

Wadge-like Degrees of Borel bqo-Valued Functions

Takayuki Kihara

Department of Mathematical Informatics, Graduate School of Informatics

Nagoya University, Japan

kihara@i.nagoya-u.ac.jp

and

Victor Selivanov

A.P. Ershov Institute of Informatics Systems SB RAS

and Kazan Federal University, Russia

vseliv@iis.nsk.su

Abstract

We unite two well known generalisations of the Wadge theory. The first one considers more general reducing functions than the continuous functions in the classical case, and the second one extends Wadge reducibility from sets (i.e., $\{0,1\}$ -valued functions) to Q -valued functions, for a better quasiorder Q . In this article, we consider more general reducibilities on the Q -valued functions and generalise some results of L. Motto Ros in the first direction and of T. Kihara and A. Montalbán in the second direction: Our main result states that the structure of the Δ_α^0 -degrees of $\Delta_{\alpha+\gamma}^0$ -measurable Q -valued functions is isomorphic to the Δ_β^0 -degrees of $\Delta_{\beta+\gamma}^0$ -measurable Q -valued functions, and these are isomorphic to the generalized homomorphism order on the γ -th iterated Q -labeled forests.

Key words: Borel hierarchy, Wadge degree, amenable reducibility, iterated labeled forest, h -quasiorder, better quasiorder.

1 Introduction

In his thesis [26], W. Wadge introduced a way of measuring the topological complexity of subsets of Baire space $\mathcal{N} = \omega^\omega$: For subsets A, B of \mathcal{N} , we say that A is *Wadge reducible* to B ($A \leq_W B$), if $A = f^{-1}(B)$ for some continuous function f on \mathcal{N} . The definition looks quite elementary; however its surprisingly well-behaved structure was never revealed without the development of deep determinacy techniques. The induced quotient poset, now called the *Wadge degrees*, turns out to be well-founded and have no three pairwise incomparable

elements. Indeed, it provides us an ultimate refinement of all known hierarchies in descriptive set theory, such as the Borel hierarchy and the Hausdorff-Kuratowski difference hierarchy. Nowadays, the notion of Wadge degrees has become important in several areas including descriptive set theory, inner model theory, computability theory, and automata theory (see also [10] for the overview of the theory of Wadge degrees).

It is straightforward to generalize the notion of Wadge reducibility to arbitrary topological spaces. However, contrary to the splendid success in the zero-dimensional case, the Wadge theory for non-zero-dimensional spaces was confronted with serious difficulties, cf. [8]. Several approaches to solving this difficulty have been proposed, and one of them is considering other natural classes of reducing functions in place of the continuous functions, cf. [18]. For a pointclass Γ , by Γ -function we mean a function f such that $f^{-1}(A) \in \Gamma$ for each $A \in \Gamma$. Since the Γ -functions are closed under composition and contain the identity function, we obtain the corresponding Γ -reducibility \leq_{Γ} . Among such reducibilities are Δ_{α}^0 -reducibilities, for each non-zero countable ordinal α . Note that Δ_1^0 -reducibility coincides with the Wadge reducibility for \mathcal{N} . We shorten the notation $\leq_{\Delta_{\alpha}^0}$ to \leq_{α} , so \leq_1 coincides with \leq_W . The Δ_{α}^0 -functions (which coincide with the Σ_{α}^0 -functions) and Δ_{α}^0 -reducibilities (among the much larger class of the so called Borel amenable reducibilities) were comprehensively investigated by L. Motto Ros (see [17] and references therein), and later by [18]. In particular, we have $(\Delta_1^1(\mathcal{N}); \leq_{\alpha}) \simeq (\Delta_1^1(\mathcal{N}); \leq_W)$, $(\Delta_{\xi}^0(\mathbb{R}); \leq_3) \simeq (\Delta_{\xi}^0(\mathcal{N}); \leq_3)$, and so on.

Another extension of the Wadge theory is the extension from the case of subsets of \mathcal{N} to the case of functions $A: \mathcal{N} \rightarrow Q$ to an arbitrary quasi-order Q . For $Q = \{0, 1, \perp\}$, this extension has already been made by Wadge [26, Section I.E]. The Wadge hierarchy for the class $Q = \text{Ord}$ of ordinals, known as the hierarchy of norms (cf. [1]) or the Steel hierarchy (cf. [5]), plays a crucial role in descriptive set theory [16]. For the other use, as explicitly described in [13], the Wadge theory is strongly tied with Martin's conjecture, one of the most prominent open problems in computability theory (see also the recent Notices article [15]), where in [13] Wadge reducibility for an arbitrary bqo Q was considered in order to obtain a better understanding on uniform universality for countable Borel equivalence relations. The necessity of the use of Q was also occurred in [3, 12], where (a variant of) the Wadge degrees of $\{0, 1, \perp\}$ -valued functions was utilized to analyze the behavior of the structure of real-valued functions, and its connection with the notion of α -rank which was originally introduced in J. Bourgain's work on refining the Odell-Rosenthal theorem in Banach space theory.

Hereafter, we identify Q -valued functions on a space X with Q -partitions of X of the form $\{A^{-1}(q)\}_{q \in Q}$ in order to stress their close relation to k -partitions (obtained when $Q = \bar{k} = \{0, \dots, k-1\}$ is an antichain with k -elements) studied by several authors. For Q -partitions A, B of X , let $A \leq_W B$ mean that there is a continuous function $f: X \rightarrow X$ such that $A(x) \leq_Q B(f(x))$ for each $x \in X$. The case of sets corresponds to the case of 2-partitions. Let $\Gamma(Q^X)$ be the set of Q -partitions A of X such that $A^{-1}(q) \in \Gamma(X)$ for all $q \in Q$. A celebrated theorem of van Engelen, Miller and Steel [25, Theorem 3.2] implies that if Q is a better quasiorder (bqo) then the Wadge ordering $\mathcal{W}_Q = (\Delta_1^1(Q^{\mathcal{N}}); \leq_W)$ on the Borel Q -partitions is a bqo, too (see also Fact 1). Although this theorem gives an important

information about the quotient-poset of \mathcal{W}_Q , it is far from a characterisation.

Many efforts (see e.g. [7, 22, 20, 23] and references therein) to characterise the quotient-poset of \mathcal{W}_Q were devoted to *k-partitions of \mathcal{N}* . The approach in [22, 20, 23] to this problem was to characterise the initial segments $(\Delta_\alpha^0(k^\mathcal{N}); \leq_W)$ for bigger and bigger ordinals $2 \leq \alpha < \omega_1$. To achieve this, the structures of iterated labeled forests with the so called homomorphism quasiorder were defined and useful properties of some natural operations on the iterated labeled forests were discovered, which have brought us a fresh look at the deep relationship between Wadge theory and wqo/bqo theory.

An important progress was recently achieved in [14] where a full characterisation of the quotient-poset of \mathcal{W}_Q for arbitrary bqo Q is obtained, with a heavy use of the (suitably extended) iterated labeled forests and of the classical computability theory.

In this paper, we unite the above-mentioned extensions of the Wadge theory by characterising the quotient-posets of $(\Delta_1^1(Q^\mathcal{N}); \leq_\alpha)$, of their variations for some other Borel amenable reducibilities (which are extended to $Q^\mathcal{N}$ in the obvious way), and of natural initial segments of such quotient-posets. A typical result (extending the above-mentioned result of L. Motto Ros) may be formulated as follows: For any bqo Q and any countable ordinals $\alpha > 0$ and $\beta \geq 3$, $(\Delta_1^1(Q^\mathcal{N}); \leq_W) \simeq (\Delta_1^1(Q^\mathcal{N}); \leq_\alpha) \simeq (\Delta_1^1(Q^\mathbb{R}); \leq_\beta)$.

We deduce this fact from the following main result (Corollary 2): for all $\alpha, \beta, \gamma < \omega_1$ with $\alpha, \beta > 0$ we have $(\Delta_{\alpha+\gamma}^0(Q^\mathcal{N}); \leq_\alpha) \simeq (\Delta_{\beta+\gamma}^0(Q^\mathcal{N}); \leq_\beta)$; which is proved by induction using the results in [14]. Among particular cases and variations of our result we mention the following:

- (1) $(\Delta_\omega^0(Q^\mathcal{N}); \leq_n) \simeq (\Delta_\omega^0(Q^\mathcal{N}); \leq_W)$ for each $2 \leq n < \omega$,
- (2) $(\bigcup_{k < \omega} \Delta_k^0(Q^\mathcal{N}); \leq_n) \simeq (\bigcup_{k < \omega} \Delta_k^0(Q^\mathcal{N}); \leq_W)$ for each $2 \leq n < \omega$,
- (3) $(\Delta_5^0(Q^\mathcal{N}); \leq_W) \simeq (\Delta_7^0(Q^\mathcal{N}); \leq_3) \simeq (\Delta_7^0(Q^\mathbb{R}); \leq_3) \simeq (\Delta_{\omega+4}^0(Q^{C^{([0,1])}}); \leq_\omega)$.

After recalling some preliminaries in the next section, we establish our main result in Section 3 which gives a characterisation of the quotient-posets of $(\Delta_1^1(Q^\mathcal{N}); \leq_{1+\xi})$, $\xi < \omega_1$, similar to that in [14]. Indeed, we prove the same characterisation for some other Borel amenable reducibilities in place of $\leq_{1+\xi}$. Finally, in Section 4 we provide an inner (i.e., using only notions for labeled forests) characterisations of the quasiorders induced on the iterated labeled forests by the quasiorders $\leq_{1+\xi}$.

2 Preliminaries

2.1 Ordinals, quasi-orders, semilattices

We assume the reader to be acquainted with the notion of ordinal. Ordinals are denoted by $\alpha, \beta, \gamma, \xi, \eta, \dots$. Every non-zero ordinal ξ is uniquely representable in the form $\xi = \omega^{\alpha_0} + \dots + \omega^{\alpha_n}$ where $n < \omega$ and $\xi \geq \alpha_0 \geq \dots \geq \alpha_n$. The first uncountable ordinal is denoted by ω_1 .

We use some standard notation and terminology on partially ordered sets (posets), cf. [2]. A *quasiorder* (qo) is a reflexive and transitive relation. A qo is *well-founded* if it has no infinite

descending chains. A *well quasiorder* (wqo) is a qo that has neither infinite descending chains nor infinite antichains. Although the wqo's are closed under many natural finitary constructions like forming finite labeled words or trees, they are not always closed under important infinitary constructions. In 1960s, C. Nash-Williams found a natural subclass of wqo's, called *better quasiorders* (bqo's) which contains most of the "natural" wqo's (in particular, all finite qo's) and has strong closure properties also for many infinitary constructions. We omit a bit technical notion of bqo which is used only in formulations. For more details on bqo's, we refer the reader to [24].

By σ -*semilattice* we mean an (upper) semilattice $(S; \sqcup)$ where supremums $\bigsqcup y_j = y_0 \sqcup y_1 \sqcup \dots$ of countable sequences of elements y_0, y_1, \dots exist. An element x of a σ -semilattice S is σ -*join-irreducible* if it cannot be represented as the countable supremum of elements strictly below x . As first stressed in [19], the σ -join-irreducible elements play a central role in the study of Wadge degrees of k -partitions. The same applies to several variations of Wadge degrees, including the Wadge degrees of Q -partitions for a countable bqo Q .

2.2 Descriptive set theory and Wadge-like reducibilities

Let ω be the space of non-negative integers with the discrete topology. By endowing $\mathcal{N} = \omega^\omega$ with the product of the discrete topologies on ω , we obtain the *Baire space*.

We assume the reader to be familiar with Borel hierarchy $\{\Delta_\alpha^0(X), \Sigma_\alpha^0(X), \Pi_\alpha^0(X)\}_{\alpha < \omega_1}$ in a Polish space X (see e.g. [9, 16]). In particular, $\Delta_1^1(X) = \bigcup \{\Sigma_{1+\alpha}^0(X) \mid \alpha < \omega_1\}$ is the class of all Borel sets. Borel hierarchy gives rise to many important classes of functions. A function $f: X \rightarrow Y$ between Polish spaces is Γ -*measurable* if $f^{-1}(U) \in \Gamma(X)$ for every open set $U \subseteq Y$ (cf. [9, 16]). The class of such functions is denoted $\Gamma(X, Y)$. If the codomain Y is a discrete space, then Σ_α^0 -measurability coincides with Δ_α^0 -measurability, i.e. $\Sigma_\alpha^0(X, Y) = \Delta_\alpha^0(X, Y)$. In our proof, we use this notion only when $X = \mathcal{N}$ and $Y \in \{\mathcal{N}, Q\}$ where Q is a bqo considered as a discrete topological space. By discreteness, $\Sigma_\alpha^0(\mathcal{N}, Q)$ coincides with the class of Δ_α^0 -partitions of \mathcal{N} (also denoted $\Delta_\alpha^0(Q^\mathcal{N})$) mentioned in Introduction. By the following topological fact, without loss of generality, one can always assume that Q is countable.

Fact 1 (cf. [24, Lemma 9.11 and Remark 9.12]). For every Borel function $f: \mathcal{N} \rightarrow Q$, the image $f(\mathcal{N})$ is separable; hence, $f(\mathcal{N})$ is countable by discreteness of Q .

For $\alpha > 1$ the class of Σ_α^0 -measurable functions on \mathcal{N} is not closed under composition (hence it does not induce a reasonable degree structure). On the other hand, there are many natural subclasses of Borel functions closed under composition. The class D_α of Δ_α^0 -functions mentioned in Introduction is such an example: It is known and easy to see that D_α is closed under composition and contains the identity function, hence the relation \leq_α is always a qo. Furthermore, $D_\alpha \subseteq D_\beta$ for all $0 < \alpha < \beta < \omega_1$, hence \leq_α is contained in \leq_β .

For any pointclass Γ and class \mathcal{F} of functions, we say that a function $f: X \rightarrow Y$ is Γ -*piecewise* \mathcal{F} if there is a partition $\{X_n\}$ of X to Γ -sets and a sequence $f_n: X_n \rightarrow Y$ of \mathcal{F} -functions with $f(x) = f_n(x)$ for any $x \in X_n$. We denote by D_α^W the class of Δ_α^0 -piecewise

continuous functions. For $\alpha > 1$, this class coincides with the Σ_α^0 -piecewise continuous functions. Note that $D_\alpha^W \subseteq D_\alpha$, D_α^W is closed under composition and contains the identity function (hence it induces a reducibility \leq_α^W on subsets of X). Furthermore, $D_\alpha^W \subseteq D_\beta^W$ for all $0 < \alpha < \beta < \omega_1$, hence \leq_α^W is contained in \leq_β^W . For more details on D_α^W , see [17, 18, 11, 6].

We will use the following basic observations:

Observation 1. (1) Let $\alpha, \beta < \omega_1$, $A \in \Sigma_{1+\beta}^0$, and let f be a $\Sigma_{1+\alpha}^0$ -measurable function on \mathcal{N} . Then $f^{-1}(A) \in \Sigma_{1+\alpha+\beta}^0$.

(2) For any $\alpha < \omega_1$, $\Sigma_{1+\alpha}^0(\mathcal{N}, \mathcal{N}) \subseteq D_{(1+\alpha)\cdot\omega}$.

(3) Let f be a Δ_α^0 -function. If g is Σ_α^0 -measurable, so is $g \circ f$.

Proof. All of them are easy to prove. The first observation is well-known (cf. [9, Section 24]). For the second one, see also [18, Proposition 3.2]. For the last one, $g^{-1}(S) \in \Sigma_\alpha^0$ for any $S \in \Sigma_1^0$. Therefore, $(g \circ f)^{-1}(S) = f^{-1}(g^{-1}(S)) \in \Sigma_\alpha^0$. \square

Along with the classes D_α , D_α^W there are other natural classes of reducing functions called Borel amenable classes of functions [17]. These classes induce corresponding reducibilities on Q -partitions of \mathcal{N} . As shown in [17, Proposition 4.3], any Borel amenable class \mathcal{G} is of the form $D_\alpha^{\mathcal{F}}$ for some \mathcal{F} and $\alpha < \omega_1$. In this case, \mathcal{G} is called a Borel amenable class of level α . As a typical example, given $\beta < \omega_1$, let us consider the class \mathcal{F} of Σ_γ^0 -measurable functions for some $\gamma < \beta \cdot \omega$. We use the symbol $D_\alpha^{<\beta\cdot\omega}$ to denote $D_\alpha^{\mathcal{F}}$, the class of all Δ_α^0 -piecewise $\Sigma_{<\beta\cdot\omega}^0$ -measurable functions. If α is an ordinal of the form $\omega^{\alpha_0} + \omega^{\alpha_1} + \dots + \omega^{\alpha_m}$ for some $\alpha_0 \geq \alpha_1 \geq \dots \geq \alpha_m$, then we define $\alpha^* = \omega^{\alpha_0}$. The class $D_{1+\alpha}^{<\alpha^*}$ naturally arises as seen below. Moreover, D_α is the largest class among Borel amenable classes of level α , cf. [17, Section 6]. The main results in [11, 6] clarify the relationship between D_α and D_α^W as follows:

$$D_{1+\alpha}^W \subseteq \dots \subseteq D_{1+\alpha}^{<1+\alpha^*} \subseteq D_{1+\alpha} \subseteq D_{1+\alpha+1}^{<1+\alpha^*}.$$

In [11, 6], it is conjectured that $D_{1+\alpha}^{<1+\alpha^*} = D_{1+\alpha}$. It is not hard to see that if α is of the form $\beta \cdot \omega$, then $\alpha^* = \alpha$, so this also provides us a fine picture between the classes mentioned in Observation 1 (2).

For a class \mathcal{F} of functions, we say that a space X is \mathcal{F} -isomorphic to Y if there is a bijection $f: X \rightarrow Y$ such that both f and f^{-1} belong to \mathcal{F} . If two spaces have the same D_ξ -isomorphism type, they have the same \leq_ξ -structure of Q -partitions:

Lemma 1. *Assume that quasi-Polish spaces X and Y are D_ξ -isomorphic. Then, for any ordinal $\theta \geq \xi$ and go Q , we have $(\Delta_\theta^0(Q^X); \leq_\xi) \simeq (\Delta_\theta^0(Q^Y); \leq_\xi)$.*

Proof. Let $h: Y \rightarrow X$ be a D_ξ -isomorphism. We show that $f \mapsto f \circ h$ induces $(\Delta_\theta^0(Q^X); \leq_\xi) \simeq (\Delta_\theta^0(Q^Y); \leq_\xi)$. Note that if $f \in \Delta_\theta^0(Q^X)$ then $f \circ h \in \Delta_\theta^0(Q^Y)$ (as $\xi \leq \theta$) by Observation 1 (3). For $f, g: X \rightarrow Q$, assume that $f \leq_\xi g$; that is, there is a Δ_ξ^0 -function ψ such that $f(x) \leq_Q g \circ \psi(x)$ for any $x \in X$. The last inequality is equivalent to the following: For any $y \in Y$, $f \circ h(y) \leq_Q g \circ \psi \circ h(y) = g \circ h \circ h^{-1} \circ \psi \circ h(y)$; hence $f \circ h \leq_\xi g \circ h$ via $h^{-1} \circ \psi \circ h$, where

note that $h^{-1} \circ \psi \circ h$ is a Δ_ξ^0 -function since D_ξ is closed under composition. For surjectivity, any $g \in \Delta_\theta^0(Q^Y)$ can be written as $(g \circ h^{-1}) \circ h$. \square

It is known that every countable dimensional uncountable Polish space is D_3^W -isomorphic to \mathcal{N} , cf. [18, Theorem 4.21], where a space is *countable dimensional* if it is a countable union of finite dimensional subspaces. Moreover, Kuratowski showed that every uncountable Polish space is D_ω -isomorphic to \mathcal{N} , cf. [18, Proposition 4.3]. Combining Lemma 1 with these facts, we get the following:

Corollary 1. (1) *Let \mathcal{X} be a countable dimensional uncountable Polish space. Then, for any ordinals $\theta \geq \xi \geq 3$, we have $(\Delta_\theta^0(Q^\mathcal{X}); \leq_\xi) \simeq (\Delta_\theta^0(Q^\mathcal{N}); \leq_\xi)$.*

(2) *Let \mathcal{X} be an uncountable Polish space. Then, for any ordinals $\theta \geq \xi \geq \omega$, we have $(\Delta_\theta^0(Q^\mathcal{X}); \leq_\xi) \simeq (\Delta_\theta^0(Q^\mathcal{N}); \leq_\xi)$.*

2.3 Tree calculus

Let ω^* be the set of finite sequences of elements of ω , including the empty sequence ε . For $\sigma, \tau \in \omega^*$, we write $\sigma \sqsubseteq \tau$ to denote that σ is an initial segment of the sequence τ . A *tree* is a non-empty set $T \subseteq \omega^*$ which is closed downwards under \sqsubseteq . For any qo Q , a Q -*tree* is a pair (T, t) consisting of a well founded tree $T \subseteq \omega^*$ and a labeling $t: T \rightarrow Q$. Let $\mathcal{T}(Q)$ be the set of Q -trees quasi-ordered by the relation: $(T, t) \leq_h (V, v)$ iff there is a monotone function $\varphi: T \rightarrow V$ with $\forall x \in T (t(x) \leq_Q v(\varphi(x)))$. Let $\mathcal{T}^\sqcup(Q)$ be defined similarly but with forests (i.e., the sub-qo $T \setminus \{\varepsilon\}$ of a tree T) in place of trees. As follows from Laver's theorem, if Q is bqo then so are also $\mathcal{T}(Q)$ and $\mathcal{T}^\sqcup(Q)$, hence \mathcal{T} is an operator on the class BQO of all bqo's. This operator was introduced in [19] and turned out useful (together with some of its iterates) for characterising some initial segments of \mathcal{W}_Q (we warn the reader that operator \mathcal{T} is denoted $\tilde{\mathcal{T}}$ in [19]). As observed in [19, 20], $\mathcal{T}^\sqcup(Q)$ is a σ -semilattice (the countable supremum operation \bigsqcup is the disjoint union of labeled forests). The σ -join-irreducible elements of $\mathcal{T}^\sqcup(Q)$ are precisely those h -equivalent to the elements of $\mathcal{T}(Q)$.

Next we recall iterations of \mathcal{T} from [20, 23]. For any $q \in Q$, let $s(q)$ be the singleton tree labeled by q , then $s: Q \rightarrow \mathcal{T}(Q)$ is an embedding of qo's. Identifying q with $s(q)$, we may think that Q is a substructure of $\mathcal{T}(Q)$. We iterate the operator \mathcal{T} as follows: $\mathcal{T}^0(Q) = Q$, $\mathcal{T}^{\alpha+1}(Q) = \mathcal{T}(\mathcal{T}^\alpha(Q))$, and $\mathcal{T}^\lambda(Q) = \bigcup_{\alpha < \lambda} \mathcal{T}^\alpha(Q)$ for a limit ordinal λ . Then $\{\mathcal{T}^\alpha(Q)\}$ is an increasing sequence of bqo's. Since all our trees are countable, $\mathcal{T}^{\omega_1}(Q)$ is a fixed point of this iteration procedure. The function s is naturally extended to a function s on \mathcal{T}^{ω_1} such that $q = s(q)$, $T \leq_h s(T)$, and $T \leq_h V$ iff $s(T) \leq_h s(V)$. This iteration procedure was extended in [14] by considering operations $(s_\alpha)_{\alpha < \omega_1}$ in place of just one function s (to unify and simplify notation, we use the notation $s_\alpha(T)$ instead of the notation $\langle T \rangle^{\omega^\alpha}$ in [14]). The idea of the iteration may be described as follows: First take the $(\omega_1$ st) fixed point $\mathcal{T}_\omega := \mathcal{T}^{\omega_1}$ closed under $s_0 = s$; then add new s_1 which enumerates fixed points for s_0 in the sense that $s_0(s_1(T)) \equiv_h s_1(T)$, and take the $(\omega_1$ st) fixed point \mathcal{T}_{ω^2} closed under s_0 and s_1 . Continue this Veblen-like procedure to produce $(\mathcal{T}_{\omega^\alpha}; s_\alpha)_{\alpha < \omega_1}$.

We now give the precise inductive definition of $(\mathcal{T}_{\omega^\alpha}, \mathcal{T}_{\omega^\alpha}^\sqcup)_{\alpha < \omega_1}$ formalizing the idea described in the previous paragraph. In [14, Definition 3.19], $\mathcal{T}_{\omega^\alpha}^\sqcup(Q)$ is defined as a set of terms in the language consisting of constant symbols $s_\alpha(q)$ for $q \in Q$, a 2-ary function symbol \cdot , an ω -ary function symbol \sqcup , and unary function symbols s_β for $\beta < \alpha$: Every constant symbol is a *singleton term*, and every singleton term is a *tree term*. If $(S_i)_{i \in \omega}$ is a sequence of tree terms, then $\sqcup_i S_i$ is a *forest term*. If S is a singleton term and F is a forest term, then $S \cdot F$ is a tree term. If T is a tree term, then $s_\beta(T)$ is a singleton term for any $\beta < \alpha$. Then $\mathcal{T}_{\omega^\alpha}(Q)$ is the set of all tree terms, and $\mathcal{T}_{\omega^\alpha}^\sqcup(Q)$ is the set of all tree and forest terms. For any non-zero countable ordinal $\xi = \omega^{\alpha_0} + \dots + \omega^{\alpha_n}$, $\alpha_0 \geq \dots \geq \alpha_n$, we define the operator $\mathcal{T}_\xi = \mathcal{T}_{\omega^{\alpha_0}} \circ \dots \circ \mathcal{T}_{\omega^{\alpha_n}}$ (let also \mathcal{T}_0 be the identity operator on BQO). Finally, let $\mathcal{T}_{\omega_1}(Q) = \bigcup_{\xi < \omega_1} \mathcal{T}_\xi(Q)$, and similarly for $\mathcal{T}_{\omega_1}^\sqcup(Q)$.

As in [14, Definition 3.20], we inductively define a qo \leq_h on $\mathcal{T}_{\omega_1}^\sqcup(Q)$ as follows: For $p, q \in Q$ and $\alpha < \omega_1$, $p \equiv_h s_\alpha(p)$, and $p \leq_h q$ iff $p \leq_Q q$. For singletons $s_\alpha(U)$ and $s_\beta(V)$, $s_\alpha(U) \leq_h s_\beta(V)$ is equivalent to $U \leq_h V$ if $\alpha = \beta$; to $s_\alpha(U) \leq_h V$ if $\alpha > \beta$; and to $U \leq_h s_\beta(V)$ if $\alpha < \beta$. For singletons A, C and forests B, D (where the empty forests are allowed, cf. [14, Definition 3.20]), define $A \cdot B \leq_h C \cdot D$ if either $A \leq_h C$ and $B \leq_h C \cdot D$, or $A \not\leq_h C$ and $A \cdot B \leq_h D$. Moreover, $B = \sqcup_i B_i \leq_h C \cdot D$ iff $B_i \leq_h C \cdot D$ for any i , and $A \cdot B \leq_h D = \sqcup_i D_i$ iff $A \cdot B \leq_h D_i$ for some i . Again, $\mathcal{T}_{\omega_1}^\sqcup(Q)$ is a σ -semilattice the σ -join-irreducible elements of which are precisely those h -equivalent to the elements of $\mathcal{T}_{\omega_1}(Q)$. See also [21].

2.4 Characterising Wadge degrees

As an extension on a number of previous works on the Wadge degrees, the complete characterisation of the Q -Wadge degrees \mathcal{W}_Q in terms of the iterated labeled forests is described in [14] as follows.

Theorem 1 ([14]). *Let $\eta < \omega_1$ and Q be a bqo. Then $(\mathcal{T}_\eta^\sqcup(Q); \leq_h) \simeq (\Delta_{1+\eta}^0(Q^\mathcal{N}); \leq_W)$ and $(\mathcal{T}_{\omega_1}^\sqcup(Q); \leq_h) \simeq (\Delta_1^1(Q^\mathcal{N}); \leq_W)$.*

The basic strategy of the proof is as follows: First assign a natural class Σ_T of functions to each $T \in \mathcal{T}_{\omega_1}^\sqcup(Q)$, which refines known hierarchies such as the Borel hierarchy $(\Sigma_\alpha^0)_{\alpha < \omega_1}$ and the difference hierarchy $(\mathcal{D}_\beta(\Sigma_\alpha^0))_{\alpha, \beta < \omega_1}$. The main task is to show that the hierarchy $(\Sigma_T)_{T \in \mathcal{T}_{\omega_1}^\sqcup(Q)}$ is *ultimate* in the sense that there is no finer hierarchy of Borel functions from the viewpoint of continuous reducibility. To achieve this, to each $T \in \mathcal{T}_{\omega_1}^\sqcup(Q)$ associate a Σ_T -complete function $\mu(T): \mathcal{N} \rightarrow Q$ (the symbol Ω_T is used in [14]).

In order to define μ , the next step of the proof strategy is to overcome the difficulty caused by the non-existence of a universal total $\Sigma_{1+\xi}^0$ -measurable function, and the notion of a conciliatory function was designed to solve this problem (cf. [4, 14]).

Let $\natural: \mathcal{N} \rightarrow \mathcal{N}$ be a function with $\natural \circ \natural = \text{id}$. We say that a function $f: \mathcal{N} \rightarrow \mathcal{N}$ is \natural -conciliatory if, for any $x, y \in \mathcal{N}$, $\natural(x) = \natural(y)$ implies $\natural \circ f(x) = \natural \circ f(y)$. Similarly, a function $A: \mathcal{N} \rightarrow Q$ is \natural -conciliatory if, for any $x, y \in \mathcal{N}$, $\natural(x) = \natural(y)$ implies $A(x) = A(y)$. We say that $f, g: \mathcal{N} \rightarrow \mathcal{N}$ are \natural -equivalent (written $f \equiv_\natural g$) if $\natural \circ f = \natural \circ g$. We will frequently use the

following basic observation: If $A: \mathcal{N} \rightarrow Q$ is \natural -conciliatory, and $f, g: \mathcal{N} \rightarrow \mathcal{N}$ are \natural -equivalent, then $A \circ f = A \circ g$. For $Q = 2$, this notion was first introduced by [4]. See also [14, Sections 2.5 and 2.6] for the idea behind these definitions. Note that, in order to avoid going back and forth between two spaces \mathcal{N} and $\hat{\omega}^\omega$ as in [14], our definitions are slightly different from the original one (\natural plays a similar role to \mathfrak{p} in [14] as seen below).

If we suitably choose \natural , then the class of \natural -conciliatory functions has the following good property [14]:

- Fact 2.** (1) Any partial continuous function g on \mathcal{N} has a *conciliatory total extension*; that is, there is a \natural -conciliatory total continuous function $\hat{g}: \mathcal{N} \rightarrow \mathcal{N}$ such that $\natural \circ g(x) = \natural \circ \hat{g}(x)$ holds for any $x \in \text{dom}(g)$.
- (2) For any countable ordinal ξ , there is a $\Sigma_{1+\xi}^0$ -measurable \natural -conciliatory function $\mathcal{U}_\xi: \mathcal{N} \rightarrow \mathcal{N}$ which is *universal*; that is, for every $\Sigma_{1+\xi}^0$ -measurable function $f: \mathcal{N} \rightarrow \mathcal{N}$, there is a continuous function $g: \mathcal{N} \rightarrow \mathcal{N}$ such that f is \natural -equivalent to $\mathcal{U}_\xi \circ g$.
- (3) Every σ -join-irreducible Borel function $f: \mathcal{N} \rightarrow Q$ is Wadge equivalent to a \natural -conciliatory function. Indeed, for any tree $T \in \mathcal{T}_{\omega_1}(Q)$, there is a Σ_T -complete \natural -conciliatory function $\mu(T): \mathcal{N} \rightarrow Q$.

For (1), see [14, Observation 2.19]; for (2), see [14, Proposition 2.15 and Proposition 3.24]; and for (3), see [14, Observation 3.15]. Regarding Fact 2 (3), even if F is not a tree, one can ensure that $\mu(F)$ is *almost \natural -conciliatory*; that is, $\natural(x) = \natural(y)$ implies $\mu(F)(nx) = \mu(F)(ny)$ for any $n \in \omega$ and $x, y \in \mathcal{N}$.

To define $\mu(T \cdot F)$ we need a conciliatory Wadge addition operation, which is denoted as $A \rightarrow B$ in [4, 14], but we use the symbol $A \cdot B$ in this paper. We do not mention the explicit definition of the operation $A \cdot B$, because we only use the following special properties:

Fact 3. If A is \natural -conciliatory, and B is almost \natural -conciliatory, then $A \cdot B$ is \natural -conciliatory. Moreover, there are \natural -conciliatory continuous functions $\pi_0, \pi_1: \mathcal{N} \rightarrow \mathcal{N}$ and an open set $J \subseteq \mathcal{N}$ such that if $x \in J$ then $(A \cdot B)(x) = B \circ \pi_1(x)$, and if $x \notin J$ then $(A \cdot B)(x) = A \circ \pi_0(x)$.

For this fact, see [14, Observation 3.11]. Let us briefly explain how to obtain such a function \natural since our notation is slightly different from [14]: For a homeomorphism $I: (\omega \cup \{\text{pass}\})^\omega \rightarrow \mathcal{N}$ and a map $z \mapsto z^{\mathfrak{p}}: (\omega \cup \{\text{pass}\})^\omega \rightarrow \omega^{\leq \omega}$ in [14, Section 2.5], consider $\delta(x) = x \mapsto (I^{-1}(x))^{\mathfrak{p}}$, which is a total surjection from \mathcal{N} to $\omega^{\leq \omega}$, and we define $\natural(x) = I(\delta(x)\text{pass}^\omega)$. Then, \natural -equivalence is the same as $\equiv_{\mathfrak{p}}$ in [14, Section 2.6], so this \natural works. Hereafter, we fix such a function \natural satisfying Facts 2 and 3, and we never use the explicit definition of \natural . We also use the terminology *conciliatory* instead of \natural -conciliatory.

For $T \in \mathcal{T}_{\omega_1}^\sqcup(Q)$, the definition of μ proceeds by induction on the rank of T (which is defined by induction scheme from the end of the previous section) so that the following holds: if $T \equiv_h s_\alpha(q)$ for $q \in Q$ then $\mu(T)$ is the constant function $\lambda x.q$ on \mathcal{N} ; if $T = s_\alpha(V)$ for some V distinct (modulo \equiv_h) from all $q \in Q$ then $\mu(T) = \mu(V) \circ \mathcal{U}_{\omega^\alpha}$; if $T = \sqcup_i S_i$ then $\mu(T) = \bigoplus_i \mu(S_i)$, where $(\bigoplus_i S_i)(nx) = S_n(x)$. if $T = S \cdot F$ for some tree S and forest F

then $\mu(T) = \mu(S) \cdot \mu(F)$. This definition fulfills Fact 2 (3), and also the following fact by [14, Lemma 3.9, Observation 3.16, and Lemma 3.22].

Fact 4. For any ordinal ξ , if $T \in \mathcal{T}_\xi^\sqcup(Q)$ then $\mu(T) \in \Delta_{1+\xi}^0(Q^\mathcal{N})$.

Isomorphisms between the quotient posets from Theorem 1 are induced by this function $\mu: \mathcal{T}_{\omega_1}^\sqcup(Q) \rightarrow \Delta_1^1(Q^\mathcal{N})$. Namely, μ is an embedding [14, Proposition 1.7], and surjective [14, Proposition 1.9] in the following sense:

Fact 5 ([14]). For any $T, V \in \mathcal{T}_{\omega_1}^\sqcup(Q)$, $T \leq_h V$ if and only if $\mu(T) \leq_W \mu(V)$. For any $\Delta_{1+\eta}^0$ -function $A: \mathcal{N} \rightarrow Q$ there is $T \in \mathcal{T}_\eta^\sqcup(Q)$ such that $A \equiv_W \mu(T)$.

3 Main result

In this section we formulate and prove the main result of this paper for the reducibilities $\leq_{1+\xi}$, $\xi < \omega_1$. This result which clearly implies all the results mentioned in Introduction, is formulated as follows.

Theorem 2. For all $\xi, \eta < \omega_1$ and bgo Q ,

$$(\mathcal{T}_\eta^\sqcup(Q); \leq_h) \simeq (\Delta_{1+\xi+\eta}^0(Q^\mathcal{N}); \leq_{1+\xi}) \simeq (\Delta_{1+\xi+\eta}^0(Q^\mathcal{N}); \leq_{1+\xi}^W).$$

Corollary 2. For all $\alpha, \beta, \eta < \omega_1$ and bgo Q ,

$$(\Delta_{1+\alpha+\eta}^0(Q^\mathcal{N}); \leq_{1+\alpha}) \simeq (\Delta_{1+\beta+\eta}^0(Q^\mathcal{N}); \leq_{1+\beta}).$$

By Corollary 1, our results extend to various spaces other than Baire space \mathcal{N} . For instance, if X is d -dimensional Euclidean space \mathbb{R}^d , then for all $\alpha, \beta, \eta < \omega_1$ with $\alpha, \beta \geq 2$, we have $(\mathcal{T}_\eta^\sqcup(Q); \leq_h) \simeq (\Delta_{1+\alpha+\eta}^0(Q^X); \leq_{1+\alpha}) \simeq (\Delta_{1+\beta+\eta}^0(Q^X); \leq_{1+\beta})$. Similarly, if X is e.g. Hilbert cube $[0, 1]^\mathbb{N}$, infinite dimensional separable Hilbert space ℓ_2 , or the function space $\mathcal{C}([0, 1], \mathbb{R})$, then for all $\alpha, \beta, \eta < \omega_1$ with $\alpha, \beta \geq \omega$, we have $(\mathcal{T}_\eta^\sqcup(Q); \leq_h) \simeq (\Delta_{1+\alpha+\eta}^0(Q^X); \leq_{1+\alpha}) \simeq (\Delta_{1+\beta+\eta}^0(Q^X); \leq_{1+\beta})$.

The proof of main result proceeds by induction on ξ . Note that for $\xi = 0$ the assertion coincides with Theorem 1. Let us explain what happens in the simplest case $\eta = 0$.

First we observe that any qo induces a kind of free σ -semilattice Q^\sqcup which we define as the qo $(Q^*; \leq^*)$ where Q^* is the set of non-empty countable subsets of Q with the so called *domination qo* defined by $S \leq^* R$ iff $\forall s \in S \exists r \in R (s \leq_Q r)$. Note that the operation \sqcup of countable supremum in Q^\sqcup is induced by the operation of countable union in Q^* (Categorical properties of $Q \mapsto Q^\sqcup$ and characterisation of some algebras expanding $(\mathcal{T}_{\omega_\alpha}^\sqcup; \leq_h)$ as free structures are considered in [21]).

Since \mathcal{T}_0 is the identity operator, the case $\eta = 0$ reduces to the following assertion.

Lemma 2. For any bgo Q , $Q^\sqcup \simeq (\Delta_{1+\xi}^0(Q^\mathcal{N}); \leq_{1+\xi}) \simeq (\Delta_{1+\xi}^0(Q^\mathcal{N}); \leq_{1+\xi}^W)$.

Proof. We have to check that $(\Delta_{1+\xi}^0(Q^{\mathcal{N}}); \leq_{1+\xi}) \simeq Q^*$. Associate with any $A \in \Delta_{1+\xi}^0(Q^{\mathcal{N}})$ the image $A(\mathcal{N}) \in Q^*$. Observe that, if $A \leq_{1+\xi} B$ via f then $A(x) \leq_Q B(f(x))$, hence $A(\mathcal{N}) \leq^* B(\mathcal{N})$. Conversely, let $A(\mathcal{N}) \leq^* B(\mathcal{N})$, then for some $g: A(\mathcal{N}) \rightarrow B(\mathcal{N})$ we have $q \leq_Q g(q)$ for any $q \in A(\mathcal{N})$. Given $q \in A(\mathcal{N})$, choose $y_q \in \mathcal{N}$ with $B(y_q) = g(q)$. Now define a function f on \mathcal{N} by $f(x) = y_{A(x)}$. Then $A(x) \leq_Q B(f(x))$. Note that the image of f is countable as $A(\mathcal{N})$ is countable by Fact 1. Moreover, for each $y \in \mathcal{N}$, since $f^{-1}(y)$ is the union of some sets $A^{-1}(q) \in \Delta_{1+\xi}^0$, $q \in A(\mathcal{N})$, and $A(\mathcal{N}) \subseteq Q$ is countable, we have $f^{-1}(y) \in \Sigma_{1+\xi}^0$. As the image of f is countable, this means that f is $\Sigma_{1+\xi}^0$ -piecewise constant. Since $\Sigma_{1+\xi}^0$ -piecewise continuity is clearly equivalent to $\Delta_{1+\xi}^0$ -piecewise continuity, this shows that $A \leq_{1+\xi}^W B$. \square

For any $\xi < \omega_1$ we define $s_\xi^*: \mathcal{T}_{\omega_1}(Q) \rightarrow \mathcal{T}_{\omega_1}(Q)$ as follows: let s_0^* be the identity transformation, and for non-zero $\xi = \omega^{\alpha_0} + \dots + \omega^{\alpha_m}$, $\alpha_0 \geq \dots \geq \alpha_m$, we set $s_\xi^* = s_{\alpha_0} \circ \dots \circ s_{\alpha_m}$. It is straightforward to check the following:

- Observation 2.** (1) For all $\xi < \omega_1$, bqo Q , and $T, V \in \mathcal{T}_{\omega_1}(Q)$, we have: $T \leq_h V$ iff $s_\xi^*(T) \leq_h s_\xi^*(V)$.
- (2) For all $\xi, \eta < \omega_1$ and bqo Q , s_ξ^* maps $\mathcal{T}_\eta(Q)$ into $\mathcal{T}_{\xi+\eta}(Q)$.

Recall from [14] that the $\Sigma_{1+\xi}^0$ -universal conciliatory function \mathcal{U}_ξ for a non-zero $\xi = \omega^{\alpha_0} + \dots + \omega^{\alpha_m}$ coincides with $\mathcal{U}_{\omega^{\alpha_m}} \circ \dots \circ \mathcal{U}_{\omega^{\alpha_0}}$ where $\mathcal{U}_{\omega^{\alpha_i}}$ is a $\Sigma_{1+\omega^{\alpha_i}}^0$ -universal conciliatory function in Fact 2 (2). We show that $T \mapsto \mu s_\xi^*(T)$ induces an embedding of $(\mathcal{T}_\eta(Q); \leq_h)$ into $(\Delta_{1+\xi+\eta}^0(Q^{\mathcal{N}}); \leq_{1+\xi})$, where recall Fact 4 for the range of μs_ξ^* .

Lemma 3. For all $\xi, \eta < \omega_1$, bqo Q , and $T, V \in \mathcal{T}_\eta(Q)$ we have: $T \leq_h V$ iff $\mu s_\xi^*(T) \leq_1 \mu s_\xi^*(V)$ iff $\mu s_\xi^*(T) \leq_{1+\xi} \mu s_\xi^*(V)$.

Proof. For the first equivalence, by Observation 2, we have $T \leq_h V$ if and only if $s_\xi^*(T) \leq_h s_\xi^*(V)$. By Fact 5, the latter is equivalent to $\mu s_\xi^*(T) \leq_1 \mu s_\xi^*(V)$. The second equivalence for $\xi = 0$ follows from Observation 2. For the second equivalence for $\xi > 0$, it suffices to show that $\mu s_\xi^*(T) \leq_{1+\xi} \mu s_\xi^*(V)$ implies $\mu s_\xi^*(T) \leq_1 \mu s_\xi^*(V)$. Let f be a $\Delta_{1+\xi}^0$ -function such that $\mu s_\xi^*(T) = \mu s_\xi^*(V) \circ f$. Then

$$\mu s_\xi^*(V) \circ f = \mu s_{\alpha_0} \circ \dots \circ s_{\alpha_m}(V) \circ f = \mu(T) \circ \mathcal{U}_{\omega^{\alpha_m}} \circ \dots \circ \mathcal{U}_{\omega^{\alpha_0}} \circ f = \mu(T) \circ \mathcal{U}_\xi \circ f.$$

As $f \in D_{1+\xi}$ and \mathcal{U}_ξ is $\Sigma_{1+\xi}^0$ -measurable, by Observation 1 (3), $\mathcal{U}_\xi \circ f$ is $\Sigma_{1+\xi}^0$ -measurable. Since \mathcal{U}_ξ is $\Sigma_{1+\xi}^0$ -universal by Fact 2 (2), there is a continuous function g on such that $\mathcal{U}_\xi \circ f$ is \natural -equivalent to $\mathcal{U}_\xi \circ g$. By Fact 2 (3), as $\mu(T)$ is conciliatory, we have $\mu(T) \circ \mathcal{U}_\xi \circ f = \mu(T) \circ \mathcal{U}_\xi \circ g$. Therefore, $\mu s_\xi^*(T) \leq_1 \mu s_\xi^*(V)$ via g . \square

The above embedding of $(\mathcal{T}_\eta(Q); \leq_h)$ into $(\Delta_{1+\xi+\eta}^0(Q^{\mathcal{N}}); \leq_{1+\xi})$ easily extends to an embedding of $(T_\eta^\sqcup(Q); \leq_h)$ into $(\Delta_{1+\xi+\eta}^0(Q^{\mathcal{N}}); \leq_{1+\xi})$. If F is of the form $T_0 \sqcup T_1 \sqcup \dots$ for some trees (T_i) , we use the symbol $s_\xi^*(F)$ to denote $s_\xi^*(T_0) \sqcup s_\xi^*(T_1) \sqcup \dots$. We also adopt the similar

convention for s_α . Then, it is easy to see that Lemma 3 extends to any forests $T, V \in \mathcal{T}_\eta^\sqcup(Q)$ as follows:

$$T \leq_h V \iff \mu s_\xi^*(T) \leq_1 \mu s_\xi^*(V) \iff \mu s_\xi^*(T) \leq_{1+\xi}^W \mu s_\xi^*(V) \iff \mu s_\xi^*(T) \leq_{1+\xi} \mu s_\xi^*(V).$$

The following small technical lemma will be used in the surjectivity proof.

Lemma 4. *Let $\beta \leq \alpha$ be ordinals. For any $\Delta_{1+\omega^\beta}^0$ -piecewise continuous function g , there is a $\Delta_{1+\omega^\alpha+\omega^\beta}^0$ -piecewise continuous function h such that $g \circ \mathcal{U}_{\omega^\alpha} \equiv_{\natural} \mathcal{U}_{\omega^\alpha} \circ h$.*

Proof. By the assumption, there is a $\Sigma_{1+\omega^\beta}^0$ -partition $(X_i)_{i \in \omega}$ of \mathcal{N} such that $g_i = g \upharpoonright X_i$ is continuous. By Fact 2 (1), there is a total conciliatory extension \hat{g}_i of g_i . By Observation 1 (1), $Y_i = \mathcal{U}_{\omega^\alpha}^{-1}(X_i)$ is $\Sigma_{1+\omega^\alpha+\omega^\beta}^0$, and $\hat{g}_i \circ \mathcal{U}_{\omega^\alpha}$ is $\Sigma_{1+\omega^\alpha}^0$ -measurable. By Fact 2 (2), as $\mathcal{U}_{\omega^\alpha}$ is $\Sigma_{1+\omega^\alpha}^0$ -universal, for any i , there is a continuous function h_i such that $\hat{g}_i \circ \mathcal{U}_{\omega^\alpha} \equiv_{\natural} \mathcal{U}_{\omega^\alpha} \circ h_i$. Then, define $h(x) = h_i(x)$ if $x \in Y_i$. Clearly, $g \circ \mathcal{U}_{\omega^\alpha} \equiv_{\natural} \mathcal{U}_{\omega^\alpha} \circ h$, and h is $\Delta_{1+\omega^\alpha+\omega^\beta}^0$ -piecewise continuous. \square

Note that h in Lemma 4 is $\Delta_{1+\omega^\alpha+\omega^\beta}^0$ -piecewise total continuous, i.e., $h \upharpoonright Y_i = h_i \upharpoonright Y_i$ for a total continuous function h_i on \mathcal{N} . Now, we prove our key lemma, which shows surjectivity of the map $T \mapsto \mu s_\xi^*(T)$ with respect to $\leq_{1+\xi}^W$.

Lemma 5. *Let ξ, η be non-zero countable ordinals and Q be a bqo. Then for any $F \in \mathcal{T}_{\xi+\eta}^\sqcup(Q)$ there is $G \in \mathcal{T}_\eta^\sqcup(Q)$ such that $\mu(F) \equiv_{1+\xi}^W \mu s_\xi^*(G)$.*

Proof. We first show the assertion for $\xi = \omega^\alpha$. Let $F \in \mathcal{T}_{\xi+\eta}(Q)$ be given. If $T = q$ for some $q \in Q$ we can take $G = \lambda x.q$. If F is not a tree, i.e., $F = F_0 \sqcup F_1 \sqcup \dots$ for some F_i , then by induction hypothesis $\mu(F_i) \equiv_{1+\xi}^W \mu s_\alpha(G_i)$ for some $G_i \in \mathcal{T}_\eta^\sqcup(Q)$. Thus, we already have $\mu(F) \equiv_{1+\xi} \mu s_\alpha(G_0) \oplus \mu s_\alpha(G_1) \oplus \dots = \mu s_\alpha(G_0 \sqcup G_1 \sqcup \dots)$, so there is nothing to do. Thus, we can assume that F is a tree.

CASE 1. $F = A \cdot B$ for some tree A and a forest B .

By induction hypothesis, there are $L, H \in \mathcal{T}_\eta^\sqcup(Q)$ such that $\mu(A) \equiv_{1+\xi}^W \mu s_\alpha(L)$ and $\mu(B) \equiv_{1+\xi}^W \mu s_\alpha(H)$. We claim that

$$\mu(F) = \mu(A) \cdot \mu(B) \equiv_{1+\xi}^W \mu s_\alpha(L) \oplus \mu s_\alpha(H) = \mu s_\alpha(L \sqcup H).$$

The direction $\mu s_\alpha(L) \oplus \mu s_\alpha(H) \leq_{1+\xi}^W \mu(A) \cdot \mu(B)$ is obvious. We show the converse direction. Let π_0, π_1, J be as in Fact 3. Then $(\mu(A) \cdot \mu(B))(x)$ is equal to $\mu(B) \circ \pi_1(x)$ if $x \in J$, and to $\mu(A) \circ \pi_0(x)$ if $x \notin J$. Let $g, h \in D_{1+\xi}^W$ witness $\mu(A) \leq_{1+\xi}^W \mu s_\alpha(L)$ and $\mu(B) \leq_{1+\xi}^W \mu s_\alpha(H)$, respectively. It is easy to see that $\mu(A) \cdot \mu(B)$ is reduced to $\mu s_\alpha(L) \oplus \mu s_\alpha(H)$ by the $D_{1+\xi}^W$ -function $x \mapsto 1h(\pi_1(x))$ on the open set J , and by $x \mapsto 0g(\pi_0(x))$ on the complement. Since J and its complements are Δ_2^0 , and $2 \leq 1 + \xi$, this reduction is $\Delta_{1+\xi}^0$ -piecewise continuous. This concludes the claim. Hence we can take $G = L \sqcup H$.

CASE 2. F is a non-trivial singleton, i.e. $F = s_\gamma(T)$ for some tree T .

If $\gamma > \alpha$ then note that the language for \mathcal{T}_ξ does not contain the symbol s_γ (as $\xi = \omega^\alpha < \omega^\gamma$), and moreover, $\mathcal{T}_{\xi+\eta}(Q) = \mathcal{T}_{\omega^\alpha} \circ \mathcal{T}_\eta(Q)$. This means that F must be constructed from

symbols $(\cdot, \sqcup, s_\beta)_{\beta < \alpha}$ and $s_\alpha(t)$ for terms t in $T_\eta(Q)$. Thus, if s_γ occurs in F , then F must be already contained in $T_\eta(Q)$. By definition, we have $s_\alpha s_\gamma(T) \equiv_h s_\gamma(T)$; hence we can take $G = F \in T_\eta(Q)$. Then, we have $F = s_\gamma(T) \equiv_h s_\alpha s_\gamma(T) = s_\alpha(F) = s_\alpha(G)$.

If $\gamma = \alpha$, then F is already of the form $s_\alpha(T)$. Again, note that the language for $\mathcal{T}_\xi = \mathcal{T}_{\omega^\alpha}$ does not contain the symbol s_α , and moreover, $\mathcal{T}_{\xi+\eta}(Q) = \mathcal{T}_{\omega^\alpha} \circ \mathcal{T}_\eta(Q)$. This again means that F must be constructed from symbols $(\cdot, \sqcup, s_\beta)_{\beta < \alpha}$ and $s_\alpha(t)$ for terms t in $T_\eta(Q)$. Hence, we must have $T \in T_\eta(Q)$. Thus, we can take $G = T$.

Finally, let $\gamma < \alpha$. By induction hypothesis, we have $\mu(T) \equiv_{1+\xi}^W \mu s_\alpha(G)$ for some $G \in T_\eta^\sqcup(Q)$. This clearly implies that $\mu s_\alpha(G) \leq_{1+\xi}^W \mu(F)$. We show the converse direction. Let $f \in D_{1+\omega^\alpha}^W$ witness $\mu(T) \leq_{1+\xi}^W \mu s_\alpha(G)$. We first assume that G is a tree. Then, we have

$$\mu(F)(x) = \mu(T)(\mathcal{U}_{\omega^\gamma}(x)) \leq_Q \mu s_\alpha(G)(f \circ \mathcal{U}_{\omega^\gamma}(x)) = \mu(G)(\mathcal{U}_{\omega^\alpha} \circ f \circ \mathcal{U}_{\omega^\gamma}(x)).$$

By Observation 1 (2), $\mathcal{U}_{\omega^\gamma} \in D_{\omega^{\gamma+1}} \subseteq D_{\omega^\alpha}$, and we also have $f \in D_{\omega^\alpha}$. As D_{ω^α} is closed under composition, we have $f \circ \mathcal{U}_{\omega^\gamma} \in D_{\omega^\alpha}$, and therefore, $\mathcal{U}_{\omega^\alpha} \circ f \circ \mathcal{U}_{\omega^\gamma}$ is still $\Sigma_{1+\omega^\alpha}^0$ -measurable by Observation 1 (3). By $\Sigma_{1+\omega^\alpha}^0$ -universality of $\mathcal{U}_{\omega^\alpha}$, there is a continuous function h such that $\mathcal{U}_{\omega^\alpha} \circ f \circ \mathcal{U}_{\omega^\gamma}$ is \downarrow -equivalent to $\mathcal{U}_{\omega^\alpha} \circ h$. Since $\mu(G)$ is conciliatory by Fact 2 (3), we obtain

$$\mu(F)(x) \leq_Q \mu(G)(\mathcal{U}_{\omega^\alpha} \circ h(x)) = \mu s_\alpha(G)(h(x)).$$

This means that $\mu(F) \leq_1 \mu s_\alpha(G)$. If G is a forest of the form $G_0 \sqcup G_1 \sqcup \dots$, then consider the set X_n of all x such that the first bit of $f \circ \mathcal{U}_{\omega^\gamma}(x)$ is n . Since $f \circ \mathcal{U}_{\omega^\gamma} \in D_{\omega^\alpha}$ as seen above, the set X_n is $\Delta_{1+\omega^\alpha}^0$. By replacing f in the above argument with $f_n: x \mapsto nx$, it is straightforward to show that $\mu(F)$ is continuously reducible to $\mu s_\alpha(G_n)$ on X_n . By combining these reductions, we get a $\Delta_{1+\omega^\alpha}^0$ -piecewise continuous reduction from $\mu(F)$ to $\mu s_\alpha(G)$. Consequently, $\mu(F) \equiv_{1+\xi}^W \mu s_\alpha(G)$.

This concludes the proof for $\xi = \omega^\alpha$. We now consider the general case $\xi = \omega^{\alpha_0} + \omega^{\alpha_1} + \dots + \omega^{\alpha_m}$ for $m > 0$. Fix $F \in \mathcal{T}_\xi^\sqcup(Q)$, and let us consider $\eta_0 = \omega^{\alpha_1} + \omega^{\alpha_2} + \dots + \omega^{\alpha_m} + \eta$. Applying the above argument, there is $G_0 \in \mathcal{T}_{\eta_0}^\sqcup(Q)$ such that $\mu(F) \equiv_{1+\omega^{\alpha_0}}^W \mu s_{\alpha_0}(G_0)$. Next, consider $\eta_1 = \omega^{\alpha_2} + \dots + \omega^{\alpha_m} + \eta$. Apply the above argument to G_0 , there is $G_1 \in \mathcal{T}_{\eta_1}^\sqcup(Q)$ such that $\mu(G_0) \equiv_{1+\omega^{\alpha_1}}^W \mu s_{\alpha_1}(G_1)$. Let us now consider $\eta_i = \omega^{\alpha_{i+1}} + \dots + \omega^{\alpha_m} + \eta$. By iterating the above procedure, we eventually obtain a sequence G_0, G_1, \dots, G_m such that $G_i \in \mathcal{T}_{\eta_i}^\sqcup(Q)$ such that $\mu(G_{i-1}) \equiv_{1+\omega^{\alpha_i}}^W \mu s_{\alpha_i}(G_i)$, where $G_{-1} = F$. We now want to show $\mu(F) \equiv_{1+\xi}^W \mu s_\xi^*(G_m) = \mu s_{\alpha_0} s_{\alpha_1} \dots s_{\alpha_m}(G_m)$.

To prove this, we claim that $\mu(F) \equiv_{1+\omega^{\alpha_0}+\omega^{\alpha_1}}^W \mu s_{\alpha_0} s_{\alpha_1}(G_1)$. For the forward direction, let $f \in D_{1+\omega^{\alpha_0}}^W$ witness $\mu(F) \leq_{1+\omega^{\alpha_0}}^W \mu s_{\alpha_0}(G_0)$ and $g \in D_{1+\omega^{\alpha_1}}^W$ witness $\mu(G_0) \leq_{1+\omega^{\alpha_1}}^W \mu s_{\alpha_1}(G_1)$. First assume that both G_0 and G_1 are trees. Then,

$$\begin{aligned} \mu(F)(x) &\leq_Q \mu s_{\alpha_0}(G_0)(f(x)) = \mu(G_0)(\mathcal{U}_{\omega^{\alpha_0}} \circ f(x)) \\ &\leq_Q \mu s_{\alpha_1}(G_1)(g \circ \mathcal{U}_{\omega^{\alpha_0}} \circ f(x)) = \mu(G_1)(\mathcal{U}_{\omega^{\alpha_1}} \circ g \circ \mathcal{U}_{\omega^{\alpha_0}} \circ f(x)). \end{aligned}$$

As $f \in D_{1+\omega^{\alpha_0}}$, the composition $\mathcal{U}_{\omega^{\alpha_0}} \circ f$ is $\Sigma_{1+\omega^{\alpha_0}}^0$ -measurable by Observation 1 (3). Similarly, as $g \in D_{1+\omega^{\alpha_1}}$, $\mathcal{U}_{\omega^{\alpha_1}} \circ g$ is $\Sigma_{1+\omega^{\alpha_1}}^0$ -measurable. Hence, $\mathcal{U}_{\omega^{\alpha_1}} \circ g \circ \mathcal{U}_{\omega^{\alpha_0}} \circ f$ is $\Sigma_{1+\omega^{\alpha_0}+\omega^{\alpha_1}}^0$ -

measurable by Observation 1 (1). By $\Sigma_{1+\omega^{\alpha_0}+\omega^{\alpha_1}}^0$ -universality of $\mathcal{U}_{\omega^{\alpha_1}} \circ \mathcal{U}_{\omega^{\alpha_0}}$, there is a continuous function h such that $\mathcal{U}_{\omega^{\alpha_1}} \circ g \circ \mathcal{U}_{\omega^{\alpha_0}} \circ f$ is \mathfrak{h} -equivalent to $\mathcal{U}_{\omega^{\alpha_1}} \circ \mathcal{U}_{\omega^{\alpha_0}} \circ h$. Since $\mu(G_1)$ is conciliatory by Fact 2 (3), we obtain

$$\mu(F)(x) \leq_Q \mu(G_1)(\mathcal{U}_{\omega^{\alpha_1}} \circ \mathcal{U}_{\omega^{\alpha_0}} \circ h(x)) = \mu s_{\alpha_0} s_{\alpha_1}(G_1)(h(x)).$$

Hence, we have $F \leq_1 \mu s_{\alpha_0} s_{\alpha_1}(G_1)$. If G_0 and G_1 are forest, we need to decompose the domain according to the first bit of $f(x)$ and that of $g \circ \mathcal{U}_{\omega^{\alpha_0}} \circ f(x)$. These functions are $\Sigma_{1+\omega^{\alpha_0}+\omega^{\alpha_1}}^0$ -measurable, and so the decomposition is $\Delta_{1+\omega^{\alpha_0}+\omega^{\alpha_1}}^0$. Hence, F is reducible to $\mu s_{\alpha_0} s_{\alpha_1}(G_1)$ by a $D_{1+\omega^{\alpha_0}+\omega^{\alpha_1}}^W$ -function.

For the converse direction, we similarly have a $D_{1+\omega^{\alpha_0}}^W$ -function f witnessing $\mu s_{\alpha_0}(G_0) \leq_{1+\omega^{\alpha_0}}^W \mu(F)$ and a $D_{1+\omega^{\alpha_1}}^W$ -function g witnessing $\mu s_{\alpha_1}(G_1) \leq_{1+\omega^{\alpha_1}}^W \mu(G_0)$. We assume that both G_0 and G_1 are trees. Then, we have

$$\mu s_{\alpha_0} s_{\alpha_1}(G_1)(x) = \mu s_{\alpha_1}(G_1)(\mathcal{U}_{\omega^{\alpha_0}}(x)) \leq_Q \mu(G_0)(g \circ \mathcal{U}_{\omega^{\alpha_0}}(x)).$$

By Lemma 4, there is a $D_{1+\omega^{\alpha_0}+\omega^{\alpha_1}}^W$ -function h such that $g \circ \mathcal{U}_{\omega^{\alpha_0}}$ is \mathfrak{h} -equivalent to $\mathcal{U}_{\omega^{\alpha_0}} \circ h$. Then, as $\mu(G_0)$ is conciliatory, we now have

$$\mu(G_0)(g \circ \mathcal{U}_{\omega^{\alpha_0}}(x)) = \mu(G_0)(\mathcal{U}_{\omega^{\alpha_0}} \circ h(x)) = \mu s_{\alpha_0}(G_0)(h(x)) \leq_Q \mu(F)(f \circ h(x)).$$

By combining the above two inequalities, we obtain $\mu s_{\alpha_0} s_{\alpha_1}(G_1)(x) \leq_Q \mu(F)(f \circ h(x))$. Since $f \circ h$ is in $D_{1+\omega^{\alpha_0}+\omega^{\alpha_1}}^W$ (as $D_{1+\omega^{\alpha_0}+\omega^{\alpha_1}}^W$ is closed under composition), this witnesses $\mu s_{\alpha_0} s_{\alpha_1}(G_1) \leq_{1+\omega^{\alpha_0}+\omega^{\alpha_1}}^W \mu(F)$. If G_0 and G_1 are forest, as in the above argument, we have similar reductions on $\Delta_{1+\omega^{\alpha_0}+\omega^{\alpha_1}}^0$ domains. This concludes the proof of our claim.

We now apply this claim to the sequence G_0, G_1, \dots, G_m , where $G_m \in \mathcal{T}_\eta(Q)$. Then, we eventually obtain $\mu(F) \equiv_{1+\xi}^W \mu s_\xi^*(G_m)$. Then, take $G = G_m$. \square

Proof of Theorem 2. The case $\xi = 0$ coincides with the second assertion in Fact 5 while the case $\eta = 0$ was considered in Lemma 2, so we assume that both ξ, η are non-zero. We show that $T \mapsto \mu s_\xi^*(T)$ induces an isomorphism $(\mathcal{T}_\eta^\sqcup(Q); \leq_h) \simeq (\Delta_{1+\xi+\eta}^0(Q^\mathcal{N}); \leq_{1+\xi})$. Note that $T \in \mathcal{T}_\eta^\sqcup(Q)$ implies $s_\xi^*(T) \in \mathcal{T}_{\xi+\eta}^\sqcup(Q)$ by Observation 2, so $\mu s_\xi^*(T) \in \Delta_{1+\xi+\eta}^0(Q^\mathcal{N})$ by Fact 4. By Lemma 3, for all $T, V \in \mathcal{T}_\eta^\sqcup(Q)$ we have: $T \leq_h V$ iff $\mu s_\xi^*(T) \leq_{1+\xi} \mu s_\xi^*(V)$ iff $\mu s_\xi^*(T) \leq_{1+\xi}^W \mu s_\xi^*(V)$, so it suffices to show that for any $A \in \Delta_{1+\xi+\eta}^0(Q^\mathcal{N})$ there is $G \in \mathcal{T}_\eta^\sqcup(Q)$ with $A \equiv_{1+\xi}^W \mu s_\xi^*(G)$ (which clearly implies $A \equiv_{1+\xi} \mu s_\xi^*(G)$). By Fact 5 again, $A \equiv_1 \mu(F)$ for some $F \in \mathcal{T}_{\xi+\eta}^\sqcup(Q)$. By Lemma 5, there is $G \in \mathcal{T}_\eta^\sqcup(Q)$ such that $\mu(F) \equiv_{1+\xi}^W \mu s_\xi^*(G)$. Thus, $A \equiv_{1+\xi}^W \mu s_\xi^*(G)$ as desired. \square

Indeed, the above proof shows that our main result holds for any $\text{qo} \preceq$ which is intermediate between $\leq_{1+\xi}^W$ and $\leq_{1+\xi}$; that is, $(\mathcal{T}_\eta^\sqcup(Q); \leq_h) \simeq (\Delta_{1+\xi+\eta}^0(Q^\mathcal{N}); \preceq)$. However, any nontrivial Borel amenable reducibility notion is induced by a class of the form $D_\xi^\mathcal{F}$ for some $\xi < \omega_1$ and \mathcal{F} , and most natural classes \mathcal{F} considered in Section 2.2 satisfy this condition (recall that D_ξ is the greatest one among level ξ classes). Consequently, our main result gives a complete combinatorial description of the structure of Borel Q -partitions w.r.t. these Borel amenable reducibilities.

4 Characterisation in terms of forests

By Fact 5, the map μ gives an isomorphism $(\mathcal{T}_{\xi+\eta}^{\sqcup}(Q); \leq_h) \simeq (\Delta_{1+\xi+\eta}^0(Q^{\mathcal{N}}); \leq_1)$. On the one hand, we have a coarser qo $\leq_{1+\xi}$ on $\Delta_{1+\xi+\eta}^0(Q^{\mathcal{N}})$, which induces a qo \leq_h^ξ on $\mathcal{T}_{\xi+\eta}^{\sqcup}(Q)$ defined by $T \leq_h^\xi S$ iff $\mu(T) \leq_{1+\xi} \mu(S)$. The quotient poset $(\mathcal{T}_{\xi+\eta}^{\sqcup}(Q); \leq_h^\xi)$ is clearly isomorphic to $(\Delta_{1+\xi+\eta}^0(Q^{\mathcal{N}}); \leq_{1+\xi})$. On the other hand, by Theorem 2, we have $(\mathcal{T}_\eta^{\sqcup}(Q); \leq_h) \simeq (\Delta_{1+\xi+\eta}^0(Q^{\mathcal{N}}); \leq_{1+\xi})$. Therefore, we have a collection $(\leq_h^\xi)_{\xi < \omega_1}$ of induced qo's satisfying $(\mathcal{T}_\eta^{\sqcup}(Q); \leq_h) \simeq (\mathcal{T}_{\alpha+\eta}^{\sqcup}(Q); \leq_h^\alpha) \simeq (\mathcal{T}_{\beta+\eta}^{\sqcup}(Q); \leq_h^\beta)$. Is it possible to characterise these induced qo's in terms of (natural operations on) labeled forests, without using the seemingly quite different Wadge-like reducibilities on Q -partitions? In this section we address this question, and show the following theorem:

Theorem 3. *For any ordinal $\xi < \omega_1$, there is an endomorphism r_ξ^* on $(\mathcal{T}_{\omega_1}^{\sqcup}(Q); \leq_h)$ such that r_ξ^* maps $\mathcal{T}_{\xi+\eta}^{\sqcup}(Q)$ into $\mathcal{T}_\eta^{\sqcup}(Q)$ for any η , and moreover, for any $T, S \in \mathcal{T}_{\omega_1}^{\sqcup}$,*

$$r_\xi^*(T) \leq_h r_\xi^*(S) \iff \mu(T) \leq_{1+\xi} \mu(S).$$

To prove Theorem 3, we first recall that, in the proof of Lemma 5, given $F \in \mathcal{T}_{\xi+\eta}^{\sqcup}(Q)$, we explicitly defined $G \in \mathcal{T}_\eta^{\sqcup}(Q)$ such that $\mu(F) \equiv_{1+\xi}^W \mu s_\xi^*(G)$. This construction induces a map $F \mapsto G$. To give a more explicit description of this map, we first consider the case $\xi = \omega^\alpha$. According to our proof of Lemma 5, the map $r_\alpha: F \mapsto G$ is defined in the following inductive manner:

- (1) $r_\alpha(q) = q$ for any $q \in Q$;
- (2) If $F = F_0 \sqcup F_1 \sqcup \dots$ for some trees F_i then $r_\alpha(F) = r_\alpha(F_0) \sqcup r_\alpha(F_1) \sqcup \dots$;
- (3) If $F = T \cdot V$ for some singleton T and forest V then $r_\alpha(F) = r_\alpha(T) \sqcup r_\alpha(V)$;
- (4) if $F = s_\beta(T)$ for some tree T then $r_\alpha(F) = r_\alpha(T)$ for $\alpha > \beta$, $r_\alpha(F) = T$ for $\alpha = \beta$, and $r_\alpha(F) = F$ for $\alpha < \beta$.

For each ξ we define a map $r_\xi^*: \mathcal{T}_{\omega_1}^{\sqcup}(Q) \rightarrow \mathcal{T}_{\omega_1}^{\sqcup}(Q)$ as follows: r_0^* is the identity map Id , and if $\xi = \omega^{\alpha_0} + \dots + \omega^{\alpha_m} > 0$, $\alpha_0 \geq \dots \geq \alpha_m$, then $r_\xi^* = r_{\alpha_m} \circ \dots \circ r_{\alpha_0}$.

Observation 3. Let $\xi, \eta < \omega_1$ and Q be a bqo. Then $r_\xi^* \circ s_\xi^* = Id$ and r_ξ^* maps $\mathcal{T}_{\xi+\eta}^{\sqcup}(Q)$ into $\mathcal{T}_\eta^{\sqcup}(Q)$.

Proof. For the first assertion, we have $r_\alpha s_\alpha(T) = T$ by definition. Thus, $r_\xi^* s_\xi^* = r_{\alpha_m} \dots r_{\alpha_0} s_{\alpha_0} \dots s_{\alpha_m}$ is the identity map. The second assertion follows from the proof of Lemma 5. \square

Lemma 6. *For any $\xi < \omega_1$, r_ξ^* is an endomorphism on $(\mathcal{T}_{\omega_1}^{\sqcup}(Q); \leq_h)$.*

Proof. It suffices to show that $T \leq_h S$ implies $r_\alpha(T) \leq_h r_\alpha(S)$. We prove the assertion by induction on the complexity of $S, T \in \mathcal{T}_{\omega_1}^\sqcup(Q)$. The base case is trivial. If $T = \sqcup_i T_i$ is a forest, then $T \leq_h S$ iff $T_i \leq_h S$ for all i , and by induction hypothesis, we have $r_\alpha(T_i) \leq_h r_\alpha(S)$ for any i . Thus, $r_\alpha(T) = \sqcup_i r_\alpha(T_i) \leq_h r_\alpha(S)$. If $S = \sqcup_i S_i$ is a forest, then $T \leq_h S$ iff $T \leq_h S_i$ for some i . By induction hypothesis, this implies $r_\alpha(T) \leq_h r_\alpha(S_i)$ for some i ; hence $r_\alpha(T) \leq_h r_\alpha(S)$.

Assume that $T = A \cdot B$ for some singleton A and forest B , and S is a singleton. By definition of \leq_h , $A \cdot B \leq_h S$ implies $A, B \leq_h S$. By induction hypothesis, we have $r_\alpha(A), r_\alpha(B) \leq_h r_\alpha(S)$, so $r_\alpha(T) = r_\alpha(A \cdot B) = r_\alpha(A) \sqcup r_\alpha(B) \leq_h r_\alpha(S)$. Assume that T is a singleton, and $S = C \cdot D$ for some singleton C and forest D . By definition of \leq_h , $T \leq_h C \cdot D$ implies either $T \leq_h C$ or $T \leq_h D$. By induction hypothesis, $r_\alpha(T) \leq_h r_\alpha(C)$ or $r_\alpha(T) \leq_h r_\alpha(D)$; hence $r_\alpha(T) \leq_h r_\alpha(C) \sqcup r_\alpha(D) = r_\alpha(C \cdot D) = r_\alpha(S)$. Assume that $T = A \cdot B$ and $S = C \cdot D$ for some singletons A, C and forests B, D . By definition, $A \cdot B \leq_h C \cdot D$ iff either $A \leq_h C$ and $B \leq_h C \cdot D$ or $A \cdot B \leq_h D$. By induction hypothesis, either $r_\alpha(A) \leq_h r_\alpha(C)$ and $r_\alpha(B) \leq_h r_\alpha(C \cdot D)$ or $r_\alpha(A \cdot B) \leq_h r_\alpha(D)$. In any case, $r_\alpha(A \cdot B) = r_\alpha(A) \sqcup r_\alpha(B) \leq_h r_\alpha(C) \sqcup r_\alpha(D) = r_\alpha(C \cdot D)$.

Finally, let us consider the case that both $S = s_\beta(U)$ and $T = s_\gamma(V)$ are singletons. Assume that $S \leq_h T$, and we want to show that $r_\alpha s_\beta(U) \leq_h r_\alpha s_\gamma(V)$. First consider the case $\beta = \gamma$. Then, $S \leq_h T$ iff $U \leq_h V$. By induction hypothesis, $r_\alpha(U) \leq_h r_\alpha(V)$. If $\beta < \alpha$, then $r_\alpha s_\beta(U) = r_\alpha(U) \leq_h r_\alpha(V) = r_\alpha s_\gamma(V)$. If $\beta = \alpha$, then $r_\alpha s_\beta(U) = U \leq_h V = r_\alpha s_\gamma(V)$. If $\beta > \alpha$, then $r_\alpha s_\beta(U) = s_\beta(U) \leq_h s_\beta(V) = s_\gamma(V) = r_\alpha s_\gamma(V)$, since $U \leq_h V$ iff $s_\beta(U) \leq_h s_\beta(V)$ by definition.

For the case $\beta < \gamma$, the assumption $s_\beta(U) \leq_h s_\gamma(V)$ is equivalent to $U \leq_h s_\gamma(V)$. By induction hypothesis, $r_\alpha(U) \leq_h r_\alpha s_\gamma(V)$. If $\beta < \alpha$, then $r_\alpha s_\beta(U) = r_\alpha(U) \leq_h r_\alpha s_\gamma(V)$. If $\beta = \alpha$, then $r_\alpha s_\beta(U) = U \leq_h s_\gamma(V) = r_\alpha s_\gamma(V)$ by $\gamma > \beta = \alpha$. If $\beta > \alpha$, then $r_\alpha s_\beta(U) = s_\beta(U) \leq_h s_\beta s_\gamma(V)$ since $U \leq_h s_\gamma(V)$ iff $s_\beta(U) \leq_h s_\beta s_\gamma(V)$. By $\beta < \gamma$, we have $s_\beta s_\gamma(V) \equiv_h s_\gamma(V) = r_\alpha s_\gamma(V)$, where the last equality follows from $\gamma > \alpha$. Thus, $r_\alpha s_\beta(U) \leq_h r_\alpha s_\gamma(V)$.

For the case $\beta > \gamma$, the assumption $s_\beta(U) \leq_h s_\gamma(V)$ is equivalent to $s_\beta(U) \leq_h V$. By induction hypothesis, $r_\alpha s_\beta(U) \leq_h r_\alpha(V)$. If $\gamma < \alpha$, then $r_\alpha s_\beta(U) \leq_h r_\alpha(V) = r_\alpha s_\gamma(V)$. If $\gamma = \alpha$, then $r_\alpha s_\beta(U) = s_\beta(U) \leq_h V = r_\alpha s_\gamma(V)$. If $\gamma > \alpha$, then $r_\alpha s_\beta(U) = s_\beta(U) \leq_h V \leq_h s_\beta(V) = r_\alpha s_\gamma(V)$. \square

Proof of Theorem 3. By Lemma 6, r_ξ^* is an endomorphism. Moreover, if $T \in \mathcal{T}_{\xi+\eta}^\sqcup(Q)$ then $r_\xi^*(T) \in \mathcal{T}_\eta^\sqcup(Q)$ by Observation 3. By Lemma 3, for any $T, V \in \mathcal{T}_{\xi+\eta}^\sqcup(Q)$, $r_\xi^*(T) \leq_h r_\xi^*(V)$ if and only if $\mu s_\xi^* r_\xi^*(T) \leq_{1+\xi} \mu s_\xi^* r_\xi^*(V)$. It is straightforward to check that, in our proof of Lemma 5, given $F \in \mathcal{T}_{\xi+\eta}^\sqcup(Q)$, G is chosen as $r_\xi^*(F)$. Thus, by Lemma 5, $\mu(T) \equiv_{1+\xi}^W \mu s_\xi^* r_\xi^*(T)$, and the similar equivalence holds for V . Consequently, $r_\xi^*(T) \leq_h r_\xi^*(V)$ if and only if $\mu(T) \leq_{1+\xi} \mu(V)$. \square

Corollary 3. *For all ordinals $\xi, \eta < \omega_1$ and bqo Q , $(\Delta_{1+\eta}^0(Q^\mathcal{N}); \leq_w)$ is a retract of $(\Delta_{1+\xi+\eta}^0(Q^\mathcal{N}); \leq_w)$.*

Proof. By Theorem 3, r_ξ^* is an endomorphism on $(\mathcal{T}_{\omega_1}^\sqcup; \leq_h)$. Thus, by Fact 5, the map $\mu(T) \mapsto \mu r_\xi^*(T)$ is well-defined on the Wadge degrees. This induces a map $\tilde{r}_\xi^*: (\Delta_{1+\xi+\eta}^0(Q^\mathcal{N}); \leq_w) \rightarrow$

$(\Delta_{1+\eta}^0(Q^{\mathcal{N}}); \leq_W)$ given by $[\mu(T)] \mapsto [\mu r_{\xi}^*(T)]$. Then, $\tilde{s}_{\xi}^*: [\mu(T)] \mapsto [\mu s_{\xi}^*(T)]$ satisfies $\tilde{r}_{\xi}^* \tilde{s}_{\xi}^* = Id$ by Observation 3; that is, $(\tilde{s}_{\xi}^*, \tilde{r}_{\xi}^*)$ is a section-retraction pair. \square

5 Open question

One of the most important open questions in bqo-Wadge theory seems to find reasonable generalisations of Theorems 1 and 2 for a certain class of non-Borel functions (under some set-theoretic assumption). For $Q = 2$, most known results have been straightforwardly extended to non-Borel sets under the axiom of determinacy. The reason why such an extension is possible is because the two-point space 2 is too easy, so its Wadge degree structure is completely determined by Wadge’s lemma, Martin-Monk’s lemma, and Steel-van Wesep’s theorem (cf. [10]). The bqo analogue of Wadge’s lemma and Martin-Monk’s lemma is van Engelen-Miller-Steel’s theorem, and a bqo analogue of Steel-van Wesep’s theorem is also known. These theorems are readily extended to non-Borel Q -partitions under a certain set-theoretic assumption, cf. [1, 13]. However, contrary to the case $Q = 2$, these theorems are far from characterising the Wadge degree structure even for $Q = 3$. Borel bqo-Wadge theory has played a significant role for unveiling hidden structures on the Wadge degrees which cannot be recognised by the Wadge theory for $Q = 2$. We expect that extending bqo-Wadge theory to non-Borel functions would lead us to new ideas revealing more deep structures in Wadge theory, and also to new constructions in wqo/bqo theory.

Acknowledgements

The first-named author was partially supported by JSPS KAKENHI Grant 19K03602, 15H03634, and the JSPS Core-to-Core Program (A. Advanced Research Networks).

References

- [1] Alexander C. Block. Operations on a Wadge-type hierarchy of ordinal-valued functions. Master’s thesis, Universiteit van Amsterdam, 2014.
- [2] B. A. Davey and H. A. Priestley. *Introduction to lattices and order*. Cambridge University Press, New York, second edition, 2002.
- [3] Adam R. Day, Rod Downey, and Linda Brown Westrick. Three topological reducibilities for discontinuous functions. arXiv: 1906.07600, 2019.
- [4] J. Duparc. Wadge hierarchy and Veblen hierarchy. I. Borel sets of finite rank. *J. Symbolic Logic*, 66(1):56–86, 2001.
- [5] J. Duparc. The Steel hierarchy of ordinal valued Borel mappings. *J. Symbolic Logic*, 68(1):187–234, 2003.

- [6] Vassilios Gregoriades, Takayuki Kihara, and Keng Meng Ng. Turing degrees in Polish spaces and decomposability of Borel functions. *arXiv: 1410.1052*, 2014.
- [7] Peter Hertling. *Topologische Komplexitätsgrade von Funktionen mit endlichem Bild*. PhD thesis, Fernuniversität Hagen, 1993.
- [8] Daisuke Ikegami, Philipp Schlicht, and Hisao Tanaka. Borel subsets of the real line and continuous reducibility. *Fund. Math.*, 244(3):209–241, 2019.
- [9] Alexander S. Kechris. *Classical descriptive set theory*, volume 156 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.
- [10] Alexander S. Kechris, Benedikt Löwe, and John R. Steel, editors. *Wadge degrees and projective ordinals. The Cabal Seminar. Volume II*, volume 37 of *Lecture Notes in Logic*. Association for Symbolic Logic, La Jolla, CA; Cambridge University Press, Cambridge, 2012.
- [11] Takayuki Kihara. Decomposing Borel functions using the Shore-Slaman join theorem. *Fund. Math.*, 230(1):1–13, 2015.
- [12] Takayuki Kihara. Topological reducibilities for discontinuous functions and their structures. *arXiv: 1906.10573*, 2019.
- [13] Takayuki Kihara and Antonio Montalbán. The uniform Martin’s conjecture for many-one degrees. *Trans. Amer. Math. Soc.*, 370(12):9025–9044, 2018.
- [14] Takayuki Kihara and Antonio Montalbán. On the structure of the Wadge degrees of bqo-valued Borel functions. *Trans. Amer. Math. Soc.*, 371(11):7885–7923, 2019.
- [15] Antonio Montalbán. Martin’s conjecture: A classification of the naturally occurring Turing degrees. *Notices Amer. Math. Soc.*, 66(8):1209–1215, 2019.
- [16] Yiannis N. Moschovakis. *Descriptive set theory*, volume 155 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2009.
- [17] Luca Motto Ros. Borel-amenable reducibilities for sets of reals. *J. Symbolic Logic*, 74(1):27–49, 2009.
- [18] Luca Motto Ros, Philipp Schlicht, and Victor Selivanov. Wadge-like reducibilities on arbitrary quasi-Polish spaces. *Math. Structures Comput. Sci.*, 25(8):1705–1754, 2015.
- [19] V. L. Selivanov. The quotient algebra of labeled forests with respect to h -equivalence. *Algebra Logika*, 46(2):217–243, 2007.
- [20] Victor Selivanov. Towards a descriptive theory of cb_0 -spaces. *Mathematical Structures in Computer Science*, 27(8):1553–1580, 2017.

- [21] Victor L. Selivanov. Q -Wadge degrees as free structures. submitted.
- [22] Victor L. Selivanov. Hierarchies of Δ_2^0 -measurable k -partitions. *MLQ Math. Log. Q.*, 53(4-5):446–461, 2007.
- [23] Victor L. Selivanov. Extending Wadge theory to k -partitions. In *Unveiling Dynamics and Complexity - 13th Conference on Computability in Europe*, volume 10307 of *Lecture Notes in Computer Science*, pages 387–399. Springer, 2017.
- [24] Stephen G. Simpson. Bqo-theory and Fraïssé’s conjecture. Chapter 9 of R. Mansfield, G. Weitkamp. *Recursive aspects of descriptive set theory*. Oxford University Press, New York, 1985., 1985.
- [25] Fons van Engelen, Arnold W. Miller, and John Steel. Rigid Borel sets and better quasi-order theory. In *Logic and combinatorics (Arcata, Calif., 1985)*, volume 65 of *Contemp. Math.*, pages 199–222. Amer. Math. Soc., Providence, RI, 1987.
- [26] William Wilfred Wadge. *Reducibility and Determinateness on the Baire Space*. ProQuest LLC, Ann Arbor, MI, 1983. Thesis (Ph.D.)–University of California, Berkeley.