cong \mathbf{X}^{\mathsf{dd}}$.

Proof. Computability of $\mathrm{id} : \mathbf{X} \to \mathbf{X}^{\mathsf{dd}}$ is available without assumptions (Corollary 5.3). For the converse direction, note that given $x \in \mathbf{X}^{\mathsf{dd}}$ we can first invoke Proposition 5.8 (since $\mathbf{X}$ is assumed to be computably Hausdorff) to obtain $x \in \mathbf{X}^{\mathsf{d}}$. We then use Lemma 5.11 to get $x \in \mathbf{X}$. Note that since $\mathbf{X}$ is computably Hausdorff, having two computable points in $\mathbf{X}$ yields two computable points in $\mathbf{X}^{\mathsf{d}}$ by Observation 5.4. $\square$

Note that combining Observation 5.4, Proposition 5.6 and Corollary 5.12 yields the effectivization of [18, Example 4.2].

Proposition 5.13. If $\mathbf{X}^{\mathsf{d}}$ contains two computable points and is computably Hausdorff, then $\mathbf{X}$ is computably compact.

Proof. To prove that $\mathbf{X}$ is computably compact, it suffices to show that $\mathtt{IsEmpty}\colon \mathcal{A}(\mathbf{X}) \to \mathbb{S}$ is computable. Fix a quotient representation $\delta\colon D \to \mathcal{A}(\mathbf{X})$ of $\mathcal{A}(\mathbf{X})$. As in the comment immediately after Lemma 2.28, consider the lift $\delta'\colon D_\bot \to \mathcal{A}(\mathbf{X})$. Now the function $\delta'' = \cap \circ (\delta \times \delta')\colon D \times D_\bot \to \mathcal{A}(\mathbf{X})$ is computable. Here, if $q = \bot$ then $\delta''(p, q) = \delta(p)$; otherwise $\delta''(p, q) = \delta(p) \cap \delta(q)$. By our assumption that $\mathbf{X}^{\mathsf{d}}$ is computably Hausdorff, the non-equality $\neq\colon \mathbf{X}^{\mathsf{d}} \times \mathbf{X}^{\mathsf{d}} \to \mathbb{S}$ is computable. As $\mathbf{X}^{\mathsf{d}}$ is a restriction of $\mathcal{A}(\mathbf{X})$, one can think of $\neq$ as a partial morphism $\nu\colon\subseteq \mathcal{A}(\mathbf{X})\times\mathcal{A}(\mathbf{X})\to\mathbb{S}$. Then the partial function $f=\lambda\langle A,B,C\rangle.\nu(A\cup B,A\cup C)\colon\subseteq \mathcal{A}(\mathbf{X})^3\to\mathbb{S}$ is computable. By Observation 2.20 together with (S3), $f\circ(\delta''\times\delta''\times\delta'')\colon\subseteq (D\times D_\perp)^3\to\mathbb{S}$ has a total extension $F\colon (D\times D_\perp)^3\to\mathbb{S}$ since $(D\times D_\perp)^3$ is a $\mathcal{B}^*$-object.

Observe that if $A\neq\emptyset$ and $A\cup B, A\cup C$ are singletons we have $f(A\cup B,A\cup C)=\perp$. Even if $A\cup B, A\cup C$ are not singletons, let us show that the same property holds at the intentional level. To be more precise, if $\mathtt{a},\mathtt{b},\mathtt{c}\in D$ are δ-names of $A,B,C\in\mathcal{A}(\mathbf{X})$, respectively, we claim that if $A\neq\emptyset$, then we must have $F(\langle\mathtt{a},\perp\rangle,\langle\mathtt{b},\perp\rangle,\langle\mathtt{c},\perp\rangle)=\perp$.

To see this, let x be an element of A. Note that $\{x\}$ is a closed set as X is T_1. Let $\dot{\mathtt{x}}$ and $\mathtt{o}$ be δ-names of the closed sets $\{x\}$ and $\emptyset$, respectively; so we now have $\mathtt{a},\dot{\mathtt{x}},\mathtt{o}\in D$ such that $\delta(\mathtt{a})=A$, $\delta(\dot{\mathtt{x}})=\{x\}$ and $\delta(\mathtt{o})=\emptyset$. Then we have $\delta''(\langle\mathtt{a},\dot{\mathtt{x}}\rangle)=A\cap\{x\}=\{x\}$ and $\delta''(\langle q,\mathtt{o}\rangle)=\emptyset$ for any $q\in D$. Observe that $\nu(\{x\},\{x\})=\perp$, so we have $f(\{x\},\emptyset,\emptyset)=\perp$. Hence, we get $F(\langle\mathtt{a},\dot{\mathtt{x}}\rangle,\langle\mathtt{b},\mathtt{o}\rangle,\langle\mathtt{c},\mathtt{o}\rangle)=\perp$. By Lemma 2.33, we obtain $F(\langle\mathtt{a},\perp\rangle,\langle\mathtt{b},\perp\rangle,\langle\mathtt{c},\perp\rangle)=\perp$. This verifies the claim.

Next, let us consider the case where $\mathtt{a}$ is a δ-name of the empty set. By our assumption, we are given names $\mathtt{y},\mathtt{z}$ of two computable points $y,z\in\mathbf{X}^{\mathsf{d}}$. As $\mathbf{X}^{\mathsf{d}}$ is a restriction of $\mathcal{A}(\mathbf{X})$, the names $\mathtt{y},\mathtt{z}$ are also names of $\{y\},\{z\}\in\mathcal{A}(\mathbf{X})$. As $y\neq z$, observe that $\nu(\{y\},\{z\})=\top$, so we have $f(\emptyset,\{y\},\{z\})=\top$. Hence, we get $F(\langle\mathtt{a},\perp\rangle,\langle\mathtt{y},\perp\rangle,\langle\mathtt{z},\perp\rangle)=\top$. Combined with the above claim, for any $\mathtt{a}\in D$, we now conclude that $\mathtt{a}$ is a δ-name of the empty set iff $F(\langle\mathtt{a},\perp\rangle,\langle\mathtt{y},\perp\rangle,\langle\mathtt{z},\perp\rangle)=\top$. This means that for the computable function $E:=\lambda\mathtt{a}.F(\langle\mathtt{a},\perp\rangle,\langle\mathtt{y},\perp\rangle,\langle\mathtt{z},\perp\rangle)\colon D\to\mathbb{S}$ we have $E(\mathtt{a})=\mathtt{IsEmpty}(\delta(\mathtt{a}))$. As δ is a quotient representation, this shows that $\mathtt{IsEmpty}\colon\mathcal{A}(X)\to\mathbb{S}$ is computable. □

Corollary 5.14. Let $\mathbf{X}$ contain two computable points. Then $\mathbf{X}^{\mathsf{dd}}$ is computably Hausdorff iff $\mathbf{X}$ is.

Proof. If $\mathbf{X}$ is computably Hausdorff, so is $\mathbf{X}^{\mathsf{top}}$ by Observation 2.26. Moreover, they have the same dual. To see this, first note that $\eta_X^{-1}\colon(\mathbf{X}^{\mathsf{top}})^{\mathsf{d}}\to\mathbf{X}^{\mathsf{d}}$ is computable by Observation 3.15. For $\mathbf{X}^{\mathsf{d}}\to(\mathbf{X}^{\mathsf{top}})^{\mathsf{d}}$, consider a restriction of the evaluation map $\langle\eta_X(y),X\setminus\{x\}\rangle\mapsto\eta_Y(y)(X\setminus\{x\})$, which is the truth value of "$x\neq y$." This yields $\langle\eta_X(y),x\rangle\mapsto(x\neq y)\colon\mathbf{X}^{\mathsf{top}}\times\mathbf{X}^{\mathsf{d}}\to\mathbb{S}$. By currying, we get $x\mapsto\{\eta_X(y):y\neq x\}\colon\mathbf{X}^{\mathsf{d}}\to\mathcal{O}(\mathbf{X}^{\mathsf{top}})$, where the latter set is $\mathbf{X}^{\mathsf{top}}\setminus\{\eta_X(y)\}$. Thus, $\eta_X\colon\mathbf{X}^{\mathsf{d}}\to(\mathbf{X}^{\mathsf{top}})^{\mathsf{d}}$ is computable; hence $\mathbf{X}^{\mathsf{d}}\simeq(\mathbf{X}^{\mathsf{top}})^{\mathsf{d}}$. Corollary 5.12 then yields $\mathbf{X}^{\mathsf{top}}\cong\mathbf{X}^{\mathsf{dd}}$, so the latter is computably Hausdorff.

Conversely, if $\mathbf{X}^{\mathsf{dd}}$ is computably Hausdorff, then by Corollary 5.9, $\mathbf{X}^{\mathsf{ddd}}$ is computably compact (we can lift a computable point from $\mathbf{X}$ to $\mathbf{X}^{\mathsf{dd}}$ by Corollary 5.3). Since $\mathbf{X}^{\mathsf{d}}\cong\mathbf{X}^{\mathsf{ddd}}$ by Corollary 3.11, $\mathbf{X}^{\mathsf{d}}$ is computably compact. Then Corollary 5.10 shows that $\mathbf{X}$ is computably Hausdorff. □

5.4 Proof of Theorem 5.1

Let us confirm that the above completes the proof of Theorem 5.1. The item (1) follows from Corollary 5.3 (2). The item (2) follows from Corollary 3.11. For the item (3), (a)$\to$(b): Corollary 5.12. (b)$\to$(c): trivial. (a)$\leftrightarrow$(c): Corollary 5.14. (a)$\leftrightarrow$(d): Corollaries 5.9 and 5.10. (a)$\leftrightarrow$(e): Observation 5.4. (a)$\to$(f): Proposition 5.8. (f)$\to$(e): Theorem 5.1 (1). For the item (4), (a)$\leftrightarrow$(c): Observation 5.5 (2) and Proposition 5.13. (b)$\leftrightarrow$(c)$\leftrightarrow$(e): Apply Theorem 5.1 (3) (d)$\leftrightarrow$(a)$\leftrightarrow$(e) to $\mathbf{X}^{\mathsf{d}}$. (a)$\leftrightarrow$(d): Observation 5.5 (1) and Proposition 5.6. The item (5) follows from Corollary 5.7.

Remark. De Groot [10] has already described his notion as follows: "*One sacrifices the Hausdorff property but gains e.g. compactness.*" It should be noted that de Groot only comments on one obvious direction, but this seems to suggest the possibility of "compact vs. Hausdorff" duality, and it would seem that subsequent studies should have eventually pinned it down.

Nevertheless, concerning Theorem 1.2, we shall note the result that the T_1 de Groot dual interchanges "compact" and "Hausdorff" is not previously known in general topology. Why? There is an obvious reason for this: this "compact vs. Hausdorff" duality is false in the category of T_1 topological spaces.

If the topological version of Theorem 1.2 also holds, then $\mathbf{X}^{\mathsf{dd}} \simeq \mathbf{X}^{\mathsf{d}}$ implies that $\mathbf{X}^{\mathsf{d}}$ is compact Hausdorff by the item (5), which implies that $\mathbf{X}$ is also compact Hausdorff by (3) and (4); hence $\mathbf{X}^{\mathsf{d}} \simeq \mathbf{X}$ by (5) again. However, it is known that there exists a T_1 topological space $\mathbf{X}$ such that $\mathbf{X}^{\mathsf{dd}} \simeq \mathbf{X}^{\mathsf{d}}$ but $\mathbf{X}^{\mathsf{d}} \not\simeq \mathbf{X}$ (see [11, Example 10] or [18, Example 4.3]).

The notion that plays an essential role in our proof is cartesian closedness, and unfortunately the category **Top** of topological spaces is not cartesian closed. Instead, Theorem 5.1 gives an analogue of Theorem 1.2 in some natural cartesian closed full subcategories of **Top**.

6 Examples

6.1 The cofinite topology on $\mathbb{N}$.

An important example to illustrate the duality between Hausdorff spaces and compact T_1-spaces is the observation that $\mathbb{N}^{\mathsf{d}} = \mathbb{N}_{\mathrm{cof}}$, where $\mathbb{N}_{\mathrm{cof}}$ are the natural numbers equipped with the cofinite topology. We then also have that $(\mathbb{N}_{\mathrm{cof}})^{\mathsf{d}} = \mathbb{N}$.

6.2 The cocylinder topology on Baire space.

As announced in Section 5, we give an example where $\mathbf{X} \simeq \mathbf{X}^{\mathsf{dd}}$ is not necessarily true even if $\mathbf{X}$ is computably compact and T_1.

Definition 6.1. The cocylinder topology τ_c on $\mathbb{N}^{\mathbb{N}}$ is generated by co-cylinders $\{X : X \not\succ \sigma\}$ where σ ranges over finite strings. We write $\mathbb{N}^{\mathbb{N}}_c = (\mathbb{N}^{\mathbb{N}}, \tau_c)$.

The space $\mathbb{N}^{\mathbb{N}}_c$ is second-countable, computably compact and T_1. It is neither Hausdorff nor sober (and thus not stably compact). We see below that $(\mathbb{N}^{\mathbb{N}}_c)^{\mathsf{d}} \simeq \mathbb{N}^{\mathbb{N}}$ and thus $(\mathbb{N}^{\mathbb{N}}_c)^{\mathsf{dd}} \simeq (\mathbb{N}^{\mathbb{N}})^{\mathsf{d}}$, but $(\mathbb{N}^{\mathbb{N}})^{\mathsf{d}}$ is not second-countable (see Section 7.3), so $(\mathbb{N}^{\mathbb{N}}_c)^{\mathsf{dd}} \not\simeq \mathbb{N}^{\mathbb{N}}_c$.

Proposition 6.2. $(\mathbb{N}^{\mathbb{N}}_c)^{\mathsf{d}} \simeq \mathbb{N}^{\mathbb{N}}$

Proof. First note that a name of $x \in \mathbb{N}^{\mathbb{N}}_c$ is an enumeration $(\sigma_n)_{n\in\mathbb{N}}$ of all non-prefixes of x. And, a name of a closed set $A \in \mathcal{A}(\mathbb{N}^{\mathbb{N}}_c)$ is a sequence $D = (D_n)_{n\in\mathbb{N}}$ of finite sets D_n of strings such that $x \in A$ iff, for any $n \in \mathbb{N}$, D_n contains a prefix of x. Thus, given an $\mathbb{N}^{\mathbb{N}}$-name of x, by putting D_n to be the singleton $\{x \upharpoonright n\}$, where $x \upharpoonright n$ is the prefix of x of length n, we get a name of $\{x\} \in \mathcal{A}(\mathbb{N}^{\mathbb{N}}_c)$. This shows that $\mathrm{id}\colon \mathbb{N}^{\mathbb{N}} \to (\mathbb{N}^{\mathbb{N}}_c)^{\mathsf{d}}$ is computable.

Conversely, assume that a name D of a closed set $A \in \mathcal{A}(\mathbb{N}^{\mathbb{N}}_c)$ is given. From such a sequence D, one may construct a finite-branching tree whose infinite paths correspond to the elements of A. To see this, we inductively construct a sequence $E = (E_n)_{n\in\mathbb{N}}$ of finite sets of strings as follows: Let E_0 be the singleton consisting of the empty string. Assume that E_n has already been constructed. For each $\sigma \in E_n$, and each $\tau \in D_n$ which is comparable with σ, put the longer of σ and τ into E_{n+1}. By leaving only shorter strings in E_{n+1}, we may assume that elements of E_{n+1} are pairwise incomparable. Note that E_{n+1} is contained in the upward closure of E_n (w.r.t. the prefix order). We claim that $x \in A$ iff E_n has a prefix of x for any $n \in \mathbb{N}$. For the backward direction, note that if E_{n+1} has a prefix of x then so does D_n. For the forward direction, if $x \in A$, one can inductively ensure that E_n contains a prefix of x. By the assumption $x \in A$, D_n also contains a prefix of x, so a prefix of x survives in E_{n+1}.

Now, the downward closure of $\bigcup_{n\in\mathbb{N}} E_n$ yields a finite-branching tree T_E. If A is a singleton $\{x\}$, by the above arguments, one can see that T_E has a unique infinite path x. However, we only have an enumeration of the tree T_E which is not pruned, so it is not straightforward to compute an $\mathbb{N}^\mathbb{N}$-name of the unique path x. To overcome this difficulty, note that only one of the elements of E_n is a prefix of x. If $\sigma \in E_n$ is not a prefix of x, we claim that there exists $m > n$ such that E_m fails to have an extension of σ. Otherwise, for any $m > n$, E_m has an extension τ_m of σ. If $(\tau_m)_{m>n}$ is eventually constant, say τ, then almost all D_m contain an initial segment of τ, so any infinite string extending τ must be a path through T_E, which is impossible. Hence, $(\tau_m)_{m\in\mathbb{N}}$ contains infinitely many different strings in T_E extending σ. Since T_E is finite-branching, König's lemma implies that T_E has an infinite path extending σ, which is again impossible by our assumption. This verifies the claim, which shows that $\sigma \in E_n$ not being a prefix of x is semidecidable. Wait for all but one string in E_n to turn out not to be a prefix of x. Then the last remaining one turns out to be a prefix of x. In this way, we can compute a $\mathbb{N}^\mathbb{N}$-name of the unique path x, which shows that $\mathrm{id}\colon (\mathbb{N}^\mathbb{N}_c)^\mathsf{d} \to \mathbb{N}^\mathbb{N}$ is computable. □

6.3 The lower reals.

The following example shows that we need to distinguish a space being isomorphic to its dual and being equal to its dual: The lower reals and the upper reals are isomorphic (with $x \mapsto -x$ being a computable isomorphism), but not equal (as $\mathrm{id} : \mathbb{R}_< \to \mathbb{R}_>$ is not computable).

Proposition 6.3. $\mathbb{R}_<^\mathsf{d} = \mathbb{R}_>$

The following shows that Observation 3.15 (about being able to reverse the direction of a computable bijection by taking the dual) does not hold once the assumption of being an order isomorphism is excluded:

Example 6.4. $\mathrm{id} : \mathbb{R} \to \mathbb{R}_<$ is a computable bijection, yet $\mathrm{id} : \mathbb{R}_<^\mathsf{d} \to \mathbb{R}^\mathsf{d}$ is not computable.

Proof. By Proposition 6.3, we have that $\mathbb{R}_<^\mathsf{d} = \mathbb{R}_>$. As explained in Subsubsection 5.2, $\mathrm{id} : \mathbb{R}^\mathsf{d} \to \mathbb{R}$ is unique closed choice on $\mathbb{R}$, which was shown to be Weihrauch equivalent to $\mathrm{C}_\mathbb{N}$ in [3]. Thus, if $\mathrm{id} : \mathbb{R}_<^\mathsf{d} \to \mathbb{R}^\mathsf{d}$ were computable, then $(\mathrm{id} : \mathbb{R}_> \to \mathbb{R}) \leq_\mathrm{W} \mathrm{C}_\mathbb{N}$. But $(\mathrm{id} : \mathbb{R}_> \to \mathbb{R}) \equiv_\mathrm{W} \lim$ and $\mathrm{C}_\mathbb{N} <_\mathrm{W} \lim$ are well-known results, leading to a contradiction. □

7 Degree Theory

7.1 Computability theoretic background

Now, let us talk about what this research was really about. Our original motivation for studying de Groot dual was not in general topology, but in the study of Π^0_1 singletons (implicit definability) in computability theory. The connection with topological de Groot dual was only an accidental discovery.

Recall that an object is implicitly definable (in arithmetic) if it is a unique solution of an (arithmetical) predicate; see e.g. Odifreddi [20, Definition XII.2.13]. One of the triggers that made this notion worth studying in logic was, for example, the following observation: Tarski's truth undefinability theorem tells us that arithmetical truth is not explicitly definable in arithmetic; nevertheless, Tarski's truth definition gives an arithmetical inductive definition of the arithmetical truth. The latter shows that the arithmetical truth is implicitly definable in arithmetic; that is, it is a unique solution of an arithmetical predicate, or more precisely, the set of codes of true sentences in first order arithmetic is an arithmetical singleton in $2^{\mathbb{N}}$ (see e.g. Odifreddi [20, Definition XII.2.13 and Proposition XII.2.19]).

In this way, implicit definability can often encode powerful information. Indeed, the arithmetical truth is implicitly Π^0_2 definable in $2^{\mathbb{N}}$. Moreover, if we write the arithmetical truth in a functional form, we can save the complexity of the formula, which is then implicitly Π^0_1 definable in $\mathbb{N}^{\mathbb{N}}$. In contrast, implicit Π^0_1 definability in $2^{\mathbb{N}}$ coincides with computability. Why does this happen, why is implicit Π^0_1 definability (Π^0_1 singleton) often tremendously strong, but sometimes not strong at all? We shall refer to this question as *Main Question*.

Of course, many researchers may have thought that this is because $2^{\mathbb{N}}$ is compact, and $\mathbb{N}^{\mathbb{N}}$ is far from compact. However, there has been almost no research that has explored this topological idea further. To summarize our situation, $\{x\}$ is Π^0_1, which means that $\{x\}$ is computable as a closed set, but x can be neither computable nor even arithmetical. In such a case, the function $\{x\} \mapsto x$ must have a very high complexity. Indeed, investigating the complexity of $\{x\} \mapsto x$ corresponds to investigating the strength of implicit Π^0_1 definability. One of the directions prompted by this idea was to study the computability-theoretic strength of unique closed choice (Brattka-de Brecht-Pauly [3]), where recall from Section 5.2 that the unique closed choice is the partial function $\mathrm{UC}_{\mathbf{X}} \colon\subseteq \mathcal{A}(\mathbf{X}) \to \mathbf{X}$ defined by $\mathrm{UC}_{\mathbf{X}}(\{x\}) = x$; that is, this is $\{x\} \mapsto x$.

While classical computability theorists concentrated on studying the details of the degree-theoretic behavior of arithmetical singletons in $\mathbb{N}^{\mathbb{N}}$, computable analysts investigated the behavior of unique closed choice in a variety of natural represented spaces, including Euclidean line $\mathbb{R}$; see [3]. This direction may not have led to a complete solution of Main Question, but it did lead us to the idea of comparing the strength of $\{x\} \mapsto x$ in various represented spaces. In Section 7.2, we add a few new results on unique closed choice.

Another direction we came up with was to study the computability-theoretic behavior of the spaces of singletons, which is the part we have already developed in this article. However, before arriving at the idea of this article, we first attempted to investigate the set of "degrees" of points in the space of singletons, in particular a comparison of $\{\deg(x) : x \in X\}$ and $\{\deg(\{x\}) : \{x\} \in X^{\mathsf{d}}\}$, where $\deg(z)$ denotes the degree of a point z in a certain sense. This direction is based on the authors' earlier work [16] on point degree spectra. We discuss the details of this direction in Section 7.3.

The two supplements (Sections 7.2 and 7.3) are interesting in their own right, independent of Main Question, the former being a new contribution to the theory of Weihrauch degrees and the latter to the theory of point degree spectra.

Let us return to Main Question. It is safe to say that Theorem 1.2 gives a complete solution to Main Question. In classical computability theory, it is considered obvious that "any computable data is implicitly Π^0_1 definable," but we now know the real reason. This is linked to computability of $x \mapsto \{x\}$, the item 3(e) in Theorem 1.2, which is equivalent to being computably Hausdorff, and classical computability theorists had only considered this problem in $2^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$, which are obviously computably Hausdorff. For the fact that "implicit Π^0_1 definability implies computability" is true in $2^{\mathbb{N}}$ but not in $\mathbb{N}^{\mathbb{N}}$, as already mentioned, most researchers had a vague sense that compactness would be involved, and indeed this was completely correct. This is linked to computability of $\{x\} \mapsto x$, the item 4(d) in Theorem 1.2, which is equivalent to being computably compact. This concludes the study of Main Question.

7.2 The strength of unique closed choice

In this section, we discuss the strength of unique closed choice in some spaces. While previous studies [3] have only discussed the strength of unique closed choice on some Polish spaces, here we give new results on the strength of unique closed choice on some non-Polish/non-second countable spaces.

We now formally introduce the notion of (unique) closed choice. A function $F\colon X \to \mathcal{P}(Y)$ is often called a multifunction, and written as $F\colon X \rightrightarrows Y$. We often think of $y \in F(x)$ as y being a solution to the xth instance of the F-problem.

Definition 7.1 (see [3,4])**.** For a represented space $\mathbf{X}$, the closed choice $\mathrm{C}_{\mathbf{X}}\colon\subseteq \mathcal{A}(\mathbf{X}) \rightrightarrows \mathbf{X}$ is a partial multifunction such that its domain is the set of all nonempty closed subsets of $\mathbf{X}$, and for any nonempty $A \in \mathcal{A}(\mathbf{X})$, any element in A is a solution to $C_{\mathbf{X}}(A)$; that is, $C_{\mathbf{X}}(A) = A$. The unique closed choice $\mathrm{UC}_{\mathbf{X}}$ is the restriction of $\mathrm{C}_{\mathbf{X}}$ to the set of all closed singletons.

For the above definition, it should be noted that an input to $C_{\mathbf{X}}$ is (an $\mathcal{A}(\mathbf{X})$-name of) a closed set A, whereas a (nondeterministic) output is (an $\mathbf{X}$-name of) a point $x \in A$. The notion used to compare the various principles is Weihraurch reducibility (see [4]).

Definition 7.2. Let $\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{W}$ be represented spaces, and $F\colon\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$ and $G\colon\subseteq \mathbf{Z} \rightrightarrows \mathbf{W}$ be partial multifunctions. We say that F is *Weihrauch reducible to* G if there exist partial computable functions $H, K\colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that

- for any name $\mathtt{x}$ of $x \in \mathrm{dom}(F)$ we have $H(\mathtt{x}) \in \mathrm{dom}(G)$,
- and for any name $\mathtt{y}$ of $y \in G(H(\mathtt{x}))$, we have $K(\mathtt{x}, \mathtt{y}) \in F(x)$.

The space we consider here is one of the most important and fundamental examples of non-second-countable spaces, the function space $\mathcal{C}(\mathbb{N}^{\mathbb{N}}, \mathbb{N})$. Here, this space is obtained as the exponential in the category of representation spaces. This space $\mathcal{C}(\mathbb{N}^{\mathbb{N}}, \mathbb{N})$ is often referred to as the second Kleene-Kreisel space.

It is not necessary to know what exactly the topology of this space is, but it is necessary to know the names of the points in this space. A *tree* is a subset of $\mathbb{N}^{<\omega}$ which is closed under taking prefixes. A tree is well-founded if there is no infinite path through it. A continuous function $f\colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ is locally constant, so a name of f can be regarded as a list of pairs $\langle \sigma, n\rangle \in \mathbb{N}^{<\mathbb{N}} \times \mathbb{N}$ that code the information that f takes the constant value n on the open neighborhood $[\sigma]$, where $[\sigma]$ is the set of all extensions of σ. If we consider this as attaching label n to string σ, we can also consider a name f as a labeled well-founded tree. Here, to ensure well-foundedness, each time a node is labeled, its (sufficiently large) extensions are removed from the tree so that it cannot be extended to an infinite path, where well-foundedness of the tree corresponds to totality of f.

This observation predicts a connection between the function space $\mathcal{C}(\mathbb{N}^{\mathbb{N}}, \mathbb{N})$ and the well-founded trees. By picking a standard bijection between $\mathbb{N}^{<\omega}$ and $\mathbb{N}$, we obtain an injective representation $\delta_{\mathrm{T}} :\subseteq 2^{\mathbb{N}} \to \mathrm{Trees}$ of the space Trees of trees. We then obtain the space $\mathrm{WT} \subseteq \mathrm{Trees}$ of well-founded trees.

Lemma 7.3. 1. $\mathbb{N}^{\mathbb{N}}$ computably embeds as a computable closed subspace into WT.

2. $\mathrm{WT}^{\mathbb{N}}$ computably embeds as a computable closed subspace into WT.

3. WT computably embeds as a computable closed subspace into $\mathcal{C}(\mathbb{N}^{\mathbb{N}}, \mathbb{N})$.
4. There is a computable closed subset $D \subseteq \mathrm{WT} \times \mathbb{N}^{\mathbb{N}}$ and a computable bijection $s : D \to \mathcal{C}(\mathbb{N}^{\mathbb{N}}, \mathbb{N})$.

Proof. (1) Map $p \in \mathbb{N}^{\mathbb{N}}$ to $\{\varepsilon\} \cup \{n0^k \mid n \in \mathbb{N} \wedge k \leq p(n)\}$. One can easily see that the non-membership relation on its image is recognizable.

(2) Map a sequence of trees $T_0, T_1, \dots$ to $\{\varepsilon\} \cup \{nv \mid n \in \mathbb{N} \wedge v \in T_n\}$. The range of the embedding consists of the non-empty trees.

(3) Pick a computable bijection $\langle \cdot, \cdot \rangle \colon \mathbb{N}^{<\omega} \to \mathbb{N}$. We map a well-founded tree T to the continuous function $f_T \colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ mapping p to $\langle p_{\leq n} \rangle$ where n is maximal such that $p_{\leq n} \in T$. Given f_T we can reconstruct T. Moreover, given some $f \colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ which is not of the form f_T we can recognize this fact: When the declaration that f takes the constant value τ on $[\sigma]$ is enumerated, we reject f if (a string coded by) τ is not an initial segment of σ.

(4) Our idea is to code a continuous function $f \in C(\mathbb{N}^{\mathbb{N}}, \mathbb{N})$ as a labeled well-founded tree, as described above. A labeled tree is a pair of a tree $T \subseteq \mathbb{N}^{<\omega}$ and a labeling function $\ell \colon T \to \mathbb{N}$; thus one may think of a labeled well-founded tree as an element of $\mathrm{WT} \times \mathbb{N}^{\mathbb{N}}$ (using a computable bijection between $\mathbb{N}$ and $\mathbb{N}^{<\omega}$). We need to choose one name for each continuous function f to get a bijection, so we consider the one for which the tree is the most minimal.

We further add a minimality witness for each tree with the following rules: Any vertex not in the tree is labeled 0. Any leaf is marked as such and labeled with an outcome $n \in \mathbb{N}$. Any inner vertex is marked as such, together with the first pair of leafs extending it that have different outcomes attached to it; that is, for any inner vertex $\sigma \in T$, its label is of the form $\ell(\sigma) = \langle \tau, \rho \rangle$ for some leafs τ and ρ extending σ such that $\ell(\tau) \neq \ell(\rho)$, and furthermore, $\langle \tau, \rho \rangle$ is the lexicographically least such pair.

Let D be the set of all labeled well-founded trees (T, ℓ) satisfying the above condition. This set D is computably closed in $\mathrm{WT} \times \mathbb{N}^{\mathbb{N}}$: If $(T, \ell) \notin D$ because of an inner vertex σ, then we can eventually recognize it by checking labels of all leafs τ, ρ extending σ. In the case of another reason, it is obviously recognizable.

Now, each labeled tree $(T, \ell) \in D$ defines a continuous function $f \colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ which maps $p \in \mathbb{N}^{\mathbb{N}}$ to the outcome label of the leaf of the tree reached by p. Defining $s(T, \ell) = f$ yields a computable bijection. □

Corollary 7.4. $\mathrm{C}_{\mathrm{WT}} \equiv_{\mathrm{W}} \mathrm{C}_{\mathcal{C}(\mathbb{N}^{\mathbb{N}}, \mathbb{N})}$ and $\mathrm{UC}_{\mathrm{WT}} \equiv_{\mathrm{W}} \mathrm{UC}_{\mathcal{C}(\mathbb{N}^{\mathbb{N}}, \mathbb{N})}$.

Proof. Using the computable closedness of the image of a computable embedding in Lemma 7.3 (3), one can easily show $\mathrm{C}_{\mathrm{WT}} \leq_{\mathrm{W}} \mathrm{C}_{\mathcal{C}(\mathbb{N}^{\mathbb{N}}, \mathbb{N})}$; see also [3, Corollary 4.3]. By the same reason, one can observe $\mathrm{UC}_{\mathrm{WT}} \leq_{\mathrm{W}} \mathrm{UC}_{\mathcal{C}(\mathbb{N}^{\mathbb{N}}, \mathbb{N})}$. Moreover, by Lemma 7.3 (4), the preimage function s^{-1} computably transforms a closed set in $\mathcal{C}(\mathbb{N}^{\mathbb{N}}, \mathbb{N})$ into a closed set in D. As D is computably closed, the latter can be considered as a closed set in $\mathrm{WT} \times \mathbb{N}^{\mathbb{N}}$. Thus, the closed choice on $\mathcal{C}(\mathbb{N}^{\mathbb{N}}, \mathbb{N})$ is Weihrauch reducible to that on $\mathrm{WT} \times \mathbb{N}^{\mathbb{N}}$. Lemma 7.3 (1),(2), the latter is Weihrauch reducible to C_{WT} by the same reason as above. As s is bijective, the above argument also shows $\mathrm{UC}_{\mathcal{C}(\mathbb{N}^{\mathbb{N}}, \mathbb{N})} \leq_{\mathrm{W}} \mathrm{UC}_{\mathrm{WT}}$. □

We also show that these principles are closed under parallelization and sequential composition. The parallelization of F is defined as $\widehat{F}(\langle x_i \rangle_{i \in \mathbb{N}}) = \langle F(x_i) \rangle_{i \in \mathbb{N}}$. The sequential composition $G \star F$ expresses the result of applying F followed by G. To be more precise, it is a representative of the greatest Weihrauch degree in $\{g \circ f : f \leq_{\mathrm{W}} F \text{ and } g \leq_{\mathrm{W}} G\}$; see [4] for the details.

Corollary 7.5. $\mathrm{C}_{\mathcal{C}(\mathbb{N}^\mathbb{N},\mathbb{N})}$ and $\mathrm{UC}_{\mathcal{C}(\mathbb{N}^\mathbb{N},\mathbb{N})}$ are closed under parallelization and compositional product.

Proof. By Corollary 7.4, it is sufficient to show both for C_{WT} and $\mathrm{UC}_{\mathrm{WT}}$. Using Lemma 7.3 (2), one can easily see that C_{WT} and $\mathrm{UC}_{\mathrm{WT}}$ are closed under parallelization. Moreover, as WT is a subspace of $\mathbb{N}^\mathbb{N}$, C_{WT} is closed under sequential composition by [3, Theorem 7.3]. The same argument applies to $\mathrm{UC}_{\mathrm{WT}}$. □

Let us discuss the Weihrauch complexity of unique choice. As discussed in [14], the unique choice on Baire space, $\mathrm{UC}_{\mathbb{N}^\mathbb{N}}$, is related to the second strongest principle ATR in the Big Five of reverse mathematics [29]. What we are now interested in is the complexity of $\mathrm{UC}_{\mathcal{C}(\mathbb{N}^\mathbb{N},\mathbb{N})}$. We shall see that it is powerful enough to determine well-foundedness of a given tree. This suggests that $\mathrm{UC}_{\mathcal{C}(\mathbb{N}^\mathbb{N},\mathbb{N})}$ is more powerful than the strongest principle in the Big Five. Let isWellfounded: Trees $\to \mathbf{2}$ be the problem determining whether the input tree is well-founded or not.

Lemma 7.6. isWellfounded $\leq_{\mathrm{W}} \mathrm{UC}_{\mathrm{WT}}$

Proof. We are given a tree T and construct a closed set A of trees containing exactly one well-founded tree. Any tree which is not a subtree of $0\mathbb{N}^{<\omega}$ or $\{1,2\}\mathbb{N}^{<\omega}$ gets rejected from A. A subtree of $0\mathbb{N}^{<\omega}$ gets rejected if and only if it differs from $0T$. A subtree of $\{1,2\}\mathbb{N}^{<\omega}$ gets rejected if it contains a vertex of the form $1v$ but not of the form $1n0^k$. Moreover, if k_n is such that $1n0^{k_n}$ is in the tree but $1n0^{k_n+1}$ is not, and we find that $k_0k_1k_2\dots k_\ell$ does not belong to T, we reject as well. Finally, the tree gets rejected unless a vertex of the form $2v$ is present if and only if v is lexicographically below $k_0k_1\dots k_{|v|-1}$ and belongs to T.

If T is well-founded, then $0T$ is the only well-founded tree belonging to A. If T is ill-founded, then the unique well-founded tree in A is of the form $\{\varepsilon,1\}\cup\{1n0^i \mid i\leq p(n)\}\cup 2T'$ where p is the left-most infinite path through T and T' in the part of T to the left of p. If A is a valid input to C_{WT}, and if S the well-founded tree in A, we can determine whether T is well-founded by inspecting whether $0\in S$ or not. □

Of course, deciding well-foundedness of a given tree corresponds to deciding the truth value of a given Π^1_1 formula φ with a real parameter. When such a formula has a number variable, the process of constructing $\{n\in\mathbb{N}:\varphi(n)\}$ is called Π^1_1 comprehension. Alternatively, the Π^1_1 comprehension principle Π^1_1-CA can be defined as the parallelization of isWellfounded. Note that this is also known as the hyperjump operator; see e.g. [26, Section II.7] or [29, Definition VII.1.5].

Corollary 7.7. Π^1_1-CA $<_{\mathrm{W}} \mathrm{UC}_{\mathcal{C}(\mathbb{N}^\mathbb{N},\mathbb{N})}$.

Proof. By Corollary 7.5, $\mathrm{UC}_{\mathcal{C}(\mathbb{N}^\mathbb{N},\mathbb{N})}$ is closed under parallelization, so parallelize both sides of Lemma 7.6 to obtain Π^1_1-CA $\leq_{\mathrm{W}} \mathrm{UC}_{\mathcal{C}(\mathbb{N}^\mathbb{N},\mathbb{N})}$. For the strictness, note that Π^1_1-CA is not closed under compositional product; that is, the double hyperjump is more powerful than the single hyperjump [26], while $\mathrm{UC}_{\mathcal{C}(\mathbb{N}^\mathbb{N},\mathbb{N})}$ is closed under compositional product by Corollary 7.5. □

The above proof indeed shows that $\mathrm{UC}_{\mathcal{C}(\mathbb{N}^\mathbb{N},\mathbb{N})}$ is stronger than the iterated hyperjump operator; that is, $\Pi^1_1\text{-CA}^\diamond \leq_{\mathrm{W}} \mathrm{UC}_{\mathcal{C}(\mathbb{N}^\mathbb{N},\mathbb{N})}$, where see [30] for the diamond operator.

7.3 The Point Degree Spectrum of $(\mathbb{N}^{\mathbb{N}})^{\mathsf{d}}$

Finally, we discuss an approach to studying the structure of a represented space by measuring the computability-theoretic complexity of its points. This idea was introduced in the authors' previous study [16], which we called the point degree spectrum. For a more comprehensive study, see also [15]. The reason why it is appropriate here to measure the complexity of points in a space is that the study of Π^0_1 singletons in classical computability theory is a degree-theoretic analysis of a point (i.e., a singleton), rather than an exploration of the entire de Groot dual.

Definition 7.8. Let $\mathbf{X}$ and $\mathbf{Y}$ be represented spaces. For $x \in \mathbf{X}$ and $y \in \mathbf{Y}$ we write $y \leq_M x$ if there exists a partial computable function $F \colon\subseteq \mathbf{X} \to \mathbf{Y}$ such that $F(x) = y$; that is, given a name of x, one can effectively find a name of y.

The structure $(\mathbf{X}, \leq_M)$ (or the set of $\equiv_M$-equivalence classes of all elements in $\mathbf{X}$) is called the point degree spectrum of $\mathbf{X}$. The point degree spectrum links the study of computability-theoretic degree structures such as the Medvedev degrees, enumeration degrees and Turing degrees to σ-homeomorphism types of topological spaces [15, 16, 23].

We show that, relative to any oracle, the point degree spectrum of the de Groot dual of $\mathbb{N}^{\mathbb{N}}$ contains non-enumeration degrees. Indeed, we show that there are "powerless" points in $(\mathbb{N}^{\mathbb{N}})^{\mathsf{d}}$ in the sense that they cannot compute any nontrivial enumeration degree.

Definition 7.9. A non-computable point $x \in \mathbf{X}$ is $\mathbb{S}^{\mathbb{N}}$-*quasi-minimal* if for any $y \in \mathbb{S}^{\mathbb{N}}$, $y \leq_M x$ implies that y is computable.

As $\mathbb{S}^{\mathbb{N}}$ is a universal second-countable T_0 space, we find that a $\mathbb{S}^{\mathbb{N}}$-quasi-minimal point is $\mathbf{Y}$-quasi-minimal for any second countable space $\mathbf{Y}$. The degrees of points in $\mathbb{S}^{\mathbb{N}}$ are exactly the enumeration degrees, so another perspective on $\mathbb{S}^{\mathbb{N}}$-quasi-minimal points is that they are non-computable points not computing any non-trivial enumeration degree.

Note that while the previous sections have focused mainly on the complexity of $\{x\} \mapsto x$, the above notion (for $\mathbf{X} = (\mathbb{N}^{\mathbb{N}})^{\mathsf{d}}$) analyzes computability of $\{x\} \mapsto y$ for all $y \in \mathbb{S}^{\mathbb{N}}$. This is also important for understanding the computability-theoretic strength of singletons.

Theorem 7.10. Relative to any oracle, there are continuum many $\mathbb{S}^{\mathbb{N}}$-quasi-minimal $(\mathbb{N}^{\mathbb{N}})^{\mathsf{d}}$-degrees.

In the following, we write $x^{\mathbf{X}}$ to emphasize that x is a point in the represented space $\mathbf{X}$. To avert superscript-overload, we will write $\mathcal{E}$ for $\mathbb{S}^{\mathbb{N}}$ and $\mathcal{B}$ for $\mathbb{N}^{\mathbb{N}}$.

Lemma 7.11. If $y^{\mathcal{E}} \leq_{\mathrm{T}} x^{\mathcal{B}^{\mathsf{d}}}$, then one of the following must hold:

1. $y^{\mathcal{E}}$ is computable.
2. $x^{\mathcal{B}} \leq_{\mathrm{T}} x^{\mathcal{B}^{\mathsf{d}}} \oplus (\mathbb{N} \setminus y)^{\mathcal{E}}$

Proof. A witness for $y^{\mathcal{E}} \leq_{\mathrm{T}} x^{\mathcal{B}^{\mathsf{d}}}$ is an enumeration operator Φ that reads sufficiently many non-prefixes of x and then enumerates numbers $n \in \mathbb{N}$ into y. The two cases we distinguish concern whether Φ will give wrong answers if fed wrong input. To be precise, the first case is that there exists some $\ell \in \mathbb{N}$ such that Φ will never enumerate some $m \notin y$ when reading only finite strings of length at least ℓ. The second case then is that for any $\ell \in \mathbb{N}$ there exists a finite list of strings, each string having length at least ℓ, which causes Φ to enumerate some $m \notin y$.

In the first case, $y^{\mathcal{E}}$ is already computable: We can inspect Φ to enumerate all numbers $m \in \mathbb{N}$ that would ever be enumerated upon reading a finite list of finite strings, each longer than ℓ. By assumption, all of these numbers actually belong to y. Moreover, $x^{\mathcal{B}^{\mathsf{d}}}$ has a name made up from only finite words longer than ℓ, which ensures that we do enumerate all $m \in y$ in the manner.

In the second case, if we do have access to $(\mathbb{N} \setminus y)^{\mathcal{E}}$ we can actually find, for each $\ell \in \mathbb{N}$, a finite sequence of words of length ℓ or more, which cause some $m \notin y$ to be enumerated by Φ. But that means that such a sequence cannot be extended to a name for $x^{\mathcal{B}^{\mathsf{d}}}$; i.e. that we have identified finitely many words of length ℓ or more with the guarantee that one of them is a prefix of $x^{\mathcal{B}}$. Doing this for all $\ell \in \mathbb{N}$ means we obtain a finitely branching tree T (with known branching factors) through which x is an infinite path. If we also have $x^{\mathcal{B}^{\mathsf{d}}}$ available to us, we can eliminate all infinite paths but x from T, and then compute $x^{\mathcal{B}}$. □

Proof of Theorem 7.10. Given an oracle z, it is easy to construct a $\Pi^0_1(z)$ singleton $\{x\}$ in $\mathbb{N}^{\mathbb{N}}$ such that $x \not\leq_{\mathrm{T}} z'$ (see [20, Exercises XII.2.14 (d), and XII.2.15 (e)]). Moreover, if $\{x\}$ is such a $\Pi^0_1(z)$, then so is $\{x \oplus z\}$. We will write $x_z := x \oplus z$ where x is constructed from z in this manner.

Now, given an oracle r, consider any $z \geq_T \mathcal{O}^r$, where $\mathcal{O}^r$ is the hyperjump of r, that is, a $\Pi^1_1(r)$-complete subset of $\mathbb{N}$. Then, $\{x_z\}$ is not a $\Pi^0_1(r)$ singleton; otherwise, x_z is Δ^1_1 in r [20, Proposition XII.2.16], and thus $x_z \leq_T \mathcal{O}^r \leq_T z$, a contradiction.

We will show that $(x_z)^{\mathcal{B}^{\mathsf{d}}}$ is $\mathcal{E}$-quasiminimal relative to r. We argued above that $(x_z)^{\mathcal{B}^{\mathsf{d}}}$ is not computable relative to r. Assume that some non-computable $y \in \mathcal{E}$ satisfies $y^{\mathcal{E}} \leq_{\mathrm{T}} (x_z)^{\mathcal{B}^{\mathsf{d}}}$ relative to r. Then it follows by Lemma 7.11 that $x_z^{\mathcal{B}} \leq_{\mathrm{T}} x_z^{\mathcal{B}^{\mathsf{d}}} \oplus (\mathbb{N} \setminus y)^{\mathcal{E}} \oplus r$. We know that z can compute $x_z^{\mathcal{B}^{\mathsf{d}}}$ and r, and thus also $y^{\mathcal{E}}$. But then z' computes z and $(\mathbb{N} \setminus y)^{\mathcal{E}}$, hence $x_z \leq_{\mathrm{T}} z'$. But we constructed x_z such that $x \not\leq_{\mathrm{T}} z'$, and thus have reached a contradiction. It follows that $(x_z)^{\mathcal{B}^{\mathsf{d}}}$ is $\mathcal{E}$-quasiminimal relative to r. As there are continuum many $z \geq_T \mathcal{O}^r$, and since $z_1 \neq z_2$ implies $x_{z_1} \neq x_{z_2}$, the claim follows. □

Having continuum many $\mathbb{S}^{\mathbb{N}}$-quasi-minimal points has a topological interpretation. For topological spaces $\mathcal{X}$ and $\mathcal{Y}$, a function $f\colon \mathcal{X} \to \mathcal{Y}$ is σ-continuous (see e.g. [16]) if it can be decomposed into countably many continuous functions; that is, there is a countable partition $\{\mathcal{X}_i\}_{i\in\mathbb{N}}$ of $\mathcal{X}$ such that $f|_{\mathcal{X}_i}$ is continuous for each $i \in \mathbb{N}$. The relevance of (the hierarchy of) σ-continuity to computability theory is discussed in depth in [9, 13, 16].

Corollary 7.12. For any second-countable T_0 space $\mathcal{Y}$, if $f\colon (\mathbb{N}^{\mathbb{N}})^{\mathsf{d}} \to \mathcal{Y}$ is σ-continuous, then there is a set A of cardinality continuum such that the image $f[A]$ is countable.

Proof. Let $f\colon \mathcal{B}^{\mathsf{d}} \to \mathcal{Y}$ be a σ-continuous function, where $\mathcal{Y}$ is a second-countable T_0 space. Then, via an embedding $\mathcal{Y} \hookrightarrow \mathbb{S}^{\mathbb{N}}$, one can think of f as a σ-continuous function $f\colon \mathcal{B}^{\mathsf{d}} \to \mathbb{S}^{\mathbb{N}}$. Then, f is σ-computable relative to some oracle r (see e.g. [16]). Note that $f(x) \leq_M x \oplus r$. Let $A \subseteq \mathcal{B}^{\mathsf{d}}$ be the set of all points which are second-countable quasi-minimal relative to r, that is, if $x \in A$ then $f(x)$ is r-computable. Then, since there are only countably many r-computable points in $\mathbb{S}^{\mathbb{N}}$, the range of $f[A]$ is countable as desired. By Theorem 7.10, A has cardinality of the continuum. □

Note that the idea of the proof of Theorem 7.10 is to exploit the difference in computability theoretic strength between explicit and implicit definability. That is, the classical observation that "implicit definability can often encode more powerful information than explicit definability" is the key to the analysis of the point degree spectrum of the de Groot dual of $\mathbb{N}^{\mathbb{N}}$.

Acknowledgements

We are grateful to Matthew de Brecht for fruitful discussions. We also thank Satoshi Nakata and Sōichiro Okuda for their explanations of synthetic domain theory. We are also grateful to the anonymous referees for valuable suggestions and comments.

References

[1] Andrej Bauer (2002): *A relationship between equilogical spaces and type two effectivity.* 48, pp. 1–15. Dagstuhl Seminar on Computability and Complexity in Analysis, 2001.

[2] Andrej Bauer, Lars Birkedal & Dana S. Scott (2004): *Equilogical spaces. Theoret. Comput. Sci.* 315(1), pp. 35–59.

[3] Vasco Brattka, Matthew de Brecht & Arno Pauly (2012): *Closed choice and a uniform low basis theorem. Ann. Pure Appl. Logic* 163(8), pp. 986–1008.

[4] Vasco Brattka, Guido Gherardi & Arno Pauly ([2021] ©2021): *Weihrauch complexity in computable analysis.* In: *Handbook of computability and complexity in analysis*, Theory Appl. Comput., Springer, Cham, pp. 367–417.

[5] Martín Escardó (2004): *Synthetic Topology: of Data Types and Classical Spaces. Electronic Notes in Theoretical Computer Science* 87, pp. 21–156.

[6] Martín Escardó & Reinhold Heckmann (2001/02): *Topologies on spaces of continuous functions.* In: *Proceedings of the 16th Summer Conference on General Topology and its Applications (New York)*, 26, pp. 545–564.

[7] Martín Escardó, Jimmie Lawson & Alex Simpson (2004): *Comparing Cartesian closed categories of (core) compactly generated spaces. Topology Appl.* 143(1-3), pp. 105–145.

[8] Jean Goubault-Larrecq (2013): *Non-Hausdorff topology and domain theory: Selected topics in point-set topology*, *New Mathematical Monographs* 22. Cambridge University Press, Cambridge.

[9] Vassilios Gregoriades, Takayuki Kihara & Keng Meng Ng (2021): *Turing degrees in Polish spaces and decomposability of Borel functions. J. Math. Log.* 21(1), pp. Paper No. 2050021, 41.

[10] J. de Groot (1967): *An isomorphism principle in general topology. Bull. Amer. Math. Soc.* 73, pp. 465–467.

[11] J. de Groot, H. Herrlich, G. E. Strecker & E. Wattel (1969): *Compactness as an operator. Compositio Math.* 21, pp. 349–375.

[12] J. M. E. Hyland (1991): *First steps in synthetic domain theory.* In: *Category theory (Como, 1990)*, *Lecture Notes in Math.* 1488, Springer, Berlin, pp. 131–156.

[13] Takayuki Kihara (2015): *Decomposing Borel functions using the Shore-Slaman join theorem. Fund. Math.* 230(1), pp. 1–13.

[14] Takayuki Kihara, Alberto Marcone & Arno Pauly (2020): *Searching for an analogue of* ATR_0 *in the Weihrauch lattice. J. Symb. Log.* 85(3), pp. 1006–1043.

[15] Takayuki Kihara, Keng Meng Ng & Arno Pauly (2019). *Enumeration degrees and non-metrizable topology.* arXiv:1904.04107.

[16] Takayuki Kihara & Arno Pauly (2022): *Point degree spectra of represented spaces. Forum Math. Sigma* 10, pp. Paper No. e31, 27.

[17] Takayuki Kihara & Arno Pauly ([2023] ©2023): *De Groot duality for represented spaces.* In: *Unity of logic and computation*, *Lecture Notes in Comput. Sci.* 13967, Springer, Cham, pp. 89–101.

[18] Martin Maria Kovár (2003): *At most 4 topologies can arise from iterating the de Groot dual. Topology Appl.* 130(2), pp. 175–182.

[19] John R. Longley (1994): *Realizability Toposes and Language Semantics.* Ph.D. thesis, University of Edinburgh.

[20] P. G. Odifreddi (1999): *Classical Recursion Theory. Vol. II*, *Studies in Logic and the Foundations of Mathematics* 143. North-Holland Publishing Co., Amsterdam.

[21] Jaap van Oosten (2008): *Realizability: an introduction to its categorical side*, *Studies in Logic and the Foundations of Mathematics* 152. Elsevier B. V., Amsterdam.

[22] Arno Pauly (2016): *On the topological aspects of the theory of represented spaces.* *Computability* 5(2), pp. 159–180.

[23] Arno Pauly (2018): *Enumeration degrees and topology.* In: *Sailing routes in the world of computation*, *Lecture Notes in Comput. Sci.* 10936, Springer, Cham, pp. 328–337.

[24] Arno Pauly & Matthew de Brecht (2015): *Descriptive set theory in the category of represented spaces.* In: *2015 30th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS 2015)*, IEEE Computer Soc., Los Alamitos, CA, pp. 438–449.

[25] Giuseppe Rosolini (1986): *Continuity and Effectiveness in Topoi.* Ph.D. thesis, University of Oxford.

[26] Gerald E. Sacks (1990): *Higher recursion theory.* Perspectives in Mathematical Logic. Springer-Verlag, Berlin.

[27] Matthias Schröder (2003): *Admissible representations for continuous computations.* Ph.D. thesis, University of Hagen, Germany.

[28] Matthias Schröder ([2021] ©2021): *Admissibly represented spaces and qcb-spaces.* In: *Handbook of computability and complexity in analysis*, Theory Appl. Comput., Springer, Cham, pp. 305–346.

[29] Stephen G. Simpson (2009): *Subsystems of second order arithmetic.* Perspectives in Logic. Cambridge University Press, Cambridge; Association for Symbolic Logic, Poughkeepsie, NY, second edition.

[30] Linda Westrick (2021): *A note on the diamond operator.* *Computability* 10(2), pp. 107–110.