The binary expansion and the intermediate value theorem in constructive reverse mathematics

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Abstract

We introduce the notion of a *convex* tree. We show that the binary expansion for real numbers in the unit interval (BE) is equivalent to weak Köning lemma (WKL) for convex trees having at most two nodes at each level, and we prove that the intermediate value theorem (IVT) is equivalent to WKL for convex tree, in the framework of constructive reverse mathematics.

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1 Introduction

In Bishop's constructive mathematics [2, 3, 4, 5], the *binary expansion* of real numbers in the unit inteval:

BE: Every real number in $[0,1]$ has a binary expansion,

and the *intermediate value theorem*:

IVT: If $f : [0,1] \to \mathbf{R}$ is a uniformly continuous function with $f(0) \leq 0 \leq f(1)$, then there exists $x \in [0,1]$ such that $f(x) = 0$, respectively imply the *lesser limited principle of omniscience* (LLPO or Σ^0_1-DML) which is an instance of De Morgan's law (DML):

$$\forall\alpha\beta[\neg(\exists n(\alpha(n)\neq 0)\wedge\exists n(\beta(n)\neq 0))\to\neg\exists n(\alpha(n)\neq 0)\vee\neg\exists n(\beta(n)\neq 0)]^1;$$

see [13, 5.9] for BE, and [4, 3.2.4] and [14, 6.1.2] for IVT; for a constructive version of IVT, see [3, 2.4.8], [4, 3.2.5] and [14, 6.1.4, 6.1.5].

In the presence of the axiom of countable choice, which is assumed in Bishop's constructive mathematics, we can show the converses, and hence BE and IVT are respectively equivalent to LLPO. Note that, in the absence of the countable choice, we are able to show, in constructive mathematics, that BE and IVT follow from *weak Köning's lemma*:

WKL: Every infinite tree has a branch.

Ishihara [6] showed that WKL is equivalent to LLPO in Bishop's constructive mathematics, and it has been noticed that most mathematical theorems equivalent to LLPO in Bishop's constructive mathematics are equivalent to WKL in the Friedman-Simpson program, called (classical) reverse mathematics; see [11] for reverse mathematics. Note that some mathematical theorems equivalent to WKL in classical reverse mathematics are equivalent to the *fan theorem* (FAN), which is a classical contraposition of and constructively weaker than WKL; see [7], [10] and [8], and also [1].

Of course, there are exceptions. Since BE and IVT are provable in the subsystem $\mathbf{RCA}_0$ of second order arithmetic, the base system for classical reverse mathematics, and $\mathbf{RCA}_0$ does not prove WKL, neither BE nor IVT proves WKL; see [11, II.6.6]. Therefore, although each of BE and IVT implies LLPO, they are strictly weaker than WKL.

Since classical reverse mathematics is formalized with classical logic, we cannot classify theorems, such as BE and IVT which are provable in $\mathbf{RCA}_0$ and theorems in intuitionistic mathematics or in constructive recursive mathematics which are inconsistent with classical logic. On the other hand, since Bishop's constructive mathematics is an informal mathematics using intuitionistic logic and is assuming some function existence axioms (the axiom

[1] Here and in the following, we follow the notational conventions in [14]: m, n, i, j, k are supposed to range over $\mathbf{N}$, a, b, c over the set $\mathbf{N}^*$ of finite sequences of $\mathbf{N}$, and $\alpha, \beta, \gamma, \delta$ over $\mathbf{N}^{\mathbf{N}}$; $|a|$ denotes the length of a finite sequence a, and $a * b$ the concatenation of two finite sequences a and b; $\overline{a}(n)$ and $\overline{\alpha}(n)$ denote the initial segments of a and α of length n, respectively, where $n \leq |a|$.

of countable choice), we cannnot directly bring theorems of Bishop's constructive mathematics into classical reverse mathematics. The aim of *constructive reverse mathematics* is to classify various theorems in intuitionistic, constructive recursive and classical mathematics by logical principles, function existence axioms and their combinations over an intuitionistic system without the axiom of countable choice which is a subsystem of $\mathbf{RCA}_0$; see [7], and [15] for intuitionistic reverse mathematics.

In this paper, we deal with, in constructive reverse mathematics, how weaker BE and IVT are than WKL, and which of them is weaker than the other. After reviewing our base system only with the quantifier-free axiom of countable choice for constructive reverse mathematics in the section 2, we introduce the notion of a *convex* tree. We show that WKL for trees having at most (exactly) two nodes at each level and WKL for convex trees having at most (exactly) two nodes at each level are equivalent in the section 3. In the section 4, we show that BE is equivalent to WKL for convex trees having at most two nodes at each level, and, in the section 5, we show that IVT implies WKL for trees having exactly two nodes at each level, and hence IVT implies BE. Finally we prove that IVT is equivalent to WKL for convex tree.

Since our base system is a subsystem of $\mathbf{RCA}_0$, we see that WKL for convex trees is derivable in $\mathbf{RCA}_0$.

2 A subsystem of elementary analysis

We adopt a subsystem $\mathbf{EL}_0$ of elementary analysis $\mathbf{EL}$ [14, 3.6] as a formal base system for constructive reverse mathematics. The language of $\mathbf{EL}$ contains, in addition to the symbols of $\mathbf{HA}$, unary function variables, denoted by $\alpha, \beta, \gamma, \delta, \ldots$, the application operator Ap, the abstraction operator λ and the recursor $\boldsymbol{r}$. We write $\varphi(t)$ for $\mathrm{Ap}(\varphi, t)$. The logic of $\mathbf{EL}$ is two-sorted intuitionistic predicate logic. As non-logical axioms we have the axioms of $\mathbf{HA}$, with induction extended to formulae of the language of $\mathbf{EL}$, the axiom for λ-conversion, the axioms for the recursor, and the *quantifier-free axiom of choice*:

$$\text{QF-AC}_{00}: \quad \forall m \exists n A(m,n) \to \exists \alpha \forall m A(m, \alpha(m)),$$

where A is a quantifier-free formula and does not contain α free; see [14, 3.6] for more details. The subsystem $\mathbf{EL}_0$ is obtained from $\mathbf{EL}$ by restricting the induction-axiom schema to quantifier-free formulae:

QF-IND: $A(0) \wedge \forall m(A(m) \to A(S(m))) \to \forall m A(m)$,

where A is a quantifier-free formula.

Note that functions in $\mathbf{EL}_0$ contain all primitive recursive functions, and are closed under primitive recursion.

Proposition 1. Σ^0_1-IND *is derivable in* $\mathbf{EL}_0$.

Proof. Let $A(m)$ be a Σ^0_1-formula of the form $\exists n B(m,n)$, where $B(m,n)$ is quantifier-free. Note that $\forall mn(B(m,n) \vee \neg B(m,n))$ as $B(m,n)$ is quantifier-free. Suppose that $A(0)$ and $\forall m(A(m) \to A(S(m)))$. Then, by intuitionistic predicate logic, we have

$$\forall mn \exists k(B(m,n) \to B(S(m),k)),$$

and hence, by QF-AC$_{00}$, there exists α such that

$$\forall mn(B(m,n) \to B(S(m), \alpha(j(m,n)))),$$

where j is a coding function of pairs of natural numbers. Since $A(0)$, there exists n_0 such that $B(0,n_0)$. Define a function γ by primitive recursion such that

$$\gamma(0) = n_0, \quad \gamma(S(m)) = \alpha(j(m, \gamma(m))).$$

Then we have

$$B(0, \gamma(0)) \wedge \forall m(B(m,\gamma(m)) \to B(S(m), \gamma(S(m)))),$$

and therefore $\forall m B(m, \gamma(m))$, by QF-IND. Thus $\forall m A(m)$. $\square$

There is no difficulty at all to establish basic theorems of arithmetic (on natural numbers) in $\mathbf{EL}_0$, as in [14, 3.2]. Using the pairing function j, we can code n-tuples of natural numbers, finite sequences of natural numbers, integers and rationals into natural numbers, develop the elementary theory of operations and relations on $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{N}^*$ and $\{0,1\}^*$, and prove their basic properites in $\mathbf{EL}_0$.

In the language of $\mathbf{EL}$ (and hence $\mathbf{EL}_0$), a *detachable subset* S of $\mathbf{N}$ is given by its characteristic function $\chi_S : \mathbf{N} \to \{0,1\}$ such that

$$\forall n(n \in S \leftrightarrow \chi_S(n) = 1).$$

We adopt a definition of real number with a fixed modulus: a *real number* is a sequence $(p_n)_n$ of rationals such that

$$\forall mn \left(|p_m - p_n| < 2^{-m} + 2^{-n}\right).$$

The relations $<$, $\leq$, and $=$ between real numbers $x = (p_n)_n$ and $y = (q_n)_n$ are defined by

$$x < y \Leftrightarrow \exists n \left(2^{-n+2} < q_n - p_n\right),$$

$x \leq y \Leftrightarrow \neg(y < x)$, and $x = y \Leftrightarrow x \leq y \wedge y \leq x$, respectively. There is no trouble to define the arithmetical operations on the reals, and to show basic theorems on them in $\mathbf{EL}_0$; see [9, Section 4] and [14, 5.2 and 5.3]. Note that for each real number $x = (p_n)_n$, we have $\forall n(|x - p_n| \leq 2^{-n})$; see [9, Lemma 4.4] and [14, Propositions 5.2.14 and 5.2.15].

Especially, the inequality relation $<$ is *cotransitive*, that is, for $x, y, z \in \mathbf{R}$

$$x < y \rightarrow x < z \vee z < y;$$

see [2, Chapter 2, Corollay], [3, Corollay 2.17], and also [14, 5.2.9]. Note that for a given n, using the cotransitivity, we can devide a detachable index set I of a set $\{x_i \mid i \in I\}$ of real numbers into disjoint detachable subsets I_-, I_0 and I^+ such that $i \in I_- \rightarrow x_i < 0$, $i \in I_0 \rightarrow |x_i| < 2^{-n}$ and $i \in I_+ \rightarrow 0 < x_i$.

A *uniformly continuous function* $f : [0,1] \rightarrow \mathbf{R}$ consists of two functions $\varphi : \mathbf{Q} \times \mathbf{N} \rightarrow \mathbf{Q}$ and $\nu : \mathbf{N} \rightarrow \mathbf{N}$ such that $f(p) = (\varphi(p,n))_n \in \mathbf{R}$, and for each k and $p, q \in \mathbf{Q}$ with $0 \leq p, q \leq 1$

$$|p - q| < 2^{-\nu(k)} \rightarrow |f(p) - f(q)| < 2^{-k}.$$

Then the uniformly continuous function $f : [0,1] \rightarrow \mathbf{R}$ is given by

$$f(x) = (\varphi(\min\{\max\{p_{\mu(n)}, 0\}, 1\}, n+1))_n,$$

where $x = (p_n)_n \in [0,1]$ and $\mu(n) = \nu(n+1)+1$, and its modulus of uniform continuity is μ; see [9, Proposition 15].

3 Weak König's lemma for convex trees

For $a, b \in \{0,1\}^*$, let $a \preceq b$ denote that a is an *initial segment* of b, that is, $a \preceq b \Leftrightarrow |a| \leq |b| \wedge \overline{b}(|a|) = a$. Note that $a \preceq c \wedge b \preceq c \rightarrow a \preceq b \vee b \preceq a$.

Let $a \sqsubset b$ denote that a is *on the left of* b (b is *on the right of* a), that is, $a \sqsubset b \Leftrightarrow \exists u \preceq a(u * \langle 0\rangle \preceq a \wedge u * \langle 1\rangle \preceq b)$. It is straightforward to show the following lemma.

Lemma 2. *1.* $\neg(a \sqsubset a)$,

2. $a \sqsubset b \wedge b \sqsubset c \to a \sqsubset c$,

3. $a \sqsubset b \vee b \sqsubset a \vee a \preceq b \vee b \preceq a$,

4. $a * \langle 0 \rangle \sqsubset b \leftrightarrow a \sqsubset b \vee a * \langle 1 \rangle \preceq b$,

5. $a * \langle 1 \rangle \sqsubset b \leftrightarrow a \sqsubset b$,

6. $a \sqsubset b * \langle 0 \rangle \leftrightarrow a \sqsubset b$,

7. $a \sqsubset b * \langle 1 \rangle \leftrightarrow a \sqsubset b \vee b * \langle 0 \rangle \preceq a$,

8. $a \sqsubset b \wedge a' \preceq a \to a' \sqsubset b \vee a' \preceq b$,

9. $a \sqsubset b \wedge b' \preceq b \to a \sqsubset b' \vee b' \preceq a$,

10. $a' \sqsubset b' \wedge a' \preceq a \wedge b' \preceq b \to a \sqsubset b$,

11. $\neg(a * \langle 0 \rangle \sqsubset b \sqsubset a * \langle 1 \rangle)$.

Let $a \sqsubseteq b \Leftrightarrow a \sqsubset b \vee a \preceq b \vee b \preceq a$. Then it is easy to see the following lemma.

Lemma 3. *1.* $a \sqsubseteq b \wedge a' \preceq a \to a' \sqsubseteq b$,

2. $a \sqsubseteq b \wedge b' \preceq b \to a \sqsubseteq b'$,

3. $a \sqsubseteq b \to a * \langle 0 \rangle \sqsubseteq b$,

4. $a \sqsubseteq b \to a \sqsubseteq b * \langle 1 \rangle$,

5. $|a| = |b| \to (a \sqsubseteq b \leftrightarrow a \sqsubset b \vee a = b)$,

6. $|a| = |b| \to (a \sqsubseteq b \wedge b \sqsubset c \to a \sqsubset c)$,

7. $|a| = |b| \to (a \sqsubseteq b \wedge b \sqsubseteq c \to a \sqsubseteq c)$.

For a detachable subset S of $\{0,1\}^*$, we write S_n for the set $\{a \in S \mid |a| = n\}$ and $|S_n|$ for the number of elements of S_n. We say that, for each n, a subset C of $\{0,1\}^n$ is *convex* if for each $a, b \in C$ and $c \in \{0,1\}^n$,

$$a \sqsubset c \sqsubset b \to c \in C,$$

and a subset S of $\{0,1\}^*$ is *convex* if S_n is convex for each n.

A *tree* T is a detachable subset of $\{0,1\}^*$ such that $\langle\rangle \in T$, and $b \in T$ and $a \preceq b$ imply $a \in T$ for each $a, b \in \{0,1\}^*$, and a tree T is *infinite* if T_n is inhabited for each n. A sequence $\alpha \in \{0,1\}^{\mathbf{N}}$ is a *branch* of a tree T if all initial segment of α are in T, that is, $\forall n(\overline{\alpha}(n) \in T)$.

A tree T *has at most (exactly) k nodes at each level* if $|T_{n+1}| \leq k$ (respectively, $|T_{n+1}| = k$) for each n. Let $\mathrm{WKL}_{\leq k}$ (WKL_k) denote WKL for trees having at most (respectively, exactly) k nodes at each level, and let WKL^c denote WKL for convex trees. Also we write $\mathrm{WKL}^c_{\leq k}$ (WKL^c_k) for WKL for convex trees having at most (respectively, exactly) k nodes at each level.

Note that, since $|T_{n+1}| \leq 2|T_n|$, we have $|T_{n+1}|/2^{n+1} \leq |T_n|/2^n$, and hence the sequence $(|T_n|/2^n)_n$ is nonincreasing.

Proposition 4. *Let T be an infinite convex tree such that*

$$|T_n|/2^n \to 0 \text{ as } n \to \infty \tag{1}$$

with a modulus of convergence. Then there exists an infinite convex subtree T' of T having at most two nodes at each level.

Proof. Let T be an infinite convex tree such that $|T_n|/2^n \to 0$ as $n \to \infty$ with a modulus $\mu : \mathbf{N} \to \mathbf{N}$ of convergence, that is, $|T_{\mu(n)}|/2^{\mu(n)} < 2^{-n}$ for each n, and let $(a_n)_n$ and $(b_n)_n$ be sequences of $\{0,1\}^*$ such that $T_n = \{c \in \{0,1\}^n \mid a_n \sqsubseteq c \sqsubseteq b_n\}$ for each n. We may assume, without loss of generality, that $n \leq \mu(n) \leq \mu(n+1)$ for each n. Define sequences $(a'_n)_n$ and $(b'_n)_n$ by $a'_n = \overline{a_{\mu(n)}}(n)$ and $b'_n = \overline{b_{\mu(n)}}(n)$. If $a'_n \sqsubset c \sqsubset b'_n$ for $c \in \{0,1\}^n$, then $a_{\mu(n)} \sqsubset c * u \sqsubset b_{\mu(n)}$ for each $u \in \{0,1\}^{\mu(n)-n}$, by Lemma 2 (10), and hence $2^{\mu(n)-n} < |T_{\mu(n)}|$, or $2^{-n} < |T_{\mu(n)}|/2^{\mu(n)}$, a contradiction. Therefore $\neg(a'_n \sqsubset c \sqsubset b'_n)$, and so $T'_n = \{a'_n, b'_n\}$ is convex and has at most two nodes. Since $a_{\mu(n)} \sqsubseteq \overline{a_{\mu(n+1)}}(\mu(n)) \sqsubseteq b_{\mu(n)}$, we have $a'_n \sqsubseteq \overline{a_{\mu(n+1)}}(n) \sqsubseteq b'_n$, by Lemma 3 (1) and (2), and hence $\overline{a_{\mu(n+1)}}(n) = a'_n$ or $\overline{a_{\mu(n+1)}}(n) = b'_n$, by Lemma 3 (5). Therefore $a'_n \preceq a'_{n+1}$ or $b'_n \preceq a'_{n+1}$. Similarly, we have $a'_n \preceq b'_{n+1}$ or $b'_n \preceq b'_{n+1}$. Thus $T' = \bigcup_{n=0}^{\infty} T'_n$ is an infinite convex subtree of T having at most two nodes at each level. $\square$

Let $\mathrm{WKL}^c_{\to 0}$ denote WKL for convex trees with the property (1) in Proposition 4. Then we have the following corollary.

Corollary 5. *The following are equivalent.*

1. $\mathrm{WKL}^c_{\leq 2}$,

2. $\mathrm{WKL}^c_{\leq k}$ *($k \geq 3$)*,

3. $\mathrm{WKL}^c_{\to 0}$.

Proof. Straightforward by Proposition 4. $\square$

Theorem 6. *The following are equivalent.*

1. $\mathrm{WKL}_{\leq 2}$,

2. WKL_2,

3. $\mathrm{WKL}^c_{\leq 2}$,

4. WKL^c_2.

Proof. Since $\mathrm{WKL}_{\leq 2} \Rightarrow \mathrm{WKL}_2 \Rightarrow \mathrm{WKL}^c_2$ and $\mathrm{WKL}_{\leq 2} \Rightarrow \mathrm{WKL}^c_{\leq 2} \Rightarrow \mathrm{WKL}^c_2$ are trivial, it suffices to show that $\mathrm{WKL}^c_2 \Rightarrow \mathrm{WKL}_{\leq 2}$.

($\mathrm{WKL}^c_2 \Rightarrow \mathrm{WKL}_{\leq 2}$): Suppose WKL^c_2, and let T be an infinite tree having at most two nodes at each level. Then there exist sequences $(a_n)_n$ and $(b_n)_n$ of $\{0,1\}^*$ such that $T_n = \{a_n, b_n\}$ and $a_n \sqsubseteq b_n$ for each n. Note that, since T is a tree, we have $a_n \preceq a_{n+1} \wedge a_n \preceq b_{n+1}$, $a_n \sqsubset b_n \wedge b_n \preceq a_{n+1} \wedge b_n \preceq b_{n+1}$, or $a_n \sqsubset b_n \wedge a_n \preceq a_{n+1} \wedge b_n \preceq b_{n+1}$ for each n. Define sequences $(a'_n)_n$ and $(b'_n)_n$ of $\{0,1\}^*$ by $a'_0 = b'_0 = \langle\rangle$ and

$$\begin{array}{lll} a'_{n+1} = a'_n * \langle 0\rangle, & b'_{n+1} = a'_n * \langle 1\rangle & \text{if } a_n \preceq a_{n+1} \wedge a_n \preceq b_{n+1}, \\ a'_{n+1} = b'_n * \langle 0\rangle, & b'_{n+1} = b'_n * \langle 1\rangle & \text{if } a_n \sqsubset b_n \wedge b_n \preceq a_{n+1} \wedge b_n \preceq b_{n+1}, \\ a'_{n+1} = a'_n * \langle 1\rangle, & b'_{n+1} = b'_n * \langle 0\rangle & \text{if } a_n \sqsubset b_n \wedge a_n \preceq a_{n+1} \wedge b_n \preceq b_{n+1}. \end{array}$$

Then it is straightforward to show, by induction on n, that $|a'_n| = |b'_n| = n$, $a'_{n+1} \sqsubset b'_{n+1}$ and $\neg\exists c(a'_n \sqsubset c \sqsubset b'_n)$ for each n, using Lemma 2 (11), (5) and (6). Therefore $T' = \bigcup_{n=0}^{\infty}\{a'_n, b'_n\}$ is an infinite convex tree having exactly two nodes at each level, and so there exists a branch α in T', by WKL^c_2. Define a mapping $f : T' \to T$ by $f(a'_n) = a_n$ and $f(b'_n) = b_n$ for each n. Then $|f(a')| = |a'|$ for each $a' \in T'$, and it is straightforward to see that $f(a') \preceq f(a' * \langle i\rangle)$ for each $a', a' * \langle i\rangle \in T'$. Thus the sequence $(f(\overline{\alpha}(n)))_n$ defines a branch in T. $\square$

4 The binary expansion

For $a \in \{0,1\}^*$, define a rational number l_a inductively by $l_{\langle\rangle} = 0$, $l_{a*\langle 0\rangle} = l_a$ and $l_{a*\langle 1\rangle} = l_a + 2^{-(|a|+1)}$, and let $r_a = l_a + 2^{-|a|}$. Note that $a \preceq b$ implies $l_a \leq l_b$ and $a \sqsubset b$ implies $2^{-|a|} \leq l_b - l_a$.

Proposition 7. *Let T be a tree, and let x be a real number such that*

$$\forall n \exists a \in T_n(|x - l_a| < 2^{-n}).$$

Then there exists an infinite convex subtree T' of T having at most two nodes at each level, and

$$\forall n \forall a' \in T'_n(|x - l_{a'}| < 2^{-n+1}).$$

Proof. Let T be a tree, and let x be a real number such that $\exists a \in T_n(|x - l_a| < 2^{-n})$ for each n. Let $x = (q_n)_n$ such that $|q_m - q_n| < 2^{-m} + 2^{-n}$ for each n and m, and let

$$T'_n = \{\overline{a}(n) \mid a \in T_{n+2} \wedge |q_{n+4} - l_a| < 2^{-(n+1)}\} \subseteq T_n$$

for each n. Then for each n, since there exists $a \in T_{n+2}$ such that $|x - l_a| < 2^{-(n+2)}$, we have

$$|q_{n+4} - l_a| \leq |q_{n+4} - x| + |x - l_a| < 2^{-(n+4)} + 2^{-(n+2)} < 2^{-(n+1)},$$

and hence T'_n is inhabited. Assume that $a * \langle i \rangle \in T'_{n+1}$. Then there exists $b \in T_{n+3}$ such that $\overline{b}(n+1) = a * \langle i \rangle$ and $|q_{n+5} - l_b| < 2^{-(n+2)}$, and hence, setting $c = \overline{b}(n+2) \in T_{n+2}$, we have $a = \overline{c}(n)$ and

$$\begin{aligned} |q_{n+4} - l_c| &\leq |q_{n+4} - q_{n+5}| + |q_{n+5} - l_b| + |l_b - l_c| \\ &< 2^{-(n+4)} + 2^{-(n+5)} + 2^{-(n+2)} + 2^{-(n+3)} < 2^{-(n+1)}. \end{aligned}$$

Therefore $a \in T'_n$. If $a' \sqsubset c \sqsubset b'$ with $a', b' \in T'_n$ and $c \in \{0,1\}^n$, then there exist $a, b \in T_{n+2}$ such that $a' = \overline{a}(n)$, $b' = \overline{b}(n)$, $|q_{n+4} - l_a| < 2^{-(n+1)}$ and $|q_{n+4} - l_b| < 2^{-(n+1)}$, and hence

$$\begin{aligned} 2^{-n+1} &= 2^{-n} + 2^{-n} \leq (l_{b'} - l_c) + (l_c - l_{a'}) \\ &= (l_{b'} - l_b) + (l_b - q_{n+4}) + (q_{n+4} - l_a) + (l_a - l_{a'}) \\ &< 0 + 2^{-(n+1)} + 2^{-(n+1)} + 2^{-n} = 2^{-n+1}, \end{aligned}$$

a contradiction. Therefore T'_n is convex and has at most two nodes. If $a' \in T'_n$, then there exists $a \in T_{n+2}$ such that $a' = \overline{a}(n)$ and $|q_{n+4} - l_a| < 2^{-(n+1)}$, and hence

$$|x - l_{a'}| \leq |x - q_{n+4}| + |q_{n+4} - l_a| + |l_a - l_{a'}| < 2^{-(n+4)} + 2^{-(n+1)} + 2^{-n} < 2^{-n+1}.$$

Thus $T' = \bigcup_{n*0}^{\infty} T'_n$ is an infinite convex subtree of T with the required properties. □

Theorem 8. *The following are equivalent.*

1. BE,
2. $\mathrm{WKL}^c_{\leq 2}$,

Proof. It suffices to show that $\mathrm{BE} \Rightarrow \mathrm{WKL}^c_2$ and $\mathrm{WKL}^c_{\leq 2} \Rightarrow \mathrm{BE}$, by Theorem 6.

($\mathrm{BE} \Rightarrow \mathrm{WKL}^c_2$): Suppose BE, and let T be an infinite convex tree having exactly two nodes at each level. Then there exist sequences $(a_n)_n$ and $(b_n)_n$ of $\{0,1\}^*$ such that $T_n = \{a_n, b_n\}$ and $a_{n+1} \sqsubset b_{n+1}$ for each n. Note that, since T is a tree, we have $a_{n+1} = a_n * \langle 0\rangle \wedge b_{n+1} = a_n * \langle 1\rangle$, $a_{n+1} = b_n * \langle 0\rangle \wedge b_{n+1} = b_n * \langle 1\rangle$, or $a_{n+1} = a_n * \langle 1\rangle \wedge b_{n+1} = b_n * \langle 0\rangle$ for each n. Define sequences $(a'_n)_n$ and $(b'_n)_n$ of $\{0,1\}^*$ by $a'_0 = b'_0 = \langle\rangle$,

$$\begin{array}{lll} a'_{2n+1} = a'_{2n} * \langle 0\rangle, & b'_{2n+1} = a'_{2n} * \langle 1\rangle & \text{if } a_{n+1} = a_n * \langle 0\rangle \wedge b_{n+1} = a_n * \langle 1\rangle, \\ a'_{2n+1} = b'_{2n} * \langle 0\rangle, & b'_{2n+1} = b'_{2n} * \langle 1\rangle & \text{if } a_{n+1} = b_n * \langle 0\rangle \wedge b_{n+1} = b_n * \langle 1\rangle, \\ a'_{2n+1} = a'_{2n} * \langle 1\rangle, & b'_{2n+1} = b'_{2n} * \langle 0\rangle & \text{if } a_{n+1} = a_n * \langle 1\rangle \wedge b_{n+1} = b_n * \langle 0\rangle, \end{array}$$

and

$$a'_{2n+2} = a'_{2n+1} * \langle 1\rangle, \quad b'_{2n+2} = b'_{2n+1} * \langle 0\rangle.$$

Then it is straightforward to show, by induction on n, that $|a'_n| = |b'_n| = n$, $a'_{n+1} \sqsubset b'_{n+1}$, and $\neg\exists c(a'_n \sqsubset c \sqsubset b'_n)$ for each n, using Lemma 2 (11), (5) and (6). Therefore $T' = \bigcup_{n=0}^{\infty}\{a'_n, b'_n\}$ is an infinite convex tree having exactly two nodes at each level. It is straightforward to see, by induction on n, that $l_{b'_{n+1}} - l_{a'_{n+1}} = 2^{-(n+1)}$ for each n, and hence $0 \leq l_{a'_{n+1}} - l_{a'_n} \leq 2^{-n}$ for each n. Therefore $(l_{a'_n})_n$ is a Cauchy sequence of rationals, and so it converges to a real number x in $[0,1]$. Note that $l_{a'_n} \leq x \leq l_{a'_n} + 2^{-n+1}$ and $x \leq l_{b'_n} + 2^{-n}$ for each n. By BE, there exists $\alpha \in \{0,1\}^{\mathbf{N}}$ such that $x = \sum_{i=0}^{\infty} \alpha(i) \cdot 2^{-(i+1)}$. Note that $l_{\overline{\alpha}(n)} \leq l_{\overline{\alpha}(n+1)}$ and $x \leq l_{\overline{\alpha}(n)} + 2^{-n}$.

We show that α is a branch in T'. Assume that $\overline{\alpha}(n) \notin T'$, and choose m so that $n \leq 2m+1$. Then $\overline{\alpha}(2m+1) \notin T'_{2m+1}$, and hence either $\overline{\alpha}(2m+1) \sqsubset a'_{2m+1}$ or $b'_{2m+1} \sqsubset \overline{\alpha}(2m+1)$, by Lemma 2 (3). In the former case, since $x \leq l_{\overline{\alpha}(2m+1)} + 2^{-(2m+1)} \leq l_{a'_{2m+1}}$, we have

$$x + 2^{-(2m+2)} \leq l_{a'_{2m+1}} + 2^{-(2m+2)} = l_{a'_{2m+2}} \leq x,$$

a contradiction. In the latter case, since $l_{b'_{2m+1}} + 2^{-(2m+1)} \leq l_{\overline{\alpha}(2m+1)} \leq x$, we have

$$x + 2^{-(2m+2)} \leq l_{b'_{2m+2}} + 2^{-(2m+2)} + 2^{-(2m+2)} = l_{b'_{2m+1}} + 2^{-(2m+1)} \leq x,$$

a contradiction. Therefore $\overline{\alpha}(n) \in T'$. Let $\beta(n) = \alpha(2n)$. Then it is straightforward to show, by simultaneous induction on n, that $\overline{\alpha}(2n) = a'_{2n}$ implies $\overline{\beta}(n) = a_n$ and $\overline{\alpha}(2n) = b'_{2n}$ implies $\overline{\beta}(n) = b_n$ for each n. Thus β is a branch in T.

($\mathrm{WKL}^c_{\leq 2} \Rightarrow \mathrm{BE}$): Suppose $\mathrm{WKL}^c_{\leq 2}$. Let $x \in [0,1]$, and let $T = \{0,1\}^*$ be the complete binary tree. Then $\exists a \in T_n(|x - l_a| < 2^{-n})$ for each n, and hence there exists an infinite convex subtree T' of T having at most two nodes at each level and $\forall a' \in T'_n(|x - l_{a'}| < 2^{-n+1})$ for each n, by Proposition 7. By $\mathrm{WKL}^c_{\leq 2}$, there exists a branch α in T', and hence in T. Since

$$|x - \textstyle\sum_{i=0}^{\infty} \alpha(i) \cdot 2^{-(i+1)}| \leq |x - l_{\overline{\alpha}(n)}| + |l_{\overline{\alpha}(n)} - \textstyle\sum_{i=0}^{\infty} \alpha(i) \cdot 2^{-(i+1)}| < 2^{-n+1} + 2^{-n}$$

for each n, we have $x = \sum_{i=0}^{\infty} \alpha(i) \cdot 2^{-(i+1)}$. □

5 The intermediate value theorem

For $a \in \{0,1\}^*$, define a rational number l'_a inductively by $l'_{\langle\rangle} = 1/3$, $l'_{a*\langle 0\rangle} = l'_a$ and $l'_{a*\langle 1\rangle} = l'_a + 2 \cdot 3^{-(|a|+2)}$, and let $r'_a = l'_a + 3^{-(|a|+1)}$. Note that $a \preceq b$ implies $l'_a \leq l'_b$ and $r'_b \leq r'_a$, and $a \sqsubset b$ implies $3^{-|a|+1} \leq l'_b - r'_a$.

Proposition 9. IVT *implies* WKL_2.

Proof. Suppose IVT, and let T be an infinite tree having exactly two nodes at each level. Then there exist sequences $(a_n)_n$ and $(b_n)_n$ of $\{0,1\}^*$ such that $a_0 = b_0 = \langle\rangle$, $T_n = \{a_n, b_n\}$ and $a_{n+1} \sqsubset b_{n+1}$ for each n. Note that

$l'_{a_n} \leq l'_{a_{n+1}} < r'_{b_{n+1}} \leq r'_{b_n}$ for each n, and hence $l'_{a_n} < r'_{b_m}$ for each n and m. For each n, define a uniformly continuous function $f_n : [0,1] \to \mathbf{R}$ by

$$f_n(x) = \max\{\min\{l'^{-1}_{a_n}(x - l'_{a_n}), 0\}, (1 - r'_{b_n})^{-1}(x - r'_{b_n})\}.$$

Note that $x < l'_{a_n}$ if and only if $f_n(x) < 0$, $l'_{a_n} \leq x \leq r'_{b_n}$ if and only if $f_n(x) = 0$, $r'_{b_n} < x$ if and only if $0 < f_n(x)$, $f_n(0) = -1$, and $f_n(1) = 1$. If $f_n(x) < 0$ and $0 < f_m(x)$ then $x < l'_{a_n} < r'_{b_m} < x$, a contradiction. Hence if $f_n(x) < 0$ for some n, then $f_m(x) \leq 0$ for each m. Similarly, if $0 < f_n(x)$ for some n, then $0 \leq f_m(x)$ for each m. Moreover, note that $|f_n(x) - f_n(y)| \leq 3|x - y|$ for each $x, y \in [0,1]$. Let

$$f(x) = \sum_{n=0}^{\infty} 2^{-(n+1)} f_n(x).$$

Then $f : [0,1] \to \mathbf{R}$ is a uniformly continuous function such that $f(0) < 0 < f(1)$, and hence there exists $x \in [0,1]$ such that $f(x) = 0$, by IVT.

We define inductively a sequence $(c_n)_n$ of T such that $|c_n| = n$, $c_n \preceq c_{n+1}$, and $\forall m \geq n \exists c \in T_m (c_n \preceq c)$ for each n. Then, trivially, the sequence $(c_n)_n$ defines a branch in T. Let $c_0 = \langle\rangle$, and suppose that c_n has been defined. If $c_n \preceq a_{n+1}$ and $\neg(c_n \preceq b_{n+1})$, then set $c_{n+1} = a_{n+1}$, and if $c_n \preceq b_{n+1}$ and $\neg(c_n \preceq a_{n+1})$, then set $c_{n+1} = b_{n+1}$. If $c_n \preceq a_{n+1}$ and $c_n \preceq b_{n+1}$, then, since $r'_{a_{n+1}} + 3^{-(n+2)} \leq l'_{b_{n+1}}$, either $r'_{a_{n+1}} < x$ or $x < l'_{b_{n+1}}$. In the former case, assume that $m \geq n+1$ and $\neg(b_{n+1} \preceq b_m)$. Then $a_{n+1} \preceq b_m$, and, since $r'_{b_m} \leq r'_{a_{n+1}} < x$, we have $0 < 2^{-(m+1)} f_m(x) \leq f(x)$, a contradiction. Therefore $b_{n+1} \preceq b_m$ for each $m \geq n+1$, and set $c_{n+1} = b_{n+1}$. In the latter case, similarly we have $a_{n+1} \preceq a_m$ for each $m \geq n+1$, and set $c_{n+1} = a_{n+1}$. $\square$

Corollary 10. IVT *implies* BE.

Proof. By Proposition 9, Theorem 6 and Proposition 8. $\square$

Lemma 11. *Let $a, b \in \{0,1\}^*$ be such that $a \sqsubset b$. Then there exist $c, d \in \{0,1\}^*$ such that $|c| = |a|$, $|d| = |b|$, $a \sqsubset c \sqsubseteq b$, $a \sqsubseteq d \sqsubset b$, $l_c = r_a$ and $r_d = l_b$.*

Proof. For $u \in \{0,1\}^*$ with $|u| > 0$, define $\mathrm{suc}(u)$ in $\{0,1\}^*$ inductively by

$$\begin{array}{ll} \mathrm{suc}(\langle 0\rangle) = \langle 1\rangle, & \mathrm{suc}(\langle 1\rangle) = \langle 1\rangle, \\ \mathrm{suc}(u * \langle 0\rangle) = u * \langle 1\rangle, & \mathrm{suc}(u * \langle 1\rangle) = \mathrm{suc}(u) * \langle 0\rangle. \end{array}$$

It is stratightforward to show, by induction on a, that $|\mathrm{suc}(a)| = |a|$. We show, by induction on a, that if $a \sqsubset b$, then $a \sqsubset \mathrm{suc}(a) \sqsubseteq b$ and $l_{\mathrm{suc}(a)} = r_a$. Suppose that $a \sqsubset b$. Then $\neg(a = \langle 1\rangle)$ and $\neg(b = \langle 0\rangle)$. If $a = \langle 0\rangle$, then $\langle 1\rangle \preceq b$, and hence $a \sqsubset \mathrm{suc}(a) \sqsubseteq b$ and $l_{\mathrm{suc}(a)} = 1/2 = r_a$. If $a = u * \langle 0\rangle$, then either $u \sqsubset b$ or $u * \langle 1\rangle \preceq b$, by Lemma 2 (4), and, by Lemma 2 (5), in both cases, we have $a \sqsubset \mathrm{suc}(a) \sqsubseteq b$ and $l_{\mathrm{suc}(a)} = l_u + 2^{-(|u|+1)} = l_{u*\langle 0\rangle} + 2^{-(|u|+1)} = r_a$. Assume that $a = u * \langle 1\rangle$. Then $u \sqsubset b$, by Lemma 2 (5), and hence $u \sqsubset \mathrm{suc}(u) \sqsubseteq b$ and $l_{\mathrm{suc}(u)} = r_u$ by induction hypothesis. Therefore $a \sqsubset \mathrm{suc}(a) \sqsubseteq b$, by Lemma 2 (5) and (6), and Lemma 3 (3), and

$$\begin{aligned} l_{\mathrm{suc}(a)} &= l_{\mathrm{suc}(u)} + 2^{-(|u|+1)} = r_u + 2^{-(|u|+1)} \\ &= l_u + 2^{-(|u|+1)} + 2^{-(|u|+1)} = l_{u*\langle 1\rangle} + 2^{-(|u|+1)} = r_a. \end{aligned}$$

For $u \in \{0,1\}^*$ with $|u| > 0$, define $\mathrm{prd}(u)$ in $\{0,1\}^*$ inductively by

$$\begin{array}{ll} \mathrm{prd}(\langle 0\rangle) = \langle 0\rangle, & \mathrm{prd}(\langle 1\rangle) = \langle 0\rangle, \\ \mathrm{prd}(u * \langle 0\rangle) = \mathrm{prd}(u) * \langle 1\rangle, & \mathrm{prd}(u * \langle 1\rangle) = u * \langle 0\rangle. \end{array}$$

Then, similarly, we see that $|\mathrm{prd}(b)| = |b|$, $a \sqsubseteq \mathrm{prd}(b) \sqsubset b$ and $r_{\mathrm{prd}(b)} = l_b$. $\square$

Theorem 12. *The following are equivalent.*

1. IVT,
2. WKL^c.

Proof. ($\mathrm{IVT} \Rightarrow \mathrm{WKL}^c$): Suppose IVT, and let T be an infinite convex tree. Then there exist sequences $(a_n)_n$ and $(b_n)_n$ of $\{0,1\}^*$ such that $T_n = \{c \in \{0,1\}^n \mid a_n \sqsubseteq c \sqsubseteq b_n\}$ for each n. For each n, define a uniformly continuous function $f_n : [0,1] \to \mathbf{R}$ by

$$f_n(x) = \max\{\min\{(l_{a_n} + 1)^{-1}(3x - l_{a_n} - 1), 0\}, (2 - r_{b_n})^{-1}(3x - r_{b_n} - 1)\}.$$

Note that if $f_n(x) = 0$, then $l_{a_n} \leq 3x - 1 \leq r_{b_n}$, if $f_n(x) < 0$ for some n, then $f_m(x) \leq 0$ for each m, and if $0 < f_n(x)$ for some n, then $0 \leq f_m(x)$ for each m. Let

$$f(x) = \sum_{n=0}^{\infty} 2^{-(n+1)} f_n(x).$$

Then $f : [0,1] \to \mathbf{R}$ is a uniformly continuous function such that $f(0) < 0 < f(1)$, and hence there exists $x \in [0,1]$ such that $f(x) = 0$, by IVT.

For each n, since $f_n(x) = 0$, we have $l_{a_n} \leq 3x - 1 \leq r_{b_n}$, and hence $\exists a \in T_n(|(3x-1) - l_a| < 2^{-n})$. Therefore there exists an infinite convex subtree T' of T having at most two nodes at each level, by Proposition 7. By Proposition 9 and Theorem 6, there exists a branch in T', and hence in T.

($\mathrm{WKL}^c \Rightarrow \mathrm{IVT}$): Suppose WKL^c, and let $f : [0,1] \to \mathbf{R}$ be a uniformly continuous function such that $f(0) \leq 0 \leq f(1)$. Then we define inductively sequences $(a_n)_n$ and $(b_n)_n$ of $\{0,1\}^*$ such that for each n

1. $|a_n| = |b_n| = n$ and $a_n \sqsubseteq b_n$,
2. $f(l_{a_n}) \leq 0 \leq f(r_{b_n})$,
3. $\forall c \in \{0,1\}^n(a_n \sqsubset c \sqsubseteq b_n \to |f(l_c)| < 2^{-n})$.

Let $a_0 = b_0 = \langle\rangle$, and suppose that a_n and b_n have been defined. Then we devide the set

$$S = \{u \in \{0,1\}^{n+1} \mid \exists v \in \{0,1\}^n(a_n \sqsubseteq v \sqsubseteq b_n \wedge v \preceq u)\}$$

into disjoint detachable subsets S_-, S_0 and S_+ such that $c \in S_- \to f(l_c) < 0$, $c \in S_0 \to |f(l_c)| < 2^{-n}$ and $c \in S_+ \to 0 < f(l_c)$. If S_- is inhabited, then choose $a_{n+1} \in S_-$ so that $\neg\exists v \in S_-(a_{n+1} \sqsubset v)$, and otherwise set $a_{n+1} = a_n * \langle 0\rangle$. If $\{u \in S_+ \mid a_{n+1} \sqsubset u\}$ is inhabited, then choose $c \in S_+$ so that $a_{n+1} \sqsubset c \wedge \neg\exists v \in S_+(a_{n+1} \sqsubset v \sqsubset c)$ and choose $b_{n+1} \in S$, by Lemma 11, so that $a_{n+1} \sqsubseteq b_{n+1} \sqsubset c$ and $r_{b_{n+1}} = l_c$, and otherwise set $b_{n+1} = b_n * \langle 1\rangle$.

It is trival that $|a_{n+1}| = |b_{n+1}| = n+1$ and $a_{n+1} \sqsubseteq b_{n+1}$. If S_- is inhabited, then, since $a_{n+1} \in S_-$, we have $f(l_{a_{n+1}}) < 0$, and otherwise, since $l_{a_{n+1}} = l_{a_n}$, we have $f(l_{a_{n+1}}) \leq 0$. If $\{u \in S_+ \mid a_{n+1} \sqsubset u\}$ is inhabited, then, since there exists $c \in S_+$ such that $r_{b_{n+1}} = l_c$, we have $0 < f(r_{b_{n+1}})$, and otherwise, since $r_{b_{n+1}} = r_{b_n}$, we have $0 \leq f(r_{b_{n+1}})$. Assume that $a_{n+1} \sqsubset c \sqsubseteq b_{n+1}$ with $c \in \{0,1\}^{n+1}$. If $c \in S_-$, then S_- is inhabited, and hence $\neg\exists v \in S_-(a_{n+1} \sqsubset v)$, a contradiction. If $c \in S_+$, then, since $\{u \in S_+ \mid a_{n+1} \sqsubset u\}$ is inhabited, there exists $c' \in S_+$ such that $\neg\exists v \in S_+(a_{n+1} \sqsubset v \sqsubset c')$ and $a_{n+1} \sqsubseteq b_{n+1} \sqsubset c'$, and hence $a_{n+1} \sqsubset c \sqsubset c'$ by Lemma 3 (6), a contradiction. Therefore $c \in S_0$.

Let $T_n = \{u \in \{0,1\}^n \mid a_n \sqsubseteq u \sqsubseteq b_n\}$ for each n, and let $T = \bigcup_{n=0}^{\infty} T_n$. Then T is an infinite convex tree, and hence there exists a branch α in T by WKL^c. Let $\mu : \mathbf{N} \to \mathbf{N}$ be a modulus of uniform continuity for f such that for each $x, y \in [0,1]$ and each n

$$|x - y| < 2^{-\mu(n)} \to |f(x) - f(y)| < 2^{-n},$$

and let $x = \sum_{i=0}^{\infty} \alpha(i) \cdot 2^{-(i+1)}$. Suppose that $|f(x)| > 0$, choose m so that $|f(x)| > 2^{-m+2}$, and let $n = \max\{m, \mu(m)\} + 1$. Then, since $|x - l_{\overline{\alpha}(n)}| \leq 2^{-n} < 2^{-\mu(m)}$, we have

$$|f(l_{\overline{\alpha}(n)})| \geq |f(x)| - 2^{-m} > 2^{-m+2} - 2^{-m} > 2^{-m+1} > 2^{-n},$$

and hence $\overline{\alpha}(n) = a_n$ and $f(l_{a_n}) < -2^{-m+1}$. If there exists $u \in \{0,1\}^n$, by Lemma 11, such that $a_n \sqsubset u \sqsubseteq b_n$, then there exists $c \in \{0,1\}^n$ such that $a_n \sqsubset c \sqsubseteq u$ and $|l_{a_n} - l_c| = |l_{a_n} - r_{a_n}| = 2^{-n} < 2^{-\mu(m)}$, and hence $a_n \sqsubset c \sqsubseteq b_n$, by Lemma 3 (7), and

$$-2^{-n} < f(l_c) \leq f(l_{a_n}) + 2^{-m} < -2^{-m+1} + 2^{-m} = -2^{-m} < -2^{-n},$$

a contradiction. Therefore $a_n = b_n$, and, since $|l_{a_n} - r_{b_n}| = 2^{-n} < 2^{-\mu(m)}$, we have

$$0 \leq f(r_{b_n}) \leq f(l_{a_n}) + 2^{-m} < -2^{-m+1} + 2^{-m} = -2^{-m},$$

a contradiction. Thus $f(x) = 0$. $\square$

Remark 13. The *fan theorem* for detachable bar:

$\mathrm{FAN_D}$: $\forall \alpha \in \{0,1\}^{\mathbf{N}} \exists n B(\overline{\alpha}n) \to \exists n \forall \alpha \in \{0,1\}^{\mathbf{N}} \exists k \leq n B(\overline{\alpha}k)$,

where B is quantifier-free, is a classical contraposition of and constructively weaker than WKL; see [14, 4.7] and [8]. Since $\mathrm{FAN_D}$ is classically equivalent to WKL, we have $\mathbf{RCA}_0 \nvdash \mathrm{FAN_D}$, and therefore, since $\mathbf{RCA}_0 \vdash \mathrm{IVT}$, we have

$$\mathbf{EL} + \mathrm{PEM} + \mathrm{IVT} \nvdash \mathrm{FAN_D},$$

where PEM denotes the principle of excluded middle. Since $\mathrm{BE} \vdash \mathrm{LLPO}$ and the *weak continuity* for numbers (WC-N) refutes LLPO (see [14, 4.6.3 and 4.6.4]), we have $\mathrm{WC\text{-}N} + \mathrm{BE} \vdash \bot$, and therefore, since $\mathrm{WC\text{-}N} + \mathrm{FAN_D}$ is consistent (see [12, 3.3.11 Theorem (ii)]), we have

$$\mathbf{EL} + \mathrm{WC\text{-}N} + \mathrm{FAN_D} \nvdash \mathrm{BE}.$$

Although $\mathrm{FAN_D}$ is incompatible with *Church's thesis* (CT) (see [14, 4.3.1 and 4.7.6]), since $\mathbf{RCA}_0 \vdash \mathrm{IVT}$, we have $\mathrm{REC} \models \mathrm{IVT}$, that is, IVT is valid in the model REC of $\mathbf{RCA}_0$ consisting of all recursive sets, and therefore, since $\mathrm{REC} \models \mathrm{CT}$, we have

$$\mathbf{EL} + \mathrm{PEM} + \mathrm{IVT} + \mathrm{CT} \nvdash \bot.$$

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