ITERATIVE USE OF THE CONVEX CHOICE PRINCIPLE

TAKAYUKI KIHARA

Abstract. In this note, we show that an iterative use of the one-dimensional convex choice principle is not reducible to any higher dimensional convex choice. This solves an open problem raised at a recent Dagstuhl meeting on Weihrauch reducibility.

1. Convex Choice

Let XC_k denote the k-dimensional convex choice. It is known that XC_1 is Weihrauch equivalent to the intermediate value theorem. In [1], it was asked whether $\mathsf{XC}_1 \star \mathsf{XC}_1 \leq_W \mathsf{XC}_1^k$ for some $k \in \omega$. It is easy to see that $\mathsf{AoUC} \leq_W \mathsf{XC}_1$ and $\mathsf{XC}_1^k \leq_W \mathsf{XC}_k$.

Theorem 1. *$\mathsf{XC}_1 \star \mathsf{AoUC} \not\leq_W \mathsf{XC}_k$ for all $k \in \mathbb{N}$.*

Proof. We will need some geometric arguments involving convexity, measure and dimension. If a convex set $X \subseteq [0,1]^k$ is at most d-dimensional, then X is included in a d-dimensional hyperplane $L \subseteq [0,1]^k$ by convexity. It is easy to define the d-dimensional Lebesgue measure λ^d on L which is consistent with the d-dimensional volume on d-parallelotopes in $[0,1]^k$.

Let $(X[s])_{s\in\omega}$ be an upper approximation of a convex closed set $X \subseteq [0,1]^k$. Even if we know that X is at most d-dimensional for some $d < k$, it is still possible that $X[s]$ can always be at least k-dimensional for all $s \in \omega$. Fortunately, however, by compactness one can ensure that for such X, say $X \subseteq L$ for some d-hyperplane L by convexity, $X[s]$ for sufficiently large s is eventually covered by a *thin* k-parallelotope $\widehat{L}$ obtained by expanding d-hyperplane L. For instance, if $X \subseteq [0,1]^3$ is included in the plane $L = \{1/2\} \times [0,1]^2$, then for all $t \in \omega$, there is $s \in \omega$ such that $X[s] \subseteq \widehat{L}(2^{-t}) := [1/2 - 2^{-t}, 1/2 + 2^{-t}] \times [0,1]^2$ by compactness. We call such $\widehat{L}(2^{-t})$ as the 2^{-t}*-thin expansion* of L.

We give a formal definition of the 2^{-t}-thin expansion of a subset Y of a hyperplane L. A d-hyperplane $L \subseteq [0,1]^k$ is named by a d-many linearly independent points in $[0,1]^k$. If $(x_i)_{i<d}$ is linearly independent, then this fact is witnessed at some finite stage. Therefore, there is a computable enumeration $(L^d_e)_{e\in\omega}$ of all rational d-hyperplanes. A rational closed subset of L is the complement of the union of finitely many rational open balls in L. Given a pair (L, Y) of (an index of) a rational d-hyperplane $L \subseteq [0,1]^k$ and a rational closed set $Y \subseteq L$, we define the 2^{-t}*-thin expansion of* Y *on* L as follows: We calculate an orthonormal basis $(\mathbf{e}_1, \dots, \mathbf{e}_{k-d})$ of the orthogonal complement of the vector space spanned by $L - v$ where $v \in L$, and define

$$\widehat{Y}(2^{-t}) = \left\{ v + \sum_{i=1}^{k-d} a_i \mathbf{e}_i : v \in Y \text{ and } -2^{-t} \leq a_i \leq 2^{-t} \text{ for any } i \leq k-d \right\},$$

Since Y is a rational closed set, we can compute the measure $\lambda^d(Y)$. Indeed, we can compute the maximum value $m^d(Y,t)$ of $\lambda^d(\widehat{Y}(2^{-t})\cap L')$ where L' ranges over all d-dimensional hyperplanes. For instance, if $Y=[0,s]\times\{y\}$, it is easy to see that $m^1(Y,2^{-t})$ is the length $\sqrt{s^2+2^{-2t+2}}$ of the diagonal of the rectangle $\widehat{Y}(2^{-t})=[0,s]\times[y-2^{-t},y+2^{-t}]$.

Let us assume that we know that a convex set $X\subseteq[0,1]^k$ is at most d-dimensional, and moreover, a co-c.e. closed subset $\tilde{X}$ of X satisfies that $\lambda^d(\tilde{X})<r$. What we will need is to find, given $\varepsilon>0$, stage s witnessing that $\lambda^d(\tilde{X})<r+\varepsilon$ under this assumption. Of course, generally, there is no stage s such that $\lambda^d(\tilde{X}[s])<r+\varepsilon$ holds since $\tilde{X}[s]$ may not even be d-dimensional for all $s\in\omega$ as discussed above. To overcome this obstacle, we again consider a thin expansion of (a subset of) a hyperplane. Indeed, given $t>0$, there must be a rational closed subset Y of a d-dimensional rational hyperplane L such that Y is very close to $\tilde{X}$, and that $\tilde{X}$ is covered by the 2^{-t}-thin expansion $\widehat{Y}(2^{-t})$ of Y. That is, by compactness, it is not hard to see that given $\varepsilon>0$, one can effectively find s,Y,t such that

$$\tilde{X}[s]\subseteq\widehat{Y}(2^{-t})\text{ and }m^d(Y,t)<r+\varepsilon.$$

In this way, if the inequality $\lambda^d(\tilde{X})<r$ holds for a co-c.e. closed subset $\tilde{X}$ of a d-dimensional convex set X, then one can effectively confirm this fact.

We next consider some nice property of an admissible representation of $[0,1]^k$. It is well-known that $[0,1]^k$ has an admissible representation δ with an effectively compact domain $[T_\delta]$ such that δ is an effectively open map (see Weihrauch). In particular, $\delta^{-1}[P]$ is compact for any compact subset $P\subseteq[0,1]^k$. Additionally, we can choose δ so that, given a finite subtree $V\subseteq T_\delta$, one can effectively find an index of the co-c.e. closed set $\mathrm{cl}(\delta[V])$, and moreover, $\lambda^d(L\cap\mathrm{cl}(\delta[V])\setminus\delta[V])=0$ for any d-dimensional hyperplane $L\subseteq[0,1]^k$ for $d\leq k$. For instance, consider a sequence of 2^{-n}-covers $(\mathcal{C}^n_k)_{k<b(n)}$ of $[0,1]^k$ consisting of rational open balls, and then each b-bounded string σ codes a sequence of open balls $(\mathcal{C}^n_{\sigma(n)})_{n<|\sigma|}$. Then we may define T_δ as the tree consisting of all b-bounded sequences that code strictly shrinking sequences of open balls, and $\delta(p)$ as a unique point in the intersection of the sequence coded by $p\in[T_\delta]$. It is not hard to verify that δ has the above mentioned properties. Hereafter, we fix such a representation $\delta:[T_\delta]\to[0,1]^k$.

Now we are ready to prove the assertion. Let $\mathrm{ITV}_{[0,1]}$ denote the subspace of $\mathcal{A}([0,1])$ consisting of nonempty closed intervals in $[0,1]$. Consider the following two partial multi-valued functions:

$$\begin{aligned}
&Z_0:=\mathsf{AoUC}\times\mathrm{id}:\mathrm{dom}(\mathsf{AoUC})\times C(2^{\mathbb{N}},\mathrm{ITV}_{[0,1]})\rightrightarrows 2^{\mathbb{N}}\times C(2^{\mathbb{N}},\mathrm{ITV}_{[0,1]}),\\
&Z_0(T,J)=\mathsf{AoUC}(T)\times\{J\},\\
&Z_1:=(\mathrm{id}\circ\pi_0,\mathsf{XC}_1\circ\mathrm{eval}):2^{\mathbb{N}}\times C(2^{\mathbb{N}},\mathrm{ITV}_{[0,1]})\rightrightarrows 2^{\mathbb{N}}\times[0,1],\\
&Z_1(x,J)=\{x\}\times\mathsf{XC}_1(J(x)).
\end{aligned}$$

Clearly, $Z_0\leq_W\mathsf{AoUC}$ and $Z_1\leq_W\mathsf{XC}_1$. We will show that $Z_1\circ Z_0\not\leq_{\mathrm{W}}\mathsf{XC}_k$.

Let $\{(P_e,\varphi_e,\psi_e)\}_{e\in\mathbb{N}}$ be an effective enumeration of all co-c.e. closed subsets of $[0,1]^k$, partial computable functions $\varphi_e:\subseteq\mathbb{N}^{\mathbb{N}}\to2^{\mathbb{N}}$ and $\psi_e:\subseteq\mathbb{N}^{\mathbb{N}}\to[0,1]$. Intuitively, (P_e,φ_e,ψ_e) is a triple constructed by the opponent Opp, who tries to show $Z_1\circ Z_0\leq_W\mathsf{XC}_k$ for some k. The game proceeds as follows: We first give an instance (T_r,J_r) of $Z_1\circ Z_0$. Then, Opp reacts with an instance P_r of XC_k, that is, a convex set $P_r\subseteq[0,1]^k$, and ensure that whenever z is a name of a solution of P_r,

$\varphi_r(z) = x$ is a path through T_r and $\psi_r(z)$ chooses an element of the interval $J_r(x)$, where Opp can use information on (names of) T_r and J_r to construct φ_r and ψ_r. Our purpose is to prevent Opp's strategy.

Hereafter, $P_e[s]$ denotes the stage s upper approximation of P_e. We identify a computable function φ_e (ψ_e, resp.) with a c.e. collection Φ_e of pairs (σ, τ) of strings $\sigma \in \mathbb{N}^{<\mathbb{N}}$ and $\tau \in 2^{<\mathbb{N}}$ (Ψ_e of pairs (σ, D) of strings $\sigma \in \mathbb{N}^{<\mathbb{N}}$ and rational open intervals in $[0,1]$, resp.) indicating that $\varphi_e(x) \succ \tau$ for all $x \succ \sigma$ ($\psi_e(x) \in D$ for all $x \succ \sigma$, resp.) We use the following notations:

$$\Phi_e^q[s] = \{(\sigma, \tau) \in \Phi_e[s] : |\tau| \geq q\},$$
$$\Psi_e^t[s] = \{(\sigma, D) \in \Psi_e[s] : \mathrm{diam}(D) < 3^{-t}\}.$$

For a relation $\Theta \subseteq X \times Y$, we write $\mathrm{Dom}\Theta$ for the set $\{x \in X : (\exists y \in Y)\ (x,y) \in \Theta\}$. We also use the following notations:

$$(\Phi_e[s])^{-1}[\rho] = \bigcup\{\sigma \in \mathrm{Dom}\Phi_e^{|\rho|}[s] : \tau \succeq \rho\},$$
$$(\Psi_e[s])_t^{-1}[I] = \bigcup\{\sigma : (\exists D)\ (\sigma, D) \in \Psi_e^t[s],\ \mathrm{diam}(D) < 3^{-t} \text{ and } D \cap I \neq \emptyset\},$$

Given e, we will construct a computable a.o.u. tree T_e and a computable function $J_e : 2^{\mathbb{N}} \to \mathrm{ITV}_{[0,1]}$ in a computable way uniformly in e. These will prevent Opp's strategy, that is, there is a name z of a solution of P_e such that if $\varphi_e(z) = x$ chooses a path through T_e then $\psi_e(z)$ cannot be an element of the interval $J_e(x)$. We will also define $\mathtt{state}(e,q,s) \in \mathbb{N} \cup \{\mathtt{end}\}$. The value $\mathtt{state}(e,q,s) = t(q)$ indicates that at stage s, the q-th substrategy of the e-th strategy executes the action *forcing the measure* $\lambda^{k-q}(\tilde{P}_e)$ *of a nonempty open subset* $\tilde{P}_e$ *of* P_e *to be less than or equal to* $2^{q-t(q)} \cdot \varepsilon_{t(q)}$ with $\varepsilon_{t(q)} := \sum_{j=0}^{t(q)+1} 2^{-j} < 2$. Therefore, if $\mathtt{state}(e,q,s)$ tends to infinity as $s \to \infty$, then the q-th substrategy eventually *forces* P_e *to be at most* $(k-q-1)$*-dimensional* under the assumption that P_e is convex. First define $\mathtt{state}(e,q,0) = 0$, and we declare that the q-th substrategy is sleeping (i.e., not active) at the beginning of stage 0. At stage s, we inductively assume that $T_e \cap 2^{s-1}$ and $\mathtt{state}(e,q,s-1)$ have already been defined, say $\mathtt{state}(e,q,s-1) = t(q)$, and if $\mathtt{state}(e,q,0) \neq \mathtt{end}$ then $T_e \cap 2^{s-1} = 2^{s-1}$.

Our strategy is as follows:

(1) At the beginning of stage s, we start to monitor the first substrategy p which is still asleep. That is, we calculate the least $p < k$ such that $\mathtt{state}(e,p,s-1) = 0$. If there is no such $p \leq k$, then go to (3). Otherwise, go to (2) with such $p \leq k$.

(2) Ask whether $\varphi_e(z)$ already computes a node of length at least $p+1$ for any name z of an element of P_e. In other words, ask whether $\delta^{-1}[P_e] \subseteq [\mathrm{Dom}\Phi_e^{p+1}[s]]$ is witnessed by stage s. By compactness, if this inclusion holds, then it holds at some stage.
   (a) If no, go to substage 0 in the item (4) after setting $\mathtt{state}(e,p,s) = 0$.
   (b) If yes, the substrategy p now starts to act (we declare that the substrategy q is *active* at any stage after s), and go to (3).

(3) For each substrategy q which is active at stage s, ask whether there is some $\tau \in 2^{q+1}$ such that any point in P_e has a name z such that $\varphi_e(z)$ does not extend τ. In other words, ask whether

$$(\exists \tau \in 2^{q+1})\ P_e \subseteq \bigcup\{\delta[\varphi_e^{-1}[\rho]] : \rho \in 2^{q+1} \text{ and } \rho \neq \tau\}$$

is witnessed by stage s. Note that $\delta[\varphi_e^{-1}[\rho]]$ is c.e. open since it is the image of a c.e. open set under an effective open map. Therefore, by compactness, if the above inclusion holds, then it holds at some stage.

(a) If no for all such q, go to substage 0 in the item (4).
(b) If yes with some q and τ, we finish the construction by setting $\mathtt{state}(e,0,s) = \mathtt{end}$ after defining T_e as a tree having a unique infinite path $\tau^\frown 0^\omega$. This construction witnesses that any point of P_e has a name z such that $\varphi_e(z) \notin [T_e]$ and hence, Opp's strategy fails.

(4) Now we describe our action at substage q of stage s. If $q \geq k$ or q is not active at stage s, go to (1) at the next stage $s+1$ after setting $T_e \cap 2^s = 2^s$. Otherwise, go to (5).
(5) Ask whether for any name z of a point of P_e, whenever $\varphi_e(z)$ extends 0^q1, the value of $\psi_e(z)$ is already approximated with precision $3^{-t(q)-2}$. In other words, ask whether

$$\delta^{-1}[P_e] \cap \varphi_e^{-1}[0^q1] \subseteq [\mathrm{Dom}\Psi_e^{t(q)+2}[s]]$$

is witnessed by stage s. Again, by compactness, this is witnessed at some finite stage.

(a) If no, go to substage $q+1$ after setting $\mathtt{state}(e,q,s) = t(q)$.
(b) If yes, go to (6).

Before describing the action (6), we need to prepare several notations. We first note that $\delta^{-1}[P_e] \cap \varphi_e^{-1}[0^q1]$ is compact, and therefore, there is a tree $V^q \subseteq T_\delta$ (where $\mathrm{dom}(\delta) = [T_\delta]$) such that $[V^q] = \delta^{-1}[P_e] \cap \varphi_e^{-1}[0^q1]$. Moreover, since we answered in the affirmative in the item (5), by compactness, there is a sufficiently large height l such that every $\sigma \in V^q$ of length l has an initial segment $\sigma' \preceq \sigma$ such that $(\sigma', D_\sigma) \in \Psi_e^{t(q)+2}$ for some interval $D_\sigma \subseteq [0,1]$ with $\mathrm{diam}(D_\sigma) < 3^{-t(q)-2}$. Given $\sigma \in V^q$ of length l, one can effectively choose such D_σ. We will define pairwise disjoint intervals I_0 and I_1 which are sufficiently separated so that if $\mathrm{diam}(D) < 3^{-t(q)-2}$ then D can only intersects with one of them. Then for every $\sigma \in V^q$ of length l, we define $h_{t(q)}(\sigma) = i$ if $D_\sigma \cap I_i \neq \emptyset$ for some $i < 2$, otherwise put $h_{t(q)}(\sigma) = 2$.

Now we inductively assume that $J_e(0^q1)[s-1]$ is a closed interval of the form $[3^{-t(q)} \cdot k, 3^{-t(q)} \cdot (k+1)]$ for some $k \in \mathbb{N}$. Then, define $I_i = [3^{-t(q)-1} \cdot (3k + 2i), 3^{-t(q)-1} \cdot (3k+2i+1)]$ for each $i < 2$. Note that I_0 and I_1 be pairwise disjoint closed subintervals of $J_e(0^q1)[s-1]$. Moreover, I_0 and I_1 satisfy the above mentioned property since the distance between I_0 and I_1 is $3^{-t(q)-1}$. Therefore, h is well-defined on $V^q \cap \omega^l$.

We consider $\mathrm{cl}(\delta[V^q]) = P_e \cap \mathrm{cl}(\delta[\varphi^{-1}[0^q1]])$. By the property of δ, the set P_e^q is co-c.e. closed, and $\lambda^{k-q}(\mathrm{cl}(\delta[V^q]) \setminus \delta[V^q]) = 0$ whenever P_e is at most $(k-q)$-dimensional. Then, define $Q_i^{t(q)}$ for each $i < 2$ as the set of all points in $\mathrm{cl}(\delta[V^q])$ all of whose names are still possible to have ψ_e-values in I_i. More formally, define $Q_i^{t(q)}$ as follows:

$$V_i^{t(q)} = V^q \cap 2^l \cap \left(h_{t(q)}^{-1}\{1-i\} \cup h_{t(q)}^{-1}\{2\}\right),$$
$$Q_i^{t(q)} = \mathrm{cl}(\delta[V^q]) \setminus \delta[V_i^{t(q)}].$$

Obviously, $V_0^{t(q)} \cup V_1^{t(q)} = V^q \cap 2^l$, and $Q_i^{t(q)}$ is effectively compact since $V_i^{t(q)}$ generates a clopen set for each $i < 2$. Moreover, we have that $\lambda^{k-q}(Q_0^{t(q)} \cap Q_1^{t(q)}) = 0$

whenever P_e is at most $(k-q)$-dimensional since $Q_0^{t(q)} \cap Q_1^{t(q)} \subseteq \mathrm{cl}(\delta[V^q]) \setminus \delta[V^q]$ and $\lambda^{k-q}(\mathrm{cl}(\delta[V^q]) \setminus \delta[V^q]) = 0$. Now, the q-th substrategy believes that we have already forced $\lambda^{k-q}(\delta[V^q]) \leq 2^{q-t(q)+1} \cdot \varepsilon_{t(q)-1}$ and therefore, $\lambda^{k-q}(Q_i^{t(q)}) \leq 2^{q-t(q)} \cdot \varepsilon_{t(q)-1}$ for some $i < 2$ since $\lambda^{k-q}(Q_0^{t(q)} \cap Q_1^{t(q)}) = 0$ as mentioned above. Here recall that we have $1 \leq \varepsilon_{t(q)-1} < \varepsilon_{t(q)} < 2$. Now, we state the action (6):

(6) Ask whether by stage s one can find a witness for the condition that $\lambda^{k-q}(Q_i^{t(q)}) \leq 2^{q-t(q)} \cdot \varepsilon_{t(q)-1}$ for some $i < 2$. That is, ask whether one can find Y, t, i by stage s such that

$$Q_i^{t(q)}[s] \subseteq \widehat{Y}(2^{-t}) \text{ and } m^{k-q}(Y,t) < 2^{q-t(q)} \cdot \varepsilon_{t(q)}.$$

  (a) If no, go to substage $q+1$ after setting $\mathtt{state}(e,q,s) = t(q)$.
  (b) If yes, define $J_e(0^q1)[s] = I_i$, and go to substage $q+1$ after setting $\mathtt{state}(e,q,s) = t(q)+1$.

Eventually, T_e is constructed as an a.o.u. tree, and $J_e(x)$ is an nonempty interval for any x. We now show that if Opp's reaction to our instance (T_e, J_e) is (P_e, φ_e, ψ_e), then Opp loses the game.

Claim. Suppose that P_e is a nonempty convex subset of $[0,1]^k$. Then, there is a realizer G of XC_k such that $(\varphi_e \circ G(\delta^{-1}[P_e]), \psi_e \circ G(\delta^{-1}[P_e])$ is not a solution to $Z_1 \circ Z_0(T_e, J_e)$, that is, $\varphi_e \circ G(\delta^{-1}[P_e]) \notin [T_e]$ or otherwise $\psi_e \circ G(\delta^{-1}[P_e]) \notin J_e \circ \varphi_e \circ G(\delta^{-1}[P_e])$.

Proof. Suppose not, that is, Opp wins the game with (P_e, φ_e, ψ_e) as a witness. We first assume that P_e is at most $(k-q)$-dimensional. By our assumption, φ_e is defined on all points in $\delta^{-1}[P_e]$. By compactness of $\delta^{-1}[P_e]$, there is stage s satisfying (2).

Suppose that there is $\tau \in 2^{q+1}$ such that $P_e[s] \subseteq \bigcup\{\delta[\varphi_e^{-1}[\rho]] : \rho \in 2^{p+1}$ and $\rho \neq \tau\}$. This means that any point $x \in P_e[s]$ has a name $\alpha_x \in \delta^{-1}\{x\}$ such that $\varphi_e(\alpha_x) \not\succ \tau$. Since $P_e \neq \emptyset$, if a realizer G chooses such a name α_x of a point $x \in P_e$, then $\varphi_e \circ G(\delta^{-1}[P_e]) \notin [T_e] = \{\tau^\frown 0^\omega\}$, that is, Opp fails to find a path of T_e, which contradicts our assumption.

Thus, $\delta^{-1}[P_e] \cap \varphi_e^{-1}[0^q1]$ is nonempty. Since this is compact and ψ_e, for any t, there is stage s satisfying (5) with $t(q) = t$. We can always assume that $\lambda^{k-q}(P_e) < 2^{q+1}$ since P_e is an at most $(k-q)$-dimensional convex set, P_e is included in a $(k-q)$-dimensional hyperplane, and the $(k-q)$-dimensional Lebesgue measure of any $(k-q)$-dimensional hyperplane in $[0,1]^k$ is at most $\sqrt{q+1} < 2^{q+1}$. Thus, if $t(q) = 0$ then $\lambda(\delta[V^q]) \leq 2^{q-t(q)+1} \cdot \varepsilon_{t(q)-1}$ holds since $\delta[V^q] \subseteq P_e$ and $\varepsilon_{-1} = 1$.

If $\lambda(\delta[V^q]) \leq 2^{q-t(q)+1} \cdot \varepsilon_{t(q)-1}$, then $\lambda(Q_i^{t(q)}) \leq 2^{q-t(q)} \cdot \varepsilon_{t(q)-1}$ for some $i < 2$ as discussed above. Therefore, by the argument discussed above, at some stage s, one can find a rational closed subset Y of a $(k-q)$-dimensional hyperplane, $t \in \mathbb{N}$, and $i < 2$ satisfying the condition in the item (6). At this stage, the q-th substrategy executes the $t(q)$-th action, that is, this defines $J_e(0^q1)[s] = I_i$. Therefore, if Opp wins, P_e has no intersection with $\delta[V_i^{t(q)}]$. This is because for any $x \in \delta[V_i^{t(q)}]$ has a name $z \in V_i^{t(q)}$, and therefore, $\varphi_e(z)$ extends 0^q1 and $\psi_e(z) \notin I_i = J_e(0^q1)$.

Consequently, we have $\delta[V^q] \subseteq Q_i^{t(q)}$, which forces that $\lambda^{k-q}(\delta[V^q]) \leq 2^{q-t(q)} \cdot \varepsilon_{t(q)} \leq 2^{q-t(q)+1}$. Eventually, we have $\lambda^{k-q}(\delta[V^q]) = 0$ as $t(q)$ tends to infinity. Since $\delta[V^q]$ is a nonempty open subset of the convex set P_e, the condition $\lambda^{k-q}(\delta[V^q]) = 0$ implies that P_e is at most $(k-q-1)$-dimensional. Eventually, this construction forces that P_e is zero-dimensional; hence, by convexity, P_e has only one point. Then, however, it must satisfy $P_e \subseteq \bigcup\{\delta[\varphi_e^{-1}[\rho]] : \rho \in 2^{p+1} \text{ and } \rho \neq \tau\}$ for some τ. Therefore, by compactness, this is witnessed at stage s, and then we answer yes to the question in (3). This witnesses the failure of Opp's strategy as before, which contradicts our assumption. □

Suppose that $Z_0 \circ Z_1 \leq_W \mathsf{XC}_k$ via H and $K = \langle K_0, K_1 \rangle$, that is, given a pair (T, J) of an a.o.u. tree T and a nonempty interval J, for any point x of an at most k-dimensional convex closed set $H(T, J)$, we have $K_0(x, T, J) = p \in [T]$ and $K_1(x, T, J) \in J(p)$. Then there is a computable function $\mathbb{N} \to \mathbb{N}$ such that $P_{f(e)} = H(T_e, J_e)$, $\varphi_{f(e)} = \lambda x.K_0(x, T_e, J_e)$ and $\psi_{f(e)} = \lambda x.K_1(x, T_e, J_e)$. By Kleene's recursion theorem, there is $r \in \mathbb{N}$ such that $(P_{f(r)}, \varphi_{f(r)}, \psi_{f(r)}) = (P_r, \varphi_r, \psi_r)$. However, by the above claim, (T_r, J_r) witnesses that Opp's strategy with (P_r, φ_r, ψ_r) fails, which contradicts our assumption. □

References

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