Computability of Subsets of Metric Spaces

Zvonko Iljazović and Takayuki Kihara

Abstract We present a survey on computability on subsets of Euclidean space and, more generally, computability concepts on metric spaces and their subsets. In particular, we discuss computability of points in co-c.e. closed sets, representations of hyperspaces, Borel codes, computability of connectedness notions, classification of Polish spaces, computability of semicomputable sets, continua and manifolds, properties of computable images of a segment, and computability structures.

1 Introduction

To investigate computability in analysis and related areas, we need a language for taking about computability of complex numbers, compact sets, manifolds, etc. There is a general consensus regarding computability of real and complex numbers. However, what do they mean by a computable compact set, a computable measurable set, a computable Borel set, a computable manifold, and so on?

There have already been a number of reasonable answers to this question. There are also various introductory materials on computability of basic concepts in analysis and related fields, cf. [80, 96, 12, 10, 76, 79]. This survey collects the answers to the above question from the modern perspective.

Computable analysis has become a flourishing field; as a result the researches are very diverse. Each researcher needs the notion of computability at an appropriate level of abstraction. Therefore, in this survey, we introduce the notion of com-

Zvonko Iljazović
Department of Mathematics, Faculty of Science, University of Zagreb, Croatia, e-mail: zilj@math.hr

Takayuki Kihara
Department of Mathematical Informatics, Graduate School of Informatics, Nagoya University, Japan, e-mail: kihara@i.nagoya-u.ac.jp

putability of sets step by step, namely, from subsets of Euclidean spaces to subsets of abstract represented spaces.

However, the real purpose of this survey is not merely introducing fundamental notions of computability in analysis. In accordance with its rapid progress, computable analysis has been becoming a matured field. A large part of the researches are no longer at the stage of discussing basic definitions and producing expected results, but at the stage of producing unexpected, surprising results with sophisticated techniques.

From the authors' own point of view, we select remarkable results related to computability of sets (the selection is by no means exhaustive, of course), and attempt to sketch how the computability notions have brought us an enormous number of highly nontrivial and astonishing results.

Here we summarize the structure of this survey. In Section 2, we discuss the notion of computability of a subset of Euclidean space. Then, in Section 3, we introduce various notions of computability of subsets of computable metric spaces. In Section 4, we give a survey on degrees of noncomputability of points in co-c.e. closed sets (also known as Π^0_1 classes) from the perspective of computable analysis. In Section 5 we introduce the notion of computability of closed and compact sets in more abstract settings. Namely, we consider the hyperspaces of closed sets and compact sets, which enable us to introduce computability of sets as computability of points in hyperspaces. We also consider computability of Borel sets in Section 5.3. In Section 6, we consider computability of path-connectivity, local connectivity, etc. In Section 7, we sketch how the structure of degrees of noncomputability of points in a Polish space is affected by the global structure of the space itself with the emphasis on topological dimension theory. In Section 8, we give a survey on various conditions under which a semicomputable set is computable. In Section 9, we state some results about computable images of a segment. In Section 10, we consider computability structures on metric spaces.

2 Computable Subsets of Euclidean Space

In this section we discuss several natural ways to define the notion of a computable subset of Euclidean space.

A real number x is computable if it can be effectively approximated by a rational number with arbitrary precision. A point x in Euclidean space $\mathbb{R}^n$ is computable if it can be effectively approximated by a rational point $q \in \mathbb{Q}^n$ with arbitrary precision. Similarly, we may say that a subset S of $\mathbb{R}^n$ is computable if it can be effectively approximated by rational points with arbitrary precision. Of course, here we need to make precise what an effective approximation by rational points means.

Let d be the Euclidean metric on $\mathbb{R}^n$. Let $A, B \subseteq \mathbb{R}^n$ and $\varepsilon > 0$. We will say that A and B are *ε-close* if for each $x \in A$ there exists $y \in B$ such that $d(x,y) < \varepsilon$ and for each $y \in B$ there exists $x \in A$ such that $d(x,y) < \varepsilon$.

It is easy to conclude that for each compact set $S \subseteq \mathbb{R}^n$ and each $\varepsilon > 0$ there exists a finite subset A of $\mathbb{Q}^n$ such that S and A are ε-close. In view of this, it is natural to define that a compact set $S \subseteq \mathbb{R}^n$ is computable if for each $k \in \mathbb{N}$ we can effectively find a finite subset A_k of $\mathbb{Q}^n$ such that S and A_k are 2^{-k}-close. Intuitively, the finite set of points with rational coordinates A_k represents the image of the set S and this image becomes sharper as k becomes larger.

Another way to define the notion of a computable subset of Euclidean space is to follow the standard definition of a computable subset of $\mathbb{N}^n$. In classical computability theory a set $S \subseteq \mathbb{N}^n$ is computable if its characteristic function $\chi_S : \mathbb{N}^n \to \mathbb{N}$ is computable. However, if $S \subseteq \mathbb{R}^n$, $S \neq \emptyset$, $S \neq \mathbb{R}$, then the function $\chi_S : \mathbb{R}^n \to \mathbb{R}$ is not continuous and hence not computable. Therefore, it does not make sense to define that a subset of Euclidean space is computable if its characteristic function is computable. But, there is a suitable replacement for the characteristic function, namely for $S \subseteq \mathbb{R}^n$, $S \neq \emptyset$, we may consider the distance function $d_S : \mathbb{R}^n \to \mathbb{R}$ defined by

$$d_S(x) = d(x, S).$$

It is reasonable to consider here closed sets since they are uniquely determined by their distance functions. We will say that a closed set $S \subseteq \mathbb{R}^n$ is computable if the function $d_S : \mathbb{R}^n \to \mathbb{R}$ is computable. Intuitively, this means that for a given $x \in \mathbb{R}^n$ we can compute how close x lies to S (although, in general, we cannot effectively determine whether $x \in S$ or $x \notin S$).

To introduce the notion of a computable subset of $\mathbb{R}^n$ we may also proceed in the following way. We first define the notion of a computably enumerable (c.e.) subset of $\mathbb{R}^n$ and then we define that $S \subseteq \mathbb{R}^n$ is computable if S and $\mathbb{R}^n \setminus S$ are c.e. (following the classical fact: $S \subseteq \mathbb{N}^n$ is computable if S and $\mathbb{N}^n \setminus S$ are c.e.).

A subset S of $\mathbb{R}^n$ may be uncountable, so it does not make much sense to define that S is c.e. if it is the image of a computable function $\mathbb{N} \to \mathbb{R}^n$. What we can do here is to define that S is c.e. if it is the closure of the image of such a function (or $S = \emptyset$). In other words, S is c.e. if $S = \emptyset$ or there exists a computable sequence in $\mathbb{R}^n$ which is dense in S. It is also reasonable to assume that S is closed (in this case the given sequence uniquely determines S).

On the other hand, if S is closed, $\mathbb{R}^n \setminus S$ is open and to define that $\mathbb{R}^n \setminus S$ is c.e. we need another notion of computable enumerability. An open set $U \subseteq \mathbb{R}^n$ will be called c.e. open if it can be effectively exhausted by open balls. More precisely, U is c.e. open if $U = \emptyset$ or

$$U = \bigcup_{i \in \mathbb{N}} B(x_i, r_i),$$

where (x_i) is a computable sequence in $\mathbb{R}^n$ and (r_i) a computable sequence of positive real numbers. Here, for $a \in \mathbb{R}^n$ and $s > 0$, $B(a, s)$ denotes the open ball of radius s centered in a. So, we will say that $S \subseteq \mathbb{R}^n$ is computable if S is c.e. closed and $\mathbb{R}^n \setminus S$ is c.e. open.

The second and third definition given in this section coincide, and all three definitions coincide if S is compact [12]. In the next section we examine computability of sets in more general ambient spaces, namely in computable metric spaces.

3 Computable Metric Spaces

To describe various computability notions in Euclidean space, such as those given in the previous section, we actually only have to fix some effective enumeration $\alpha : \mathbb{N} \to \mathbb{Q}^n$ of $\mathbb{Q}^n$ or, more generally, some computable sequence $\alpha : \mathbb{N} \to \mathbb{R}^n$ whose image is dense in $\mathbb{R}^n$. This motivates the study of the notion of a computable metric space.

A *computable metric space* is a triple (X,d,α), where (X,d) is a metric space and $\alpha = (\alpha_i)$ is a sequence in X whose image is dense in (X,d) and such that the function $\mathbb{N}^2 \to \mathbb{R}$, $(i,j) \mapsto d(\alpha_i,\alpha_j)$, is computable. If d is a complete metric, then we also say that (X,d,α) is a *computable Polish space*. For an introduction to computable metric spaces, we refer the reader to [5, 95, 12, 70, 29].

Let (X,d,α) be a computable metric space. A point $x \in X$ is said to be *computable* in (X,d,α) if there exists a computable function $f : \mathbb{N} \to \mathbb{N}$ such that $d(x,\alpha_{f(k)}) < 2^{-k}$ for each $k \in \mathbb{N}$. A sequence (x_i) in X is said to be *computable* in (X,d,α) if there exists a computable function $f : \mathbb{N}^2 \to \mathbb{N}$ such that $d(x_i,\alpha_{f(i,k)}) < 2^{-k}$ for all $i,k \in \mathbb{N}$.

Example 3.1. Let $n \in \mathbb{N}$, $n \geq 1$, let $\alpha : \mathbb{N} \to \mathbb{Q}^n$ be a computable surjection and let d be the Euclidean metric on $\mathbb{R}^n$. Then $(\mathbb{R}^n,d,\alpha)$ is a computable metric space. It is easy to conclude that $x \in \mathbb{R}^n$, $x = (x_1,\dots,x_n)$, is a computable point in $(\mathbb{R}^n,d,\alpha)$ if and only if $x_1,\dots,x_n$ are computable numbers. Moreover, a sequence (x_i) in $\mathbb{R}^n$ is computable in $(\mathbb{R}^n,d,\alpha)$ if and only if the component sequences of (x_i) are computable as functions $\mathbb{N} \to \mathbb{R}$. We say that $(\mathbb{R}^n,d,\alpha)$ is the *computable Euclidean space*.

A *computable normed space* $(X,\|\cdot\|,e)$ is a separable normed space $(X,\|\cdot\|)$ together with a numbering $e : \mathbb{N} \to X$ such that the linear span of $\mathrm{rng}(e)$ is dense in X, and the induced metric space is a computable metric space. A complete computable normed space is called a *computable Banach space*. If a computable normed space is also a Hilbert space, then it is called a *computable Hilbert space*. For basics on these notions, see also Pour-El and Richards [80].

3.1 Computable Compact and Closed Sets

From now on, let (Δ_j) be some fixed effective enumeration of all finite subsets of $\mathbb{N}$. If (X,d,α) is a computable metric space and $j \in \mathbb{N}$, let

$$\Lambda_j = \{\alpha_i \mid i \in \Delta_j\}. \tag{1}$$

Clearly, (Λ_j) is an enumeration of all finite subsets of $\{\alpha_i \mid i \in \mathbb{N}\}$.

Let (X,d) be a metric space. That two subsets A and B of X are ε-*close* can be defined in the same way as in the case of Euclidean spaces (Section 2). For nonempty compact sets A and B in (X,d) we define their *Hausdorff distance*

$$d_H(A,B) = \inf\{\varepsilon > 0 \mid A \text{ and } B \text{ are } \varepsilon\text{-close}\}. \tag{2}$$

It is not hard to conclude that $d_H(A,B) < \varepsilon$ if and only if A and B are ε-close.

If (X,d,α) is a computable metric space and S a nonempty compact set in (X,d), then for each $\varepsilon > 0$ there exists $j \in \mathbb{N}$ such that $d_H(S,\Lambda_j) < \varepsilon$.

Let (X,d,α) be a computable metric space and let S be a compact set in (X,d). We say that S is a *computable compact set* in (X,d,α) if $S = \emptyset$ or there exists a computable function $f : \mathbb{N} \to \mathbb{N}$ such that $d_H(S,\Lambda_{f(k)}) < 2^{-k}$ for each $k \in \mathbb{N}$.

Example 3.2. Let (X,d,α) be a computable metric space and let $\mathscr{K}$ be the set of all nonempty compact sets in (X,d). Then the function $d_H : \mathscr{K} \times \mathscr{K} \to \mathbb{R}$ defined by (2) is a metric on $\mathscr{K}$ [73]. Let $\Lambda = (\Lambda_j)$ be the sequence defined by (1). It is easy to conclude that Λ is a dense sequence in the metric space $(\mathscr{K},d_H)$. Furthermore, the function $\mathbb{N}^2 \to \mathbb{R}$, $(i,j) \mapsto d_H(\Lambda_i,\Lambda_j)$, is computable (see e.g. Proposition 2.5 in [41]). Hence $(\mathscr{K},d_H,\Lambda)$ is a computable metric space. Note that computable points in $(\mathscr{K},d_H,\Lambda)$ are exactly nonempty computable compact sets in (X,d,α).

A computable metric space (X,d,α) is said to be *effectively compact* if X is a computable compact set in (X,d,α).

If (X,d) is a metric space, $x \in X$ and $r > 0$, by $B(x,r)$ we will denote the open ball in (X,d) of radius r centered in x and by $\overline{B}(x,r)$ the corresponding closed ball.

Let (X,d,α) be a computable metric space, $n \in \mathbb{N}$ and $r \in \mathbb{Q}$, $r > 0$. We say that $B(\alpha_n,r)$ is a *rational open ball* in (X,d,α) and $\overline{B}(\alpha_n,r)$ a *rational closed ball* in (X,d,α). Let $\tau_1,\tau_2 : \mathbb{N} \to \mathbb{N}$ and $q : \mathbb{N} \to \mathbb{Q}$ be some fixed computable functions such that the image of q is the set of all positive rational numbers and $\{(\tau_1(i),\tau_2(i)) \mid i \in \mathbb{N}\} = \mathbb{N}^2$. For $i \in \mathbb{N}$ we define $\lambda_i = \alpha_{\tau_1(i)}$, $\rho_i = q_{\tau_2(i)}$ and

$$I_i = B(\lambda_i,\rho_i), \ \ \hat{I}_i = \overline{B}(\lambda_i,\rho_i).$$

Then (I_i) is an enumeration of all rational open balls and $(\hat{I}_i)$ is an enumeration of all rational closed balls in (X,d,α).

Let (X,d,α) be a computable metric space and let S be a closed set in (X,d). We say that S is a *computably enumerable closed* set in (X,d,α) (or merely a *computably enumerable* set in (X,d,α)) if $\{i \in \mathbb{N} \mid I_i \cap S \neq \emptyset\}$ is a c.e. subset of $\mathbb{N}$.

Suppose (X,d,α) is a computable metric space, S a closed set in (X,d) and (x_j) a computable sequence in (X,d,α) which is dense in S, i.e. such that $S = \overline{\{x_j \mid j \in \mathbb{N}\}}$. (Here, by $\overline{A}$, for $A \subseteq X$, we denote the closure of A in the metric space (X,d).) Then S is a c.e. set in (X,d,α) [12]. So the following implication holds:

$$S \text{ contains a dense computable sequence } \Rightarrow S \text{ c.e.} \tag{3}$$

The converse of implication (3) does not hold in general [4].

Let (X,d) be a metric space and $S \subseteq X$. We say that S is a *complete* set in (X,d) if $S = \emptyset$ or $S \neq \emptyset$ and $(S,d|_{S\times S})$ is a complete metric space.

If S is a complete set in a metric space (X,d), then S is closed in (X,d). Conversely, a closed set in (X,d) need not be complete, however if the metric space (X,d) is complete, then each closed set in (X,d) is complete.

Although the converse of the implication (3) does not hold in general, it does hold if S is a nonempty complete set. Hence, if (X,d,α) is a computable metric space and S is a nonempty c.e. set in this space which is complete in (X,d), then S contains a dense sequence which is computable in (X,d,α) (see [46]). In particular, if (X,d) is a complete metric space, then each nonempty c.e. set in (X,d,α) contains a dense computable sequence [12].

Let (X,d,α) be a computable metric space and let $U \subseteq X$. We say that U is a *computably enumerable open* set in (X,d,α) if there exists a c.e. set $A \subseteq \mathbb{N}$ such that $U = \bigcup_{i\in A} I_i$. We say that S is *co-computably enumerable* (co-c.e.) *closed* set in (X,d,α) if $X \setminus S$ is a c.e. open set in (X,d,α).

Let (X,d,α) be a computable metric space and let $S \subseteq X$. We say that S is a *computable closed* set in (X,d,α) if S is c.e. and co-c.e. closed in (X,d,α).

Let (X,d,α) be a computable metric space, $n \geq 1$ and $B_1,\dots,B_n$ rational open balls in this space. Then we say that $B_1 \cup \dots \cup B_n$ is a *rational open set* in (X,d,α).

If (X,d,α) is a computable metric space and $j \in \mathbb{N}$, let

$$J_j = \bigcup_{i\in\Delta_j} I_i.$$

Then $\{J_j \mid j \in \mathbb{N}\}$ is the family of all rational open sets in (X,d,α).

Let (X,d,α) be a computable metric space and let K be a compact set in (X,d). We say that K is a *semicomputable compact set* in (X,d,α) if the set $\{j \in \mathbb{N} \mid K \subseteq J_j\}$ is c.e.

Less formally, K is semicomputable compact if we can effectively enumerate all rational open sets which cover K.

Let (X,d,α) be a computable metric space and let $K \subseteq X$. Then the following equivalence holds (see [41]):

$$K \text{ computable compact } \iff K \text{ c.e. and } K \text{ semicomputable compact.} \tag{4}$$

The notion of a semicomputable compact set can be generalized in the following way. Let (X,d,α) be a computable metric space and let $S \subseteq X$ be such that

(i) $S \cap B$ is a compact set in (X,d) for each closed ball B in (X,d);
(ii) the set $\{(i,j) \in \mathbb{N}^2 \mid S \cap \hat{I}_i \subseteq J_j\}$ is c.e.

Then we say that S is a *semicomputable set* in (X,d,α).

If S is compact in (X,d) and semicomputable in (X,d,α), then it is easy to conclude that S is semicomputable compact in (X,d,α). The converse of this implication also holds (see Proposition 3.3 in [14]), hence the following equivalence holds:

$$S \text{ compact and } S \text{ semicomputable } \iff S \text{ semicomputable compact.} \tag{5}$$

So, the notion of a semicomputable set generalizes the notion of a semicomputable compact set. In view of (4), we extend the notion of a computable compact set.

Let (X,d,α) be a computable metric space and let $S \subseteq X$. We say that S is a *computable set* in (X,d,α) if S is c.e. and semicomputable.

By (4) and (5) we have

$$S \text{ computable compact } \iff S \text{ compact and } S \text{ computable.}$$

Condition (i) from the definition of a semicomputable set easily implies that each semicomputable set in (X,d,α) is closed in (X,d). Moreover, we have the following result (see Proposition 3.5 in [14]).

Proposition 3.3. *Let (X,d,α) be a computable metric space. Then each semicomputable set in this space is co-c.e. closed. Consequently, each computable set in (X,d,α) is a computable closed set in (X,d,α).*

In general, a co-c.e. closed set need not be semicomputable. Also, a computable closed set need not be a computable set. Namely, in Example 3.2 in [40] a computable metric space $([0,b],d,\alpha)$ was constructed, where b is a positive real number and d is the Euclidean metric on $[0,b]$, such that $\{b\}$ is a co-c.e. closed set, but b is not a computable point in this space. In general, it is easy to conclude that in a computable metric space a point x is computable if and only if the set $\{x\}$ is semicomputable (see page 10 in [14]). Therefore, $\{b\}$ is not a semicomputable set in $([0,b],d,\alpha)$. Moreover, $[0,b]$ is a computable closed set in this space (in general, if (X,d,α) is a computable metric space, then X is clearly a computable closed set in (X,d,α)), but $[0,b]$ is not a computable set in this space: it is not semicomputable, which follows from Example 3.2 in [40].

However, under certain conditions on the ambient space, the notions of a semicomputable set and a co-c.e. closed set coincide.

Let (X,d,α) be a computable metric space such that the set $\{(i,j)\in\mathbb{N}^2 \mid \hat{I}_i \subseteq J_j\}$ is c.e. Then we say that (X,d,α) has the *effective covering property* [12].

The following theorem gives a sufficient condition that a computable metric space has the effective covering property (see [37]).

Theorem 3.4. *Let (X,d,α) be a computable metric space such that each closed ball in (X,d) is compact. Suppose that there exists a computable point a_0 and a computable sequence (x_i) in this space and a computable function $F:\mathbb{N}^2\to\mathbb{N}$ such that $B(a_0,m)\subseteq\bigcup_{0\le i\le F(m,k)}B(x_i,2^{-k})$ for all $m,k\in\mathbb{N}$, $m\ge 1$. Then (X,d,α) has the effective covering property.*

Using Theorem 3.4, it is easy to conclude that the computable Euclidean space has the effective covering property.

Example 3.5. Let I^∞ denote the set of all sequences in $[0,1]$. It is known that the metric d on I^∞ defined by $d((x_i),(y_i))=\sum_{i=0}^{\infty}\frac{1}{2^i}|x_i-y_i|$ induces a topology which coincides with the product topology on I^∞. The metric space (I^∞,d) is compact (Tychonoff's theorem) and it is called the *Hilbert cube*.

Let $r:\mathbb{N}\to\mathbb{Q}$ be a computable function whose range is $[0,1]\cap\mathbb{Q}$. Let $\sigma:\mathbb{N}^2\to\mathbb{N}$ and $\eta:\mathbb{N}\to\mathbb{N}$ be computable functions such that each nonempty finite sequence in $\mathbb{N}$ equals $(\sigma(i,0),\dots,\sigma(i,\eta(i)))$ for some $i\in\mathbb{N}$ (such functions certainly exist). We define $\alpha:\mathbb{N}\to I^\infty$ by $\alpha_i=(r_{\sigma(i,0)},\dots,r_{\sigma(i,\eta(i))},0,0,\dots)$. Then (I^∞,d,α) is a computable metric space. Using Theorem 3.4 it is not hard to conclude that (I^∞,d,α) has the effective covering property (see [37]).

The proof of the next proposition can be found in [14] (Proposition 3.6).

Proposition 3.6. *Let (X,d,α) be a computable metric space which has the effective covering property and compact closed balls. Let $S \subseteq X$. Then S is co-c.e. closed if and only if S is semicomputable. Consequently, S is a computable closed set if and only if S is a computable set.*

4 Non-Computability of Points in Co-C.E. Closed Sets

4.1 Basis Theorems in Computability Theory

In classical computability theory, a lot of energy has been devoted to the study of the Turing degrees of points in subsets of an underlying space (mostly $2^{\mathbb{N}}$ or $\mathbb{N}^{\mathbb{N}}$). This field was pioneered by Kleene in 1950s, who showed that

1. There is a nonempty co-c.e. closed subset of $\mathbb{R}$ with no computable points.
2. There is a nonempty co-c.e. closed subset of $\mathbb{N}^{\mathbb{N}}$ with no Δ^1_1 points.

The above results are sometimes referred as Kleene's non-basis theorems. These theorems were a starting point of the long-running study of degrees of points in co-c.e. closed sets. As a second step, Kreisel proved the following basis theorems.

3. Every nonempty co-c.e. closed subset of $\mathbb{R}$ has a $\mathbf{0}'$-computable point.
4. Every nonempty co-c.e. singleton in $\mathbb{R}$ is computable.

These basis theorems fail for non-σ-compact spaces such as $\mathbb{N}^{\mathbb{N}}$. Indeed, Jockusch-McLaughlin [49] pointed out that for any computable ordinal α,

5. there is a co-c.e. singleton $\{x\}$ in $\mathbb{N}^{\mathbb{N}}$ such that x is not $\mathbf{0}^{(\alpha)}$-computable,

where $\mathbf{0}^{(\alpha)}$ is the α-th Turing jump. This kind of bad behavior of a co-c.e. closed set led us to the notion of semicomputability. For further studies on the degrees of co-c.e. singletons in $\mathbb{N}^{\mathbb{N}}$, see [18, 87] and [75, Chapters XII and XIII].

Regarding (3), the Kreisel basis theorem actually shows that the *leftmost* point of a nonempty co-c.e. closed set $P \subseteq [0,1]$ is *left-c.e.*, that is, the supremum of a computable sequence of rationals. It should be carefully noted that the notion "left-c.e." makes no sense at all in $[0,1]^n$ for $n \geq 2$.

In the higher dimensional case, the following analog of left-c.e. is useful. For $n \leq \omega$, a point $x = (x_i)_{i<n} \in [0,1]^n$ is *n-left-CEA* if x_0 is left-c.e., and x_{i+1} is left-c.e. relative to x_i uniformly in i. More formally, there is a computable sequence $(g_i)_{i<n}$ of computable functions $g_i : [0,1]^i \to \mathbb{Q}^{\mathbb{N}}$ such that $x_i = \sup_n g_i(x_0,\dots,x_{i-1})(n)$.

Given $n \leq \omega$ and a nonempty closed set $P \subseteq [0,1]^n$, inductively define the *leftmost point* $(x_i)_{i<n}$ of P as follows. Define x_k as the smallest value such that P has a point whose first $k+1$ coordinates are $(x_0,\dots,x_k)$. By compactness of P, such a point exists. Then, it is easy to get a higher-dimensional analog of Kreisel's basis theorem.

Proposition 4.1 (see Kihara-Pauly [57]). *For any $n \leq \omega$, the leftmost point in a nonempty co-c.e. closed subset of $[0,1]^n$ is n-left-CEA.*

The Kreisel basis theorem has been refined by Jockusch-Soare's so-called low basis theorem. The importance of the low basis theorem is that the standard proof is applicable for any effectively compact computable metric space.

Given a computable metric space $\mathscr{X}$, its presentation automatically involves a computable list $(G_e)_{e\in\mathbb{N}}$ of c.e. open sets in $\mathscr{X}$. Then, the *Turing jump* of a point $x \in \mathscr{X}$ is defined by $x' = \{e \in \mathbb{N} : x \in G_e\}$. This generalization of the Turing jump has desirable properties; see Gregoriades-Kihara-Ng [28]. We say that a point $x \in \mathscr{X}$ is *low* if x' is Turing reducible to $\mathbf{0}'$.

Theorem 4.2 (Low Basis Theorem; Jockusch-Soare [50]). *Every nonempty co-c.e. closed set in an effectively compact computable metric space contains a low point.*

Proof. Let P be a nonempty co-c.e. closed subset of an effectively compact computable metric space $\mathscr{X}$. We construct a $\mathbf{0}'$-computable decreasing sequence $(Q_e)_{e\in\mathbb{N}}$ of co-c.e. closed sets in $\mathscr{X}$. Define $Q_0 = P$. By effective compactness, we can decide $Q_e \subseteq G_e$ using $\mathbf{0}'$ uniformly in e. Put $Q_{e+1} = Q_e$ if $Q_e \subseteq G_e$; otherwise, put $Q_{e+1} = Q_e \setminus G_e$. For any $z \in \bigcap_{e\in\mathbb{N}} Q_e$, clearly, $z'(e) = 1$ if and only if $Q_e \subseteq G_e$. This concludes that $z' \leq_T \mathbf{0}'$ since the latter condition is $\mathbf{0}'$-computable. □

A uniform version of the low basis theorem has also been proved by Brattka et al. [8]. As a historical remark, the original low basis theorem [50] has been proved in the context of degrees of theories. A *PA-degree* is a Turing degree $\mathbf{d}$ such that every co-c.e. closed subset of $2^{\mathbb{N}}$ has a $\mathbf{d}$-computable point.

We shall emphasize that our introduction of basis theorems only scratches the surface of extremely deep studies on co-c.e. closed sets (also known as Π^0_1 classes). We refer the interested reader to Cenzer [15] and Diamondstone et al. [22] for more detailed introduction on degree-theoretic analysis of co-c.e. closed sets. Detailed analysis of basis theorems has also been carried out from the perspective of *Medvedev degrees* and *Muchnik degrees*, cf. [86, 35, 34].

4.2 Basis Theorems in Computable Analysis

In computable analysis, we deal with a variety of geometric and topological properties of co-c.e. closed sets. When restricting our attention to co-c.e. closed sets possessing such global properties, basis and non-basis theorems often exhibit an interesting behavior. We first introduce a classical example in this direction. In the early stages of computability theory, speaking of global properties, they were always associated with *measure* and *category*. For instance,

Theorem 4.3 (Kreisel-Lacombe [58]). *There is a co-c.e. closed subset of $[0,1]$ of positive Lebesgue measure that contains no computable point.*

A co-c.e. closed set constructed in Theorem 4.3 is totally disconnected; otherwise, it contains a nonempty interval, and has a computable point. This trivial observation has became a source of new basis and non-basis theorems. From the geometric viewpoint, an interval is *convex*. From the topological viewpoint, an interval is *connected*. For the geometric side, Le Roux-Ziegler [61] observed that a nonempty convex co-c.e. closed set in $\mathbb{R}^n$ contains a computable point. In fact,

Theorem 4.4 (Neumann [74]). *Every nonempty convex co-c.e. closed subset of a finite dimensional computable Banach space contains a computable point.*

Surprisingly, however, it is not true in infinite dimensional spaces. This fact is first implicitly mentioned by Miller [68]. Later it is shown that there is a computable dynamical system without computable invariant measures [25], where the set of invariant measures in a computable system forms a compact convex co-c.e. set.

Theorem 4.5 ([68, 25, 74]; see also Theorem 7.2). *There exists a nonempty convex co-c.e. closed subset of the Hilbert cube $[0,1]^{\mathbb{N}}$ containing no computable points.*

For the topological side, it naturally raises the question whether every *connected* co-c.e. closed set contains a computable point. It is trivially false as pointed out by Le Roux-Ziegler [61].

Example 4.6. If A is a co-c.e. closed subset of $[0,1]$ with no computable element, then the *Cantor tartan* given by $([0,1]\times A)\cup(A\times[0,1])$ is a connected co-c.e. closed subset of $[0,1]^2$ with no computable points. Similarly, $([0,1]^2\times A)\cup([0,1]\times A\times[0,1])\cup(A\times[0,1]^2)$ is a simply connected co-c.e. closed subset of $[0,1]^3$ with no computable points.

A topological space X is *n-connected* if it is pathwise connected and $\pi_i(X)\equiv 0$ for any $1\leq i\leq n$, where $\pi_i(X)$ is the i-th homotopy group of X. A space X is *simply connected* if X is 1-connected. By a similar construction as in Example 4.6, one can get a nonempty n-connected, but not $(n+1)$-connected, co-c.e. closed set in $[0,1]^{n+2}$ which contains no computable points.

A space X is *contractible* if the identity map on X is null-homotopic. Note that, if X is contractible, then X is n-connected for each $n\geq 1$. A higher dimensional variant of a Cantor tartan is never contractible. Thus, to construct a contractible co-c.e. closed set with no computable points, we need a different approach. By a *curve*, we mean a one-dimensional nondegenerate continuum. By a *continuum*, we mean a compact and connected metric space.

Theorem 4.7 (Kihara [53]). *There exists a contractible, co-c.e., planar curve which contains no computable points.*

Proof (Sketch). Let $C\subseteq[0,1]$ be a co-c.e. closed set with no computable points, and A be a computable arc, one of whose endpoints is non-computable (see Miller [67, Example 4.1]). Imagine the cone space $(C\times A)/(C\times\{a\})$, where a is a unique non-computable end-point of A. This gives us a Cantor fan with no computable points although it is unclear if it is co-c.e. or if it computably embeds into the Euclidean plane. However, a slight modification of this construction makes a fan co-c.e. in $[0,1]^2$. For more details, see [53]. □

As indicated in the above sketch, the example given by Kihara [53] is topologically homeomorphic to the Cantor fan. All known finite-dimensional co-c.e. closed sets with no computable points are not locally connected.

Question 4.8. Does there exist a nonempty, locally connected, co-c.e. closed subset of $[0,1]^n$ for some $n \in \mathbb{N}$ containing no computable points?

Note that Theorem 4.5 implies the existence of a nonempty, locally connected, co-c.e. closed subset of the Hilbert cube $[0,1]^{\mathbb{N}}$ which contains no computable points since every convex set is locally connected.

As a historical remark, basis theorems in classical computable analysis have sometimes been associated with mass problems. A *mass problem* is a subset of a (represented) space, which appears as the set of solutions of a mathematical problem. Several mathematical problems in algebra, analysis, combinatorics, etc. have been found to be represented as co-c.e. closed subsets of certain computable metric spaces (cf. Cenzer-Remmel [16]). Degrees of difficulty of mass problems are often measured by Medvedev and Muchnik reducibility. We refer the reader to Simpson [84, 86] for Muchnik degrees of co-c.e. closed sets. The concept of mass problems is strongly tied with Reverse Mathematics [85], and the study of Weihrauch degrees as well (see the last Chapter in this handbook).

5 Represented Spaces and Uniform Computability

In previous sections, we only took account of *non-uniform computability*. In this section, we introduce the notion of a represented space, which gives us a language for talking about *uniform computability*. For basics on represented spaces, see Weihrauch [96]. We also refer the reader to Pauly [76] for an excellent introduction to the theory of represented spaces.

Let $\mathscr{X} = (X, d, \alpha)$ be a computable metric space. Then, a *Cauchy name* of a point $x \in X$ is a sequence $p \in \mathbb{N}^{\mathbb{N}}$ such that $d(x, \alpha_{p(k)}) < 2^{-k}$ for any $k \in \mathbb{N}$. This notion induces a partial surjection $\delta :\subseteq \mathbb{N}^{\mathbb{N}} \to X$ defined by

$$\delta(p) = x \iff p \text{ is a Cauchy name of } x.$$

This surjection δ is called the *Cauchy representation of* X (induced from (d, α)).

In general, a *represented space* is a pair $\mathscr{X} = (X, \delta_X)$ of a set X and a partial surjection $\delta_X :\subseteq \mathbb{N}^{\mathbb{N}} \to X$. Such δ_X is called a *representation of* X. If $\delta_X(p) = x$, then p is called a δ_X*-name of* x (or simply, a *name of* x if δ_X is clear from the context). A point $x \in \mathscr{X}$ is *computable* if x has a computable name. A function $f : \mathscr{X} \to \mathscr{Y}$ is *computable* (*continuous*, resp.) if there is a partial computable (continuous, resp.) function on $\mathbb{N}^{\mathbb{N}}$ which, given a name of $x \in \mathscr{X}$, returns a name of $f(x) \in \mathscr{Y}$. In general, a partial function $\Phi :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is a *realizer* of f if for any name p of $x \in \mathscr{X}$, $\Phi(p)$ is a name of $f(x)$. By definition, f is computable if and only if f has a computable realizer.

Remark 5.1. The notion of computability on computable metric spaces coincides with the notion of computability on represented spaces (w.r.t. the induced Cauchy representation). Thus, the theory of represented spaces generalizes the classical theory on computable metric spaces. Indeed, this epoch-making theory makes it possible to develop computability theory on an extremely wide class of topological spaces including various non-Hausdorff spaces, non-second-countable spaces, etc. cf. [76]. More precisely, Schröder [81, 82] showed that a T_0 space is admissibly represented if and only if it has a countable cs-network (a *cs-network* is a variant of Arhangel'skii's notion of a network introduced by Guthrie [31]).

Remark 5.2. As a related concept, the notion of a numbered set has been extensively studied in the theory of numbering (see Ershov [23]). A *numbered set* is a represented space (X, δ_X) such that the domain of δ_X is (effectively homeomorphic to) the natural numbers $\mathbb{N}$. There is a unification of these concepts. In realizability theory [93, 1], a represented space is called a *modest set*, and a multi-represented space is called an *assembly*. To be precise, a represented space is a modest set over Kleene's second (relative) algebra, i.e., the (relative) partial combinatory algebra given by Kleene's functional realizability, and a numbered set is a modest set over Kleene's first algebra, i.e., the pca given by Kleene's number realizability. We refer the reader to Bauer [1] for more details. This unification is useful since many generalized computation models such as infinite time Turing machines induce pcas [2], and thus we do not have to reinvent the wheel for generalized computable analysis.

5.1 Represented Hyperspaces

The notion of a representation provides us an abstract way of introducing computability on subsets of a space by considering a *represented hyperspace*.

By $A(X)$, we denote the set of all closed subsets of a computable metric space $\mathscr{X} = (X, d, \alpha)$. Recall that $(I_i)_{i\in\mathbb{N}}$ is a list of rational open balls in $\mathscr{X}$. For $p \in \mathbb{N}^{\mathbb{N}}$, we write $\mathrm{rng}(p) = \{p(n) - 1 : p(n) > 0\}$. We first introduce two representations ψ_+ and ψ_- of $A(X)$ capturing c.e. closed sets and co-c.e. closed sets, respectively.

$$\psi_+(p) = S \iff \mathrm{rng}(p) = \{n : S \cap I_n \neq \emptyset\},$$
$$\psi_-(p) = S \iff S = X \setminus \bigcup \{I_n : n \in \mathrm{rng}(p)\}.$$

The representations ψ_+ and ψ_- correspond to the lower Fell topology and the upper Fell topology on the hyperspace $A(X)$ of closed subsets of X (cf. [12]). We then consider represented spaces $\mathscr{A}_+(\mathscr{X}) = (A(X), \psi_+)$, and $\mathscr{A}_-(\mathscr{X}) = (A(X), \psi_-)$. It is clear that the computable points in $\mathscr{A}_+(\mathscr{X})$ and $\mathscr{A}_-(\mathscr{X})$ are exactly the c.e. closed sets and the co-c.e. closed sets, respectively. One can also get a representation capturing computable closed sets as follows.

$$\psi_\pm(p \oplus q) = S \iff \psi_+(p) = \psi_-(q) = S,$$

where $(p \oplus q)(2n) = p(n)$ and $(p \oplus q)(2n+1) = q(n)$. Then, the computable points in $\mathscr{A}_\pm(\mathscr{X})$ are exactly the computable closed sets. Note that some authors use $\mathscr{A}(\mathscr{X})$ to denote $\mathscr{A}_\pm(\mathscr{X})$, while some other authors use $\mathscr{A}(\mathscr{X})$ to denote $\mathscr{A}_-(\mathscr{X})$.

We next introduce a representation of the hyperspace $K(X)$ of compact subsets of X. For a computable metric space $\mathscr{X} = (X, d, \alpha)$, recall that $(J_j)_{j\in\mathbb{N}}$ is the list of rational open sets in $\mathscr{X}$. Then, we define

$$\kappa_-(p) = S \iff \mathrm{rng}(p) = \{j \in \mathbb{N} : S \subseteq J_{p(j)}\}.$$

We also define $\kappa_\pm(p \oplus q) = S$ if and only if $\psi_+(p) = \kappa_-(q) = S$. We then define $\mathscr{K}_-(\mathscr{X}) = (K(X), \kappa_-)$ and $\mathscr{K}_\pm(\mathscr{X}) = (K(X), \kappa_\pm)$. The computable points in $\mathscr{K}_-(\mathscr{X})$ and $\mathscr{K}_\pm(\mathscr{X})$ are exactly the semicomputable compact sets and the computable compact sets, respectively.

The notion of a represented hyperspace enables us to discuss *uniform* computability of operations on closed and compact subsets of a computable metric space. For instance, consider the union and the intersection of co-c.e. closed sets. It is clear that if A and B are co-c.e. closed subsets of $\mathscr{X}$, so are $A \cup B$ and $A \cap B$. Indeed, the union and the intersection $\cup, \cap : \mathscr{A}_-(\mathscr{X}) \times \mathscr{A}_-(\mathscr{X}) \to \mathscr{A}_-(\mathscr{X})$ are computable, that is, given names of A and B, one can effectively find names of $A \cup B$ and $A \cap B$. In this wise, the notion of computability on represented spaces automatically involves uniformity.

From the uniform perspective, the negative representation is quite well-behaved. Actually, most basic operations on $\mathscr{A}_-(\mathscr{X})$ are known to be computable. For the positive representation, as shown in Brattka-Weihrauch [13], even the intersection $\cap : \mathscr{A}_\pm(\mathbb{R}^n)^2 \to \mathscr{A}_+(\mathbb{R}^n)$ is not computable. Indeed, for a T_1-space $\mathscr{X}$, $\cap : \mathscr{A}_+(\mathscr{X})^2 \to \mathscr{A}_+(\mathscr{X})$ is computable, iff $\mathscr{X}$ is computably discrete [76].

There are a number of results regarding uniform computability of operations on hyperspaces. For instance, let chull be the map which, given a closed set, returns its *convex hull*. Then, $\mathrm{chull} : \mathscr{A}_+(\mathbb{R}^n) \to \mathscr{A}_+(\mathbb{R}^n)$ and $\mathrm{chull} : \mathscr{A}_-([0,1]^n) \to \mathscr{A}_-([0,1]^n)$ are computable [104, 60]. This useful result gives us a computable enumeration of all co-c.e. closed convex sets in $[0,1]^n$ while there is no limit computable way of deciding convexity of a co-c.e. closed set (cf. [79]). For further studies on computability on operation on hyperspaces, see Brattka-Presser [12]. For further reading on computability on other hyperspaces, we refer the reader to [32, 103, 104] for regular closed sets, and to [100, 101, 99] for measurable sets.

5.2 Represented Function Spaces

There is a way of viewing a hyperspace of closed sets as a function space. To see this, we first explain an important nature of represented spaces: *The category of represented spaces and (relatively) computable functions is cartesian closed*. This follows from a more general fact that the category $Mod(\underset{\sim}{\Sigma}, \Sigma)$ of modest sets over a relative pca $\langle \underset{\sim}{\Sigma}, \Sigma \rangle$ is cartesian closed (cf. Bauer [1]). The following is the details.

By Φ_e^z, we denote the e-th partial computable function on $\mathbb{N}^\mathbb{N}$ relative to an oracle $z \in \mathbb{N}^\mathbb{N}$. Let $e^\frown z$ denote the concatenation of e and z, that is, $(e^\frown z)(0) = e$ and $(e^\frown z)(n+1) = z(n)$. If $\mathscr{X}$ and $\mathscr{Y}$ are represented spaces, the set of relatively computable functions from $\mathscr{X}$ to $\mathscr{Y}$ is represented as follows.

$$\eta(e^\frown z) = f \iff \text{if } p \text{ is a name of } x \in \mathscr{X}, \text{ then } \Phi_e^z(p) \text{ is a name of } f(x) \in \mathscr{Y}.$$

In other words, Φ_e^z is a realizer of f. By $\mathscr{C}(\mathscr{X},\mathscr{Y})$ we denote the space of relatively computable functions from $\mathscr{X}$ to $\mathscr{Y}$ represented by η. Clearly, the computable points in $\mathscr{C}(\mathscr{X},\mathscr{Y})$ are exactly the computable functions from $\mathscr{X}$ to $\mathscr{Y}$.

Consider the set $S = \{\top, \bot\}$ represented by

$$\delta_S(p) = \begin{cases} \top & \text{if } (\exists n)\ p(n) \neq 0, \\ \bot & \text{if } (\forall n)\ p(n) = 0. \end{cases}$$

We call $\mathbb{S} = (S, \delta_S)$ the *represented Sierpiński space* [81]. Assume that $\mathscr{X}$ is a represented space. Then, we can think of the function space $\mathscr{C}(\mathscr{X},\mathbb{S})$ as the *hyperspace of open sets in* $\mathscr{X}$, by identifying a function $f : \mathscr{X} \to \mathbb{S}$ with the open set $f^{-1}\{\top\} \subseteq \mathscr{X}$. Similarly, this space can also be viewed as the *hyperspace of closed sets*, by identifying a function $f : \mathscr{X} \to \mathbb{S}$ with the closed set $f^{-1}\{\bot\} \subseteq \mathscr{X}$. Via this identification, the representation η of the space $\mathscr{C}(\mathscr{X},\mathbb{S})$ yields the *Sierpiński representation* ψ_{Sier} *of* the hyperspace $A(X)$ of closed subsets of X as follows.

$$\psi_{\mathrm{Sier}}(p) = S \iff S = \eta(p)^{-1}\{\bot\}$$

We now claim that ψ_{Sier} is equivalent to ψ_-. Given representations δ and η of a set X, we say that δ is *reducible to* η if there is a computable function which, given a δ-name of $x \in X$, returns an η-name of x. In other words, $\mathrm{id} : (X, \delta) \to (X, \eta)$ is computable. We say that δ is *equivalent to* η if δ is bireducible to η. Under this definition, one can show the following.

Proposition 5.3 (Brattka-Presser [12]). *The negative representation ψ_- of the hyperspace of closed subsets of a computable metric space is equivalent to the Sierpiński representation ψ_{Sier}.*

The represented hyperspace $\mathscr{K}_-(\mathscr{X})$ of compact sets can also be viewed as a function space. Let $\mathscr{O}(\mathscr{X})$ be the hyperspace of open sets in $\mathscr{X}$ represented as above, i.e., $\mathscr{O}(\mathscr{X}) \simeq \mathscr{C}(\mathscr{X},\mathbb{S})$. A space $\mathscr{X}$ is compact, iff the universal quantifier $\forall_\mathscr{X} : \mathscr{O}(\mathscr{X}) \to \mathbb{S}$ is continuous, where $\forall_\mathscr{X}(X) = \top$ and $\forall_\mathscr{X}(U) = \top$ for $U \neq X$. Thus, a subset Y of $\mathscr{X}$ is compact, iff $A_Y : \mathscr{O}(\mathscr{X}) \to \mathbb{S}$ is continuous, where

$$A_Y(U) = \begin{cases} \top & \text{if } Y \subseteq U, \\ \bot & \text{if } Y \not\subseteq U. \end{cases}$$

In other words, $Y \subseteq \mathscr{X}$ is compact, iff $A_Y \in \mathscr{O}\mathscr{O}(\mathscr{X})$. Note that $A_Y = A_Z$ iff Y and Z have the same saturation (cf. [76]). Thus, this notion yields a representation

$\kappa_\forall$ of saturated compact sets:

$$\kappa_\forall(p) = K \iff p \text{ is an } \mathscr{O}\mathscr{O}(\mathscr{X})\text{-name of } A_K.$$

One can easily see that $\kappa_\forall$ is equivalent to κ_- (for the hyperspace of compact subsets of a computable metric space).

The dual notion of compactness is known as overtness [90]. A space $\mathscr{X}$ is *overt*, iff the existential quantifier $\exists_{\mathscr{X}} : \mathscr{O}(\mathscr{X}) \to \mathbb{S}$ is continuous, where $\exists_{\mathscr{X}}(U) = \top$ for $U \neq \emptyset$ and $\exists_{\mathscr{X}}(\emptyset) = \bot$. Thus, a subset Y of $\mathscr{X}$ is overt, iff $E_Y : \mathscr{O}(\mathscr{X}) \to \mathbb{S}$ is continuous, where

$$E_Y(U) = \begin{cases} \top & \text{if } Y \cap U \neq \emptyset, \\ \bot & \text{if } Y \cap U = \emptyset. \end{cases}$$

Although every subset $Y \subseteq \mathscr{X}$ is known to be overt, this definition yields a nontrivial (multi-)represented space $\mathscr{V}(\mathscr{X})$ of overt subsets of $\mathscr{X}$ by identifying $Y \subseteq \mathscr{X}$ with $E_Y \in \mathscr{O}\mathscr{O}(\mathscr{X})$. Obviously, $E_Y = E_Z$ iff Y and Z have the same topological closure; hence it induces a representation $\psi_\exists$ of the hyperspace of closed subsets of $\mathscr{X}$:

$$\psi_\exists(p) = Y \iff p \text{ is an } \mathscr{O}\mathscr{O}(\mathscr{X})\text{-name of } E_Y.$$

It is clear that $\psi_\exists$ is equivalent to the positive representation ψ_+ (for the hyperspace of closed subsets of a computable metric space).

In this way, computability theory on hyperspaces is absorbed into computability theory on function spaces. It should be carefully noted that the function-space representations ψ_{Sier}, $\kappa_\forall$, and $\psi_\exists$ are defined for the hyperspaces of *any* represented spaces, while the hyperspace representations ψ_-, ψ_+, κ_-, etc. make sense only for the hyperspaces of computable metric spaces. The representations introduced in this section are essentially due to Schröder [81]. The term "overt" is due to Taylor [90]. This framework has become fundamental in various contexts such as Escardó's *synthetic topology* [24] and Taylor's *abstract Stone duality* [90, 91]. See also Pauly [76] for more detailed study on represented hyperspaces in the language of function spaces.

5.3 Borel Codes

A representation of Borel subsets of $\mathbb{R}$ (widely known as *Borel codes*) was first introduced by Solovay [88] to define the notion of a *random real over a model*. Solovay further explored the theory of Borel codes in his monumental work [89] on a model of Zermelo-Fraenkel (ZF) set theory in which all sets of reals are Lebesgue measurable. Since then, his representation of Borel sets has been a fundamental notion almost everywhere in set theory.

Here we only deal with Borel sets of finite rank. Let $\mathscr{X}$ be a computable metric space. We define representations σ^0_n and π^0_n of $\utilde{\Sigma}^0_n$ and $\utilde{\Pi}^0_n$ subsets of $\mathscr{X}$ as follows.

$$\pi^0_1(p) = \psi_-(p), \qquad \sigma^0_1(p) = \mathscr{X} \setminus \pi^0_1(p),$$
$$\pi^0_n(p) = \mathscr{X} \setminus \sigma^0_n(p), \qquad \sigma^0_{n+1}(p) = \bigcup_{i\in\mathbb{N}} \pi^0_n(p_i),$$

where recall that ψ_- is the negative representation of the hyperspace of closed subsets of $\mathscr{X}$. By $\boldsymbol{\Sigma}^0_n(\mathscr{X})$ and $\boldsymbol{\Pi}^0_n(\mathscr{X})$, we denote the hyperspaces of $\boldsymbol{\Sigma}^0_n$ and $\boldsymbol{\Pi}^0_n$ subsets of $\mathscr{X}$ represented by σ^0_n and π^0_n, respectively. By definition, $\boldsymbol{\Pi}^0_1(\mathscr{X})$ is identical with $\mathscr{A}_-(\mathscr{X})$. In particular, the computable points in the spaces $\boldsymbol{\Sigma}^0_1(\mathscr{X})$ and $\boldsymbol{\Pi}^0_1(\mathscr{X})$ are the c.e. open sets and the co-c.e. closed sets, respectively. In general, a computable point in $\boldsymbol{\Sigma}^0_n(\mathscr{X})$ ($\boldsymbol{\Pi}^0_n(\mathscr{X})$, resp.) is called a Σ^0_n set (a Π^0_n set, resp.)

A function $f : \mathscr{X} \to \mathscr{Y}$ is *$\boldsymbol{\Sigma}^0_n$-measurable* if the preimage of an open set under f is $\boldsymbol{\Sigma}^0_n$. By second-countability of $\mathscr{Y}$, this is equivalent to saying that $f^{-1} : \boldsymbol{\Sigma}^0_1(\mathscr{Y}) \to \boldsymbol{\Sigma}^0_n(\mathscr{X})$ is continuous. We say that $f : \mathscr{X} \to \mathscr{Y}$ is *Σ^0_n-computable* if $f^{-1} : \boldsymbol{\Sigma}^0_1(\mathscr{Y}) \to \boldsymbol{\Sigma}^0_n(\mathscr{X})$ is computable (see Brattka [6]). Clearly, $\boldsymbol{\Sigma}^0_1$-measurability and Σ^0_1-computability are equivalent to continuity and computability, respectively. The correspondence between the Borel hierarchy and the Baire hierarchy (the Banach-Hausdorff-Lebesgue theorem) was effectivized by Brattka as follows.

Theorem 5.4 (Brattka [6]). *Let $\mathscr{X}$ and $\mathscr{Y}$ be computable metric spaces, and let $k \geq 2$. Then, any Σ^0_{k+1}-computable function $f : \mathscr{X} \to \mathscr{Y}$ is the pointwise limit of a computable sequence of Σ^0_k-computable functions. For $\mathscr{X} = \mathbb{N}^\mathbb{N}$ this holds true in case $k = 1$ as well.*

In other words, Σ^0_{n+1}-computability is equivalent to n-th iterated limit computability (which can be thought of as a uniform version of the Shoenfield Limit Lemma). By using this notion, we can talk about the degrees of noncomputability of operations on represented spaces. Borel complexity of operations on represented hyperspaces has been studied by Gherardi [26] and Brattka-Gherardi [9].

Example 5.5 (Gherardi [26]). The intersection $\cap : \mathscr{A}(\mathbb{R}^n) \times \mathscr{A}(\mathbb{R}^n) \to \mathscr{A}(\mathbb{R}^n)$ is Σ^0_2-computable, but not computable. The intersection $\cap : \mathscr{A}_+(\mathbb{R}^n) \times \mathscr{A}_+(\mathbb{R}^n) \to \mathscr{A}_+(\mathbb{R}^n)$ is Σ^0_3-computable, but not Σ^0_2-computable.

In his work on functional analysis, Jayne [47] introduced a finer hierarchy of Borel functions. For $f : \mathscr{X} \to \mathscr{Y}$, we write $f^{-1}\boldsymbol{\Sigma}^0_m \subseteq \boldsymbol{\Sigma}^0_n$ if the preimage of a $\boldsymbol{\Sigma}^0_m$ set under f is $\boldsymbol{\Sigma}^0_n$. In this terminology, Σ^0_n-measurability is described as $f^{-1}\boldsymbol{\Sigma}^0_1 \subseteq \boldsymbol{\Sigma}^0_n$. By using Louveau's separation theorem [62] in effective descriptive set theory, Gregoriades-Kihara-Ng [28] showed that the property $f^{-1}\boldsymbol{\Sigma}^0_m \subseteq \boldsymbol{\Sigma}^0_n$ is equivalent to that $f^{-1} : \boldsymbol{\Sigma}^0_m(\mathscr{Y}) \to \boldsymbol{\Sigma}^0_n(\mathscr{X})$ has a Borel realizer. However, it is open whether the property $f^{-1}\boldsymbol{\Sigma}^0_m \subseteq \boldsymbol{\Sigma}^0_n$ is equivalent to that $f^{-1} : \boldsymbol{\Sigma}^0_m(\mathscr{Y}) \to \boldsymbol{\Sigma}^0_n(\mathscr{X})$ is continuous.

The Jayne-Rogers theorem [48] states that for a function f from an analytic subset $\mathscr{X}$ of a Polish space to a separable metric space, $f^{-1}\boldsymbol{\Sigma}^0_2 \subseteq \boldsymbol{\Sigma}^0_2$ if and only if it is closed-piecewise continuous, that is, there is a closed cover $(P_n)_{n\in\mathbb{N}}$ of $\mathscr{X}$ such that $f \upharpoonright P_n$ is continuous for any $n \in \mathbb{N}$.

We consider effective versions of Jayne's Borel hierarchy and piecewise continuity. A *computable Π^0_n cover* of $\mathscr{X}$ is a computable sequence $(P_n)_{n\in\mathbb{N}}$ of Π^0_n

subsets of $\mathscr{X}$ such that $\mathscr{X} = \bigcup_n P_n$. We say that $f : \mathscr{X} \to \mathscr{Y}$ is *Π^0_n-piecewise Σ^0_m-computable* if there is a computable Π^0_n cover of $\mathscr{X}$ such that the restriction $f \upharpoonright P_n$ is Σ^0_m-computable uniformly in $n \in \mathbb{N}$. If $m = 1$, we simply say that f is Π^0_n-piecewise computable. The notion of Π^0_1-piecewise computability is equivalent to *computability with finite mindchanges*, which has turned out to be a very important notion in computable analysis (cf. [8, 21]). Then, the Jayne-Rogers theorem is effectivized as:

Theorem 5.6 (Pauly-de Brecht [77]). *Let $f : \mathscr{X} \to \mathscr{Y}$ be a function between computable metric spaces $\mathscr{X}$ and $\mathscr{Y}$. Then, $f^{-1} : \underset{\sim}{\Sigma}^0_2(\mathscr{Y}) \to \underset{\sim}{\Sigma}^0_2(\mathscr{X})$ is computable if and only if f is Π^0_1-piecewise computable.*

Soon after, Kihara [54] found that Theorem 5.6 can be generalized to higher Borel ranks whenever $\mathscr{X}$ and $\mathscr{Y}$ are finite dimensional. The notion of topological dimension suddenly appeared out of nowhere! The reason is later clarified by Kihara-Pauly [57]: In [54], the Shore-Slaman join theorem in Turing degree theory was a key tool for generalizing Theorem 5.6; however, the degree structure of an infinite dimensional computable metric space is generally different from the Turing degrees (see Sections 7.2 and 7.3). After this discovery, Gregoriades-Kihara-Ng [28] introduced a variant of the Kumabe-Slaman forcing to generalize the Shore-Slaman join theorem in the setting of "infinite dimensional" Turing degree theory, and then succeeded to remove the dimension-theoretic restriction from the former result [54].

Theorem 5.7 (Gregoriades-Kihara-Ng [28]). *Let $f : \mathscr{X} \to \mathscr{Y}$ be a function between computable Polish spaces $\mathscr{X}$ and $\mathscr{Y}$, and assume that $n < 2m$. Then, $f^{-1} : \underset{\sim}{\Sigma}^0_{m+1}(\mathscr{Y}) \to \underset{\sim}{\Sigma}^0_{n+1}(\mathscr{X})$ is computable if and only if f is Π^0_n-piecewise Σ^0_{n-m+1}-computable.*

An open problem is whether we can remove the assumption $n < 2m$ from Theorem 5.7. For other directions of investigation on piecewise computability, it is also important to think about decomposition into finitely many computable functions. For $k \in \mathbb{N}$, we say that $f : \mathscr{X} \to \mathscr{Y}$ is *k-Γ-piecewise computable* if there is an increasing sequence $(G_i)_{i<k}$ of Γ sets such that $\mathscr{X} = \bigcup_{i<k} G_i$ and $f \upharpoonright G_i \setminus G_{i-1}$ is computable. Then, $(k+1)$-Π^0_1-piecewise computability corresponds to *computability with at most k mindchanges* [21], and $(k+1)$-Δ^0_2-piecewise computability corresponds to *computability with finite mindchanges and at most k errors* [33]. For further reading on piecewise computability, see also de Brecht [21] and Kihara [55].

The notion of Borel codes in the context of represented spaces has also been studied in Gregoriades et al. [29] in detail. Borel codes are also used to introduce (multi-)representations of Borel-generated σ-ideals such as Lebesgue null sets and meager sets (cf. [56]). Representations of such σ-ideals are evidently useful when talking about randomness, genericity, forcing, etc. as Solovay did. This way of thinking has become ubiquitous in modern set theory.

For further direction, Pauly-de Brecht [78] recently proposed *synthetic descriptive set theory* as a reinterpretation of descriptive set theory (DST) in the category-theoretic context. One of the core ideas of synthetic DST is the use of endofunctors. An endofunctor is a functor from a category to itself. They introduced specific endofunctors relevant for the study of DST, e.g. the finite mindchange endofunctor $^\nabla$

and the jump endofunctor $'$. Applying $^\nabla$ and $'$ to the hyperspace of open sets yields the hyperspaces of $\utilde{\Delta}^0_2$ sets and $\utilde{\Sigma}^0_2$ sets in the Borel hierarchy, respectively. In this way, synthetic DST provides a language for talking about descriptive set theoretic concepts in a unified category-theoretic manner.

6 Computability of Connectedness Notions

The notion of represented spaces is useful for introducing effective versions of various topological concepts. In this section, we will use represented spaces to introduce the notions of effective pathwise connectivity, effective local connectivity, etc., and then we will address a few computability-theoretic works involving these notions.

6.1 Effective Connectivity Properties

Consider the notion of pathwise connectivity and arcwise connectivity. Every arcwise connected space is pathwise connected, and the converse also holds for Hausdorff spaces. However, Miller [67] first observed that the notions of effective pathwise connectivity and effective arcwise connectivity do not coincide even for computable closed subsets of $[0,1]^2$.

Example 6.1 (Miller [67, Example 5.1]). There is a planar arc A with computable end points such that A is computable closed, A is the image of a computable function $f:[0,1]\to[0,1]^2$, but A cannot be the image of a computable injection $g:[0,1]\to[0,1]^2$.

Informally, we say that a space $\mathscr{X}$ is computably pathwise connected if, given (names of) points $x,y\in\mathscr{X}$, one can effectively find (a name of) a continuous function $f:[0,1]\to\mathscr{X}$ such that $f(0)=x$ and $f(1)=y$.

This yields the notion of computability of a *multi-valued* function. For represented spaces $\mathscr{X}$ and $\mathscr{Y}$, a multi-valued function $F:\mathscr{X}\rightrightarrows\mathscr{Y}$ is computable if there is a computable (single-valued) function Φ which, given a name of a point $x\in\mathscr{X}$, returns a name of an element of $F(x)\in\mathscr{X}$. Note that Φ does not necessarily induce a function from $\mathscr{X}$ to $\mathscr{Y}$, that is, even if p_0 and p_1 are names of the same point $x\in\mathscr{X}$, $\Phi(p_0)$ and $\Phi(p_1)$ can be names of different points $y_0\neq y_1$ in $F(x)$.

Brattka [7] formalized the notion of effective pathwise connectivity as follows. We say that a computable metric space $\mathscr{X}$ is *effectively pathwise connected* if the multi-valued function $F:\mathscr{X}^2\rightrightarrows\mathscr{C}([0,1],\mathscr{X})$ defined by

$$F(x,y)=\{f\in\mathscr{C}([0,1],\mathscr{X}):f(0)=x\text{ and }f(1)=y\}$$

is computable. Note that if F has a single-valued continuous selection, then $\mathscr{X}$ has to be contractible. Thus, multi-valuedness of the above definition is essential!

One can also define various different effectivizations of path/arcwise connectivity. For instance, a computable metric space $\mathscr{X}$ is *[co-c.e. arc]-connected* if the multi-valued function $F : \mathscr{X}^2 \rightrightarrows \mathscr{A}_-(\mathscr{X})$ defined by

$$F(x,y) = \{S \in \mathscr{A}_-(\mathscr{X}) : x,y \in S \text{ and } S \text{ is an arc}\}$$

is computable. It is easy to see that there is a planar curve A such that A is computable closed, effective pathwise connected, but not [co-c.e. arc]-connected [53].

6.2 Computable Graph Theorem

In classical computability theory a function $f : \mathbb{N} \to \mathbb{N}$ is computable if and only if its graph is computable. On the other hand, if X and Y are topological spaces such that Y is compact and Hausdorff, then a function $f : X \to Y$ is continuous if and only if its graph is a closed set in $X \times Y$ [72]. The question is what can be said about a computable version of this result, i.e. if X and Y are computable metric spaces and $f : X \to Y$, under what assumptions the computability of f is equivalent to the fact that the graph of f is a computable closed set.

If (X,d,α) and (Y,d',α') are computable metric spaces, we define their *product* as the computable metric space $(X \times Y, d'', \alpha'')$ defined by

$$d''((x,y),(x',y')) = \max\{d(x,x'),d'(y,y')\} \text{ and } \alpha''\langle i,j\rangle = (\alpha(i),\alpha'(j)).$$

Let (X,d,α) be a computable metric space and let $\mathscr{O}$ be the family of all open sets in (X,d). Let δ be the representation of $\mathscr{O}$ defined by

$$\delta(p) = \bigcup \{I_n \mid n \in \operatorname{rng}(p)\}.$$

We say that (X,d,α) is *effectively locally connected* [7] if there exists a computable multivalued function $C :\subseteq X \times \mathbb{R} \rightrightarrows \mathscr{O}$ such that for all $x \in X$ and $r > 0$ the set $C(x,r)$ is nonempty and each $U \in C(x,r)$ is a connected set such that $x \in U \subseteq B(x,r)$.

Theorem 6.2 (Brattka [7]). *Let X and Y be computable metric spaces. Then the function*

$$\operatorname{graph} : \mathscr{C}(X,Y) \to \mathscr{A}_\pm(X \times Y), \;\; f \mapsto \operatorname{graph}(f) \qquad (6)$$

is computable.

(i) If Y is effectively compact, then the partial inverse $\operatorname{graph}^{-1} :\subseteq \mathscr{A}_-(X,Y) \to \mathscr{C}(X,Y)$ *of the map (6) is computable.*
(ii) If X is effectively locally connected and $Y = \mathbb{R}^n$ for some $n \geq 1$, then the partial inverse $\operatorname{graph}^{-1} :\subseteq \mathscr{A}_\pm(X,Y) \to \mathscr{C}(X,Y)$ *of the map (6) is computable.*
(iii) If X is effectively pathwise connected and $n \geq 1$, then the map

$$F :\subseteq \mathscr{A}_-(X \times \mathbb{R}^n) \times X \times \mathbb{R}^n \to \mathscr{C}(X,\mathbb{R}^n), \;\; (A,a,b) \mapsto \operatorname{graph}^{-1}(A),$$

which is defined for all (A,a,b) such that $A = \mathrm{graph}(f)$ and $f(a) = b$ for some $f \in \mathscr{C}(X,\mathbb{R}^n)$, is computable.

As a consequence of Theorem 6.2 we get the following result.

Corollary 6.3 (Brattka [7]). *Let X and Y be computable metric spaces. Suppose Y is effectively compact and $f : X \to Y$. Then f is computable if and only if $\mathrm{graph}(f)$ is co-c.e. closed and if and only if $\mathrm{graph}(f)$ is computable closed. We get the same conclusion if we assume that X is effectively pathwise connected and $Y = \mathbb{R}^n$ for some $n \geq 1$.*

The additional assumptions on the computable metric spaces in the statement of Corollary 6.3 cannot be omitted: in general it is possible that f is not computable although it is continuous and $\mathrm{graph}(f)$ is computable closed [7].

A computable metric space (X,d,α) is said to be *locally computable* if for each compact set A in (X,d) there exists a computable compact set K in (X,d,α) such that $A \subseteq K$. In the following theorem we get the same conclusion as in Corollary 6.3 but with different assumptions.

Theorem 6.4 (Brattka [7]). *Let X and Y be computable metric spaces such that X is compact and Y is locally computable. Let $f : X \to Y$ be a continuous function. Then f is computable if and only if $\mathrm{graph}(f)$ is co-c.e. closed and if and only if $\mathrm{graph}(f)$ is computable closed.*

6.3 Degrees of Difficulty

A new paradigm brought from the theory of representation enables us to talk about the degrees of difficulty of problems involving hyperspaces. For instance,

1. Find a connected component of a given nonempty co-c.e. closed subset of $[0,1]^n$.
2. Find a nontrivial subcontinuum of a given co-c.e. closed subset of $[0,1]^{\mathbb{N}}$ of positive dimension.

The problem (1) has been studied by Le Roux-Ziegler [61] and Brattka et al. [11]. If a compact metric space is not zero-dimensional, it always has a nondegenerate subcontinuum. The problem (2) has been studied by Kihara [52].

In general topology, there are various strengthening of connectivity. One of those is the notion of a Cantor manifold, which is introduced by Urysohn as one of the most fundamental notions in topological dimension theory. We say that a topological space $\mathscr{X}$ is disconnected by $A \subseteq \mathscr{X}$ if $\mathscr{X} \setminus A$ is a union of disjoint open sets. It is clear that a space is disconnected iff it is disconnected by the empty set $\emptyset$. An *n-dimensional Cantor manifold* is an n-dimensional compact space which is not disconnected by an at most $(n-2)$-dimensional subset.

Kihara [52] recently noticed that the notion of a Cantor manifold has an application in the study of degrees of points in computable metric spaces. A key tool is the Hurewicz-Tumarkin Cantor manifold theorem, which says that every n-dimensional compact metric space contains an n-dimensional Cantor manifold (cf. [36, 92]). An interesting open question is to determine the degree of difficulty of the Cantor manifold theorem. For instance,

Question 6.5. Let $P \subseteq [0,1]^{\mathbb{N}}$ be a co-c.e. closed set of positive dimension. Does every PA-degree computes an $\mathscr{A}_{-}$-name of a Cantor submanifold of P?

7 Classification of Polish Spaces

7.1 Borel Isomorphism Theorem

Kuratowski's Borel isomorphism theorem is one of the most fundamental theorems on Polish spaces, which says that every uncountable Polish space is Borel isomorphic to $\mathbb{R}$. For an effective counterpart, if we replace "uncountable" with "perfect," it is known that every perfect computable Polish space is Δ^1_1-isomorphic to $\mathbb{R}$ (see Moschovakis [70, Section 3I]). However, Gregoriades [27] showed that perfectness is essential for effectivity.

Theorem 7.1 (Gregoriades [27]). *There exists a zero-dimensional, uncountable, computable Polish space which is not Δ^1_1-isomorphic to $\mathbb{R}$.*

Proof. Let $T \subseteq \omega^{<\omega}$ be Kleene's tree, none of whose infinite paths is Δ^1_1, and let $[T]$ be the set of infinite paths through T. Consider $\mathscr{T} = T \cup [T]$, where a basic open set is the set of all extensions of a finite string in T, or the singleton consisting of a finite string. It is easy to give a computable Polish metrization of $\mathscr{T}$. Then T is a c.e. open set consisting of isolated points in $\mathscr{T}$. By our choice of T, the Δ^1_1 points in $\mathscr{T}$ are exactly T; hence, c.e. open in $\mathscr{T}$. Therefore, $\mathscr{T}$ is not Δ^1_1-isomorphic to $\mathbb{R}$ since the set of all Δ^1_1 points in $\mathbb{R}$ is not Δ^1_1. $\square$

Gregoriades [27] then studied the Δ^1_1-embeddability order on (zero-dimensional) computable Polish spaces, and showed that every countable partial order can be embedded into the order. The proof requires a detailed analysis of (hyper)degrees of points in computable Polish spaces. All spaces that appear in his proof computably embed into $\mathbb{N}^{\mathbb{N}}$, and thus, one can just adopt classical degree theory. However, as we will see later, exploring degree theory in an arbitrary computable metric space leads us to the discovery of a new connection between computability and dimension.

7.2 Continuous Degree Theory

Recall that every points in a computable metric space is named by elements in $\mathbb{N}^{\mathbb{N}}$ via the Cauchy representation. We estimate how complicated a point in a computable metric space is by considering the *degree of difficulty of calling a name* of the point. Of course, it is possible for each point to have many names, and this nature yields the phenomenon that there is a point with no easiest names with respect to Turing reducibility.

Let $\mathscr{X}$ and $\mathscr{Y}$ be computable metric spaces. A point $y \in \mathscr{Y}$ *is point-Turing reducible to* $x \in \mathscr{X}$ if there is a computable function Φ that, given a name p of x, returns a name $\Phi(p)$ of y. In other words, there is a partial computable function $f :\subseteq \mathscr{X} \to \mathscr{Y}$ such that $f(x) = y$. This notion was introduced by Miller [68] under the name of *representation reducibility* and *continuous degrees*. We say that a point $x \in \mathscr{X}$ *has a* $\mathscr{Y}$*-degree* if x is point-Turing equivalent to a point in $\mathscr{Y}$. Pour-El and Richards [80] observed that any point in Euclidean space has a $2^{\mathbb{N}}$-degree.

Miller's discovery of non-$2^{\mathbb{N}}$-degrees and the connection between such degrees and Scott ideals is an astounding achievement of the noughties, which has brought a new paradigm in computability theory.

A *Turing ideal* is a set $\mathscr{I} \subseteq 2^{\mathbb{N}}$ which forms an ideal w.r.t. Turing reducibility $\leq_T$, that is, $x \leq_T y \in \mathscr{I}$ implies $x \in \mathscr{I}$, and $x, y \in \mathscr{I}$ implies $x \oplus y \in \mathscr{I}$. A *Scott ideal* is a Turing ideal $\mathscr{I}$ such that for any $x \in \mathscr{I}$, if $P \subseteq 2^{\mathbb{N}}$ is nonempty and co-c.e. closed relative to x (that is, P is computable relative to x as a point in $\mathscr{A}_-(2^{\mathbb{N}})$), then $P \cap \mathscr{I}$ is nonempty.

Theorem 7.2 (Miller [68]).

1. *The Hilbert cube* $[0,1]^{\mathbb{N}}$ *has a point of non-*$2^{\mathbb{N}}$*-degree.*
2. *If* $x \in [0,1]^{\mathbb{N}}$ *has a non-*$2^{\mathbb{N}}$*-degree, then the set of all* $y \in 2^{\mathbb{N}}$ *that are point-Turing reducible to* x *forms a Scott ideal.*
3. *For every countable Scott ideal* $\mathscr{I}$*, there is a point* $x \in [0,1]^{\mathbb{N}}$ *such that* $y \in \mathscr{I}$ *if and only if* y *is point-Turing reducible to* x*.*

Proof (for 1). Let Φ_e be the e-th partial computable function from $[0,1]^{\mathbb{N}}$ to $[0,1]$. By approximating partial computations, one can easily construct a computable function $\psi : [0,1]^{\mathbb{N}} \times \mathbb{N} \to \mathscr{A}_-([0,1])$ such that $\psi(x,e) = \{\Phi_e(x)\}$ if $x \in \mathrm{dom}(\Phi_e)$; otherwise, $\psi(x,e)$ is a nondegenerate interval. Then, $\Psi : [0,1]^{\mathbb{N}} \to \mathscr{A}_-([0,1]^{\mathbb{N}})$ defined by $\Psi(x) = \prod_e \psi(x,e)$ is a computable function with a co-c.e. closed graph such that $\Psi(x)$ is nonempty and convex for any $x \in [0,1]^{\mathbb{N}}$. By Kakutani's fixed point theorem, Ψ has a fixed point $x \in \Psi(x)$. Note that there is a computable multi-valued function $F : [0,1]^{\mathbb{N}} \rightrightarrows [0,1]$ such that $F(x) \neq x(e)$ for any $e \in \mathbb{N}$. In particular, any name of x computes some $y \in [0,1]$ such that $y \notin \{x(e) : e \in \mathbb{N}\}$. If x has a $2^{\mathbb{N}}$-degree, x computes its name, and thus, computes such a y; however, by the property $x \in \Psi(x)$, if x computes y, then $y = x(e)$ for some e, a contradiction. □

Later, Day-Miller [20] observed a similar phenomenon in the theory of algorithmic randomness. The space of Borel probability measures on $2^{\mathbb{N}}$ is computably metrizable (e.g. via the Prokhorov metric). A probability measure μ is called *neutral* if every infinite binary sequence is Martin-Löf random w.r.t. μ. Day-Miller [20] noticed that a neutral measure cannot have a $2^{\mathbb{N}}$-degree, and hence, its lower Turing cone forms a Scott ideal as in Theorem 7.2 (2). They also showed an analog of Theorem 7.2 (3): For every Scott ideal $\mathscr{I}$, there is a neutral measure μ such that $y \in \mathscr{I}$ iff y is point-Turing reducible to μ.

7.3 Computable Aspects of Infinite Dimensionality

The previous works on continuous degrees [68, 20] make crucial use of the Kakutani fixed point theorem (for infinite dimensional spaces). This leads us to the conjecture that the notion of topological dimension is essential in degree theory on computable metric spaces. It becomes more and more plausible by the recent works [54, 28] extending Theorem 5.6 and also by the following result.

Theorem 7.3 (Kihara-Pauly [57]). *The following are equivalent for computable metric spaces $\mathscr{X}$ and $\mathscr{Y}$:*

1. *Every $\mathscr{X}$-degree is a $\mathscr{Y}$-degree.*
2. *There is a countable partition $(\mathscr{X}_i)_{i\in\mathbb{N}}$ of $\mathscr{X}$ such that each $\mathscr{X}_i$ computably embeds into $\mathscr{Y}$.*

By Theorem 7.3 with $\mathscr{Y} = 2^{\mathbb{N}}$, we can characterize the Turing degrees in terms of topological dimension theory. A topological space is called *countable dimensional* if it is a countable union of finite dimensional subspaces. If $\mathscr{X}$ is Polish, it is equivalent to having transfinite small inductive dimension in the sense of Menger-Urysohn (cf. [36]). Then, relative to some oracle, all points in $\mathscr{X}$ have $2^{\mathbb{N}}$-degrees, iff $\mathscr{X}$ is countable dimensional. In particular, Theorem 7.2 (1) is a corollary of Theorem 7.3 since it is known that the Hilbert cube is not countable dimensional (cf. [36, 92]). This simple observation completely solves a mystery about the occurrence of non-$2^{\mathbb{N}}$-degrees in the Hilbert cube (and the space of probability measures).

There is another circumstantial evidence that topological dimension theory is crucial to making a deep study of computable metric spaces. For $n \leq \omega$, an *n-left-CEA operator* is a function $\Gamma : \mathbb{N}^{\mathbb{N}} \to [0,1]^n$ such that $\Gamma(x)$ is n-left-CEA uniformly relative to x. A *universal n-left-CEA operator* is an n-left-CEA operator $\Gamma : \mathbb{N}^{\mathbb{N}} \to [0,1]^n$ such that for any n-left-CEA operator Λ, there is e such that $\Gamma(e^\frown x) = \Lambda(x)$ for any x. Kihara-Pauly [57] showed that the graph of a universal n-left-CEA operator (as a subspace of the Hilbert cube) has an interesting dimension-theoretic property.

1. The graph $\mathbb{G}_n$ of a universal n-left-CEA operator is a totally disconnected n-dimensional Polish space, whose countable product $\mathbb{G}_n^{\mathbb{N}}$ is also n-dimensional.
2. The graph $\mathbb{G}_\omega$ of a universal ω-left-CEA operator is a totally disconnected infinite dimensional Polish space.

We say that $\mathscr{X}$ is finite-level Borel isomorphic to $\mathscr{Y}$ if there is a bijection $f : \mathscr{X} \to \mathscr{Y}$ such that $f^{-1}\underset{\sim}{\Sigma}^0_n \subseteq \underset{\sim}{\Sigma}^0_n$ and $f\underset{\sim}{\Sigma}^0_n \subseteq \underset{\sim}{\Sigma}^0_n$ for some $n \in \mathbb{N}$. We also say that $\mathscr{X}$ finite-level Borel embeds into $\mathscr{Y}$ if $\mathscr{X}$ is finite-level Borel isomorphic to a Borel subset of $\mathscr{Y}$ of finite rank. By applying Theorems 5.7, 7.2 and 7.3, one can show that the graph of a universal ω-left-CEA operator has an intermediate finite-level Borel isomorphism type.

Theorem 7.4 (Kihara-Pauly [57]). *The Hilbert cube does not finite-level Borel embed into $\mathbb{G}_\omega$, and $\mathbb{G}_\omega$ does not finite-level Borel embed into any countable dimensional Polish space.*

Despite of its importance, there are only a few results on *computable topological dimension theory*. The very first step was done by Kenny [51]. Effectivizations of the existence of a Henderson compactum and the Cantor manifold theorem are discussed in Kihara [52]. Recently, McNicholl and Rute took an important next step. They introduced the notion of a *uniform degree*, which is a generalization of truth-table degrees in the setting of computable metric spaces. Then, for instance, they proved that a point in $\mathbb{R}^2$ is contained in a computable arc if and only if it has an $\mathbb{R}$-uniform degree. The notion of a uniform degree is connected to various notions in topological dimension theory. We have the impression that the further development on the generalized truth-table degrees would give new insights into the computability-theoretic nature of topological dimension theory.

8 Computability of Semicomputable Sets

Each computable closed set (and in particular computable) set is clearly co-c.e. closed. Conversely, a co-c.e. closed set need not be computable closed. Moreover, as noted in Section 4, there exists a nonempty co-c.e. subset of $\mathbb{R}$ which does not contain a computable point. Hence, co-c.e. sets can be "far away from being computable". However, it turns out that under certain assumptions we can conclude that a co-c.e. closed set is computable. The pioneer work in this area was made by Miller.

Theorem 8.1 (Miller [67]).

1. *If $S \subseteq \mathbb{R}^m$ is co-c.e. closed and $S \cong \mathbb{S}^n$ for some $n \geq 1$, then S is computable.*
2. *If $D \subseteq \mathbb{R}^m$ is co-c.e. closed and there exists, for some $n \geq 1$, a homeomorphism $f : \mathbb{B}^n \to D$ such that $f(\mathbb{S}^{n-1})$ is also co-c.e. closed, then D is computable.*

Here $\mathbb{S}^n$ denotes the unit sphere in $\mathbb{R}^{n+1}$, $\mathbb{B}^n$ denotes the unit closed ball in $\mathbb{R}^n$ and $X \cong Y$ denotes that topological spaces X and Y are homeomorphic.

By Theorem 8.1 in Euclidean space each co-c.e. closed topological circle is computable and each co-c.e. arc with computable endpoints is computable. The second claim of Theorem 8.1 does not hold in general if we omit the assumption that $f(\mathbb{S}^{n-1})$ is co-c.e. closed. At the same time, the fact that $f(\mathbb{S}^{n-1})$ is co-c.e. closed is not actually necessary for $f(\mathbb{B}^n)$ to be computable.

Theorem 8.2 (Miller [67]).

1. *There is a co-c.e. arc in $\mathbb{R}^2$ which is not computable.*
2. *There is a computable arc in $\mathbb{R}^2$ with non-computable endpoints.*

So, although a co-c.e. closed set need not be computable, it makes sense to ask the following general question: under what conditions does the implication

$$S \text{ co-c.e. closed} \implies S \text{ computable closed} \tag{7}$$

hold in a computable metric space (X, d, α)? Miller's work shows that topology plays an important role regarding conditions under which (7) holds. In Theorem

8.1, however, the ambient space is Euclidean space. It was later shown by Iljazović [39] that the claim of that theorem holds in more general computable metric spaces:

Theorem 8.3. *Let (X,d,α) be a computable metric space which has the effective covering property and compact closed balls. Suppose $S \subseteq X$ is such that $S \cong \mathbb{S}^n$ for some $n \geq 1$ or $S \cong \mathbb{B}^n$ by a homeomorphism $f : \mathbb{B}^n \to S$ such that $f(\mathbb{S}^{n-1})$ is a co-c.e. closed set in (X,d,α). Then the implication (7) holds.*

In particular, (7) holds if S is a topological circle or an arc with computable endpoints (in a computable metric space which has the effective covering property and compact closed balls).

8.1 Semicomputable Chainable and Circularly Chainable Continua

Arcs and topological circles are just representatives of more general topological spaces: chainable and circularly chainable continua [73].

Let (X,d) be a metric space, $\varepsilon > 0$ and $C_0,\dots,C_m$ a finite sequence of nonempty open sets in (X,d) such that $\operatorname{diam} C_i < \varepsilon$ for each $i \in \{0,\dots,m\}$. We say that $C_0,\dots,C_m$ is an *ε-chain* in (X,d) if, for all $i,j \in \{0,\dots,m\}$, $C_i \cap C_j = \emptyset$ if and only if $1 < |i-j|$. We say that $C_0,\dots,C_m$ is an *ε-circular chain* in (X,d) if, for all $i,j \in \{0,\dots,m\}$, $C_i \cap C_j = \emptyset$ if and only if $1 < |i-j| < m$. We say that a finite sequence of sets $C_0,\dots,C_m$ covers a set X if $X \subseteq C_0 \cup \cdots \cup C_m$.

A continuum (X,d) will be called a *chainable continuum* if for each $\varepsilon > 0$ there exists an ε-chain in (X,d) which covers X. A continuum (X,d) will be called a *circularly chainable continuum* if for each $\varepsilon > 0$ there exists an ε-circular chain in (X,d) which covers X. If (X,d) is a continuum and $a,b \in X$, we say that (X,d) is a *continuum chainable from a to b* if for each $\varepsilon > 0$ there exits an ε-chain $C_0,\dots,C_m$ in (X,d) which covers X and such that $a \in C_0$ and $b \in C_m$.

The segment $[0,1]$ is a continuum chainable from 0 to 1. Consequently, if X is an arc with endpoints a and b, then X is a continuum chainable from a to b. The unit circle $\mathbb{S}^1$ is a circularly chainable continuum and therefore each topological circle is also a circularly chainable continuum. On the other hand, the space $K = (\{0\} \times [-1,1]) \cup \{(x, \sin\frac{1}{x}) \mid x \in (0,1]\}$, known as the topologist's sine curve, is an example of chainable continuum (K is chainable from a to c and also from b to c, where $a = (0,-1)$, $b = (0,1)$ and $c = (1, \sin 1)$) which is not an arc. Furthermore, the space $W = K \cup (\{0\} \times [-2,-1]) \cup ([0,1] \times \{-2\}) \cup (\{1\} \times [-2, \sin 1])$, known as the Warsaw circle, is an example of an circularly chainable continuum which is not a topological circle.

Theorem 8.4 (Iljazović [37]). *Let (X,d,α) be a computable metric space which has the effective covering property and compact closed balls. Let $S \subseteq X$.*

1. *If S is (as a subspace of (X,d)) a circularly chainable continuum which is not chainable, then (7) holds.*

2. *If S is a continuum chainable from a to b, where a and b are computable points in (X,d,α), then (7) holds.*
3. *Suppose S is a co-c.e. closed set. If S is a chainable and* decomposable *continuum, then for each $\varepsilon > 0$ there exists a subcontinuum K of S such that K is computable and $d_H(S,K) < \varepsilon$. Moreover, K can be chosen so that it is chainable from a to b, where a and b are computable points.*

That a continuum K is *decomposable* means that there exist proper subcontinua K_1 and K_2 of K such that $K = K_1 \cup K_2$. For example, $[0,1]$ is decomposable since $[0,1] = [0,\frac{1}{2}] \cup [\frac{1}{2},1]$.

Each topological circle is a circularly chainable continuum which is not chainable. Therefore, Theorem 8.4 is a generalization of Theorem 8.3 for $n = 1$.

In Theorem 8.3 and Theorem 8.4 we assume that the computable metric space (X,d,α) has the effective covering property and compact closed balls. The natural question is: are these additional assumptions on a computable metric space necessary? The answer is affirmative: we cannot omit these assumptions. By [40] there exists a computable metric space such that the following hold:

1. There exists a co-c.e. closed topological circle S in (X,d,α) which does not contain a computable point (and which in particular is not computable closed);
2. There exists a co-c.e. closed arc S in (X,d,α) which is chainable from a to b, where a and b are computable points, but which is not computable closed;
3. There exists a co-c.e. closed arc S in (X,d,α) which does not contain a computable point (and which in particular cannot be approximated by a computable subcontinuum).

Moreover, (X,d,α) can be chosen so that either it has compact closed balls (but not the effective covering property) or it has the effective covering property (but not compact closed balls).

Recall that even a one-point co-c.e. closed set need not be computable (the discussion after Proposition 3.3). This indicates that we have to restrict ourselves to some special computable metric spaces if we are looking for topological conditions under which (7) holds.

On the other hand, by Theorem 3.6, in computable metric spaces which have the effective covering property and compact closed balls conditions under which (7) holds are same as conditions under which the implication

$$S \text{ semicomputable} \implies S \text{ computable} \tag{8}$$

holds. It turns out, however, that it is more convenient to search for conditions under which (8) holds than for conditions under which (7) holds since we do not need any additional assumptions on the ambient space. For example, (8) holds in any computable metric space if $S \cong \mathbb{S}^n$ or $S \cong \mathbb{B}^n$ by a homeomorphism $f : \mathbb{B}^n \to S$ such that $f(\mathbb{S}^{n-1})$ is a semicomputable set [41]. This is a generalization of Theorem 8.3. Similarly, if in Theorem 8.4 we remove the assumptions on the computable metric space, replace “co-c.e. set” by “semicomputable set” and replace (7) by (8), we get the claim which also holds [43] and which generalizes Theorem 8.4.

Actually, we have a much more general result than Theorem 8.3: (8) holds if S is a compact manifold with computable boundary.

8.2 Semicomputable Manifolds

Let $\mathbb{H}^n = \{(x_1,\dots,x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$ and $\mathrm{Bd}\,\mathbb{H}^n = \{(x_1,\dots,x_n) \in \mathbb{R}^n \mid x_n = 0\}$ (for $n \geq 1$).

A second countable Hausdorff space X is said to be an *n–manifold with boundary* if for each $x \in X$ there exists a neighborhood N of x in X such that N is homeomorphic to $\mathbb{R}^n$ or there exists a homeomorphism $f\colon \mathbb{H}^n \to N$ such that $x \in f(\mathrm{Bd}\,\mathbb{H}^n)$.

If X is an n–manifold with boundary, we define ∂X to be the set of all points $x \in X$ which have no neighborhood homeomorphic to $\mathbb{R}^n$. We say that ∂X is the *boundary* of the manifold X.

If X is an n-manifold with boundary such that $\partial X = \emptyset$, then we say that X is an *n-manifold*. Hence a second countable Hausdorff space X is an n-manifold if and only if each point $x \in X$ has a neighborhood homeomorphic to $\mathbb{R}^n$.

Let $n \in \mathbb{N}$. Then $\mathbb{S}^n$ is an n-manifold and $\mathbb{B}^n$ is an n-manifold with boundary, its boundary is $\mathbb{S}^{n-1}$ [71]. Consequently, if $X \cong \mathbb{S}^n$, then X is an n-manifold. Furthermore, if $f : \mathbb{B}^n \to X$ is a homeomorphism, then X is an n-manifold with boundary and $\partial X = f(\mathbb{S}^{n-1})$.

If (X,d) is a metric space, $A \subseteq X$ and $r > 0$, then we will denote by $N_r(A)$ the *r-neighborhood* of A, i.e.

$$N_r(A) = \bigcup_{x\in A} B(x,r).$$

Note that for $A,B \subseteq X$ and $r > 0$ we have that A and B are r-close if and only if $A \subseteq N_r(B)$ and $B \subseteq N_r(A)$. The following notion is useful in the proof of the fact that (8) holds for compact manifolds with computable boundaries.

Let (X,d,α) be a computable metric space and $A,B \subseteq X$, $A \subseteq B$. We say that A is *computable up to B* if there exists a computable function $f : \mathbb{N} \to \mathbb{N}$ such that

$$A \subseteq N_{2^{-k}}(\Lambda_{f(k)}) \text{ and } \Lambda_{f(k)} \subseteq N_{2^{-k}}(B)$$

for each $k \in \mathbb{N}$ (recall definition (1)). Clearly, if S is a nonempty compact set in (X,d), then S is computable if and only if S is computable up to S. Furthermore, it is easy to prove the following fact (see [41]): if $A_1,\dots,A_n$ are sets computable up to S, then $A_1 \cup \dots \cup A_n$ is computable up to S.

If $S \subseteq X$ and $x \in S$, we say that S is *computable at x* if there exists a neighborhood N of x in S such that N is computable up to S. The proof of the following proposition is straightforward (see [41]).

Proposition 8.5. *Let (X,d,α) be a computable metric space and let $S \subseteq X$ be a compact set. Then S is computable if and only if S is computable at x for each $x \in S$.*

A connection between topology and computability is apparent in the following result: if x has a Euclidean neighborhood in S, then S is computable at x.

Theorem 8.6 (Iljazović [41]). *Let (X,d,α) be a computable metric space and let S be a semicomputable set in this space.*

1. *Suppose $x \in S$ is a point which has a neighborhood in S homeomorphic to $\mathbb{R}^n$ for some $n \in \mathbb{N} \setminus \{0\}$. Then S is computable at x.*
2. *Let T be a semicomputable set such that $T \subseteq S$. Suppose $x \in S$ is a point which has a neighborhood N in S with the following property: there exists $n \in \mathbb{N} \setminus \{0\}$ and a homeomorphism $f : \mathbb{H}^n \to N$ such that $f(\mathrm{Bd}\,\mathbb{H}^n) = N \cap T$. Then S is computable at x.*

Actually, the original result from [41] assumes that S is compact, but this assumption can be easily removed (see Theorem 5.2 in [46]).

Proposition 8.5 and Theorem 8.6 imply that (8) holds if S is a compact manifold with computable boundary. This can be stated in the following way.

Theorem 8.7 ([41]). *Let (X,d,α) be a computable metric space and let S be a semicomputable set in this space which is, as a subspace of (X,d), a compact manifold with boundary. Then the following implication holds:*

$$\partial S \text{ computable} \implies S \text{ computable}.$$

In particular, each semicomputable compact manifold is computable.

In general, $S \subseteq \mathbb{R}^n$ is co-c.e. closed if and only if $S = f^{-1}(\{0\})$ for some computable function $f : \mathbb{R}^n \to \mathbb{R}$ [12]. Hence, if $f : \mathbb{R}^n \to \mathbb{R}$ is a computable function, then $f^{-1}(\{0\})$ need not be a computable set, moreover $f^{-1}(\{0\})$ need not contain a computable point even if $f^{-1}(\{0\}) \neq \emptyset$.

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function of class C^1. Suppose $y \in \mathbb{R}^m$ is a regular value of f (which means that the differential $D(f)(x) : \mathbb{R}^n \to \mathbb{R}^m$ of f in x is surjective for each $x \in f^{-1}(\{y\})$) and $f^{-1}(\{y\}) \neq \emptyset$. Then it is known from differential topology (see e.g. [83]) that $f^{-1}(\{y\})$ is an $(n-m)$-manifold. Additionally, if f is computable, then $f^{-1}(\{y\})$ is co-c.e. closed and therefore semicomputable (Proposition 3.6). The following is a consequence of Theorem 8.7.

Corollary 8.8. *If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a computable function of class C^1 and $y \in \mathbb{R}^m$ is a regular value of f such that $f^{-1}(\{y\})$ is bounded, then $f^{-1}(\{y\})$ is computable.*

For example, Corollary 8.8 easily implies that the set of all $(x,y,z) \in \mathbb{R}^3$ such that $x^2(1+e^x)+y^2(1+e^y)+z^2(1+e^z) = 1$ is computable (see [41]).

The claim of Theorem 8.7 need not hold if we omit the assumption that S is compact. This is shown by the following simple example from [14].

Example 8.9. Let A be a c.e. subset of $\mathbb{N}$ which is not computable. Then the set $B = \mathbb{N} \setminus A$ is co-c.e. closed in $\mathbb{R}$ and therefore the set $S = B \times \mathbb{R}$ is co-c.e. closed in $\mathbb{R}^2$. Since $B \subseteq \mathbb{N}$, S is a 1-manifold. So S is a semicomputable 1-manifold in $\mathbb{R}^2$, but S is not computable in $\mathbb{R}^2$ which can be deduced from the fact that B is not computable in $\mathbb{N}$.

In general, if X is an n-manifold with boundary, then each connected component of X is also an n-manifold with boundary. Semicomputable 1-manifolds with boundaries have been studied in [14]. It is a well known fact (see e.g. [83]) that if X is a connected 1-manifold with boundary, then X is a topological line (i.e. $X \cong \mathbb{R}$) or a topological ray (i.e. $X \cong [0,\infty)$) or a topological circle or an arc.

Theorem 8.10 (Burnik-Iljazović [14]). *Let (X,d,α) be a computable metric space and let S be a semicomputable set in this space which is, as a subspace of (X,d), a 1-manifold with boundary. Let K be a connected component of S.*

1. *If K is a topological line or circle, then K is c.e. closed in (X,d,α).*
2. *If K is a topological ray with computable endpoint or an arc with computable endpoints, then K is c.e. closed.*
3. *If ∂S is semicomputable, then each connected component of S is c.e. closed.*

Since the union of finitely many c.e. closed sets is c.e. closed, an immediate consequence of Theorem 8.10 is the following theorem.

Theorem 8.11 ([14]). *Let (X,d,α) be a computable metric space and let S be a semicomputable set in this space which is a 1-manifold with boundary. Suppose that S has finitely many connected components. Then the following implication holds:*

$$\partial S \text{ computable} \implies S \text{ computable}.$$

In particular, each semicomputable 1-manifold with finitely many connected components is computable.

It should be mentioned that the uniform versions of Theorems 8.3, 8.7 and 8.11 do not hold in general: there exists a sequence of topological circles in $\mathbb{R}^2$ (in fact in $[0,1]^2$) which is uniformly semi-computable, but not uniformly computable (Example 7 in [37]).

If S is a co-c.e. closed set in a computable metric space (X,d,α) such that $X \setminus S$ is disconnected (for example, this holds by the Generalized Jordan curve theorem if S is homeomorphic to $\mathbb{S}^n$ and $X = \mathbb{R}^{n+1}$), then it is possible to conclude that, under some additional assumptions, S is computable closed or at least contains a computable point. Such conditions have been studied in [37, 42].

Finally, let us mention that semicomputable manifolds in *computable topological spaces* have been studied in [44]. For a study of computable topological spaces see [98, 97].

8.3 Inner Approximation

Let $\mathscr{A}$ be a collection of continua, and let B be a continuum. We say that B *is inner approximated by* $\mathscr{A}$ if, for any $\varepsilon > 0$, there exists $A \in \mathscr{A}$ such that $A \subseteq B$ and $d_H(A,B) < \varepsilon$, where we recall that d_H is the Hausdorff distance. Classically, every arcwise connected continuum is inner approximated by locally connected continua.

In the computability-theoretic setting, claim 3 of Theorem 8.4 says the following: every co-c.e. chainable and decomposable continuum is inner approximated by computable chainable continua (in appropriate computable metric spaces; in fact, by [43], in any computable metric space any semicomputable chainable and decomposable continuum is inner approximated by computable chainable continua).

However, we do not always have a computable inner approximation. Recall that X is contractible if the identity map on X is null-homotopic, and that a curve means a one-dimensional continuum.

Theorem 8.12 (Kihara [53]).

1. *There exists a contractible, locally contractible, co-c.e., planar curve which is not inner approximated by computable continua.*
2. *There exists a contractible, computable, planar curve which is not inner approximated by locally connected, co-c.e. continua.*

8.4 Density of Computable Points in Semicomputable Sets

Let (X,d,α) be a computable metric space, $S \subseteq X$ and $x \in S$. Suppose that there exists a neighborhood N of x in S such that N is computable in (X,d,α). Then N, as a subset of S, is clearly computable up to S. So if x has a computable neighborhood in S, then S is computable at x.

Theorem 8.13 (Iljazović-Validžić [46]). *Let (X,d,α) be a computable metric space and S a complete set in (X,d). Suppose S is computable at x. Then there exists a neighborhood N of x in S such that N is computable compact in (X,d,α). Moreover, for each $\varepsilon > 0$ there exists such N with the property that* $\operatorname{diam} N < \varepsilon$.

If S is a nonempty compact set in a metric space (X,d) and $\varepsilon > 0$, then $S \cap \overline{B}(x, \frac{\varepsilon}{3})$ is a compact set for each $x \in X$, and it follows readily that there exist compact sets $K_1,\dots,K_n$ whose union is S and whose diameters are less than ε. On the other hand, if S is a computable compact set, $i \in \mathbb{N}$ and $r \in \mathbb{Q}$, $r > 0$, then the intersection $S \cap \overline{B}(\alpha_i, r)$ need not be computable compact even if $\overline{B}(\alpha_i, r)$ is computable compact (see [46]). Nevertheless, we have the following result.

Corollary 8.14 ([46]). *Let (X,d,α) be a computable metric space and let S be a nonempty computable compact set in this space. Then for each $\varepsilon > 0$ there exist nonempty computable compact sets $K_1,\dots,K_n$ such that $S = K_1 \cup \dots \cup K_n$ and* $\operatorname{diam} K_i < \varepsilon$ *for each $i \in \{1,\dots,n\}$.*

Proof. Let $\varepsilon > 0$. Since S is computable, it is computable at each point $x \in S$ and therefore, by Theorem 8.13, each point of S has a neighborhood in S which is computable and has the diameter less than ε. This, together with the compactness of S, proves the claim of the corollary. □

Using Corollary 8.14 it is easy to deduce the following facts (see [46]): if (F,G) is a separation of a computable compact set, then F and G are computable compact sets; if S is a computable compact set which has finitely many components, then each of these components is a computable compact set.

A connected component of a computable compact set need not be computable in general. For example, there are uncountably many components of the Cantor set, so at least one of them is not computable.

Let S be a semicomputable set in a computable metric space. Then S may not be computable but, at the same time, computable points which lie in S may be dense in S. Therefore, the general question is under what conditions the implication

$$S \text{ semicomputable} \implies \text{computable points are dense in } S \tag{9}$$

holds in a computable metric space? If (8) holds under certain conditions, then under same conditions (9) also holds. The converse does not hold. For example, (9) holds if S is an arc in Euclidean space, but (8) need not hold if S is an arc in Euclidean space [67]. Miller has proved in [67] that (9) holds in Euclidean space if $S \cong \mathbb{B}^n$ for some $n \in \mathbb{N}$.

The following result follows from Theorem 8.6 and Theorem 8.13.

Theorem 8.15 (Iljazović-Validžić [46]). *Let (X,d,α) be a computable metric space and let S be a semicomputable set in this space. Suppose that S is a manifold with boundary or S has the topological type of a polyhedron. Then the set of all computable points which belong to S is dense in S.*

That S has the topological type of a polyhedron means that $S \cong P$ for some polyhedron P. A polyhedron in a space obtained from simplices (line segments, triangles, tetrahedra, and their higher dimensional analogues) by gluing them together along their faces (see [71] for the definition).

Suppose S is a nonempty compact set in which computable points are dense. Then for each $\varepsilon > 0$ there exist computable points $x_0,\dots,x_n \in S$ such that S and $\{x_0,\dots,x_n\}$ are ε-close. So, for each $\varepsilon > 0$ there exists a computable set K such that $K \subseteq S$ and $d_H(S,K) < \varepsilon$. In particular, if S is a nonempty semicomputable compact set which satisfies the conditions of Theorem 8.15, then S can be approximated by its computable subset with arbitrary precision. However, if S is a nonempty semicomputable compact manifold with boundary, then we can find an even better approximation, namely for each $\varepsilon > 0$ there exists a computable subset K of S which is ε-close to S and covers the entire set S except for some part of S which lies in an ε-neighborhood of ∂S (see Theorem 5.4 in [46]).

At the end of this section, let us mention the following problem. Suppose (X,d,α) is a computable metric space and U and V are c.e. open sets in this space. Let $S = X \setminus (U \cup V)$. Suppose A is a computable compact set which is connected and which intersects both U and V. Then clearly $A \cap S \neq \emptyset$. The question is: does $A \cap S$ have to contain a computable point? It is not hard to see that the answer in general is negative [43]. An affirmative answer to this question is given in [43] in the case when A is an arc and, under some additional assumptions, in the case when A is a chainable continuum. These results can be considered as generalizations of the computable intermediate value theorem: if $f : [0,1] \to \mathbb{R}$ is a computable function such that $f(0) < 0$ and $f(1) > 0$, then f has a computable zero-point [80].

Let K be the unit square in the plane, i.e. $K = [0,1] \times [0,1]$, and let $\mathring{K}$ be the corresponding open unit square. Suppose $f, g : [0,1] \to K$ are continuous functions such that $f(0) = (0,0)$, $f(1) = (1,1)$, $g(0) = (0,1)$, $g(1) = (1,0)$ and $f(t), g(t) \in \mathring{K}$ for each $t \in [0,1] \setminus \{0,1\}$. Then the images of f and g intersect. This nontrivial fact can be proved using the Jordan curve theorem (see e.g. [94]). Manukyan has proved that a constructive version of this result does not hold in general [63, 59]. The open question was: does a computable version of this result hold? Weihrauch has recently solved this problem; his result can also be considered as a generalization of the computable intermediate value theorem.

Theorem 8.16 (Weihrauch [94]). *Let $f, g : [0,1] \to K$ be computable functions such that $f(0) = (0,0)$, $f(1) = (1,1)$, $g(0) = (0,1)$, $g(1) = (1,0)$ and $f(t), g(t) \in \mathring{K}$ for each $t \in [0,1] \setminus \{0,1\}$. Then the images of f and g intersect in a computable point.*

The image of a computable function $[0,1] \to \mathbb{R}^n$ is easily seen to be a computable compact set. On the other hand, the intersection of a semicomputable set and a co-c.e. closed set is a semicomputable set (in any computable metric space, see [43]). So Theorem 8.16 and the above results from [43] can be viewed as results which provide conditions under which a semicomputable set contains a computable point.

9 Computable Images of a Segment

As said, if $f : [0,1] \to \mathbb{R}^n$ is a computable function, then $f([0,1])$ is a computable compact set in $\mathbb{R}^n$. The question is what can be said about various forms of the converse of this statement. By Example 6.1, there exists a computable arc A in $\mathbb{R}^2$ with computable endpoints such that A is not the image of any computable injection $[0,1] \to \mathbb{R}^2$ (in fact, A is the image of a computable function $[0,1] \to \mathbb{R}^2$). Moreover, we have the following result.

Theorem 9.1 (Gu-Lutz-Mayordomo [30]). *There exists an arc A in $\mathbb{R}^2$ with the following properties:*

1. *A is rectifiable (i.e. A has finite length) and smooth except at one endpoint;*
2. *there exists a computable function $f : [0,1] \to \mathbb{R}^2$ whose image is A (moreover, f, the velocity function f' and the acceleration function f'' are polynomial time computable);*
3. *for any computable function $f : [0,1] \to \mathbb{R}^2$ whose image is A and for every $m \in \mathbb{N}$, $m \geq 1$, there exist disjoint closed subintervals $I_0, \dots, I_m$ of $[0,1]$ such that the arc $f(I_0)$ has positive length and $f(I_i) = f(I_0)$ for each $i \in \{1, \dots, m\}$.*

If $f : [0,1] \to \mathbb{R}^2$ is a computable injection, then the arc $f([0,1])$ need not be rectifiable. On the other hand, if $f : [0,1] \to \mathbb{R}^2$ is a computable function such that the curve $f([0,1])$ is rectifiable, then $f([0,1])$ clearly need not be an arc. The following result points out a significant difference between these two types of sets.

Theorem 9.2 (McNicholl [64]).

1. *There exists a computable injection* $f : [0,1] \to \mathbb{R}^2$ *and a point* $x \in f([0,1])$ *such that there exists no computable function* $g : [0,1] \to \mathbb{R}^2$ *with the following property:* $g([0,1])$ *is a rectifiable curve and* $x \in g([0,1])$.
2. *There exists a computable function* $g : [0,1] \to \mathbb{R}^2$ *such that* $g([0,1])$ *is a rectifiable curve and a point* $x \in g([0,1])$ *with the following property: there exists no computable injection* $f : [0,1] \to \mathbb{R}^2$ *such that* $x \in f([0,1])$.

Of course, the arc $f([0,1])$ in 1. cannot be rectifiable and the function g in 2. cannot be injective.

A metrizable space which is locally connected, connected and compact is called a *Peano continuum*. The Hahn-Mazurkiewicz theorem (see e.g. [73, 17]) says that a Hausdorff space X is a Peano continuum if and only if there is a continuous surjection $[0,1] \to X$.

If X is a subset of the plane, it makes sense to ask whether a computable version of the Hahn-Mazurkiewicz theorem holds.

Theorem 9.3 (Couch-Daniel-McNicholl [19]). *There is a computable compact set* X *in* $\mathbb{R}^2$ *which is a Peano continuum and such that there exists no computable function* $[0,1] \to \mathbb{R}^2$ *whose image is* X.

However, the situation changes under the assumption that X is effectively locally connected. If $X \subseteq \mathbb{R}^n$, a *local connectivity operator* for X is a continuous operator that, given a name of a point $p \in X$ in $\mathbb{R}^n$ and a name of a rational rectangle R in $\mathbb{R}^n$ which contains p gives a name of an open set U in $\mathbb{R}^n$ such that $U \cap X$ is connected and $p \in U \cap X \subseteq R$ (see [19]). The set X is in [19] defined to be *effectively locally connected* if it has a computable local connectivity operator.

Theorem 9.4 (Couch-Daniel-McNicholl [19]). *Let* $X \subseteq \mathbb{R}^n$ *be a computable Peano continuum. Suppose* X *is effectively locally connected. Then there exists a computable function* $g : [0,1] \to \mathbb{R}^n$ *whose image is* X. *Moreover, a name of such a function* g *can be uniformly computed from a name of a computable compact set* X *and a name of a local connectivity operator for* X.

The following theorem is a computable version of another well know topological result: each compact metric space is a continuous image of the Cantor set (see Theorem 6.C.12 in [17]).

Theorem 9.5 (Couch-Daniel-McNicholl [19]). *Let* C *be the Cantor middle-third set. Let* X *be a nonempty computable compact set in* $\mathbb{R}^n$. *Then there exists a computable surjection* $C \to X$. *Moreover, a name of such a surjection can be uniformly computed from a name of a computable compact set* X.

10 Computability Structures

If (X,d) is a metric space and α a sequence in X such that (X,d,α) is a computable metric space, then we say that α is an *effective separating sequence* in (X,d) [102].

Suppose α and β are effective separating sequences in a metric space (X,d). We say that α and β are *equivalent* if α is a computable sequence in (X,d,β) and β is a computable sequence in (X,d,α).

If (X,d,α) is a computable metric space, let $\mathscr{S}_\alpha$ denote the set of all computable sequences in (X,d,α). It is easy to conclude that effective separating sequences α and β in (X,d) are equivalent if and only if $\mathscr{S}_\alpha = \mathscr{S}_\beta$ [45].

The notions of a computable point, a computable sequence, a c.e. closed set, a co-c.e. closed set, a computable compact set in a computable metric space (X,d,α) depend, by definition, on the sequence α. However, it is easy to conclude that these notions coincide in (X,d,α) and (X,d,β) if α and β are equivalent effective separating sequences. This means that these notions can be viewed as notions defined related to the entire set $\mathscr{S}_\alpha$ and not just to α itself. Therefore, we can take the sets of the form $\mathscr{S}_\alpha$ as a basis for computability concepts on a metric space (X,d), which leads to the notion of a computability structure on a metric space.

A *computability structure* $\mathscr{S}$ on a metric space (X,d) is a set of sequences in X such that the following holds [69, 102, 80, 45]:

1. if $(x_i),(y_j) \in \mathscr{S}$, then the function $\mathbb{N}^2 \to \mathbb{R}$, $(i,j) \mapsto d(x_i,y_j)$, is computable;
2. if $(x_i) \in \mathscr{S}$ and (y_i) is a sequence in X such that $d(y_i,x_{F(i,k)}) < 2^{-k}$ for all $i,k \in \mathbb{N}$, where $F : \mathbb{N}^2 \to \mathbb{N}$ is a computable function, then $(y_i) \in \mathscr{S}$.

A computability structure $\mathscr{S}$ on a metric space (X,d) is said to be *separable* if there exists $(x_i) \in \mathscr{S}$ such that (x_i) is a dense sequence in (X,d).

If (X,d) is a metric space, then there exist computability structures on (X,d): we can take any $a \in X$, and then $\{(a,a,a,\dots)\}$ is trivially a computability structure on (X,d). On the other hand, a separable computability structure on a metric space (X,d) need not exist. It certainly does not exist if (X,d) is not a separable metric space, but even if (X,d) is separable, a separable computability structure on (X,d) need not exist. For example, we can take $X = \{0,\gamma\}$, where γ is an incomputable real number, and the Euclidean metric d on X.

The general question is: if (X,d) is a metric space, how many separable computability structures do exist on (X,d)?

A computable metric space (X,d,α) is said to be *effectively totaly bounded* if there exists a computable function $f : \mathbb{N} \to \mathbb{N}$ such that

$$X = B(\alpha_0, 2^{-k}) \cup \dots \cup B(\alpha_{f(k)}, 2^{-k})$$

for each $k \in \mathbb{N}$. If (X,d,α) is effectively totaly bounded, then (X,d) is obviously totaly bounded. Conversely, if (X,d) is totaly bounded, (X,d,α) need not be effectively totaly bounded [38].

If α and β are equivalent effective separating sequences on a metric space (X,d,α), then it is not hard to conclude that (X,d,α) is effectively totaly bounded if and only if (X,d,β) is effectively totaly bounded. However, this claim holds even if α and β are not equivalent.

Theorem 10.1 (Iljazović [38]). *If α and β are effective separating sequences in a metric space (X,d), then (X,d,α) is effectively totally bounded if and only if (X,d,β) is effectively totally bounded.*

Using Theorem 3.4 and Proposition 3.6 it is easy to conclude that a computable metric space (X,d,α) is effectively compact if and only if (X,d) is compact and (X,d,α) is effectively totaly bounded.

Theorem 10.2 (Iljazović [38]). *Let (X,d,α) be an effectively compact computable metric space such that there exist only finitely many isometries of the metric space (X,d). Then there exists a unique separable computability structure on (X,d).*

For example, Theorem 10.2 implies that there exists a unique separable computability structure on $[0,1]$.

Let (X,d) be a metric space, $\mathscr{S}$ a set of sequences in X and $f : X \to X$ an isometry. Let $f(\mathscr{S})$ denotes the set $\{(f(x_i)) \mid (x_i) \in \mathscr{S}\}$. Then $\mathscr{S}$ is a (separable) computability structure on (X,d) if and only if $f(\mathscr{S})$ is a (separable) computability structure on (X,d).

A metric space (X,d) is said to be *computably categorical* if for all separable computability structures $\mathscr{S}$ and $\mathscr{T}$ on (X,d) there exists an isometry $f : X \to X$ such that $f(\mathscr{S}) = \mathscr{T}$ [66].

Theorem 10.3 (Melnikov [66]).

1. *Every separable Hilbert space is computably categorical (as a metric space).*
2. *The space $C[0,1]$ of all continuous functions $[0,1] \to \mathbb{R}$ with the metric of uniform convergence (supremum metric) is not computably categorical.*
3. *Cantor space $\{0,1\}^{\mathbb{N}}$ with the metric $d((x_n),(y_n)) = \max\{2^{-n} \mid x_n \neq y_n\}$ is computably categorical.*

Theorem 10.4 (McNicholl [65]). *Let $p \in \mathbb{R}$, $p \geq 1$. Let l^p be the set of all sequences (x_i) of complex (or real) numbers such that $\sum_{i=0}^{\infty} |x_i|^p < \infty$. Then l^p, with the metric induced by the norm $\|(x_i)\| = (\sum_{i=0}^{\infty} |x_i|^p)^{\frac{1}{p}}$, is computably categorical if and only if $p = 2$.*

The general question is: if (X,d) is a separable metric space, does there exist a metric space (Y,d') which is homeomorphic to (X,d) and which has a separable computability structure? A similar question is this: if S is a (compact) set in a computable metric space, does there exist a computable compact set T in the same space such that S and T are homeomorphic? Bosserhoff and Hertling have studied a similar problem in Euclidean space and they got the following result.

Theorem 10.5 (Bosserhoff-Hertling [3]). *Let $n \in \mathbb{N}$, $n \geq 1$.*

1. *There exists a c.e. closed set K in $\mathbb{R}^n$ such that K is compact, $K \subseteq [0,1]^n$, and such that $f(K)$ is not a computable compact set in $\mathbb{R}^n$ for any homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$.*
2. *There exists a co-c.e. closed set K in $\mathbb{R}^n$ such that K is compact, $K \subseteq [0,1]^n$, and such that $f(K)$ is not a computable compact set in $\mathbb{R}^n$ for any homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$.*

A computability structure on a metric space is called *maximal* if it is maximal with respect to inclusion. Each separable computability structure is maximal, but the converse does not hold in general. The question is under what conditions a maximal computability structure is separable. Another question is under what conditions a maximal computability structure on a metric space is unique.

Let $\mathscr{S}$ be a computability structure on a metric space (X,d) and let $a \in X$. We say that a is a *computable point* in $\mathscr{S}$ if there exists $(x_i) \in \mathscr{S}$ and $i \in \mathbb{N}$ such that $x_i = a$.

For $X \subseteq \mathbb{R}^n$, $X \neq \emptyset$, let $\dim X$ be the largest number $k \in \mathbb{N}$ such that there exist geometrically independent points $a_0, \dots, a_k \in X$.

Theorem 10.6 (Iljazović-Validžić [46]).

1. *Each maximal computability structure on $\mathbb{R}^n$ is separable.*
2. *If $X \subseteq \mathbb{R}^n$, $\dim X = k$, $k \geq 1$, and $a_0, \dots, a_{k-1} \in X$ are geometrically independent points such that $d(x_i, x_j)$ is a computable number for all $i, j \in \{0, \dots, k-1\}$, where d is the Euclidean metric on X, then there exists a unique maximal computability structure on (X,d) in which $a_0, \dots, a_{k-1}$ are computable points.*
3. *Let $\gamma > 0$. For $a \in [0,\gamma]$ let $\mathscr{M}_a$ be the unique maximal computability structure on $[0,\gamma]$ in which a is a computable point (such a computability structure exists by claim 2). Then $\mathscr{M}_a$ is a separable computable structure if and only if a and $\gamma - a$ are left computable numbers.*

References

1. Andrej Bauer. *The Realizability Approach to Computable Analysis and Topology*. PhD thesis, School of Computer Science, Carnegie Mellon University, Pittsburgh, 2000.
2. Andrej Bauer. An injection from the Baire space to natural numbers. *Mathematical Structures in Computer Science*, 25:1484–1489, 10 2015.
3. Volker Bosserhoff and Peter Hertling. Effective subsets under homeomorphisms of R^n. *Inform. and Comput.*, 245:197–212, 2015.
4. Vasco Brattka. Random numbers and an incomplete immune recursive set. In *Automata, Languages and Programming, 29th International Colloquium, ICALP 2002, Malaga, Spain, July 8-13, 2002, Proceedings*, pages 950–961, 2002.
5. Vasco Brattka. Computability over topological structures. In S. Barry Cooper and Sergey S. Goncharov, editors, *Computability and Models*, pages 93–136. Kluwer Academic Publishers, New York, 2003.
6. Vasco Brattka. Effective Borel measurability and reducibility of functions. *Mathematical Logic Quarterly*, 51(1):19–44, 2005.
7. Vasco Brattka. Plottable real number functions and the computable graph theorem. *SIAM Journal on Computing*, 38(1):303–328, 2008.
8. Vasco Brattka, Matthew de Brecht, and Arno Pauly. Closed choice and a uniform low basis theorem. *Annals of Pure and Applied Logic*, 163:986–1008, 2012.
9. Vasco Brattka and Guido Gherardi. Borel complexity of topological operations on computable metric spaces. *Journal of Logic and Computation*, 19(1):45–76, 2009.
10. Vasco Brattka, Peter Hertling, and Klaus Weihrauch. A tutorial on computable analysis. In S. Barry Cooper, Benedikt Löwe, and Andrea Sorbi, editors, *New Computational Paradigms: Changing Conceptions of What is Computable*, pages 425–491. Springer, New York, 2008.

11. Vasco Brattka, Stéphane Le Roux, Joseph S. Miller, and Arno Pauly. The Brouwer fixed point theorem revisited. In *Pursuit of the universal*, volume 9709 of *Lecture Notes in Comput. Sci.*, pages 58–67. Springer, [Cham], 2016.
12. Vasco Brattka and Gero Presser. Computability on subsets of metric spaces. *Theoretical Computer Science*, 305:43–76, 2003.
13. Vasco Brattka and Klaus Weihrauch. Computability on subsets of Euclidean space I: Closed and compact subsets. *Theoretical Computer Science*, 219:65–93, 1999.
14. Konrad Burnik and Zvonko Iljazović. Computability of 1-manifolds. *Logical Methods in Computer Science*, 10(2):2:8, 28, 2014.
15. Douglas Cenzer. Π^0_1–classes in computability theory. In Edward R. Griffor, editor, *Handbook of Computability Theory*, volume 140 of *Studies in Logic and the Foundations of Mathematics*, pages 37–85. Elsevier, Amsterdam, 1999.
16. Douglas Cenzer and J.B. Remmel. Π^0_1–classes in mathematics. In Yu. L. Ershov, S.S. Goncharov, A. Nerode, J.B. Remmel, and V.W. Marek, editors, *Handbook of Recursive Mathematics*, volume 139 of *Studies in Logic and the Foundations of Mathematics*, pages 623–821. Elsevier, Amsterdam, 1998. Volume 2, Recursive Algebra, Analysis and Combinatorics.
17. Charles O. Christenson and Wiliam L. Voxman. *Aspects of Topology*. Marcel Dekker, Inc., New York, 1977.
18. Peter Clote. On recursive trees with a unique infinite branch. *Proc. Amer. Math. Soc.*, 93(2):335–342, 1985.
19. P. J. Couch, B. D. Daniel, and Timothy H. McNicholl. Computing space-filling curves. *Theory Comput. Syst.*, 50(2):370–386, 2012.
20. Adam R. Day and Joseph S. Miller. Randomness for non-computable measures. *Trans. Amer. Math. Soc.*, 365(7):3575–3591, 2013.
21. Matthew de Brecht. Levels of discontinuity, limit-computability, and jump operators. In Vasco Brattka, Hannes Diener, and Dieter Spreen, editors, *Logic, Computation, Hierarchies*, Ontos Mathematical Logic, pages 93–122. Walter de Gruyter, Boston, 2014.
22. David E. Diamondstone, Damir D. Dzhafarov, and Robert I. Soare. Π^0_1 classes, Peano arithmetic, randomness, and computable domination. *Notre Dame J. Form. Log.*, 51(1):127–159, 2010.
23. Yuri L. Ershov. Theory of numberings. In *Handbook of computability theory*, volume 140 of *Stud. Logic Found. Math.*, pages 473–503. North-Holland, Amsterdam, 1999.
24. Martín Escardó. Synthetic topology of data types and classical spaces. *Electronic Notes in Theoretical Computer Science*, 87:21–165, 2004.
25. Stefano Galatolo, Mathieu Hoyrup, and Cristobal Rojas. Dynamics and abstract computability: computing invariant measures. *Discrete Contin. Dyn. Syst.*, 29(1):193–212, 2011.
26. Guido Gherardi. Effective Borel degrees of some topological functions. *Mathematical Logic Quarterly*, 52(6):625–642, 2006.
27. Vassilios Gregoriades. Classes of Polish spaces under effective Borel isomorphism. *Mem. Amer. Math. Soc.*, 240(1135):vii+87, 2016.
28. Vassilios Gregoriades, Takayuki Kihara, and Keng Meng Ng. Turing degrees in Polish spaces and decomposability of Borel functions. submitted. arXiv:1410.1052.
29. Vassilios Gregoriades, Tamás Kispéter, and Arno Pauly. A comparison of concepts from computable analysis and effective descriptive set theory. *Math. Structures Comput. Sci.*, 27(8):1414–1436, 2017.
30. Xiaoyang Gu, Jack H. Lutz, and Elvira Mayordomo. Curves that must be retraced. *Inform. and Comput.*, 209(6):992–1006, 2011.
31. J. A. Guthrie. A characterization of $\aleph_0$-spaces. *General Topology and Appl.*, 1(2):105–110, 1971.
32. Peter Hertling. A comparison of certain representations of regularly closed sets. In Vasco Brattka, Matthias Schröder, and Klaus Weihrauch, editors, *CCA 2002 Computability and Complexity in Analysis*, volume 66 of *Electronic Notes in Theoretical Computer Science*, Amsterdam, 2002. Elsevier. 5th International Workshop, CCA 2002, Málaga, Spain, July 12–13, 2002.

33. Kojiro Higuchi and Takayuki Kihara. Inside the Muchnik degrees I: Discontinuity, learnability and constructivism. *Annals of Pure and Applied Logic*, 165(5):1058–1114, 2014.
34. Kojiro Higuchi and Takayuki Kihara. Inside the Muchnik degrees II: The degree structures induced by the arithmetical hierarchy of countably continuous functions. *Annals of Pure and Applied Logic*, 165(6):1201–1241, 2014.
35. Peter G. Hinman. A survey of Mučnik and Medvedev degrees. *Bull. Symbolic Logic*, 18(2):161–229, 2012.
36. Witold Hurewicz and Henry Wallman. *Dimension Theory*. Princeton Mathematical Series, v. 4. Princeton University Press, Princeton, N. J., 1941.
37. Zvonko Iljazović. Chainable and circularly chainable co-r.e. sets in computable metric spaces. *J.UCS*, 15(6):1206–1235, 2009.
38. Zvonko Iljazović. Isometries and computability structures. *J.UCS*, 16(18):2569–2596, 2010.
39. Zvonko Iljazović. Co-c.e. spheres and cells in computable metric spaces. *Log. Methods Comput. Sci.*, 7(3):3:05, 21, 2011.
40. Zvonko Iljazović. Local computability of computable metric spaces and computability of co-c.e. continua. *Glas. Mat. Ser. III*, 47(67)(1):1–20, 2012.
41. Zvonko Iljazović. Compact manifolds with computable boundaries. *Log. Methods Comput. Sci.*, 9(4):4:19, 22, 2013.
42. Zvonko Iljazović and Bojan Pažek. Co-c.e. sets with disconnected complements. *Theory of Computing Systems*, doi:10.1007/s00224-017-9781-x, 2017.
43. Zvonko Iljazović and Bojan Pažek. Computable intersection points. *Computability*, 7(1):57–99, 2018.
44. Zvonko Iljazović and Igor Sušić. Semicomputable manifolds in computable topological spaces. *Journal of Complexity*, doi.org/10.1016/j.jco.2017.11.004, 2017.
45. Zvonko Iljazović and Lucija Validžić. Maximal computability structures. *Bulletin of Symbolic Logic*, 22(4):445–468, 2016.
46. Zvonko Iljazović and Lucija Validžić. Computable neighbourhoods of points in semicomputable manifolds. *Annals of Pure and Applied Logic*, 168(4):840–859, 2017.
47. J. E. Jayne. The space of class α Baire functions. *Bull. Amer. Math. Soc.*, 80:1151–1156, 1974.
48. J. E. Jayne and C. A. Rogers. First level Borel functions and isomorphisms. *J. Math. Pures Appl. (9)*, 61(2):177–205, 1982.
49. C. G. Jockusch, Jr. and T. G. McLaughlin. Countable retracing functions and Π^0_2 predicates. *Pacific J. Math.*, 30:67–93, 1969.
50. Carl G. Jockusch, Jr. and Robert I. Soare. Π^0_1 classes and degrees of theories. *Trans. Amer. Math. Soc.*, 173:33–56, 1972.
51. Robert Kenny. Effective zero-dimensionality for computable metric spaces. *Logical Methods in Computer Science*, 11:1:11,25, 2015.
52. Takayuki Kihara. Effective forcing with Cantor manifolds. preprint. arXiv:1702.02630.
53. Takayuki Kihara. Incomputability of simply connected planar continua. *Computability*, 1(2):131–152, 2012.
54. Takayuki Kihara. Decomposing Borel functions using the Shore-Slaman join theorem. *Fundamenta Mathematicae*, 230:1–13, 2015.
55. Takayuki Kihara. Borel-piecewise continuous reducibility for uniformization problems. *Log. Methods Comput. Sci.*, 12(4):Paper 4, 35, 2016.
56. Takayuki Kihara. Higher randomness and lim-sup forcing within and beyond hyperarithmetic. In Sy-David Friedman, editor, *Sets and Computations*, volume 33 of *Lecture Notes Series, Institute for Mathematical Sciences, National University of Singapore*, pages 117–155. World Scientific, 2017.
57. Takayuki Kihara and Arno Pauly. Point degree spectra of represented spaces. submitted. arXiv:1405.6866.
58. Georg Kreisel and Daniel Lacombe. Ensembles récursivement mesurables et ensembles récursivement ouverts et fermés. *Comptes Rendus Académie des Sciences Paris*, 245:1106–1109, 1957.

59. Boris Abramovich Kušner. Markov's constructive analysis; a participant's view. *Theoretical Computer Science*, 219:267–285, 1999.
60. Stéphane Le Roux and Arno Pauly. Finite choice, convex choice and finding roots. *Logical Methods in Computer Science*, 11(4):4:6, 31, 2015.
61. Stéphane Le Roux and Martin Ziegler. Singular coverings and non-uniform notions of closed set computability. *Mathematical Logic Quarterly*, 54(5):545–560, 2008.
62. Alain Louveau. A separation theorem for Σ^1_1 sets. *Trans. Amer. Math. Soc.*, 260(2):363–378, 1980.
63. S.N. Manukyan. O nekotorykh topologicheskikh osobennostyakh konstruktivnykh prostykh dug (in Russian). In B.A. Kushner and A.A. Markov, editors, *Issledovaniya po teorii algorifmov i matematicheskoy logike, volume 2*, pages 122–129. Vychislitelny Tsentr AN SSSR, Moscow, 1976.
64. Timothy H. McNicholl. The power of backtracking and the confinement of length. *Proc. Amer. Math. Soc.*, 141(3):1041–1053, 2013.
65. Timothy H. McNicholl. A note on the computable categoricity of ℓ^p spaces. In Arnold Beckmann, Victor Mitrana, and Mariya Soskova, editors, *Evolving Computability*, volume 9136 of *Lecture Notes in Computer Science*, pages 268–275, Cham, 2015. Springer. 11th Conference on Computability in Europe, CiE 2015, Bucharest, Romania, June 29–July 3, 2015.
66. Alexander G. Melnikov. Computably isometric spaces. *The Journal of Symbolic Logic*, 78(4):1055–1085, December 2013.
67. Joseph S. Miller. Effectiveness for embedded spheres and balls. In Vasco Brattka, Matthias Schröder, and Klaus Weihrauch, editors, *CCA 2002 Computability and Complexity in Analysis*, volume 66 of *Electronic Notes in Theoretical Computer Science*, Amsterdam, 2002. Elsevier. 5th International Workshop, CCA 2002, Málaga, Spain, July 12–13, 2002.
68. Joseph S. Miller. Degrees of unsolvability of continuous functions. *The Journal of Symbolic Logic*, 69(2):555–584, 2004.
69. Takakazu Mori, Yoshiki Tsujii, and Mariko Yasugi. Computability structures on metric spaces. In Douglas S. Bridges, Cristian S. Calude, Jeremy Gibbons, Steve Reeves, and Ian H. Witten, editors, *Combinatorics, Complexity, and Logic*, Discrete Mathematics and Theoretical Computer Science, pages 351–362, Singapore, 1997. Springer. Proceedings of DMTCS'96.
70. Yiannis N. Moschovakis. *Descriptive set theory*, volume 155 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2009.
71. James R. Munkres. *Elements of Algebraic Topology*. Addison-Wesley Publishing Company, Menlo Park, California, 1984.
72. James R. Munkres. *Topology*. Prentice Hall, 2000.
73. Sam B. Nadler. *Continuum theory*. Marcel Dekker, Inc., New York, 1992.
74. Eike Neumann. Computational problems in metric fixed point theory and their Weihrauch degrees. *Logical Methods in Computer Science*, 11:4:20,44, 2015.
75. P. G. Odifreddi. *Classical recursion theory. Vol. II*, volume 143 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 1999.
76. Arno Pauly. On the topological aspects of the theory of represented spaces. *Computability*, 5(2):159–180, 2016.
77. Arno Pauly and Matthew de Brecht. Non-deterministic computation and the Jayne-Rogers theorem. In Benedikt Löwe and Glynn Winskel, editors, *Proceedings 8th International Workshop on Developments in Computational Models, DCM 2012, Cambridge, United Kingdom, 17 June 2012.*, volume 143 of *Electronic Proceedings in Theoretical Computer Science*, pages 87–96, 2014.
78. Arno Pauly and Matthew de Brecht. Descriptive set theory in the category of represented spaces. In *30th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pages 438–449, 2015.
79. Michelle Porter, Adam Day, and Rodney Downey. Notes on computable analysis. *Theory Comput. Syst.*, 60(1):53–111, 2017.

80. Marian B. Pour-El and J. Ian Richards. *Computability in Analysis and Physics*. Perspectives in Mathematical Logic. Springer, Berlin, 1989.
81. Matthias Schröder. *Admissible Representations for Continuous Computations*. PhD thesis, Fachbereich Informatik, FernUniversität Hagen, 2002.
82. Matthias Schröder. Extended admissibility. *Theoretical Computer Science*, 284(2):519–538, 2002.
83. Anant R. Shastri. *Elements of Differential Topology*. CRC Press, Taylor and Francis Group, 2011.
84. Jakob G. Simonsen. On local non-compactness in recursive mathematics. *Mathematical Logic Quarterly*, 52(4):323–330, 2006.
85. Stephen G. Simpson. *Subsystems of Second Order Arithmetic*. Perspectives in Logic, Association for Symbolic Logic. Cambridge University Press, Poughkeepsie, second edition, 2009.
86. Stephen G. Simpson. Mass problems associated with effectively closed sets. *The Tohoku Mathematical Journal. Second Series*, 63(4):489–517, 2011.
87. Stephen G. Simpson. Implicit definability in arithmetic. *Notre Dame J. Form. Log.*, 57(3):329–339, 2016.
88. Robert M. Solovay. On the cardinality of Σ^1_2 sets of reals. In *Foundations of Mathematics (Symposium Commemorating Kurt Gödel, Columbus, Ohio, 1966)*, pages 58–73. Springer, New York, 1969.
89. Robert M. Solovay. A model of set-theory in which every set of reals is Lebesgue measurable. *Ann. of Math. (2)*, 92:1–56, 1970.
90. Paul Taylor. A lambda calculus for real analysis. *J. Log. Anal.*, 2:Paper 5, 115, 2010.
91. Paul Taylor. Foundations *for* computable topology. In *Foundational theories of classical and constructive mathematics*, volume 76 of *West. Ont. Ser. Philos. Sci.*, pages 265–310. Springer, Dordrecht, 2011.
92. Jan van Mill. *The infinite-dimensional topology of function spaces*, volume 64 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, 2001.
93. Jaap van Oosten. *Realizability: an introduction to its categorical side*, volume 152 of *Studies in Logic and the Foundations of Mathematics*. Elsevier B. V., Amsterdam, 2008.
94. Klaus Weihrauch. Computable planar paths intersect in a computable point. preprint. arXiv:1708.07460.
95. Klaus Weihrauch. Computability on computable metric spaces. *Theoretical Computer Science*, 113:191–210, 1993. Fundamental Study.
96. Klaus Weihrauch. *Computable Analysis*. Springer, Berlin, 2000.
97. Klaus Weihrauch. Computable separation in topology, from T_0 to T_2. *J.UCS*, 16(18):2733–2753, 2010.
98. Klaus Weihrauch and Tanja Grubba. Elementary computable topology. *J.UCS*, 15(6):1381–1422, 2009.
99. Klaus Weihrauch and Nazanin Tavana-Roshandel. Representations of measurable sets in computable measure theory. *Logical Methods in Computer Science*, 10:3:7,21, 2014.
100. Yongcheng Wu and Decheng Ding. Computability of measurable sets via effective metrics. *Mathematical Logic Quarterly*, 51(6):543–559, 2005.
101. Yongcheng Wu and Decheng Ding. Computability of measurable sets via effective topologies. *Archive for Mathematical Logic*, 45(3):365–379, 2006.
102. Mariko Yasugi, Takakazu Mori, and Yoshiki Tsujii. Effective properties of sets and functions in metric spaces with computability structure. *Theoretical Computer Science*, 219:467–486, 1999.
103. Martin Ziegler. Computability on regular subsets of Euclidean space. *Mathematical Logic Quarterly*, 48(Suppl. 1):157–181, 2002.
104. Martin Ziegler. Computable operators on regular sets. *Mathematical Logic Quarterly*, 50(4,5):392–404, 2004.