# UNIFIED CHARACTERIZATIONS OF LOWNESS PROPERTIES VIA KOLMOGOROV COMPLEXITY

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ABSTRACT. Consider a randomness notion $\mathcal{C}$. A uniform test in the sense of $\mathcal{C}$ is a total computable procedure that each oracle $X$ produces a test relative to $X$ in the sense of $\mathcal{C}$. We say that a binary sequence $Y$ is $\mathcal{C}$-random uniformly relative to $X$ if $Y$ passes all uniform $\mathcal{C}$ tests relative to $X$.

Suppose now we have a pair of randomness notions $\mathcal{C}$ and $\mathcal{D}$ where $\mathcal{C} \subseteq D$, for instance Martin-Löf randomness and Schnorr randomness. Several authors have characterized classes of the form $\mathrm{Low}(\mathcal{C}, \mathcal{D})$ which consist of the oracles $X$ that are so feeble that $\mathcal{C} \subseteq \mathcal{D}^X$. Our goal is to do the same when the randomness notion $\mathcal{D}$ is relativized uniformly: denote by $\mathrm{Low}^\star(\mathcal{C}, \mathcal{D})$ the class of oracles $X$ such that every $\mathcal{C}$-random is uniformly $\mathcal{D}$-random relative to $X$.

(1) We show that $X \in \mathrm{Low}^\star(\mathrm{MLR}, \mathrm{SR})$ if and only if $X$ is c.e. tt-traceable if and only if $X$ is anticomplex if and only if $X$ is Martin-Löf packing measure zero with respect to all computable dimension functions.

(2) We also show that $X \in \mathrm{Low}^\star(\mathrm{SR}, \mathrm{WR})$ if and only if $X$ is computably i.o. tt-traceable if and only if $X$ is not totally complex if and only if $X$ is Schnorr Hausdorff measure zero with respect to all computable dimension functions.

## 1. INTRODUCTION

1.1. **Lowness and triviality.** In the theory of algorithmic randomness there are three main approaches to define a randomness notion: typicalness, unpredictability and incompressibility. Some randomness notions have characterizations of these three types. Similarly, there are two different approaches to define a non-randomness notion. One is *computational weakness*, that is, the class of sets that are too computationally weak to derandomize another random set. The other is *strong compressibility*. The main theme of this paper is the correspondence between these two approaches. The first result of this type was the equivalence between lowness for ML-randomness and $K$-triviality.

In computability theory a set $A \in 2^\omega$ is called *low* if the jump $A'$ of $A$ is Turing reducible to the halting problem $\emptyset'$. This means that the set is computationally weak and is not useful as an oracle. This class is an important class and there are many results relating to it. Similarly, in the theory of algorithmic randomness we consider a set that does not have computational power to derandomize a random set. Formally, a set is called *low for ML-randomness* if every ML-random set is ML-random relative to $A$, which was introduced in Zambella [56]. This class is also an important class and there are many results relating to it. For instance, the existence of a noncomputable set that is low for ML-randomness was shown in

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Kučera and Terwijn [36]. Naturally, the class of such sets is closed downward under Turing reducibility.

Another important notion in the theory of algorithmic randomness is $K$-triviality. First recall that the Levin-Schnorr theorem says that, a set $A \in 2^\omega$ is Martin-Löf random if and only if $K(A\upharpoonright n) > n - O(1)$, where $K$ is the prefix-free Kolmogorov complexity. This theorem means that typicalness of a set in the sense of Martin-Löf is equivalent to incompressibility of the initial segments of the set. The notion of $K$-triviality was introduced in Chaitin [10] as the opposite of incompressibility, that is, sets with this property are easy to describe. Formally, a set $A$ is called *$K$-trivial* if $K(A \upharpoonright n) \leq K(n) + O(1)$. Solovay [49] showed that there is a noncomputable $K$-trivial set, that is, the class of $K$-trivial sets is a strict superclass of the class of computable sets. See [56, 9, 15] for the improvement after that.

We further introduce two classes. A set $A$ is called *low for $K$* if $K(\sigma) \leq K^A(\sigma) + O(1)$, which was introduced by An. A. Muchnick (unpublished). Note that this also means computational weakness. A set $A$ is called a *base for* MLR if there is a set $X \geq_T A$ such that $X$ is ML-random relative to $A$, which was introduced by Kučera [35]. The following theorem is a surprising achievement in this field:

**Theorem 1.1** (Nies [41], Hirschfeldt, Nies and Stephan [27]). *The following are equivalent for a set $A$:*

(i) *$A$ is $K$-trivial,*
(ii) *$A$ is low for ML-randomness,*
(iii) *$A$ is low for $K$,*
(iv) *$A$ is a base for* MLR.

We can consider analogical notions for other randomness notions. According to Downey-Griffiths [16], a set $A \in 2^\omega$ is Schnorr random if and only if $K_M(A \upharpoonright n) > n - O(1)$ for every computable measure machine $M$. Here $M$ is said to be *a computable measure machine* if $M$ is prefix-free and its halting probability is a computable real. Also note that $K_M$ is the Kolmogorov complexity with respect to machine $M$. Then, Downey-Griffiths [16] introduced the notion of Schnorr triviality. A set $A$ is called *Schnorr trivial* if for every computable measure machine $M$, there exists a computable measure machine $N$ such that $K_N(A\upharpoonright n) \leq K_M(n)+O(1)$. As a counterpart of Theorem 1.1, Franklin and Stephan [20] showed that a set is Schnorr trivial if and only if it is low for uniform Schnorr randomness (for the definition, see Section 2). This is another equivalence between computational weakness and strong compressibility.

1.2. **Traceability and complexity.** The notion of lowness can be defined for any pair of randomness notions. We give some further notions introduced in the study of variants of the notions above.

Let W2R, MLR, CR, SR and WR be the classes of sets that are weakly 2-random, ML-random, computably random, Schnorr random and Kurtz random, respectively. Let $\mathcal{C}$ and $\mathcal{D}$ be classes of sets given by relativizable definitions. A set $A$ is low for $\mathcal{C}$ versus $\mathcal{D}$ (denoted by $A \in \mathrm{Low}(\mathcal{C}, \mathcal{D})$) if $\mathcal{C} \subseteq \mathcal{D}^A$. This notation was introduced by Kjos-Hanssen, Nies and Stephan [34]. We say that a set $A$ is low for $\mathcal{C}$ to mean $A \in \mathrm{Low}(\mathcal{C}) = \mathrm{Low}(\mathcal{C}, \mathcal{C})$. Note that all these classes are closed downward under Turing reducibility.

One of the most important concepts in the study of the lowness for randomness is traceability. Traceability is another formulation of computational weakness. (See

Definition 3.1 for the definition.) This notion was first introduced, inspired by a notion in set theory, by Terwijn and Zambella [51], who showed that a set $A$ is computably traceable if and only if it is low for Schnorr tests.

We give some characterization of lowness for a pair of randomness notions. As noted previously, $A \in \mathrm{Low}(\mathrm{MLR}, \mathrm{MLR})$ iff $A$ is $K$-trivial [41]. Actually $A \in \mathrm{Low}(\mathrm{MLR}, \mathrm{CR})$ iff $A$ is $K$-trivial [41]. Replacing MLR with W2R, the class has not changed: $A \in \mathrm{Low}(\mathrm{W2R}, \mathrm{MLR})$ iff $A$ is $K$-trivial [14]; $A \in \mathrm{Low}(\mathrm{W2R}, \mathrm{CR})$ iff $A$ is $K$-trivial [42]; $A \in \mathrm{Low}(\mathrm{W2R}, \mathrm{W2R})$ iff $A$ is $K$-trivial [14, 32, 42]. Thus, many lowness classes are characterized as $K$-triviality.

Most classes contain a non-computable set but there is an exception: a set $A \in \mathrm{Low}(\mathrm{CR}, \mathrm{CR})$ iff $A$ is computable [41].

When the second component is SR, the classes are characterized by traceability: $A \in \mathrm{Low}(\mathrm{SR}, \mathrm{SR})$ iff $A$ is computably traceable [51, 34]; $A \in \mathrm{Low}(\mathrm{MLR}, \mathrm{SR})$ iff $A$ is c.e. traceable [34]; $A \in \mathrm{Low}(\mathrm{CR}, \mathrm{SR})$ iff $A$ is computably traceable [34]; $A \in \mathrm{Low}(\mathrm{W2R}, \mathrm{SR})$ iff $A$ is c.e. traceable [2].

When the second component is WR, we have the following: $A \in \mathrm{Low}(\mathrm{WR}, \mathrm{WR})$ iff $A$ is hyperimmune free and not DNC [50, 25]; $A \in \mathrm{Low}(\mathrm{CR}, \mathrm{WR})$ iff $A$ is not high or DNC [25]; $A \in \mathrm{Low}(\mathrm{SR}, \mathrm{WR})$ iff $A$ is not high or DNC [25]; $A \in \mathrm{Low}(\mathrm{MLR}, \mathrm{WR})$ iff $A$ is not DNC [11, 25]. See also Table 3 in Section 7 for the relationship among these lowness notions.

We employ several complexity notions concerning the growth speed of the Kolmogorov complexity to give characterizations of lowness notions. *Complexity* and *autocomplexity* were introduced by Kanovich [29] and the relation with DNC functions and traceability was studied in [31]. Franklin, Greenberg, Stephan and Wu [18] introduced *anticomplexity* and studied the relation with traceability. We will give some characterizations of variants of complexity via traceability in Section 3. Some of them are already in Hölzl and Merkle [28].

1.3. **Uniform relativization.** If we say that a set $A \in 2^\omega$ is *computable relative to* a set $A \in 2^\omega$, then it usually means that $A$ is Turing reducible to $B$, that is, $A \leq_T B$. On the other hand, we can consider many variants such as $A \leq_{tt} B$.

Given a randomness notion $R$ and a set $A \in 2^\omega$, we sometimes define $A$-relativized $R$-randomness notion by allowing the access to $A$ as an oracle. Again we can consider variants of the way of relativization.

Relativized ML-randomness is a natural notion because of van Lambalgen's theorem [52], which says that $A \oplus B$ is ML-random if and only if $A$ is ML-random and $B$ is ML-random relative to $A$. In contrast, van Lambalgen's theorem with the usual relativization does not hold for Schnorr randomness, computable randomness [37, 55], Kurtz randomness [21] or weak 2-randomness [1].

Miyabe [39] and Miyabe and Rute [40] introduced *uniform relativization*. Note that a test relative to a set $X$ can be identified with an oracle procedure that produces a test relative to $X$. Here the procedure may not produces a test for a set $Y \neq X$. A uniform test requires the procedure to produce a test relative to $X$ for each oracle $X$. This is another type of relativization and is called uniform relativization. See Section 2.1 for the concrete definition. They showed that van Lambalgen's theorem holds for uniform Schnorr randomness and uniformly computable randomness (in a weaker sense). Furthermore, Kihara and Miyabe [30] showed that van Lambalgen's theorem holds for uniform Kurtz randomness in another weaker sense.

Thus, uniformly relativized randomness notions are more natural notions in some cases.

Uniformly relativized randomness notion is a weaker notion than a randomness notion with the usual relativization. Namely, if $A$ is Schnorr random relative to $B$, then $A$ is Schnorr random uniformly relative to $B$. The converse does not hold (for instance, choose a high Schnorr trivial set $B$ [23], and a Schnorr random set $A \leq_T B$ [42]). Of course, if $B$ is computable, then $A$ is $R$-random if and only if $A$ is $R$-random relative to $B$ if and only if $A$ is $R$-random uniformly relative to $B$.

The usual way to define a relativized randomness notion is the following: we first define tests (or complexity, martingales) and randomness notions, and relativize them after that. When we define uniformly relativized randomness notions, it is more appropriate to define them directly because we need to talk about how to relativize them explicitly. Thus, it is more appropriate to think that a randomness notion is always equipped with its relativization. For instance, uniform Schnorr randomness means that Schnorr randomness with uniform relativization.

The importance of the way of relativization goes for the study of lowness. Franklin and Stephan [20] have already studied the truth-table version of relativization of Schnorr randomness, which is equivalent to uniform Schnorr randomness, and have shown that Schnorr triviality is equivalent to lowness for uniform Schnorr randomness. This is another evidence that uniform relativization is a natural notion.

With this background, we will consider the uniformly relativized versions of lowness and will characterize them via variants of traceability and complexity, namely, truth-table traceability and total complexity.

1.4. **Overview of the paper.** In Section 3, we characterize lowness notions relating to ML-randomness, Schnorr randomness and Kurtz randomness via traceability. In Section 4, we prove the last equivalence between lowness and traceability by applying the characterization of strong measure zero in the set theory of the real line. In Section 5, we see the relationship between traceability and variants of complexity such as complex sets and autocomplex sets. The results in Sections 3 and 5 will be summarized in Table 2. In Section 6, we study the relationship between variants of complexity and dimension-theoretic notions such as Hausdorff dimension and packing dimension. In Section 7, we summarize our results and give all implications among twelve lowness properties. Our main results concerning the relationship between lowness for randomness notions and Kolmogorov complexity are illustrated in Tables 2 and 3.

## 2. Preliminaries

We refer to [44, 45, 48] for background in computability theory and to the books [13, 42] for the one in algorithmic randomness. *Cantor space* $2^\omega$ is the set of infinite binary sequences equipped with the canonical product topology. A basic open set on $2^\omega$ is a cylinder $[\sigma] = \{X \in \omega \ : \ \sigma \prec X\}$ for a finite binary string $\sigma \in 2^{<\omega}$. The open set generated by a set $S \subseteq 2^{<\omega}$ of strings is denoted by $[\![S]\!]$, i.e., $[\![S]\!] = \bigcup_{\sigma \in S}[\sigma]$. We fix a computable enumeration $\{B_n\}_{n\in\omega}$ of all basic open sets. An open set $U$ is *c.e.* if $U = \bigcup_{n\in\omega} B_{p(n)}$ where $p : \omega \to \omega$ is a computable function, or equivalently, $U = [\![S]\!]$ for some c.e. set $S \subseteq 2^{<\omega}$ of strings. Let $\varepsilon$ denote the empty string. A real $x \in \mathbb{R}$ is *computable* if there is a computable sequence $\{q_n\}$ of rationals such that $|x - q_n| < 2^{-n}$ for each $n$.

Note that the class of clopen sets is the class of finite unions of cylinders. Then, we fix a computable enumeration $\{C_n\}_{n\in\omega}$ of all clopen sets.

2.1. **Randomness notions and their relativization.** An *ML-test* is a sequence $\{U_n\}$ of uniformly c.e. open sets such that $\mu(U_n) \leq 2^{-n}$ for each $n$. A set $X \in 2^\omega$ is *ML-random* if $X \notin \bigcap_n U_n$ for every ML-test. A *Schnorr test* is an ML-test $\{U_n\}$ such that $\mu(U_n)$ is uniformly computable. A set $X$ is *Schnorr random* if $X \notin \bigcap_n U_n$ for each Schnorr test. A *Kurtz null test* is a ML test $\{U_n\}$ for which there is a computable function $f : \omega \to (2^{<\omega})^{<\omega}$ such that $U_n = [\![f(n)]\!]$ for all $n$. A set $X$ is *Kurtz random* if $X \notin \bigcap_n U_n$ for each Kurtz null test $\{U_n\}$. We use MLR, SR and WR to mean the sets of ML-random, Schnorr random and Kurtz random reals. We have the following proper inclusions:

$$\mathrm{MLR} \subsetneq \mathrm{SR} \subsetneq \mathrm{WR}.$$

We give two kinds of relativization of these randomness notions. There is a way of relativization that has been frequently used in the literature. *Uniform relativization* is a different way of relativization. Miyabe and Rute [40] defined uniform Schnorr randomness and uniformly computable randomness and Kihara and Miyabe [30] defined uniform Kurtz randomness. Here, we give a unified setting of these relativized randomness notions.

Before introducing the notion of uniform relativization, recall that every Borel set (in particular, every $G_\delta$ null set) can be identified with a real, so-called a *Borel code*. In this paper, we use a coding of tests rather than the standard Borel coding of null sets. Here, a *test* is a sequence $\{U_n\}$ of open sets such that $\lim_n \mu(U_n) = 0$. Let $\mathcal{T}$ be the class of all tests. We define three kinds of representations (i.e., codings) of subsets of $\mathcal{T}$.

The *Martin-Löf representation* (or ML-representation) is the partial function $\rho_{\mathrm{MLR}} :\subseteq \omega^\omega \to \mathcal{T}$ such that

$$\rho_{\mathrm{MLR}}(p) = \{U_n\}_{n\in\omega} \text{ where } U_n = \bigcup_m B_{p(n,m)} \text{ and } \mu(U_n) \leq 2^{-n}.$$

The *Schnorr representation* is the partial function $\rho_{\mathrm{SR}} :\subseteq \omega^\omega \to \mathcal{T}$ such that

$$\rho_{\mathrm{SR}}(p) = \{U_n\}_{n\in\omega} \text{ where}$$
$$U_n = \bigcup_m B_{p(0,n,m)},\ \mu(U_n) \leq 2^{-n} \text{ and } |\mu(U_n) - q_{p(1,n,m)}| < 2^{-m},$$

where $q_k$ is the $k$-th rational number. The *Kurtz representation* is the partial function $\rho_{\mathrm{WR}} :\subseteq \omega^\omega \to \mathcal{T}$ such that

$$\rho_{\mathrm{WR}}(p) = \{U_n\}_{n\in\omega} \text{ where } U_n = C_{p(n,m)} \text{ and } \mu(U_n) \leq 2^{-n}.$$

Note that these representations are not surjective. Clearly, for each representation $\rho \in \{\rho_{\mathrm{MLR}}, \rho_{\mathrm{SR}}, \rho_{\mathrm{WR}}\}$, the set $\bigcap \rho(p)$ is null for every $p \in \mathrm{dom}(\rho)$. Our representations of tests are essentially same as the standard Borel coding of $G_\delta$ sets, except we require that each test decoded by a code should rapidly converges to a null set. On a general theory of representations in computable analysis, see also [53, 8, 7, 54].

**Example 2.1.** Assume that $p \in \omega^\omega$ is computable in a set $A$. If $p \in \mathrm{dom}(\rho_{\mathrm{MLR}})$, then $\rho_{\mathrm{MLR}}(p)$ is usually called a Martin-Löf test relative to $A$.

Let $\mathrm{R} \in \{\mathrm{MLR}, \mathrm{SR}, \mathrm{WR}\}$. An R-*test relative to* $A \in 2^\omega$ is $\rho_{\mathrm{R}}(f(A))$ for a (partial) computable function $f :\subseteq 2^\omega \to \omega^\omega$. A *uniform* R-*test* is a function

$X \mapsto \rho_{\rm R}(f(X))$ for a total computable function $f : 2^\omega \to \omega^\omega$. We say that a set $B$ is R-*random relative to* $A$ (denoted by $B \in {\rm R}(A)$) if $B \notin \bigcap \rho_{\rm R}(f(A))$ for every partial computable function $f :\subseteq 2^\omega \to \omega^\omega$. We say that a set $B$ is R-*random uniformly relative to* $A$ (denoted by $B \in {\rm R}^\star(A)$) if $B \notin \bigcap \rho_{\rm R}(f(A))$ for every total computable function $f : 2^\omega \to \omega^\omega$. It is not hard to check that these definitions are equivalent to the definitions in the literature. As usual, SR and SR$^\star$ do not change when we replace the definition of $\rho_{\rm SR}$ with

$$\rho_{\rm SR}(p) = \{U_n\}_{n\in\omega} \text{ where } U_n = \bigcup_m B_{p(n,m)} \text{ and } \mu(U_n) = 2^{-n},$$

on Cantor space with the uniform measure.

As already seen in the introduction, we write $A \in {\rm Low}(\mathcal{C}, \mathcal{D})$ if $\mathcal{C} \subseteq \mathcal{D}(A)$. Similarly, we write $A \in {\rm Low}^\star(\mathcal{C}, \mathcal{D})$ if $\mathcal{C} \subseteq \mathcal{D}^\star(A)$. Since the inclusion $\mathcal{D}(A) \subseteq \mathcal{D}^\star(A)$ generally holds for randomness notions $\mathcal{D}$, we always have

$${\rm Low}(\mathcal{C}, \mathcal{D}) \subseteq {\rm Low}^\star(\mathcal{C}, \mathcal{D}).$$

A survey article on the lowness property for randomness notions is to be found in Franklin [19].

## 3. Lowness and Traceability

Many known lowness notions are characterized by using the notion of traceability. The transition from lowness to traceability provides an important change in perspective, that is, it is a transition from the ($\Pi^1_1$) property for others to the (arithmetical) property for a set itself. In this section we characterize lowness notions via traceability in a unified form.

**Definition 3.1.** A *trace* is a sequence $(T_n)_{n\in\omega}$ of finite sets. A function $f$ is *traced* by $(T_n)_{n\in\omega}$ if $f(n) \in T_n$ for every $n$. A function $f$ is *infinitely often* (abbreviated as i.o.) traced by $(T_n)_{n\in\omega}$ if $f(n) \in T_n$ for infinitely often $n$. We say that $f$ is *computably often* (abbreviated as c.o.) traced by $(T_n)_{n\in\omega}$ if $f(n) \in T_n$ for computably often $n$, that is, there is a computable order $l$ such that for every $k$, $f(n) \in T_n$ for some $n \in [l(k), l(k+1))$. If a trace $(T_n)_{n\in\omega}$ is called *c.e.* if it is uniformly c.e., and it is called *computable* if the canonical index of $T_n$ is computable uniformly in $n$. A trace $(T_n)_\omega$ is *bounded* by an order $p$ if $\#T_n \leq p(n)$ for every $n$. Such a $p$ is called a *bound* for $(T_n)_\omega$.

A set $A$ is *c.e. traceable* (*computably traceable*) if there is a computable order $p$ such that every $f \leq_T A$ is traced by a c.e. (resp. computable) trace with bound $p$. If the condition $f \leq_T$ is replaced with $f \leq_{tt} A$, then we say that $A$ is *c.e. tt-traceable* and *computably tt-traceable*, respectively. We can also introduce *c.e. i.o. traceability*, *computably c.o. tt-traceability* and any other combination in a straightforward manner.

**Remark 3.2.** In Kihara-Miyabe [30], computably c.o. *tt*-traceability is called Kurtz *tt*-traceablility.

First recall that $A \in {\rm Low}({\rm SR}, {\rm SR})$ iff $A$ is computably traceable by Terwijn-Zambella [51, 34]. Its uniform-relativization version is that $A \in {\rm Low}^\star({\rm SR}, {\rm SR})$ iff $A$ is computably *tt*-traceable, which is shown by Franklin-Stephan [20].

3.1. **Characterizing** $\mathrm{Low}^\star(\mathrm{MLR},\mathrm{SR})$**.** A set $A \in \mathrm{Low}(\mathrm{MLR},\mathrm{SR})$ iff $A$ is c.e. traceable by Kjos-Hanssen-Nies-Stephan [34]. Here we give the uniform-relativization version of this result.

**Theorem 3.3.** *A set* $A \in \mathrm{Low}^\star(\mathrm{MLR},\mathrm{SR})$ *iff* $A$ *is c.e. tt-traceable.*

We show this by giving a series of lemmas following the argument in [2, 38].

For a set $W \subseteq 2^{<\omega}$, we denote by $W^\omega$ the set of all sets of the form $\sigma_0\sigma_1\sigma_2\dots$ such that $\sigma_i \in W$ for every $i \in \omega$. The following lemma is obtained as a special case of the combination of Lemma 12 and Proposition 13 in Bienvenu-Miller [2] with the class $\mathcal{C}$ of all bounded c.e. open sets and the family $\{\mathcal{T}^{(e)}\}$ of all Martin-Löf tests.

**Lemma 3.4** (A rephrase of [2, Lemma 12 and Proposition 13])**.** *Let* $W$ *be a prefix-free subset of* $2^{<\omega}$*. Suppose that* $[W]$ *cannot be covered by any bounded c.e. open set. Then,* $W^\omega$ *contains a Martin-Löf random set.*

Let $\mathcal{O}$ be the class of open sets on $2^\omega$. An open set $U \subseteq 2^\omega$ is *bounded* if $\mu(U) < 1$. We say that a computable function $g : 2^\omega \to \mathcal{O}$ is a *uniformly Schnorr function* if the function $X \mapsto \mu(g(X))$ is computable. A computable function $g : 2^\omega \to \mathcal{O}$ is *strictly bounded* if $\sup_{X\in 2^\omega} \mu(g(X))$ is strictly less than 1.

**Lemma 3.5.** *Suppose that* $A \in \mathrm{Low}^\star(\mathrm{MLR},\mathrm{SR})$*. Then,* $A$ *satisfies the following property* (I)*:*

- (I) $f(A)$ *is covered by a bounded c.e. open set for every strictly bounded uniformly Schnorr function* $f$*.*

*Proof.* We show the contrapositive. Let $f$ be a strictly bounded and uniformly Schnorr function such that $f(A)$ is not covered by any bounded c.e. open set. Let $g$ be a c.e. function from $2^\omega$ to a subset of $2^{<\omega}$ such that $g(X)$ is prefix-free and $[\![g(X)]\!] = f(X)$ for each $X$. By Lemma 3.4, there exists a ML-random set $Z \in (g(A))^\omega$, which is not Schnorr random uniformly relative to $A$. $\square$

A function $g : \omega \to \mathbb{R}^+$ is called *summable* if $\sum_n g(n) < \infty$. Consider the following property (II) for a set $A$:

- (II) For every total computable function $f : 2^\omega \times \omega \to \mathbb{R}^+$ such that $X \mapsto \sum_n f(X,n)$ is finite and computable, there exists a left-c.e. summable function $g : \omega \to \mathbb{R}^+$ such that $f(A,n) \le g(n)$ for all $n$.

**Lemma 3.6.** *Suppose that* $A$ *satisfies the property (I) in Lemma 3.5. Then,* $A$ *also satisfies the property (II).*

The following proof is almost identical to that of Proposition 25 in [2].

*Proof.* Let $f : 2^\omega \times \omega \to \mathbb{R}^+$ such that $X \mapsto \sum_n f(X,n)$ is computable. We can assume that $f(n) \le 1$ for all $n$.

Let $B_{n,\alpha} = \{X \in [0,1]^\omega \ :\ X_n \in [0,\alpha)\}$. Consider the function $U : 2^\omega \to \mathcal{O}$ defined by $U(X) = \bigcup_n B_{n,f(X,n)}$. Then, $U$ is a strictly bounded and uniformly Schnorr function. By the assumption of (c), $U(A)$ is covered by some bounded c.e. open set $V$. Let $g(n) = \sup\{\alpha \in [0,1] \ :\ B_{n,\alpha} \subseteq V\}$. Then, $f(A,n) \le g(n)$ for all $n$, and $g$ is left-c.e. The sum $\sum_n g(n)$ is bounded because $1 - \mu(\bigcup_n B_{n,g(n)}) \ge 1 - \mu(V) > 0$ and $\prod_n (1 - g(n)) > 0$. $\square$

**Lemma 3.7.** *Suppose that* $A$ *satisfies the property (II). Then,* $A$ *satisfies the following property* (III)*:*

(III) *for each uniformly computable measure machine $M$, we have $K(\sigma) \leq K_{M^A}(\sigma) + O(1)$.*

The following proof is almost identical to that of Proposition 28 in [2]

*Proof.* Let $M$ be a uniformly computable measure machine. Let $f : 2^\omega \times 2^{<\omega} \to \mathbb{R}^+$ be the function defined by $f(X, \sigma) = 2^{-K_{M^X}(\sigma)}$. Then $f$ is a left-c.e. funtion and $X \mapsto \sum_n f(X, n)$ is computable. By the assumption of (d), there is a left-c.e. summable function $g : 2^{<\omega} \to \mathbb{R}^+$ such that $f(A, n) \leq g(n)$ for all $n$.

Let $c \in \omega$ be a constant such that $\sum_\sigma g(\sigma) \leq 2^c$. Then, $L = \{(k, \sigma) : g(\sigma) \geq 2^{-k+c+1}\}$ is a KC-set. By the KC-theorem, $K \leq -\log g + c + 1 \leq -\log f + c + 1 \leq K_{M^A} + c + 1$. □

**Lemma 3.8.** *Suppose that $A$ satisfies the property (III) in Lemma 3.7. Then, $A$ is c.e. tt-traceable.*

*Proof.* Let $\Phi : 2^\omega \times \omega \to 2^{<\omega}$. We define an oracle machine $M$ by $M^X(0^n 1) = \Phi^A(n)$. Then $M$ is a uniformly computable measure machine. By the assumption (e), $K(\sigma) \leq K_{M^A}(\sigma) + c$. Let $T_n = \{\sigma : K(\sigma) \leq n + c + 1\}$. Then, $\{T_n\}$ is a c.e. trace with $|T_n| \leq 2^{n+c+1}$. Note that $K(\Phi^A(n)) \leq K_{M^A}(\Phi^A(n)) + c \leq n + c + 1$. Hence, $\Phi^A(n) \in T_n$ for all $n$. Hence, $A$ is c.e. tt-traceable. □

**Lemma 3.9.** *Let $A$ be a c.e. tt-traceable set. Then, for each uniform Schnorr test $f$, $\bigcap_n f(A, n)$ is covered by a ML-test. Thus, $A \in \mathrm{Low}^\star(\mathrm{MLR}, \mathrm{SR})$.*

*Proof of (f)⇒(b).* Let $f$ be a uniform Schnorr test. Then, there is a tt-reduction $\Phi : 2^\omega \times \omega \to (2^{<\omega})^{<\omega}$ such that $\bigcup_m [\![\Phi(X, \langle n, m\rangle)]\!] = f(X, n)$ and $\mu([\![\Phi(X, \langle n, m\rangle)]\!]) \leq 2^{-n-m}$. Since $A$ is c.e. tt-traceable, there is a c.e. trace $\{T_k\}_{k\in\omega}$ such that $|T_{\langle n,m\rangle}| \leq m$ and $\Phi(X, \langle n, m\rangle) \in T_{\langle n,m\rangle}$ for each $n, m \in \omega$. We can assume that, if $W \in T_{\langle n,m\rangle}$, then $W \in (2^{<\omega})^{<\omega}$, and $\mu([\![W]\!]) \leq 2^{-n-m}$. Let

$$U_n = \bigcup_m \bigcup_{W \in T_{\langle n+c,m\rangle}} [\![W]\!]$$

where $c$ will be specified soon. Then, $\{U_n\}$ is a sequence of uniformly c.e. open sets. Furthermore,

$$\mu(U_n) \leq \bigcup_m \sum_{W \in T_{\langle n,m\rangle}} [\![W]\!] \leq \sum_m m 2^{-n-c-m} \leq 2^{-n}$$

for all $n$ for a sufficiently large $c$. Then, $\{U_n\}$ is a ML-test. Finally, note that, if $Z \in \bigcap_n f(A, n)$, then $Z \in \bigcap_n U_n$. □

3.2. **Characterizing** $\mathrm{Low}(\mathrm{MLR}, \mathrm{WR})$ **and** $\mathrm{Low}^\star(\mathrm{MLR}, \mathrm{WR})$**.** Note that the class $\mathrm{Low}(\mathrm{MLR}, \mathrm{WR})$ has already characterized as non-DNC by Greenberg-Miller [25]. Here we give a characterization via traceability.

**Theorem 3.10.** *A set $A \in \mathrm{Low}(\mathrm{MLR}, \mathrm{WR})$ iff $A$ is c.e. i.o. traceable.*

*Proof.* By Greenberg-Miller [25], $\mathrm{Low}(\mathrm{MLR}, \mathrm{WR})$ is characterized as non-DNC. Hölzl-Merkle [28, Theorem 18] showed that non-DNC is equivalent to c.e. i.o. traceability. □

Its uniform-relativization version is as follows.

**Theorem 3.11.** *A set $A \in \mathrm{Low}^\star(\mathrm{MLR}, \mathrm{WR})$ iff $A$ is c.e. i.o. tt-traceable.*

Next we characterize $\mathrm{Low}^\star(\mathrm{MLR}, \mathrm{WR})$ by modifying [13, Lemma 8.10.2].

**Lemma 3.12.** *If $A \in \mathrm{Low}^\star(\mathrm{MLR}, \mathrm{WR})$, then for every strictly increasing $f \leq_{tt} A$, the range of $f$ is not a subset of a Martin-Löf random set.*

*Proof.* The proof is a straightforward modification of that of (iii)⇒(iv) of Theorem 6.1 in Kihara-Miyabe [30]. For every strictly increasing $f \leq_{tt} A$, there is a total computable function $\Psi$ such that $\Psi^A = f$. Without loss of generality, we may safely assume that $\Psi^Z$ is strictly increasing for every $Z \in 2^\omega$. Let $E^Z$ be the set of all supersets of the range of $\Psi^Z$. Then, $Z \mapsto E^Z$ forms a uniform Kurtz test. Since $A$ is low for Martin-Löf randomness versus uniform Kurtz randomness, $E^A$ does not contain a Martin-Löf random element. In other words, the range of $f = \Psi^A$ is not a subset of a Martin-Löf random set. □

**Lemma 3.13.** *Suppose that, for every strictly increasing $f \leq_{tt} A$, the range of $f$ is not a subset of a Martin-Löf random set. Then, $A$ is c.e. i.o. tt-traceable.*

*Proof.* Let $R_1$ be a nonempty $\Pi^0_1$ class consisting of ML random reals. Let $\{P_e\}$ be a computable enumeration of all $\Pi^0_1$ classes. Then there is a $\Pi^0_1$ class $R \subseteq R_1$ of positive measure and a constant $c \in \omega$ such that $R \cap P_e \neq \emptyset$ implies $\mu(R \cap P_e) \geq 2^{-e-c}$ (see [13, Lemma 8.10.1]). Then, for each finite set $D \subset \omega$, by $R^D$ we denote the set of all supersets of $D$ contained in $R$, i.e.,

$$R^D = \{X \in R : D \subseteq X\}.$$

Choose a computable function $e$ such that $e(D)$ is an index of $R^D \subseteq R$, i.e., $P_{e(D)} = R^D$. Then, we have $\mu(R^D) \geq 2^{-e(D)-c}$ if $R^D$ is nonempty. Consider the following set $W^D \subseteq \omega$ for any finite set $D \subset \omega$.

$$W^D = \{n \in \omega : n \leq \max D\} \cup \{n \in \omega : R^{D \cup \{n\}} = \emptyset\}.$$

By compactness, $W^D$ is c.e., uniformly in $D$. If $X \in R^D$, then $X(n) = 0$ for any $n \in W^D \setminus \{n : n \leq \max D\}$. Hence,

$$2^{-e(D)-c} \leq \mu(R^D) \leq 2^{-|W^D| + \max D + 1}.$$

Therefore, $|W^D|$ is bounded by the order $h(D) = e(D) + c + \max D + 1$.

Assume that $A$ is not c.e. i.o. *tt*-traceable. Then there is a total computable function $\Gamma$ such that $\Gamma^A(D) \notin W^D$ for all finite sets $D \subseteq \omega$, where we identify each finite set $D$ with its canonical index. Note that $\Gamma^A(D) > \max D$ for all finite sets $D \subseteq \omega$. Then, inductively define the total computable function $\Theta$ as follows.

$$\Theta^Z(n) = \Gamma^Z(\{\Theta^Z(m) : m < n\}).$$

Clearly, $\Theta^A$ is strictly increasing, and by induction, we can easily see that $R^{\mathrm{rng}(\Theta^A)}$ is nonempty, that is, the range of $\Theta^A$ is infinite and has a superset which is contained in $R$. In other words, the range of $f = \Theta^A \leq_{tt} A$ is an infinite subset of a Martin-Löf random set. □

We say that a total function $f : \omega \to \omega$ is *infinitely often (i.o.) equal to* a partial function $g : \omega \to \omega$ if $f(n) = g(n)$ for infinitely many $n \in \mathrm{dom}(g)$. If not, we say that $f$ is *eventually always different* from $g$ (see [31]).

**Lemma 3.14.** *Let $A$ be a c.e. i.o. tt-traceable set. Then, every $f \leq_{tt} A$ is i.o. equal to a partial computable function.*

*Proof.* Assume that $A$ is c.e. i.o. $tt$-traceable. Then, there is a computable order $p$ such that every $g \leq_{tt} A$ is i.o. traced by a c.e. trace $(T_n)_{n\in\omega}$ with $\#T_n \leq p(n)$ for every $n \in \omega$. Define a computable order $q$ by $q(0) = 0$ and $q(n+1) = q(n) + p(n)$. Assume that $f \leq_{tt} A$ is given. Let $u$ be the use of the computation, i.e., $\Phi^{A\upharpoonright u(n)}(n) = f(n)$ for every $n$. Then we must have a computable trace $(T_n)_{n\in\omega}$ for $n \mapsto A\upharpoonright u(q(n+1))$. Enumerate $T_n$ as $\{\sigma_k : k \in [q(n), q(n+1))\}$ for every $n$, where $\sigma_k$ may be undefined. Then, we define a partial computable function $g$ by $g(k) = \Phi^{\sigma_k}(k)$ for every $k$. It is not hard to see that $f$ is i.o. equal to $g$. □

**Lemma 3.15.** *Suppose that every $f \leq_{tt} A$ is i.o. equal to a partial computable function. Then, $A \in \mathrm{Low}^\star(\mathrm{MLR}, \mathrm{WR})$.*

*Proof.* Fix any uniform Kurtz test $\{\Phi^{Z\upharpoonright u(n)}(n)\}_{n\in\omega}$, where $u$ is a computable order and $\mu(\Phi^{Z\upharpoonright u(n)}) \leq 2^{-n}$ for all $Z \in 2^\omega$. Then, $f(n) = A\upharpoonright u(n)$ is truth-table reducible to $A$. Assume that $f \leq_{tt} A$ is i.o. equal to a partial computable function, that is, there is a partial computable function $\rho$ such that $|\rho(n)| \geq u(n)$ for any $n \in \mathrm{dom}(\rho)$; and $A\upharpoonright u(n) = \rho(n)$ for infinitely many $n \in \mathrm{dom}(\rho)$. Then, consider the following sequence $\{V_n\}_{n\in\omega}$:

$$V_n^\rho = \bigcup\{[\![\Phi^{\rho(m)}(m)]\!] : m > n,\ m \in \mathrm{dom}(\rho), \text{ and } \mu(\Phi^{\rho(m)}(m)) \leq 2^{-m}\}.$$

Clearly, $\{V_n^\rho\}_{n\in\omega}$ is uniformly c.e., and $\mu(V_n) \leq 2^{-n}$. Hence, $\{V_n^\rho\}_{n\in\omega}$ is a ML test, and clearly $\bigcap_m \Phi^{A\upharpoonright u(m)}(m) \subseteq \bigcap_m V_m^\rho$. Therefore, if a real is not $A$-$tt$-Kurtz random, then it is not ML random. In other words, $A \in \mathrm{Low}^\star(\mathrm{MLR}, \mathrm{WR})$. □

3.3. **Characterizing** $\mathrm{Low}(\mathrm{SR}, \mathrm{WR})$**.** By combining previous works, we have the following result.

**Theorem 3.16.** *A set $A \in \mathrm{Low}(\mathrm{SR}, \mathrm{WR})$ iff $A$ is computably i.o. traceable.*

*Proof.* Again, by Greenberg-Miller [25], $\mathrm{Low}(\mathrm{SR}, \mathrm{WR})$ is characterized as non-DNC and non-high. Hölzl-Merkle [28, Theorem 18] showed that non-DNC is equivalent to c.e. i.o. traceability. Furthermore, Hölzl-Merkle [28, Theorem 5] showed that c.e. i.o. traceability and non-high is equivalent to computable i.o. traceability. □

We will characterize its uniform-relativization version $\mathrm{Low}^\star(\mathrm{SR}, \mathrm{WR})$ in Section 4.

3.4. **Characterizing** $\mathrm{Low}(\mathrm{WR}, \mathrm{WR})$**.** A set $A \in \mathrm{Low}^\star(\mathrm{WR}, \mathrm{WR})$ iff $A$ is computably c.o. $tt$-traceable by Kihara-Miyabe [30, Theorem 6.1]. We give its usual-relativization version.

**Theorem 3.17.** *A set $A \in \mathrm{Low}(\mathrm{WR}, \mathrm{WR})$ iff $A$ is computably c.o. traceable*

*Proof.* Kihara-Miyabe [30, Corollary 6.9] characterized $\mathrm{Low}(\mathrm{WR}, \mathrm{WR})$ as being computably c.o. $tt$-traceable and hyperimmune-free. The hyperimmune-freeness ensures that every $f \leq_T A$ is indeed $f \leq_{tt} A$. Hence, every $A \in \mathrm{Low}(\mathrm{WR}, \mathrm{WR})$ must be computably c.o. traceable.

Conversely, assume that $A$ is computably c.o. traceable. Every function $f \leq_T A$ is dominated by a strictly increasing function $g \leq_T A$. By computably c.o. traceablity, there are a computable trace $\{D_n\}_{n\in\omega}$ and a computable function $l$ such that for every $k$, $g(n) \in D_n$ for some $n \in [l(k), l(k+1))$. Now it is easy to see that $g$ is dominated by the function $1 + \max\bigcup_{n\in[l_{k+1}, l_{k+2})} D_n$. □

Now we summarize the relationship between lowness for randomness notions and traceability in Table 1.

| | MLR | SR | WR |
|---|---|---|---|
| MLR | unknown | c.e. traceable [34] | c.e. i.o. traceable (3.10) |
| SR | | computably traceable [51, 34] | computably i.o. traceable (3.16) |
| WR | | | computably c.o. traceable (3.17) |

| | MLR$^\star$ | SR$^\star$ | WR$^\star$ |
|---|---|---|---|
| MLR | unknown | c.e. *tt*-traceable (3.3) | c.e. i.o. *tt*-traceable (3.11) |
| SR | | computably *tt*-traceable [20] | computably i.o. *tt*-traceable (4.1) |
| WR | | | computably c.o. *tt*-traceable [30] |

TABLE 1. Characterization of Lowness Properties by Traceability

## 4. Effective Strong Measure Zero

In this section we characterize another lowness notion via traceability.

**Theorem 4.1.** *A set* $A \in \mathrm{Low}^\star(\mathrm{SR}, \mathrm{WR})$ *iff* $A$ *is computably i.o. tt-traceable.*

We will see in Theorems 5.3 and 6.3 that computably i.o. traceability is characterized as being Schnorr $\mathcal{H}^h$-null for every computable order $h$. Here, see Section 6 for the definition of Schnorr $\mathcal{H}^h$-nullness. It is not hard to see that this property is essentially equivalent to the notion of effectively strongly measure zero in the sense of Higuchi-Kihara [26]. The following proof is inspired by the proof of Pawlikowski's characterization [46] of strong measure zero in set theory.

**Theorem 4.2.** *The following are equivalent for* $V \subseteq 2^\omega$.

(i) $V$ *is Schnorr* $\mathcal{H}^h$*-null for every computable order* $h$.
(ii) $E[V] = \bigcup_{A\in V} E(A)$ *is covered by a Schnorr test for every uniform Kurtz test* $E$.

*Proof.* (i)⇒(ii): Every uniform Kurtz test $E$ can be thought of as a truth table functional $\Psi$ such that $E(Z) = \bigcap_n [\![\Psi^Z(n)]\!]$, and $\mu([\![\Psi^Z(n)]\!]) \leq 2^{-n}$. Let $u$ be a computable modulus of uniform computability of $\Psi$, that is, for all $Z \in 2^\omega$ and all $n \in \omega$, the value $\Psi^{Z\upharpoonright u(n)}(n)$ is determined.

Let $h$ be a sufficiently slow-growing computable order fulfilling $2^{-h(u(n))} \geq 1/(n+1)$ for all $n \in \omega$. Assume that $A$ is Schnorr $h$-dimensional measure zero. By our assumption, we have a computable sequence $\{W_n\}_{n\in\omega}$ of c.e. sets of strings such that $V \subseteq [\![W_n]\!]$ and $\sum_{\sigma\in W_n} 2^{-h(|\sigma|)} < 1/(n+1)$ is computable uniformly in $n \in \omega$. Thus, each $\sigma \in W_n$ has length greater than $u(n)$, and moreover $W_n$ contains at most $k$ strings of length $\leq u(n+k)$, since, otherwise,

$$\sum_{\sigma\in W_n} 2^{-h(|\sigma|)} \geq (k+1)2^{-h(u(n+k))} \geq \frac{k+1}{n+k+1} \geq \frac{1}{n+1}.$$

For every $\sigma$, let $k_\sigma$ be the greatest $k$ such that $|\sigma| \geq u(k)$. Then define $N_n = \bigcup_{\sigma\in W_n} [\![\Psi^\sigma(k_\sigma)]\!]$. Then $\mu(N_n) \leq 2^{-n}$ since $W_n$ contains at most $k$ strings of length $\leq u(n+k)$ as seen before, and $\mu([\![\Psi^\sigma(k_\sigma)]\!]) \leq 2^{-k_\sigma}$. It is also not hard to see that $\mu(N_n)$ is computable uniformly in $n$. Therefore, $\{N_n\}_{n\in\omega}$ forms a Schnorr test. Moreover, $V \subseteq \bigcap_n [\![W_n]\!]$ clearly implies $E[V] \subseteq \bigcap_n N_n$ as desired.

(ii)⇒(i): We first define a special Kurtz test $D$ which has some probabilistic independence property.

**Construction of a Kurtz test $D$.** Given a computable order $g$, inductively define two computable orders $h$ and $h^+$ by $h(0) = 0$, $h^+(n) = h(n) + g(n)$, and $h(n+1) = h^+(n) + 2^{g(n)}$. Then let $D_k \subseteq 2^{h(k+1)}$ be the set of all strings of the form $\tau^\frown\sigma_i{}^\frown\rho$ such that $|\tau| = h(k)$, $\sigma_i$ is the $i$-th string with length $g(n)$, $|\rho| = 2^{g(n)}$ and the $i$-th bit of $\rho$ is 0 (equivalently $\rho(i) = 0$). Then $D = \bigcap_n [\![D_n]\!]$ is a Kurtz test, since the $\mu$-measure of $\bigcap_{m<n}[\![D_m]\!]$ and $[\![D_n]\!]$ are $2^{-n}$ and $2^{-1}$ respectively.

Now it is easy to see that $E : Z \mapsto Z + D = \{Z + Y : Y \in D\}$ is a uniform Kurtz test, where $Z + Y \in 2^\omega$ is defined by $(Z + Y)(n) \equiv Z(n) + Y(n) \bmod 2$. Therefore, the assertion (ii) ensures that $E[V] = V + D = \bigcup_{A\in V}(A+D)$ is covered by a Schnorr test $\{W_n\}_{n\in\omega}$. Let $W = W_3$. By using the property $V + D \subseteq W$, we approximate $V$ by a sequence of clopen sets $V[k]$ such that $\{\bigcup_{k>n} V[k]\}_{n\in\omega}$ forms a Schnorr $\mathcal{H}^f$-test that covers $V$, that is, $V \subseteq \bigcap_n \bigcup_{k>n} V[k]$, where $f$ will be introduced later.

We say that *a string $\sigma$ approaches $W$ over a string $\tau$ except for $\varepsilon$* if

$$(1-\varepsilon)\cdot\mu(W|\tau\sigma) > \mu(W|\tau)$$

is satisfied. Note that Kolmogorov's inequality implies that $\mu(\{\sigma : t\mu(W|\sigma) > \mu(W)\}) < t$. Hence, given a string $\tau$, the probability of the event that a string approaches $W$ over $\tau$ except for $2^{-(k+1)}$ is less than $1 - 2^{-(k+1)}$. We also say that *a string $\sigma \in 2^{[h(k),h(k)^+)}$ approaches $V$ except for $\varepsilon$* if $\sigma$ has an extension $\sigma^+ \in 2^{[h(k),h(k+1))}$ such that every string in $\sigma + D_k$ approaches $W$ over a string of length $h(k)$ except for $\varepsilon$. By $V_\tau[k] \subseteq 2^{[h(k),h(k)^+)}$ we denote the set of all strings that approach $V$ over $\tau$ except for $2^{-(k+1)}$, and put $V[k] = \bigcup_{\tau\in 2^{h(k)}} V_\tau[k]$.

**Claim.** For every $k \in \omega$, there are at most $(k+1)\cdot 2^{h(k)}$ many strings which approach $V$ except for $2^{-(k+1)}$. In other words, $\#V[k] \leq (k+1)\cdot 2^{h(k)}$.

By definition, every string in $V_\tau[k]+D_k$ approaches $W$ over $\tau$ except for $2^{-(k+1)}$. By probabilistic independence of $D_k$, the $\mu$-measure of $V_\tau[k]$ is $1 - 2^{-|V_\tau[k]|}$, while the probability of approaching $W$ over $\tau$ except for $2^{-(k+1)}$ is less than $1-2^{-(k+1)}$ as mentioned before. This implies $\#V_\tau[k] \leq k+1$. Hence, the desired value is computed by multiplying the above number by the number of strings $\tau \in 2^{h(k)}$.

**Claim.** For every $A \in V$, the segment $A\upharpoonright[h(k), h(k+1))$ approaches $V$ itself except for $2^{-(k+1)}$ for infinitely many $k$. In other words, $V \subseteq \bigcap_l \bigcup_{k>l} V[k]$.

Suppose for the sake of contradiction that some $A \in V$ approaches $V$ except for $2^{-(k+1)}$ for at most finitely many $k$. Then there exists $k_0$ such that for every $k \geq k_0$ and every $\tau \in 2^{h(k)}$, some $\sigma \in A\upharpoonright[h(k), h(k+1)) + D_k$ does not approach $V$ except for $2^{-(k+1)}$. Therefore, inductively, we can construct $Z \in \mathbb{Q} + A + D \subseteq W$ fulfilling

$$(1-2^{-(k+1)})\cdot\mu(W|Z\upharpoonright h(k+1)) \leq \mu(W|Z\upharpoonright h(k)).$$

Here, $Z \upharpoonright h(k_0)$ may satisfy $\mu(W|Z \upharpoonright h(k_0)) \leq \mu(W)$. Since $W$ is open, there exists $k \geq k_0$ such that $Z\upharpoonright h(k) \subseteq W$. Hence, we have $\mu(W|Z\upharpoonright h(k)) = 1$ and

$$\mu(W) \geq \mu(W|Z\upharpoonright h(k_0)) \geq \prod_{i=k_0}^{k-1}(1-2^{-(i+1)})\cdot\mu(W|Z\upharpoonright h(k)) > 1/8.$$

This contradicts our assumption $\mu(W) = 1/8$.

Given a computable order $f$, take $g$ as a sufficiently fast-growing computable order satisfying $f(h(k) + g(k)) - h(k) \geq 2k + 2$. Then, for $U_n = \bigcup_{k>n} V[k]$, the collection $\{U_n\}_{n\in\omega}$ can be covered by a Schnorr $\mathcal{H}^f$-test, since for every $k \in \omega$,

$$\mathrm{dwt}_f(V[k]) \leq (k+1) \cdot 2^{h(k)} \cdot 2^{-f(h(k)+g(k))} \leq 2^{-(k+1)}.$$

Consequently, $A$ is Schnorr $\mathcal{H}^f$-null for every computable order $f$. □

It remains to show that the assertion (ii) in Theorem 4.2 is equivalent to $A \in \mathrm{Low}^\star(\mathrm{SR}, \mathrm{WR})$. To verify this, we transform every Kurtz test into a "nice" Kurtz test. Call $H \subseteq 2^\omega$ *infinitely often (i.o.) homogeneous* if for infinitely many $n$ and for any $\sigma, \tau \in 2^n$, if $H \cap [\sigma]$ and $H \cap [\tau]$ are nonempty, then these sets are equivalent above level $n$. Such an $n$ is called a homogeneity level for $H$.

**Lemma 4.3.** *Every Kurtz test (uniformly) relative to $A$ is covered by an i.o. homogeneous Kurtz test (uniformly) relative to $A$.*

*Proof.* Let $E^A = \bigcap_n [\![E^A_n]\!]$ be a Kurtz test (uniformly) relative to $A$. Note that $\mu([\![E^A_n]\!]) \leq 2^{-n}$ for every $n$. We inductively define a computable order $l$ as follows. Put $l(0) = 0$, and $l(n+1)$ be the maximal length of strings contained in $E^A_{n\cdot l(n)}$. Then, for every $n$, let $\mathrm{dup}_{l(n)}(E^A_{n\cdot l(n)})$ be the set obtained by duplicating $E^A_{n\cdot l(n)}$ at level $l(n)$, that is,

$$\mathrm{dup}_{l(n)}(E^A_{n\cdot l(n)}) = \{\tau^\frown\rho : \tau \in 2^{l(n)} \text{ and } \sigma^\frown\rho \in E^A_{n\cdot l(n)} \text{ for some } \sigma \in 2^{l(n)}\}.$$

Put $D^A_n = \mathrm{dup}_{l(n)}(E^A_{n\cdot l(n)})$ for every $n$. Since $\mu([\![E^A_{n\cdot l(n)}]\!]) \leq 2^{-n\cdot l(n)}$, we have $\mu([\![D^A_n]\!]) \leq 2^{l(n)} 2^{-n\cdot l(n)} = 2^{-n}$. Then, clearly $D^A = \bigcap_n [\![D^A_n]\!]$ is an i.o. homogeneous Kurtz test, and $D^A$ covers $E^A$. □

Let $\mathcal{I}$ be a class of subsets of $2^\omega$. The class $\mathcal{I}$ is said to be *closed under finite duplication* provided $N \in \mathcal{I}$ implies $\mathrm{dup}_{|\sigma|}(N \cap [\sigma]) \in \mathcal{I}$ for every string $\sigma$. Here,

$$\mathrm{dup}_{|\sigma|}(N \cap [\sigma]) = \{\tau^\frown g : \tau \in 2^{|\sigma|} \text{ and } \sigma^\frown g \in N\}.$$

Note that the class of all Schnorr tests is closed under finite duplication.

**Lemma 4.4.** *Let $\mathcal{I}$ be a countable class closed under finite duplication. For every i.o. homogeneous closed set $H \subseteq 2^\omega$, if $H$ is not covered by any $N \in \mathcal{I}$, then $H$ is not covered by $\bigcup \mathcal{I}$.*

*Proof.* Suppose for the sake of contradiction that $H$ is i.o. homogeneous, and $H$ is not covered by any $N \in \mathcal{I}$. We first claim that, for every $N \in \mathcal{I}$, the set $H \setminus N$ is dense in $H$. Otherwise, there is a string $\sigma$ extendable in $H$ such that $H \cap [\sigma] \subseteq N$. Without loss of generality, we may assume that $|\sigma|$ is a homogeneity level of $H$. However, the homogeneity of $H$ implies $H \subseteq \mathrm{dup}_{|\sigma|}(N \cap [\sigma])$, and since $\mathcal{I}$ is closed under finite duplication, we have $\mathrm{dup}_{|\sigma|}(N \cap [\sigma]) \in \mathcal{I}$. This contradicts our assumption. Thus, $H \setminus N$ is dense in $H$, and since $\mathcal{I}$ has at most countably many elements, the Baire category theorem in $H$ ensures the existence of an element in $H$ that is not contained in any $N \in \mathcal{I}$. In other words, $H$ is not covered by $\bigcup \mathcal{I}$. □

**Corollary 4.5.** *The following are equivalent for $A \in 2^\omega$.*

(i) *$E(A)$ is covered by a Schnorr test for every uniform Kurtz test $E : 2^\omega \to \mathcal{E}$.*
(ii) $A \in \mathrm{Low}^\star(\mathrm{SR}, \mathrm{WR})$.

*Proof.* (i)⇒(ii): Assume that the assertion (i) holds. If $Z$ is not Kurtz random uniformly relative to $A$, we have a uniform Kurtz test $E$ such that $Z \in E(A)$. By our assumption, $E(A)$ is covered by a Schnorr test, and then $Z$ is contained in such a Schnorr test. Therefore, $Z$ is not Schnorr random. Hence, $A \in \mathrm{Low}^\star(\mathrm{SR}, \mathrm{WR})$.

(ii)⇒(i): We show the contrapositive. Assume that there exists a uniform Kurtz test $E$ such that $E(A)$ is not covered by a Schnorr test. By Lemma 4.3, we may safely assume that $E(A)$ is i.o. homogeneous. Since the class of all Schnorr tests is countable and closed under finite duplication, by Lemma 4.4, $E(A)$ is not covered by the intersection of all Schnorr tests, that is, $E(A)$ contains a Schnorr random element. Consequently, $A \notin \mathrm{Low}^\star(\mathrm{SR}, \mathrm{WR})$. □

## 5. Traceability and Complexity

If an infinite binary sequence is traceable in some sense, the sequence would be expected to be so compressible that the compression rate can be bounded by any computable function. Implicitly, this expectation was verified by Kjos-Hanssen, Merkle and Stephan [31], who used the notion of a complex set to characterize the class of diagonally noncomputable functions. More directly, Hölzl-Merkle [28] gave characterizations of traceability notions by complexity concepts with respect to prefix-free machines and total machines, where a partial computable function $M : 2^{<\omega} \to 2^{<\omega}$ is said to be a *total machine* if $\mathrm{dom}(M)$ is total. Franklin-Greenberg-Stephan-Wu [18] also introduced the notion of anticomplex.

**Definition 5.1** ([31, 28, 18])**.** A set $A \in 2^\omega$ is *complex* (resp. *autocomplex*) if there exists a computable (resp. $A$-computable) order $g$ such that $K(A \upharpoonright g(n)) \geq n$ for almost all $n$. A set $A$ is *totally complex* if there exists a computable order $g$ such that for every total machine $M$, we have $K_M(A \upharpoonright g(n)) \geq n$ for almost all $n$. A set $A$ is *anticomplex* if for every computable order $g$, we have $K(A \upharpoonright g(n)) \leq n$ for almost all $n$.

We introduce variants of complexity. The c.o. version is inspired by Theorem 5.2 in [30].

**Definition 5.2.** A set $A$ is *totally auto-anticomplex* if, for every $A$-computable order $g$, there is a total machine $M$ such that $K_M(A \upharpoonright g(n)) \leq n$ for almost all $n$. A set $A$ is *totally c.o. anticomplex* if for every computable order $g$, there exists a total machine $M$ making $K_M(A \upharpoonright g(n)) \leq n$ to be true computably often, that is, there exists a computable order $h$ such that for every $k \in \omega$, $K_M(A \upharpoonright g(n)) \leq n$ for some $n \in [h(k), h(k+1))$. Moreover, one can also define any combination of these notions such as *totally c.o. auto-anticomplex* in a straightforward manner.

Hölzl-Merkle [28, Theorem 26] characterized Schnorr triviality as totally anticomplex sets, while the former concept refers to a computable measure machine and the latter concept refers to a total machine. We claim that one can replace "total machine" in the definition of a totally complex set with "computable measure machine" or "decidable machine". Here, recall that a partial computable function $M : 2^{<\omega} \to 2^{<\omega}$ is *a computable measure machine* if it is prefix-free and $\mu(\bigcup_{\sigma \in \mathrm{dom}(M)} [\![\sigma]\!])$ is computable, and $M$ is *a decidable machine* if $\mathrm{dom}(M)$ is computable.

We can easily see that every computable measure machine is indeed a prefix-free decidable machine. Moreover, if $M$ is a prefix-free decidable machine, then one can easily construct a total machine $N$ such that $K_N(\sigma) \leq K_M(\sigma)$ for every $\sigma$ by adding $N(\tau) = \varepsilon$ for every $\tau \notin \mathrm{dom}(M)$. Given a total machine $M$, we construct a machine $N$ by setting $N(0^{|\tau|}1\tau) = M(\tau)$ for every $\tau$. Then it is easy to see that $K_M(\sigma) \leq n$ implies $K_N(\sigma) \leq 2n+1$, and the measure of $\mathrm{dom}(N)$ is computable. Hence, we obtain the claim.

Similarly, the replacement does not change the notion of a totally autocomplex set and so on.

We review some known results. A set $A$ is c.e. traceable iff $A$ is auto-anticomplex by Hölzl-Merkle [28, Theorem 22]. A set $A$ is c.e. $tt$-traceable iff $A$ is anticomplex by Hölzl-Merkle [28, Theorem 23]. A set $A$ is computably $tt$-traceable iff $A$ is totally anticomplex by Hölzl-Merkle [28, Theorem 26]. A set $A$ is c.e. i.o. traceable iff $A$ is not autocomplex by Kjos-Hanssen–Merkle–Stephan [31]; see also Hölzl-Merkle [28, Theorem 18]. A set $A$ is c.e. i.o. $tt$-traceable iff $A$ is not complex by Kjos-Hanssen–Merkle–Stephan [31]; see also Hölzl-Merkle [28, Theorem 20]. A set $A$ is computably i.o. $tt$-traceable iff $A$ is not totally complex by Hölzl-Merkle [28, Theorem 25]. A set $A$ is computably c.o. $tt$-traceable iff $A$ is not totally c.o. anticomplex by Kihara-Miyabe [30, Theorem 6.1].

We show some characterizations of remaining possible combinations.

**Theorem 5.3.** *For any set $A \in 2^\omega$,*

(i) *$A$ is computably traceable iff $A$ is totally auto-anticomplex.*
(ii) *$A$ is computably i.o. traceable iff $A$ is not totally autocomplex.*
(iii) *$A$ is computably c.o. traceable iff $A$ is totally c.o. auto-anticomplex.*

Here we only show the first one because the other two are not hard to see by the same argument.

*Proof.* Assume that $A$ is computably traceable. Then, there is a computable order $p$ such that every $g \leq_T A$ is traced by a computable trace $(T_n)_{n\in\omega}$ with $\#T_n \leq p(n)$ for every $n \in \omega$. Let $g$ be any $A$-computable order. Then we must have a computable trace $(T_n)_{n\in\omega}$ for $n \mapsto A\upharpoonright g(n)$. Define a computable order $q$ by $q(0) = 0$ and $q(n+1) = q(n) + p(n)$, and then enumerate $T_n$ as $\{\sigma^n_k : k \in [q(n), q(n+1))\}$ for every $n$. We construct a computable measure machine $M$ by $M(0^k1) = \sigma^n_k$ for $k \in [q(n), q(n+1))$. Then, clearly $K_M(A\upharpoonright g(n)) \leq q(n+1)+1$ for all $n$. Note that the computable order $q$ does not depend on $g$. Thus we can conclude that $A$ is totally auto-anticomplex.

Conversely, assume that $A$ is totally auto-anticomplex. To see that $A$ is computably traceable, assume that $g \leq_T A$ is given. Let $u \leq_T A$ be the $A$-use in the computation of $g \leq_T A$, i.e., $g(n) = \Phi^{A\upharpoonright u(n)}(n)$ for some computation $\Phi$. It suffices to construct a computable trace which traces $n \mapsto A \upharpoonright u(n)$. Since $A$ is totally auto-anticomplex, there exists a computable measure machine $M$ such that $K_M(A\upharpoonright u(n)) \leq n$ for all $n$. Then, consider the set $T_n = \{\sigma \in 2^{<\omega} : K_M(\sigma) \leq n\}$. Clearly, $\#T_n \leq 2^n$, and $(T_n)_{n\in\omega}$ is computable uniformly in $n \in \omega$ since the measure of the domain of $M$ is computable. Thus, $(T_n)_{n\in\omega}$ is a computable trace with bound $n \mapsto 2^n$ which traces $n \mapsto A\upharpoonright u(n)$ as desired. $\square$

Now, the lowness notions are completely characterized by Kolmogorov complexity as in Table 2, where $M$ ranges over all computable measure machines, and the quantifiers $\forall^\infty$, $\exists^\infty$ and $\exists^{\mathrm{c.o.}}$ are interpreted as "for all but finitely many", "infinitely often" and "computably often (recall Definitions 3.1 and 5.2)", respectively.

| | MLR | SR | WR |
|---|---|---|---|
| MLR | $K$-trivial: $\forall n\, K(A\upharpoonright n) \leq^+ K(n)$ | auto-anticomplex: $\forall g \leq_{\mathrm{T}} A\, \forall^\infty n\, K(A\upharpoonright g(n)) \leq n$ | not autocomplex: $\forall g \leq_{\mathrm{T}} A\, \exists^\infty n\, K(A\upharpoonright g(n)) \leq n$ |
| SR | | totally auto-anticomplex: $\forall g \leq_{\mathrm{T}} A\, \exists M \forall^\infty n\, K_M(A\upharpoonright g(n)) \leq n$ | not totally autocomplex: $\forall g \leq_{\mathrm{T}} A\, \exists M \exists^\infty n\, K_M(A\upharpoonright g(n)) \leq n$ |
| WR | | | totally c.o. auto-anticomplex: $\forall g \leq_{\mathrm{T}} A\, \exists M \exists^{\text{c.o.}} n\, K_M(A\upharpoonright g(n)) \leq n$ |

| | MLR* | SR* | WR* |
|---|---|---|---|
| MLR | $K$-trivial: $\forall n\, K(A\upharpoonright n) \leq^+ K(n)$ | anticomplex: $\forall g \leq_{\mathrm{T}} \emptyset\, \forall^\infty n\, K(A\upharpoonright g(n)) \leq n$ | not complex: $\forall g \leq_{\mathrm{T}} \emptyset\, \exists^\infty n\, K(A\upharpoonright g(n)) \leq n$ |
| SR | | tot. anticomplex (Schnorr trivial): $\forall g \leq_{\mathrm{T}} \emptyset\, \exists M \forall^\infty n\, K_M(A\upharpoonright g(n)) \leq n$ $\equiv \forall N \exists M \forall n\, K_M(A\upharpoonright n) \leq K_N(n)$ | not totally complex: $\forall g \leq_{\mathrm{T}} \emptyset\, \exists M \exists^\infty n\, K_M(A\upharpoonright g(n)) \leq n$ |
| WR | | | totally c.o. anticomplex: $\forall g \leq_{\mathrm{T}} \emptyset\, \exists M \exists^{\text{c.o.}} n\, K_M(A\upharpoonright g(n)) \leq n$ |

TABLE 2. Characterization of Lowness Properties by Kolmogorov Complexity

## 6. Complexity and Dimension

In the previous section, the notion of traceability has been characterized in the context of Kolmogorov complexity. The compression rate of an infinite binary sequence is closely related to the notions of effective Hausdorff dimension and martingales (see for instance, Downey-Hirschfeldt [13, Chapter 13]). Kihara-Miyabe [30, Section 5] introduced the notion of Kurtz dimension (with respect to any computable dimension function) as a clopen version of effective dimension, and characterized c.o. traceability by using the notion of Kurtz dimension. In this section, we introduce the notion of effective dimension scaled by a computable function, and then characterize the notion by complexity and martingales.

6.1. **Hausdorff Dimension.** For an order $h : \omega \to \omega$, a set $E \subseteq 2^\omega$ is *effective Hausdorff $h$-dimensional measure zero* or *Martin-Löf $\mathcal{H}^h$-null* if there is a computable sequence $\{W_n\}_{n\in\omega}$ of c.e. sets of strings such that

$$E \subseteq [\![W_n]\!] \text{ and } \sum_{\sigma \in W_n} 2^{-h(|\sigma|)} \leq 2^{-n} \text{ for all } n \in \omega.$$

For every set $W$ of strings, the value $\sum_{\sigma\in W} 2^{-h(|\sigma|)}$ is called *the direct weight of $W$ with respect to scale $h$*, or simply, *the direct $h$-weight of $W$*, and abbreviated as $\mathrm{dwt}_h(W)$ hereafter. If $\{\mathrm{dwt}_h(W_n)\}_{n\in\omega}$ is a computable sequence, then $E$ is called *Schnorr $\mathcal{H}^h$-null.* If $\{W_n\}_{n\in\omega}$ is a computable sequence of finite sets of strings, then $E$ is called *Kurtz $\mathcal{H}^h$-null.* We also say that $A \in 2^\omega$ is *Martin-Löf (resp. Schnorr, and Kurtz) $\mathcal{H}^h$-null* if $\{A\}$ is Martin-Löf (resp. Schnorr, and Kurtz) $\mathcal{H}^h$-null.

**Remark 6.1.** If it is the case that $h$ is of the form $n \mapsto sn$, the notions of Martin-Löf and Schnorr $\mathcal{H}^h$-nullness have been widely studied in the terminology of effective dimension and Schnorr dimension (see Downey-Hirschfeldt [13, Chapter 13]). The notion of Martin-Löf $\mathcal{H}^h$-nullness for any dimension function $h$ has also been introduced in Reimann [47] in a slightly different form. The notion of Kurtz $\mathcal{H}^h$-nullness for any dimension function $h$ has been introduced in Kihara-Miyabe [30].

The following theorem generalizes a well-known characterization of effective dimension (see also Downey-Hirschfeldt [13, Chapter 13]).

**Theorem 6.2.** *Let $h$ be any computable order. Then, the following are equivalent for a set $E \subseteq 2^\omega$.*

(i) *$E$ is Martin-Löf $\mathcal{H}^h$-null.*
(ii) *There is a c.e. martingale $d$ such that for all $A \in E$,*

$$\limsup_{n\to\infty} \frac{d(A\upharpoonright n)}{2^{n-h(n)}} = \infty.$$

(iii) *For all $A \in E$,*

$$\liminf_{n\to\infty} (K(A\upharpoonright n) - h(n)) = -\infty.$$

As a corollary, a set is not complex if and only if it is Martin-Löf $\mathcal{H}^h$-null for every computable order $h$. We omit the proof of Theorem 6.2 since the proof is straightforward. Instead, we give the proof of the next theorem, a Schnorr version of Theorem 6.2. The following theorem generalizes a characterization of Schnorr dimension by Downey-Merkle-Reimann [17] (see also Downey-Hirschfeldt [13, Section 13.15]).

**Theorem 6.3.** *Let $h$ be any computable order. Then, the following are equivalent for a set $E \subseteq 2^\omega$.*

(i) *$E$ is Schnorr $\mathcal{H}^h$-null.*
(ii) *There is a computable martingale $d$ such that for all $A \in E$,*

$$\limsup_{n\to\infty} \frac{d(A\upharpoonright n)}{2^{n-h(n)}} = \infty.$$

(iii) *There is a computable measure machine $M$ such that for all $A \in E$,*

$$\liminf_{n\to\infty} (K_M(A\upharpoonright n) - h(n)) = -\infty.$$

*Proof.* (i)⇒(ii): Suppose that $E \subseteq 2^\omega$ is Schnorr $\mathcal{H}^h$-null via a sequence $\{W_n\}_{n\in\omega}$. For each $\sigma$, let $B_\sigma$ be a martingale defined by

$$B_\sigma(\tau) = \begin{cases} 2^{|\tau|} & \text{if } \tau \preceq \sigma \\ 2^{|\sigma|} & \text{if } \sigma \prec \tau \\ 0 & \text{otherwise.} \end{cases}$$

Then $d = \sum_n \sum_{\sigma\in W_{2n+1}} 2^{n-h(|\sigma|)} B_\sigma$ is a martingale with the initial capital

$$\sum_n \sum_{\sigma\in W_{2n+1}} 2^n \cdot 2^{-h(|\sigma|)} \leq \sum_n 2^n \cdot \mathrm{dwt}_h(W_{2n+1}) \leq \sum_n 2^{-n-1} = 1.$$

To see that $d$ is computable, it suffices to approximate $\sum_{\sigma\in W_{2n+1}} 2^{n-h(|\sigma|)} B_\sigma$ with any precision $2^{-t}$. Wait for stage $s$ to get $\mathrm{dwt}_h(W_{2n+1,>s}) \leq 2^{-n-|\tau|-t}$, where $W_{2n+1,>s} = W_{2n+1} \setminus W_{2n+1,s}$. Then,

$$\sum_{\sigma\in W_{2n+1,>s}} 2^{n-h(|\sigma|)} B_\sigma(\tau) = \sum_{\sigma\in W_{2n+1,>s}} 2^{n+|\tau|-h(|\sigma|)} = 2^{n+|\tau|} \cdot \mathrm{dwt}_h(W_{2n+1,>s}) \leq 2^{-t}$$

Therefore, $d$ is computable.

Now fix $A \in E$. For each $n$, let $k_n$ be a number such that $A\upharpoonright k_n \in W_{2n+1}$. Then,

$$d(A\upharpoonright k_n) \geq 2^{n-h(k_n)} B_{A\upharpoonright k_n}(A\upharpoonright k_n) = 2^n \cdot 2^{k_n-h(k_n)}.$$

Consequently, we have the desired condition.

(ii)⇒(iii): Let $d$ be a computable martingale satisfying (ii). Without loss of generality, we may assume that $d(\epsilon) = 1$. Consider the following c.e. set:

$$W_n = \{\sigma \in 2^{<\omega} : d(\sigma) > 2^{2n} 2^{|\sigma| - h(|\sigma|)}\}.$$

Let $V_n$ be a maximal c.e. subantichain of $W_n$. Then

$$\sum_{\sigma \in V_n} 2^{n - h(|\sigma|)} \leq \sum_{\sigma \in V_n} 2^{n - h(|\sigma|)} \frac{2^{-2n} d(\sigma)}{2^{|\sigma| - h(|\sigma|)}} = 2^{-n} \sum_{\sigma \in V_n} 2^{-|\sigma|} d(\sigma) \leq 2^{-n}.$$

Here, the last inequality follows from Kolmogorov's inequality (see [13, Theorem 6.3.3]) with our assumption $d(\epsilon) = 1$. Then, the KC theorem [13, Theorem 3.6.1] implies the existence of a machine $M$ such that $K_M(\sigma) \leq h(|\sigma|) - n + c$ holds for each $n$ and $\sigma \in V_n$. In particular, $\liminf_{n \to \infty}(K_M(A \upharpoonright n) - h(n)) = -\infty$ for all $A \in E$.

It remains to show that $M$ is indeed a computable measure machine. To see this, we approximate the value $\sum_{\sigma \in V_n} 2^{n - h(|\sigma|)}$ within a given precision $2^{-t}$. Given $t$, we choose $s$ such that $2^{s - h(s)} \geq 2^{t-n}$. If $\tau \in V_n$ and $|\tau| \geq s$, we have

$$d(\tau) \geq 2^{2n} \cdot 2^{|\tau| - h(|\tau|)} \geq 2^{2n + s - h(s)} \geq 2^{t+n}.$$

By Kolmogorov's inequality, the probability of $\{\sigma : d(\sigma) \geq 2^{t+n}\}$ is not greater than $2^{-t-n}$. Thus,

$$\sum_{\sigma \in V_n \cap 2^{\geq s}} 2^{n - h(|\sigma|)} \leq 2^n \sum_{\sigma \in V_n \cap 2^{\geq s}} 2^{-|\sigma|} \leq 2^{-t}.$$

Hence, the value is shown to be computable, and this ensures that the measure of $\mathrm{dom}(M)$ is computable.

(iii)⇒(i): Assume that there exists a computable measure machine $M$ such that $\liminf_{n \to \infty}(K_M(A \upharpoonright n) - h(n)) = -\infty$ for all $A \in E$. Consider the sequence $\{W_n\}_{n \in \omega}$ of c.e. open sets defined by

$$W_n = \{\sigma \in 2^{<\omega} : K_M(\sigma) \leq h(|\sigma|) - n\}.$$

Then $A \in \bigcap_n W_n$, and

$$\mathrm{dwt}_h(W_n) = \sum_{\sigma \in W_n} 2^{-h(|\sigma|)} \leq 2^{-n} \sum_{\sigma \in W_n} 2^{-K_M(\sigma)} \leq 2^{-n}.$$

To see that $\mathrm{dwt}_h(W_n)$ is computable uniformly in $n$, given precision $t$, we wait for stage $s$ to meet $\mu([\![\mathrm{dom}(M_{>s})]\!]) \leq 2^{n-t}$. Then

$$\mathrm{dwt}_h(W_{n,>s}) \leq 2^{-n} \sum_{\sigma \in W_{n,>s}} 2^{-K_M(\sigma)} \leq 2^{-n} \sum_{\tau \in \mathrm{dom}(M_{>s})} 2^{-|\tau|} \leq 2^{-t}$$

Hence, $E$ is Schnorr $h$-dimensional measure zero. □

As a corollary, a set is not totally complex if and only if it is Schnorr $\mathcal{H}^h$-null for every computable order $h$.

The Kurtz version of Theorems 6.2 and 6.3 has already been proved in Kihara-Miyabe [30].

**Theorem 6.4** (Kihara-Miyabe [30]). *Let $h$ be any computable order. Then, the following are equivalent for a set $E \subseteq 2^\omega$.*

(i) *$E$ is Kurtz $\mathcal{H}^h$-null.*

(ii) *There are a computable martingale $d$ and a computable order $g$ such that for all $A \in E$,*

$$(\forall n \in \omega)(\exists k \in [g(n), g(n+1))) \quad d(A \upharpoonright k) \geq 2^n \cdot 2^{k-h(k)}.$$

(iii) *There are a computable measure machine $M$ and a computable order $g$ such that for all $A \in E$,*

$$(\forall n \in \omega)(\exists k \in [g(n), g(n+1))) \quad K_M(A \upharpoonright k) \leq h(k) - n.$$

6.2. **Packing Dimension.** The notion of packing dimension from fractal geometry is also known to be strongly related to Kolmogorov complexity (see Downey-Hirschfeldt [13, Section 13.11]). We next introduce the effective version of packing dimension for any dimension function $h$. For an order $h : \omega \to \omega$, given a tree $T \subseteq 2^{<\omega}$, define $\mathcal{P}^h_n(T) \in [0, \infty]$ as follows.

$$\mathcal{P}^h_n(T) = \sup \left\{ \sum_{\sigma \in W} 2^{-h(|\sigma|)} : W \subseteq T \cap 2^{\geq n} \right\},$$

where $W$ ranges over all prefix-free subsets of $2^{<\omega}$. Then, we denote the value $\lim_{n\to\infty} \mathcal{P}^h_n(T)$ by $\mathcal{P}^h_\infty(T)$. The *packing $h$-dimensional measure* of a set $E \subseteq 2^\omega$ is defined as follows.

$$\mathcal{P}^h(E) = \inf \left\{ \sum_{i=0}^{\infty} \mathcal{P}^h_\infty(T_i) : E \subseteq \bigcup_{i \in \omega} [T_i] \right\},$$

where $T_i$ ranges over all subtrees of $2^{<\omega}$, and we denote by $[T]$ the closed set generated by a tree $T$, i.e., $[T] = \{A \in 2^\omega : (\forall n)\ A \upharpoonright n \in T\}$.

A set $E \subseteq 2^\omega$ is said to be *packing $h$-null* if $\mathcal{P}^h(E) = 0$. If $\{T_i\}_{i \in \omega}$ can be chosen as a computable sequence of c.e. trees (resp. computable trees), then $E$ is called *Martin-Löf $\mathcal{P}^h$-null* (resp. *Schnorr $\mathcal{P}^h$-null*). We also say that $A \in 2^\omega$ is *Martin-Löf (resp. Schnorr) $\mathcal{P}^h$-null* if $\{A\}$ is Martin-Löf (resp. Schnorr) $\mathcal{P}^h$-null.

**Remark 6.5.** It is not hard to see that if there is a real $s$ such that $h(n) = sn$ for every $n \in \omega$, the value $\mathcal{P}^h(E)$ is known as the *$s$-dimensional packing measure* of $E$ in the usual sense. If $h$ is such a function, the notions of Martin-Löf and Schnorr $\mathcal{P}^h$-nullness have been studied in the terminology of *effective packing dimension* and *Schnorr packing dimension* (see Downey-Hirschfeldt [13, Chapter 13]), though it seems that the notion of Schnorr packing dimension has been only defined in the terminology of martingales.

It is known that for every set $E \subseteq 2^\omega$, the packing dimension $\dim_{\mathrm{P}}(E)$ coincides with the modified upper box-counting dimension $\overline{\dim}_{\mathrm{MB}}(E)$. Here, recall that the upper box-counting dimension $\overline{\dim}_{\mathrm{B}}(E)$ of a set $E \subseteq 2^\omega$ is given by

$$\overline{\dim}_{\mathrm{B}}(E) = \limsup_{n\to\infty} \frac{\log |E \upharpoonright n|}{n},$$

where $E \upharpoonright n = \{\sigma \in 2^n : E \cap [\sigma] \neq \emptyset\}$. We introduce the function-scaled version of the upper box-counting dimension. For an order $h : \omega \to \omega$, a tree $T \subseteq 2^{<\omega}$ is said to be *upper box-counting $h$-null* if

$$\limsup_{n\to\infty} (\log |T \upharpoonright n| - h(n)) = -\infty,$$

where $T \upharpoonright n = \{\sigma \in 2^n : \sigma \in T\}$.

A set $E \subseteq 2^\omega$ is *modified upper box-counting $h$-null* if there is a sequence $\{T_i\}_{i\in\omega}$ of trees such that $E \subseteq \bigcup_{i\in\omega}[T_i]$ and $T_i$ is upper box-counting $h$-null for every $i \in \omega$. We simply say that such a set $E$ is *very $h$-small*. We say that a set $E \subseteq 2^\omega$ is *Martin-Löf (resp. Schnorr) very $h$-small* if there is a computable sequence $\{T_i\}_{i\in\omega}$ of c.e. (resp. computable) trees $T_i$ such that $E \subseteq \bigcup_{i\in\omega}[T_i]$ and $T_i$ is upper box-counting $h$-null for every $i \in \omega$.

**Remark 6.6.** An effectivization of box-counting dimension has been introduced by Reimann [47], see Downey-Hirschfeldt [13, Section 13.11]. Our terminology "very $h$-smallness" is inspired by the notions of smallness and very smallness in Binns [3]. Here, Binns [3] introduced these notions in the following manner: For a closed set $P \subseteq 2^\omega$, we define $b_P(n)$ to be the least $k$ such that $|P\upharpoonright k| \geq n$. The set $P$ is *small* if $b_P$ is not dominated by a computable function, and *very small* if $b_P$ dominates all computable functions. It is easy to see that a $\Pi^0_1$ class is very small if and only if it is Martin-Löf (Schnorr) very $h$-small for every computable order $h$. Binns [3, 4] has investigated nontrivial behavior of the Muchnik degrees of small and very small $\Pi^0_1$ classes. The relationship between smallness and traceability has been mentioned in Binns and Kjos-Hanssen [5, 6] in the context of reverse mathematics.

We say that an order $h$ is *suitable* if $h(n+1) \leq h(n)+1$ holds. We also consider its inverse function $h^{-1}$, where for every $n \in \omega$, the value $h^{-1}(n)$ is defined to be the least $t$ such that $h(t) \geq n$ holds. Note that the inverse of a suitable order is strictly increasing.

**Theorem 6.7.** *Let $E$ be a subset of $2^\omega$, and let $h$ be a computable suitable order. Then we have (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).*

(i) *There is a computable order $u$ such that, for all $A \in E$,*
$$K(A\upharpoonright h^{-1}(n)) \leq n - u(n), \text{ for almost all } n \in \omega.$$
(ii) *$E$ is Martin-Löf $\mathcal{P}^h$-null.*
(iii) *$E$ is Martin-Löf very $h$-small.*
(iv) *For all $A \in E$, there is a constant $c \in \omega$ such that*
$$C(A\upharpoonright h^{-1}(n)) \leq n + c, \text{ for almost all } n \in \omega.$$

*Proof.* (i)$\Rightarrow$(ii): Put $g = h^{-1}$. Consider the following c.e. tree:
$$T_k = \{\sigma \in 2^{<\omega} : (\forall n \geq k)\ g(n) < |\sigma| \ \rightarrow\ K(\sigma\upharpoonright g(n)) \leq n - u(n)\}.$$
Clearly, $E \subseteq \bigcup_k [T_k]$. Let $W$ be a prefix-free subset of $T_k \cap 2^{\geq n}$ for $n > k$ with $u(n) \geq t$. Then, we can find a prefix-free set $V \subseteq T_k \cap \bigcup_{s\geq n} 2^{g(s)}$ such that $[V] = [W]$ by enumerating all extensions $\tau$ of a string $\sigma \in W$, where $\tau$ is of length $g(s)$ for the least $s$ such that $g(s) \geq |\sigma|$. Then, $h(n+1) \leq h(n)+1$ implies that
$$\sum_{\sigma\in W} 2^{-h(|\sigma|)} \leq \sum_{\sigma\in V} 2^{-h(|\sigma|)} = \sum_{n=k+1}^{\infty} \sum_{\sigma\in V\cap 2^{g(n)}} 2^{-h(g(n))} \leq \sum_{n=k+1}^{\infty} \sum_{\sigma\in V\cap 2^{g(n)}} 2^{-n}$$
$$\leq \sum_{n=k+1}^{\infty} \sum_{\sigma\in V\cap 2^{g(n)}} 2^{-K(\sigma)-u(n)} \leq \sum_{\sigma\in V} 2^{-t}2^{-K(\sigma)} \leq 2^{-t}.$$
Therefore, $\mathcal{P}^h_\infty(T_k) = 0$, for all $k \in \omega$. Hence, $\mathcal{P}^h_\infty(E) = 0$.

(ii)$\Rightarrow$(iii): Clearly, $\mathcal{P}^h_n(T) \geq 2^{-h(m)}|T\upharpoonright m|$ for all $m \geq n$. Therefore,
$$\log \mathcal{P}^h_n(T) \geq \log(2^{-h(m)}|T\upharpoonright m|) = \log|T\upharpoonright m| - h(m).$$

Hence, $\log \mathcal{P}^h_\infty(T) \geq \log \mathcal{P}^h_n(T) \geq \limsup_{n\to\infty}(\log|T\upharpoonright n| - h(n))$. Thus, if $E$ is packing $h$-null, then it is very $h$-small.

(iii)⇒(iv): Assume that $E$ is Martin-Löf very $h$-small via a computable sequence $\{T_i\}_{i\in\omega}$ of c.e. trees. Then, for every $i \in \omega$ there is $k_i \in \omega$ such that $\log|T_i\upharpoonright n| \leq h(n)$, i.e., $|T_i\upharpoonright n| \leq 2^{h(n)}$ for all $n \geq k_i$. Then, we have $|T_i\upharpoonright h^{-1}(n)| \leq 2^n$, whenever $h^{-1}(n) \geq k_i$. Since $h^{-1}$ is strictly increasing, one can easily construct a machine $M(i)$ such that $C_{M(i)}(\sigma) = n$ for all $\sigma \in T_i\upharpoonright h^{-1}(n)$ with $h^{-1}(n) \geq k_i$. □

**Corollary 6.8.** *The following conditions are pairwise equivalent for $E \subseteq 2^\omega$.*

(i) *For all computable $g$, for all $A \in E$,*

$$K(A\upharpoonright g(n)) \leq n, \text{ for almost all } n \in \omega.$$

(ii) *$E$ is Martin-Löf $\mathcal{P}^h$-null, for all computable orders $h$.*
(iii) *$E$ is Martin-Löf very $h$-small, for all computable orders $h$.*

In particular, $A \in 2^\omega$ is anticomplex if and only if it is Martin-Löf $\mathcal{P}^h$-null for every computable order $h$.

**Theorem 6.9.** *Let $E$ be a subset of $2^\omega$, and let $h$ be a computable suitable order. Then we have (i) ⇒ (ii) ⇒ (iii) ⇒ (iv).*

(i) *There are a computable measure machine $M$ and a computable order $u$ such that, for all $A \in E$,*

$$K_M(A\upharpoonright h^{-1}(n)) \leq n - u(n), \text{ for almost all } n \in \omega.$$

(ii) *$E$ is Schnorr $\mathcal{P}^h$-null.*
(iii) *$E$ is Schnorr very $h$-small.*
(iv) *There is a computable measure machine $M$ such that for all $A \in E$, there is a constant $c \in \omega$ such that*

$$C(A\upharpoonright h^{-1}(n)) \leq n + c, \text{ for almost all } n \in \omega.$$

*Proof.* Straightforward. □

**Corollary 6.10.** *The following conditions are pairwise equivalent for $E \subseteq 2^\omega$.*

(i) *For all computable $g$, there is a computable measure machine $M$ such that for all $A \in E$,*

$$K_M(A\upharpoonright g(n)) \leq n, \text{ for almost all } n \in \omega.$$

(ii) *$E$ is Schnorr $\mathcal{P}^h$-null, for all computable orders $h$.*
(iii) *$E$ is Schnorr very $h$-small, for all computable orders $h$.*

In particular, $A \in 2^\omega$ is totally anticomplex (or equivalently, Schnorr trivial) if and only if it is Schnorr $\mathcal{P}^h$-null for every computable order $h$.

## 7. Implication and Separation

Table 3 shows the relationship among lowness properties for pairs of randomness notions. In this section, we show that Table 3 is complete, that is, no further arrows could be added to Table 3.

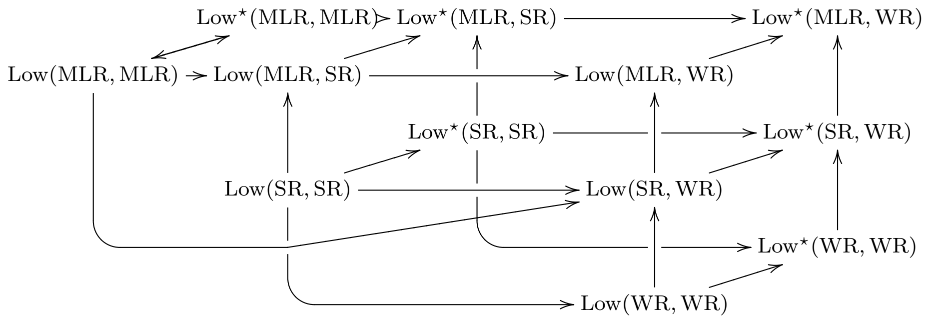


TABLE 3. Implications of Lowness Properties

7.1. **Implications.** Table 3 contains three nontrivial inclusions. We first check that these implications can be deduced from known results.

**Proposition 7.1.** $\mathrm{Low}(\mathrm{MLR},\mathrm{MLR}) \subseteq \mathrm{Low}(\mathrm{SR},\mathrm{WR})$.

*Proof.* Greenberg-Miller [25] characterized $\mathrm{Low}(\mathrm{SR},\mathrm{WR})$ as being neither DNC nor high. If $A \in \mathrm{Low}(\mathrm{MLR},\mathrm{MLR})$, then $A \in \mathrm{Low}(\mathrm{MLR},\mathrm{WR})$, whence $A$ is not DNC. Furthermore, $K$-triviality implies superlow [41], therefore non-high. □

The inclusion $\mathrm{Low}(\mathrm{SR},\mathrm{SR}) \subseteq \mathrm{Low}(\mathrm{WR},\mathrm{WR})$ holds because computable traceability implies computable c.o. traceability. The same argument shows the inclusion $\mathrm{Low}^\star(\mathrm{SR},\mathrm{SR}) \subseteq \mathrm{Low}^\star(\mathrm{WR},\mathrm{WR})$. It is also known that several lowness properties coincide inside non-high or hyperimmune-free Turing degrees as listed below.

(i) $A \in \mathrm{Low}(\mathrm{SR},\mathrm{SR})$ iff $A \in \mathrm{Low}(\mathrm{MLR},\mathrm{SR})$ and $A$ is hyperimmune-free ([34]).
(ii) $A \in \mathrm{Low}(\mathrm{SR},\mathrm{WR})$ iff $A \in \mathrm{Low}(\mathrm{MLR},\mathrm{WR})$ and $A$ is not high ([25]).
(iii) $A \in \mathrm{Low}(\mathrm{WR},\mathrm{WR})$ iff $A \in \mathrm{Low}(\mathrm{SR},\mathrm{WR})$ and $A$ is hyperimmune-free ([25]).
(iv) $A \in \mathrm{Low}(\mathrm{SR},\mathrm{SR})$ iff $\mathrm{Low}^\star(\mathrm{SR},\mathrm{SR})$ and $A$ is hyperimmune-free ([22]).
(v) $A \in \mathrm{Low}(\mathrm{WR},\mathrm{WR})$ iff $\mathrm{Low}^\star(\mathrm{WR},\mathrm{WR})$ and $A$ is hyperimmune-free ([30]).

Indeed, by the characterizations via traceability, it is easy to see that for every hyperimmune-free $A$, $A \in \mathrm{Low}(\mathrm{R},\mathrm{S})$ if and only if $A \in \mathrm{Low}^\star(\mathrm{R},\mathrm{S})$ for any randomness notions $\mathrm{R},\mathrm{S} \in \{\mathrm{MLR},\mathrm{SR},\mathrm{WR}\}$. Therefore, we indeed have the following.

(vi) $A \in \mathrm{Low}(\mathrm{SR},\mathrm{SR})$ iff $A \in \mathrm{Low}^\star(\mathrm{MLR},\mathrm{SR})$ and $A$ is hyperimmune-free.
(vii) $A \in \mathrm{Low}(\mathrm{WR},\mathrm{WR})$ iff $A \in \mathrm{Low}^\star(\mathrm{MLR},\mathrm{WR})$ and $A$ is hyperimmune-free.

However, there must be a difference between $\mathrm{Low}(\mathrm{SR},\mathrm{SR})$ and $\mathrm{Low}(\mathrm{WR},\mathrm{WR})$ within hyperimmune-free Turing degrees. We also have another correspondence within non-high degrees.

**Proposition 7.2.** *If $A$ is not high, then $A \in \mathrm{Low}^\star(\mathrm{MLR},\mathrm{WR})$ if and only if $A \in \mathrm{Low}^\star(\mathrm{SR},\mathrm{WR})$.*

*Proof.* Assume that $A$ is not high, and $A \in \mathrm{Low}^\star(\mathrm{MLR},\mathrm{WR})$. Then $A$ is c.e. i.o. $tt$-traceable. It suffices to show that $A$ is computably i.o. $tt$-traceable. Fix a computable function $g \leq_{tt} A$. Let $(T_n)_{n\in\omega}$ be a c.e. trace of $n \mapsto g(n)$. Given a computable enumeration $(T_{n,s})_{n,s\in\omega}$ and $k$, we can $A$-computably find the least stage $t(k)$ such that $g(n) \in T_{n,t(k)}$ for at least $k+1$ many $n \in \omega$. Since $A$ is not high, we have a computable function $v$ that is not dominated by $t$. We may assume that $v$ is non-decreasing.

Then, we construct a computable trace $T^* = (T^*_n)_{n\in\omega}$ by putting $T^*_n = T_{n,v(n)}$ for every $n \in \omega$. For any $n \in \omega$, if $v(n) \geq t(n)$, there are at least $n+1$ many $s$ such that $g(s) \in T_{s,v(n)}$. In particular, there must be $s \geq n$ such that $g(s) \in T_{s,v(n)} \subseteq T_{s,v(s)}$. By our choice of $v$, such $n$ occur infinitely often. Hence, $g$ is traced by $T^*$ infinitely often. □

7.2. **Separations.** Franklin [24] noted the following separations.

(i) $\mathrm{Low}(\mathrm{SR},\mathrm{SR}) \not\subseteq \mathrm{Low}(\mathrm{MLR},\mathrm{MLR})$.
(ii) $\mathrm{Low}(\mathrm{MLR},\mathrm{MLR}) \not\subseteq \mathrm{Low}^\star(\mathrm{SR},\mathrm{SR})$.

Assertion (i) can be easily verified by looking at the cardinalities of the two sets, and assertion (ii) is verified by constructing a 1-generic $K$-trivial set, where such a generic cannot be Schnorr trivial. By looking at the distribution of generic reals, we easily see the following separation result.

**Proposition 7.3.** $\mathrm{Low}(\mathrm{SR},\mathrm{WR}) \not\subseteq \mathrm{Low}^\star(\mathrm{WR},\mathrm{WR})$.

*Proof.* A weak 1-generic real cannot also be in $\mathrm{Low}^\star(\mathrm{WR},\mathrm{WR})$, since weak 1-genericity implies Kurtz randomness. However, every 1-generic real is diagonally computable, and no 2-generic real is high. Hence, every 2-generic real should be contained in $\mathrm{Low}(\mathrm{SR},\mathrm{WR})$. □

Note that the above genericity argument clearly implies the following.

(iii) $\mathrm{Low}(\mathrm{MLR},\mathrm{MLR}) \not\subseteq \mathrm{Low}^\star(\mathrm{WR},\mathrm{WR})$.

**Proposition 7.4.** $\mathrm{Low}(\mathrm{WR},\mathrm{WR}) \not\subseteq \mathrm{Low}^\star(\mathrm{MLR},\mathrm{SR})$.

*Proof.* Higuchi-Kihara [26] constructed a small perfect $\Pi^0_1$ class $P \subseteq 2^\omega$ containing no anticomplex element. Here, recall the definition of smallness from Remark 6.6. It was shown in Higuchi-Kihara [26] that if a $\Pi^0_1$ classes is small, then it is Martin-Löf $\mathcal{H}^h$-null for every computable order $h$. By effective compactness of $P$, if it is covered by a c.e. open set $W_n$, we can effectively find a clopen subset of $W_n$ that covers $P$ uniformly in $n$. Hence, $P$ is indeed Kurtz $\mathcal{H}^h$-null for every computable order $h$. Therefore, $P$ is included in $\mathrm{Low}^\star(\mathrm{WR},\mathrm{WR})$ by Kihara-Miyabe [30, Theorem 6.1]. By hyperimmune-free basis theorem (see for instance, Downey-Hirschfeldt [13, Theorem 2.19.11]), $P$ has a hyperimmune-free element, and every hyperimmune-free element $A \in P$ is contained in $\mathrm{Low}(\mathrm{WR},\mathrm{WR})$ as seen in Section 7.1. The property of $P$ implies that $A$ is not anticomplex. Hence, we have $A \notin \mathrm{Low}^\star(\mathrm{MLR},\mathrm{SR})$. □

**Proposition 7.5.** $\mathrm{Low}^\star(\mathrm{SR},\mathrm{SR}) \not\subseteq \mathrm{Low}(\mathrm{MLR},\mathrm{WR})$.

*Proof.* Recall that $\mathrm{Low}(\mathrm{MLR},\mathrm{WR})$ is characterized as being not autocomplex, and $\mathrm{Low}^\star(\mathrm{SR},\mathrm{SR})$ is characterized by Schnorr triviality by Franklin-Stephan [20]. Kanovich showed that a c.e. set $A$ is autocomplex if and only if it is Turing equivalent to the halting problem (see Downey-Hirschfeldt [13, Theorem 8.16.7]). Downey, Griffiths and LaForte [12, Theorem 8] showed that there exists a c.e. Schnorr trivial set which is Turing equivalent to the halting problem. Hence, an autocomplex Schnorr trivial real exists. □

**Proposition 7.6.** $\mathrm{Low}(\mathrm{MLR},\mathrm{SR}) \not\subseteq \mathrm{Low}^\star(\mathrm{SR},\mathrm{WR})$.

*Proof.* Kjos-Hanssen, Nies and Stephan [34] characterized the class Low(MLR, SR) by c.e. traceability. Kjos-Hanssen and Nies [33] showed that there exists a super-high jump traceable set. Thus, there exists a high c.e. traceable real (see also Franklin, Greenberg, Stephan and Wu [18, Section 4.1]). Note that c.e. traceability is invariant under the Turing equivalence. Furthermore every high degree contains a Schnorr random real ([43]; see also Downey and Hirschfeldt [13, Theorem 8.11.6]). In particular, such a real cannot be contained in $\mathrm{Low}^\star(\mathrm{SR}, \mathrm{WR})$. ☐

Consequently, we conclude that Table 3 is complete.

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