

ON THE STRUCTURE OF THE WADGE DEGREES OF BQO-VALUED BOREL FUNCTIONS

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ABSTRACT. In this article, we give a full description of the Wadge degrees of Borel functions from ω^ω to a better quasi ordering \mathcal{Q} . More precisely, for any countable ordinal ξ , we show that the Wadge degrees of $\Delta_{1+\xi}^0$ -measurable functions $\omega^\omega \rightarrow \mathcal{Q}$ can be represented by countable joins of the ξ -th transfinite nests of \mathcal{Q} -labeled well-founded trees.

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1. INTRODUCTION

In his doctorate thesis [Wad83], Wadge proposed a notion of reducibility between sets of reals that is not only natural, but also surprisingly well behaved, as opposed to most computability theoretic reducibilities which have a rather messy structure.

Definition 1.1 (Wadge [Wad83]). Given $\mathcal{A}, \mathcal{B} \subseteq \omega^\omega$, we say that \mathcal{A} is *Wadge reducible* to \mathcal{B} , and write $\mathcal{A} \leq_w \mathcal{B}$, if there is a continuous function $f: \omega^\omega \rightarrow \omega^\omega$ such that $X \in \mathcal{A} \iff f(X) \in \mathcal{B}$ for all $X \in \omega^\omega$.

The relation \leq_w is a pre-ordering, and, as usual, it induces an equivalence \equiv_w and a degree structure. Wadge showed the Wadge degrees are semi-linearly-ordered in the sense that all anti-chains have size at most 2. Then, Martin and Monk showed they are well-founded. (This is all assuming Γ -determinacy when dealing with sets in a pointclass Γ .) Furthermore, each Wadge degree is in a sense *natural*, and can be assigned a *name* using an ordinal less than Θ and a symbol from $\{\Delta, \Sigma, \Pi\}$ ([VW78]; see also the Cabal volume [KLS12]), a name from which we can understand the nature of that Wadge degree. Based on this perspective, Duparc [Dup01, Dup] gave an explicit description of each Borel Wadge degree of a subset of ω^ω .

The Wadge degrees were later extended in various directions. We can encapsulate all those extensions within the following framework:

Definition 1.2. Let $(\mathcal{Q}; \leq_{\mathcal{Q}})$ be a partial ordering. For \mathcal{Q} -valued functions $\mathcal{A}, \mathcal{B}: \omega^\omega \rightarrow \mathcal{Q}$, we say that \mathcal{A} is *\mathcal{Q} -Wadge reducible to \mathcal{B}* (written $\mathcal{A} \leq_w \mathcal{B}$) if there is a continuous function $\theta: \omega^\omega \rightarrow \omega^\omega$ such that

$$(\forall X \in \omega^\omega) \mathcal{A}(X) \leq_{\mathcal{Q}} \mathcal{B}(\theta(X)).$$

The original Wadge degrees are the case $\mathcal{Q} = 2$ in the definition above, coding sets by their characteristic functions $\omega^\omega \rightarrow 2$ and viewing 2 as the partial ordering with two incomparable elements 0 and 1.

The first extension already considered by Wadge [Wad83, Section 1.E], was to *partial* functions $\omega^\omega \rightarrow \{0, 1\}$, or equivalently, total functions $\omega^\omega \rightarrow \{\perp, 0, 1\}$, where \perp is thought of as being below both 0 and 1, which are incomparable with each other. The degree structure we obtain is also semi-well-ordered, but slightly different than the structure of the Wadge degrees. These degrees are connected to recent work of Day, Downey, and Westrick [DDW17], as observed by Kihara [Kih17].

Shortly after, Steel studied the Wadge degrees of ordinal-valued functions with domain ω^ω , and showed they are well-ordered (see [Dup03, Theorem 1]). Later, together with van Engelen and Miller [vEMS87], they employed bqo theory to unify these results, and showed that if \mathcal{Q} is better-quasi-ordered (bqo, see Definition 2.1), then so is the poset of the Wadge degrees of \mathcal{Q} -valued Borel functions. We delay the definition of better-quasi-ordering until Definition 2.1, and for now let us just say that better-quasi-orderings are well-founded, have no infinite antichains, and have very good closure properties. van Engelen, Miller and Steel's results is even more surprising than Wadge–Martin–Monk's semi-well-orderness of the 2-Wadge degrees. Naturally defined better-quasi-orders always have a nice structure.

For a bqo \mathcal{Q} , the \mathcal{Q} -Wadge degrees are recently found to play an important role in computability theory. The authors [KM] showed that there is a natural isomorphism

between the structure of \mathcal{Q} -Wadge degrees and that of the “natural” many-one degrees of \mathcal{Q} -valued problems. Hence, exploring \mathcal{Q} -Wadge degrees is the same thing as exploring natural \mathcal{Q} -many-one degrees. The objective of this paper is to describe the structure of the \mathcal{Q} -Wadge degrees by showing that it is isomorphic to another partial ordering that is easier to visualize and understand.

In the last decade, Selivanov [Sel07, Sel11] started studying the case of k -partitions, that is, the case when $\mathcal{Q} = k$, the poset with k incomparable elements for finite k . Selivanov [Sel07] gave a full description of the Wadge degrees of Δ_2^0 k -partitions, naming each such degree by a k -labeled well-founded forest, in a way that the name describes the nature of the k -Wadge degree. What he does is essentially a generalization of the Hausdorff-Kuratowski hierarchy from $k = 2$ to larger k 's, where the structure becomes much richer. More precisely, for a set \mathcal{Q} , let $\text{Tree}(\mathcal{Q})$ be the set of all \mathcal{Q} -labeled well-founded countable trees, and let ${}^\cup\text{Tree}(\mathcal{Q})$ be the set of all \mathcal{Q} -labeled well-founded countable forests. Note that every such a forest F can be thought of as a collection (or a disjoint union) of countably many \mathcal{Q} -labeled well-founded countable trees. Selivanov introduced a quasi-order \leq on ${}^\cup\text{Tree}(k)$, where given by $S \leq T$ if there is a homomorphism from S to T which preserves inclusion of strings (\subseteq) and preserves labels (as defined in 3.1).

Theorem 1.3 (Selivanov [Sel07]). *Let $k \in \omega$. The Wadge quasi-ordering on the Δ_2^0 -measurable k -valued functions is isomorphic to the quasi-ordering (${}^\cup\text{Tree}(k); \leq$) on well-founded k -labeled forests.*

We have recently learned that Selivanov has extended his result to the class of Δ_3^0 k -partitions, using forests labeled with labeled trees [Sel16, Sel17]. His techniques are very different from ours.

The objective of this paper is to give a description of the Wadge degrees of Borel functions $\omega^\omega \rightarrow \mathcal{Q}$, where \mathcal{Q} is any better-quasi-ordering (bqo), generalizing Selivanov results from Δ_3^0 to all Borel functions and from finite k to all bqos \mathcal{Q} .

To name the Wadge degrees of Δ_{n+1}^0 -measurable \mathcal{Q} -valued functions, we will use trees labeled by trees labeled by trees ... labeled by \mathcal{Q} . That is, we will define $\text{Tree}^n(\mathcal{Q})$ as $\text{Tree}(\text{Tree}(\dots \text{Tree}(\mathcal{Q}) \dots))$ iterated n times, then define ${}^\cup\text{Tree}^n(\mathcal{Q})$ as the disjoint unions of these trees (see Section 3.1.1). We think of each forest $T \in {}^\cup\text{Tree}^n(\mathcal{Q})$ as a process of mind-changes which captures a natural class Σ_T of Δ_{n+1}^0 -measurable functions. Based on this viewpoint, we will then define a quasi-order \leq on ${}^\cup\text{Tree}^n(\mathcal{Q})$ that matches Wadge reducibility on the classes of functions described by these forests.

Theorem 1.4. *Let \mathcal{Q} be a bqo. Then, the Wadge quasi-ordering on the Δ_{n+1}^0 -measurable \mathcal{Q} -valued functions is isomorphic to (${}^\cup\text{Tree}^n(\mathcal{Q}), \leq$).*

To extend this result through the Borel hierarchy, we will introduce the ξ -th iterated version ${}^\cup\text{Tree}^\xi(\mathcal{Q})$ for each countable ordinal ξ , and show the following transfinite version:

Theorem 1.5. *Let \mathcal{Q} be a bqo, and ξ be a countable ordinal. Then, the Wadge quasi-ordering on the $\Delta_{1+\xi}^0$ -measurable \mathcal{Q} -valued functions is isomorphic to (${}^\cup\text{Tree}^\xi(\mathcal{Q}), \leq$).*

We deal with functions of finite Borel rank and prove Theorem 1.4 in Sections 3–5. We will then extend those ideas to infinite Borel rank and prove Theorem 1.5 in Section 6.

The main steps for the proof are as follows. First, we need to formally define $\sqcup\text{Tree}^\xi(\mathcal{Q})$ and the ordering \trianglelefteq . Then, as suggested above, in Section 3.2, we will assign a pointclass Σ_T of \mathcal{Q} -valued functions to each forest $T \in \sqcup\text{Tree}^\xi(\mathcal{Q})$. For instance, $\Sigma_{\langle 0 \rangle \rightarrow \langle 1 \rangle}$ is the class of characteristic functions of Σ_1^0 sets, and $\Sigma_{\langle 0 \rangle \rightarrow \langle 1 \rangle \rightarrow \langle 0 \rangle}$ is the class of characteristic functions of sets which are differences of two open sets.

Proposition 1.6. *For every $T \in \sqcup\text{Tree}^\xi(\mathcal{Q})$, every function in Σ_T is $\Delta_{1+\xi}^0$ -measurable.*

These pointclasses will match the ordering \trianglelefteq on forests in the following sense:

Proposition 1.7. *For $S, T \in \sqcup\text{Tree}^\xi(\mathcal{Q})$, $S \trianglelefteq T$ if and only if every Σ_S function is Wadge reducible to some Σ_T function.*

For pointclass Σ_T , we will define a Σ_T -complete function Ω_T , that is, Ω_T is in Σ_T and any other function in Σ_T is Wadge reducible to Ω_T .

Proposition 1.8. *For each $T \in \sqcup\text{Tree}^\xi(\mathcal{Q})$, there is a Σ_T -complete function Ω_T .*

We can then restate Proposition 1.7 as $S \trianglelefteq T \iff \Omega_S \leq_w \Omega_T$. This gives us an embedding of $\sqcup\text{Tree}^\xi(\mathcal{Q})$ into the $\Delta_{1+\xi}$ \mathcal{Q} -Wadge degrees. The last step is to show that this embedding is onto.

Proposition 1.9. *Every $\Delta_{1+\xi}^0$ -measurable function $\omega^\omega \rightarrow \mathcal{Q}$ is Wadge equivalent to a Σ_T -complete function for some $T \in \text{Tree}^\xi(\mathcal{Q})$.*

2. THE \mathcal{Q} -WADGE DEGREES

Let us start by describing what we knew about the structure of the \mathcal{Q} -Wadge degrees.

2.1. The Borel hierarchy of functions. We should be careful here, as that there are several different definitions of the Borel hierarchy (specifically, at limit ranks). We adopt the following definition: For $\alpha > 0$, a set $\mathcal{S} \subseteq \omega^\omega$ is Σ_α^0 if \mathcal{S} can be written as $\mathcal{S} = \bigcup_{n \in \omega} \mathcal{S}_n$ where each \mathcal{S}_n is $\Pi_{\beta_n}^0$ for some $\beta_n < \alpha$. Then, we define Π_α^0 and Δ_α^0 in a usual manner. For a countable ordinal ξ , and a topological space \mathcal{X} , a function $\mathcal{A}: \omega^\omega \rightarrow \mathcal{X}$ is Σ_ξ^0 -measurable if $\mathcal{A}^{-1}[U]$ is Σ_ξ^0 for each open set $U \subseteq \mathcal{X}$. In particular: $\mathcal{A}: \omega^\omega \rightarrow \omega^\omega$ is Σ_ξ^0 -measurable if $\mathcal{A}^{-1}[\sigma]$ is Σ_ξ^0 for each $\sigma \in \omega^{<\omega}$, where $[\sigma] = \{X \in \omega^\omega : \sigma \subseteq X\}$; $\mathcal{A}: \omega^\omega \rightarrow \mathcal{Q}$ is Σ_ξ^0 -measurable if it is with respect to the discrete topology on \mathcal{Q} . If \mathcal{Q} is a discrete space, the class of total Σ_ξ^0 -measurable functions $\omega^\omega \rightarrow \mathcal{Q}$ is the same as that of Δ_ξ^0 -measurable functions. Note that the range of a Borel function from ω^ω to a discrete space is countable; otherwise ZFC would prove the existence of 2^{\aleph_1} many pairwise different Borel subsets of ω^ω . Thus, $\mathcal{A}: \omega^\omega \rightarrow \mathcal{Q}$ is Δ_ξ^0 -measurable if and only if the range of \mathcal{A} is countable, and $\mathcal{A}^{-1}[\{q\}]$ is Δ_ξ^0 for any $q \in \mathcal{Q}$. Since we will be dealing with Borel functions $\mathcal{A}: \omega^\omega \rightarrow \mathcal{Q}$, they will always have countable range. We can thus assume from the rest of the paper that \mathcal{Q} is actually countable, even though all the results will extend to uncountable \mathcal{Q} for functions with countable range.

Continuous functions are exactly the Σ_1^0 -measurable functions. For each continuous function G there is a partial computable operator $\Phi_e: \omega^{\leq\omega} \rightarrow \omega^{\leq\omega}$ and an oracle $C \in \omega^\omega$ such that $G(X) = \Phi_e(C \oplus X)$ for all $X \in \omega^\omega$. Also, we will often identify a continuous function $\omega^\omega \rightarrow \omega^\omega$ with its corresponding approximation function $\omega^{\leq\omega} \rightarrow \omega^{\leq\omega}$.

For functions $\mathcal{A}, \mathcal{B}: \omega^\omega \rightarrow \omega^\omega$, we say that \mathcal{A} is *Wadge reducible* to \mathcal{B} if there is a continuous function θ such that $\mathcal{A} = \mathcal{B} \circ \theta$. Note that this matches Definition 1.2 if we think of \mathcal{Q} as ω^ω where every two reals are incomparable under $\leq_{\mathcal{Q}}$.

2.2. Wadge degrees and games. Wadge [Wad83, Theorem B8] introduced a perfect-information, infinite, two-player game, known as the *Wadge game*, which can be used to define Wadge reducibility. For \mathcal{Q} -valued functions $\mathcal{A}, \mathcal{B}: \omega^\omega \rightarrow \mathcal{Q}$, here is the \mathcal{Q} -valued version $G_w(\mathcal{A}, \mathcal{B})$ of the Wadge game: At n -th round of the game, Player I chooses $x_n \in \omega$ and II chooses $y_n \in \omega \cup \{\text{pass}\}$ (where $\text{pass} \notin \omega$). Eventually Players I and II produce infinite sequences $X = (x_n)_{n \in \omega}$ and $Y = (y_n)_{n \in \omega}$, respectively. Let Y^{P} denote the result dropping all **passes** from Y . We say that *Player II wins the game* $G_w(\mathcal{A}, \mathcal{B})$ if

$$Y^{\text{P}} \text{ is an infinite sequence, and } \mathcal{A}(X) \leq_{\mathcal{Q}} \mathcal{B}(Y^{\text{P}}).$$

As in Wadge [Wad83, Theorem B8], one can easily check that $\mathcal{A} \leq_w \mathcal{B}$ holds if and only if Player II wins the game $G_w(\mathcal{A}, \mathcal{B})$. We will often identify a winning strategy with a continuous function generated by it.

2.3. Better quasi orderings. To define bqos, we need to introduce some notation. Let $[\omega]^\omega$ be the set of all strictly increasing sequences on ω , whose topology is inherited from ω^ω . We also assume that a quasi-order \mathcal{Q} is equipped with the discrete topology. Given $X \in [\omega]^\omega$, let X^- denote the result of dropping the first entry from X (or equivalently, $X^- = X \setminus \{\min X\}$, if we think of $X \in [\omega]^\omega$ as an infinite subset of ω).

Definition 2.1 (Nash-Williams [NW65]). A quasi-order \mathcal{Q} is a *better-quasi-order* (abbreviated as bqo) if, for any continuous function $f: [\omega]^\omega \rightarrow \mathcal{Q}$, there is $X \in [\omega]^\omega$ such that $f(X) \leq_{\mathcal{Q}} f(X^-)$.

The formulation of the definition above is due to Simpson [Sim85]. He also show that one can use Borel functions f in the definition and obtain the same notion.

Example 2.2. For a natural number k , the discrete order $\mathcal{Q} = (k; =)$, denoted by k , is a bqo. More generally, every finite partial ordering is a bqo.

Every bqo is also a *well-quasi-order* (often abbreviated as wqo), that is, that it is well-founded and has no infinite antichain. Bqo's were introduced by Nash-Williams to prove wqo results, as bqo's have better closure properties than wqo's under infinitary operations. For instance, Laver [Lav78] showed that if \mathcal{Q} is a bqo, then so are $\text{Tree}(\mathcal{Q})$ ordered by the \trianglelefteq , and the class of scattered \mathcal{Q} -labeled linear orderings ordered by $\leq_{\mathcal{Q}}$ -preserving embeddability. The most relevant such result for us is the following:

Theorem 2.3 (van Engelen–Miller–Steel [vEMS87, Theorem 3.2]). *If \mathcal{Q} is a bqo, then the Wadge degrees of \mathcal{Q} -valued Borel functions on ω^ω form a bqo too.*

2.4. Self-duality and join-reducibility. Two important notions when trying to understand the notion of the \mathcal{Q} -Wadge degrees is that of σ -join-reducibility and self-duality.

Definition 2.4. We say that a \mathcal{Q} -Wadge degree \mathbf{a} is *σ -join-reducible* if \mathbf{a} is the least upper bound of a countable collection $(\mathbf{b}_i)_{i \in \omega}$ of \mathcal{Q} -Wadge degrees such that $\mathbf{b}_i <_w \mathbf{a}$. Otherwise, we say that \mathbf{a} is *σ -join-irreducible*.

Definition 2.5 (Louveau and Saint-Raymond [LSR90]). We say that a function $\mathcal{A}: \omega^\omega \rightarrow \mathcal{Q}$ is *self-dual* if there is a continuous function $\theta: \omega^\omega \rightarrow \omega^\omega$ such that $\mathcal{A}(\theta(X)) \not\leq_{\mathcal{Q}} \mathcal{A}(X)$ for all $X \in \omega^\omega$.

For example, in the case $\mathcal{Q} = 2$, the Δ Wadge degrees are the self-dual ones, and the Σ 's and the Π 's are not. Also, each Δ degree is the least upper bound of the Σ degree and the Π degree immediately below it.

Before stating the equivalence of these two notions, the following definition gives us a useful tool to study the Wadge degree of a function $\mathcal{A}: \omega^\omega \rightarrow \mathcal{Q}$. For $\sigma \in \omega^{<\omega}$, define the function $\mathcal{A} \upharpoonright [\sigma]$ by $(\mathcal{A} \upharpoonright [\sigma])(X) = \mathcal{A}(\sigma \frown X)$ for any $X \in \omega^\omega$ (see also Observations 3.5 and 3.6), where $\sigma \frown X$ is the concatenation of σ and X . Notice that for each $\sigma \in \omega^{<\omega}$, $\mathcal{A} \upharpoonright [\sigma] \leq_w \mathcal{A}$ by essentially the identity operation. For some of these σ we will have $\mathcal{A} \upharpoonright [\sigma] \equiv_w \mathcal{A}$ and for some $\mathcal{A} \upharpoonright [\sigma] <_w \mathcal{A}$. Define

$$\mathcal{F}(\mathcal{A}) = \{X : (\forall n) \mathcal{A} \upharpoonright [X \upharpoonright n] \equiv_w \mathcal{A}\}.$$

One more definition, given $\mathcal{A}_n: \omega^\omega \rightarrow \mathcal{Q}$, $\bigoplus_{n \in \omega} \mathcal{A}_n$ is defined by

$$\left(\bigoplus_{n \in \omega} \mathcal{A}_n\right)(n \frown X) = \mathcal{A}_n(X).$$

Proposition 2.6. *Let \mathcal{Q} be a bqo and $\mathcal{A}: \omega^\omega \rightarrow \mathcal{Q}$ a Borel function. The following are equivalent*

- (1) \mathcal{A} is σ -join-reducible.
- (2) $\mathcal{A} \equiv_w \bigoplus_{n \in \omega} \mathcal{A}_n$, for some \mathcal{A}_n which are σ -join-irreducible and $\mathcal{A}_n <_w \mathcal{A}$.
- (3) $\mathcal{F}(\mathcal{A})$ is empty.
- (4) \mathcal{A} is self-dual

Proof. The equivalence between (1) and (4) was proved by Block [Blo14, Proposition 3.5.4], and is a generalization of Steel–van Wesep’s theorem [VW78] from $\mathcal{Q} = 2$ to general \mathcal{Q} .

Let us prove (3) \Rightarrow (1). Suppose $\mathcal{F}(\mathcal{A})$ is empty, and let V be the set of minimal stings in $\omega^{<\omega}$ such that $\mathcal{A} \upharpoonright [\sigma] <_w \mathcal{A}$. Then $\{[\sigma] : \sigma \in V\}$ is a clopen partition of ω^ω . It is not hard to see that $\mathcal{A} \equiv_w \bigoplus_{\sigma \in V} \mathcal{A} \upharpoonright [\sigma]$, and hence that \mathcal{A} is σ -join-reducible.

For the direction (1) \Rightarrow (2), suppose that \mathcal{A} is σ -join-reducible, and that its Wadge degree is the least upper bound of \mathcal{B}_i , for $i \in \omega$, with $\mathcal{B}_i <_w \mathcal{A}$. Since $\mathcal{B}_j \leq_w \bigoplus_{i \in \omega} \mathcal{B}_i$ for all $j \in \omega$, we get that $\mathcal{A} \leq_w \bigoplus_{i \in \omega} \mathcal{B}_i$. Furthermore, since \mathcal{Q} -Wadge degrees are bqo, we can use transfinite induction and assume that each \mathcal{B}_i is either σ -join-irreducible or a sum of σ -join-irreducibles. We would then get that \mathcal{A} is itself equivalent to a sum of σ -join-irreducibles.

For (2) \Rightarrow (3), let θ witness that $\mathcal{A} \leq_w \bigoplus_{i \in \omega} \mathcal{A}_i$. For each $X \in \omega^\omega$, there exists n such that $\theta(X \upharpoonright n)$ is non-empty. If i is the first entry of $\theta(X \upharpoonright n)^p$, we then get that θ witnesses that $\mathcal{A} \upharpoonright [X \upharpoonright n] \leq_w \mathcal{A}_i <_w \mathcal{A}$. It follows that $X \notin \mathcal{F}(\mathcal{A})$ and hence that $\mathcal{F}(\mathcal{A})$ is empty. \square

2.5. Conciliatory functions. There is another way of characterizing non-self-dual functions, and it is using conciliatory functions. Essentially, these are functions whose domain is $\omega^{\leq\omega}$ instead of just ω^ω . For a Borel function $\mathcal{A}: \omega^\omega \rightarrow \mathcal{Q}$, it will follow from our results that \mathcal{A} is non-self-dual if and only if it can be extended to a function

$\hat{\mathcal{A}}: \omega^{\leq\omega} \rightarrow \mathcal{Q}$ that is Wadge equivalent to \mathcal{A} (as defined below). This was proved by Duparc [Dup01] for $\mathcal{Q} = 2$ — he actually introduced the notion of a conciliatory set. We generalize the notion of a conciliatory set in the \mathcal{Q} -valued setting and prove this result as a consequence of Proposition 1.9 and Observation 3.15.

To be able to deal with Wadge reducibility and with complexity pointclasses, we will use the following representation of conciliatory functions. Fix a symbol ‘pass’ and define

$$\hat{\omega} = \omega \cup \{\text{pass}\}.$$

Given $X \in \hat{\omega}^\omega$, we use the notation $X^p \in \omega^{\leq\omega}$ to denote the string obtained by removing all pass’s from X (see also the definition of the Wadge game; Section 2.2).

Definition 2.7. A function $\mathcal{A}: \hat{\omega}^\omega \rightarrow \mathcal{Q}$ is *conciliatory* if

$$(\forall X, Y \in \hat{\omega}^\omega) [X^p = Y^p \implies \mathcal{A}(X) = \mathcal{A}(Y)].$$

A function $\Psi: \hat{\omega}^\omega \rightarrow \hat{\omega}^\omega$ is *conciliatory* if

$$(\forall X, Y \in \hat{\omega}^\omega) [X^p = Y^p \implies \Psi(X)^p = \Psi(Y)^p].$$

Conciliatory functions are in one-to-one correspondence with functions $\omega^{\leq\omega} \rightarrow \mathcal{Q}$ and $\omega^{\leq\omega} \rightarrow \omega^{\leq\omega}$ respectively. However, when we think of their Wadge degrees and of their complexity, it is better to think of them as maps defined on $\hat{\omega}^\omega$. The obvious topology to give to $\hat{\omega}^\omega$ is the product topology of the discrete space $\hat{\omega}$, which is homeomorphic to ω^ω (just because there is a bijection between $\hat{\omega}$ and ω). We will thus treat $\hat{\omega}^\omega$ exactly as we treat ω^ω when we define complexity classes of sets and functions. For instance, a Wadge reduction between conciliatory functions $\mathcal{A}: \hat{\omega}^\omega \rightarrow \mathcal{Q}$ and $\mathcal{B}: \hat{\omega}^\omega \rightarrow \mathcal{Q}$, would be continuous function $\theta: \hat{\omega}^\omega \rightarrow \hat{\omega}^\omega$ which is not necessarily conciliatory. Thus, this function θ is not necessarily well-defined as a function on $\omega^{\leq\omega}$.

Via the identification between $\hat{\omega}^\omega$ and ω^ω , conciliatory functions are just a special class of regular functions. Then, for instance, we can then transform a conciliatory function $\mathcal{A}: \hat{\omega}^\omega \rightarrow \mathcal{Q}$ into a function $\underline{\mathcal{A}}: \omega^\omega \rightarrow \mathcal{Q}$ which is Wadge equivalent to \mathcal{A} . Thus, the conciliatory Wadge degrees are just a subset of the standard Wadge degrees of functions on ω^ω . However, they will be very useful to us when we define the Σ_T -complete functions Ω_T .

Observation 2.8. Every conciliatory function is σ -join-irreducible.

Proof. If \mathcal{A} is conciliatory, it is easy to see that $\text{pass}^\omega \in \mathcal{F}(\mathcal{A})$, where pass^ω is the infinite sequence consisting only of pass. Thus, by Proposition 2.6, \mathcal{A} is σ -join-irreducible. \square

It is the converse direction of this observation that is hard to prove.

The following lemmas and observations will help us get gain some intuition on conciliatory functions, even though they will not be used in the rest of the paper.

Observation 2.9. Every partial computable operator Φ_e can be viewed as a conciliatory function. Essentially, it just outputs passes while it is waiting either for a new value of the oracle, or a new computation to converge. By the same reason, every continuous function $\omega^\omega \rightarrow \omega^\omega$ can be extended to a conciliatory function as we mentioned at the end of Section 2.

Lemma 2.10. *A function $G: \omega^{\leq\omega} \rightarrow \omega^{\leq\omega}$ can be represented as a continuous conciliatory function $\hat{\omega}^\omega \rightarrow \hat{\omega}^\omega$ if and only if $\sigma \subseteq \tau$ implies $G(\sigma) \subseteq G(\tau)$ for every $\sigma, \tau \in \omega^{<\omega}$, and $G(X) = \bigcup_n G(X \upharpoonright n)$ for every $X \in \omega^\omega$.*

Sketch of the proof. For the left-to-right implication, suppose \hat{G} is continuous conciliatory function such that $\hat{G}(X)^{\mathsf{P}} = G(X^{\mathsf{P}})$ for all $X \in \hat{\omega}^\omega$. Suppose $\tau = \sigma \hat{\wedge} \gamma$. Every initial segment of $G(\sigma)$ must be an initial segment of $G(\tau)$ because every initial segment of $G(\sigma)^{\mathsf{P}}$ is contained in $\hat{G}(\sigma \hat{\wedge} \text{pass pass} \cdots \text{pass})^{\mathsf{P}}$ for some number of passes. Then,

$$G(\tau) = \hat{G}(\sigma \hat{\wedge} \text{pass pass} \cdots \text{pass} \hat{\wedge} \gamma \hat{\wedge} \text{pass}^\omega)^{\mathsf{P}}.$$

It follows that $G(\sigma) \subseteq G(\tau)$. By the same argument, if $\sigma \subseteq X$, then $G(\sigma) \subseteq G(X)$. We leave the remaining details to the reader. \square

One can show that a function $G: \omega^{\leq\omega} \rightarrow \mathcal{Q}$ can be represented as a continuous conciliatory function $\hat{\omega}^\omega \rightarrow \mathcal{Q}$ if and only if it is constant. (Just think of \mathcal{Q} as ω , being the first entry of the output of a function as in the lemma.) The case of Σ_2^0 function gets more interesting.

Lemma 2.11. *A function $G: \omega^{\leq\omega} \rightarrow \omega^{\leq\omega}$ can be represented as a Σ_2^0 conciliatory function if and only if for every $X \in \omega^\omega$, $G(X)$ is the pointwise limit of $G(X \upharpoonright m)$ in the following sense: for every $\sigma \in \omega^{<\omega}$,*

$$\sigma \subseteq G(X) \iff \exists n \forall m > n (\sigma \subseteq G(X \upharpoonright m)).$$

In particular, a function $G: \omega^{\leq\omega} \rightarrow \mathcal{Q}$ is Σ_2^0 conciliatory if and only if $G(X) = \lim_n G(X \upharpoonright n)$ for every $X \in \omega^\omega$.

Sketch of the proof. For the left-to-right implication, suppose \hat{G} is Σ_2^0 conciliatory function such that $\hat{G}(X)^{\mathsf{P}} = G(X^{\mathsf{P}})$ for all $X \in \hat{\omega}^\omega$. By definition, the predicate $\tau \subseteq \hat{G}(X)$ is Σ_2^0 -definable with parameters. For $\sigma \in \omega^{<\omega}$ and $X \in \omega^\omega$, note that the predicate $\sigma \subseteq \hat{G}(X)^{\mathsf{P}}$ is equivalent to the existence of $\tau \in \hat{\omega}^{<\omega}$ such that $\tau^{\mathsf{P}} = \sigma$ and $\tau \subseteq \hat{G}(X)$. The latter condition is also Σ_2^0 -definable with parameters. Thus, there is an R such that

$$\sigma \subseteq \hat{G}(X)^{\mathsf{P}} \iff \exists n \forall m > n R(\sigma, n, m, X \upharpoonright m)$$

for $\sigma \in \omega^{<\omega}$ and $X \in \hat{\omega}^\omega$. Suppose, toward a contradiction that $\sigma \subseteq G(X)$, but there exists $k_0 < k_1 < \cdots$ such that $\sigma \not\subseteq G(X \upharpoonright k_n)$. We will then define $Y \in \hat{\omega}^\omega$ with $Y^{\mathsf{P}} = X^{\mathsf{P}}$ such that $\sigma \not\subseteq G(Y) = G(X)$. We define Y by finite approximations $Y_0 \subseteq Y_1 \subseteq Y_2 \cdots$ so that $Y_n^{\mathsf{P}} = X \upharpoonright k_n^{\mathsf{P}}$. At each stage n , since $\sigma \not\subseteq G(X \upharpoonright k_n)$, there is an m_n such that $\neg R(\sigma, n, m_n, Y_n \hat{\wedge} \text{pass}^k)$, where k is so that $|Y_n \hat{\wedge} \text{pass}^k| = m_n$. Define Y_{n+1} to be $Y_n \hat{\wedge} \text{pass}^k \hat{\wedge} X \upharpoonright [k_n + 1, k_{n+1}]$, so that $Y_{n+1}^{\mathsf{P}} = X \upharpoonright k_{n+1}^{\mathsf{P}}$. We then have $\forall n \neg R(\sigma, n, m_n, Y \upharpoonright m_n)$, and hence that $\sigma \not\subseteq G(Y)$.

We leave the converse direction to the reader. It is a standard argument in computability theory. \square

The following lemma is also quite standard. It is just a uniform version of the limit lemma.

Lemma 2.12. *Every partial Σ_2^0 function $G: \omega^\omega \rightarrow \omega^\omega$ can be extended to a Σ_2^0 conciliatory function $\hat{G}: \hat{\omega}^\omega \rightarrow \hat{\omega}^\omega$, so that $\hat{G}(X) = G(X)$ for all $X \in \omega^\omega$.*

2.6. Universal Σ_2^0 conciliatory functions. First, let us observe that there is no universal total Σ_2^0 -measurable function on ω^ω , as it would be Δ_2^0 , and there is no greatest Δ_2^0 Wadge degree. This is the main reason we need to deal with conciliatory functions in this paper. Hereafter, for functions $\mathcal{A}, \mathcal{B} : \mathcal{X} \rightarrow \hat{\omega}^\omega$ for $\mathcal{X} \in \{\omega^\omega, \hat{\omega}^\omega\}$, we write $\mathcal{A} \equiv_p \mathcal{B}$ if $\mathcal{A}(X)^p = \mathcal{B}(X)^p$ for all $X \in \mathcal{X}$.

Definition 2.13. Let $\mathcal{U} : \hat{\omega}^\omega \rightarrow \hat{\omega}^\omega$ be a conciliatory function. We say that \mathcal{U} is Σ_2^0 -universal if it is Σ_2^0 -measurable, if for every Σ_2^0 -measurable conciliatory function $G : \hat{\omega}^\omega \rightarrow \hat{\omega}^\omega$, there exists a continuous function $\theta : \hat{\omega}^\omega \rightarrow \hat{\omega}^\omega$ such that $G \equiv_p \mathcal{U} \circ \theta$.

Let us define a Σ_2^0 -universal function \mathcal{U} . Let $\{\sigma_n : n \in \omega\}$ be an effective enumeration of $\omega^{<\omega}$. Think of an input Y to \mathcal{U} as a code for a sequence of strings $\sigma_{Y(0)}, \sigma_{Y(1)}, \sigma_{Y(2)}, \dots$ and $\mathcal{U}(Y)$ as the pointwise limit of these strings. That is, we would like to define $\mathcal{U}(Y)(j) = \lim_{i \rightarrow \infty} \sigma_{Y(i)}(j)$ if the limit exists, and let it be undefined otherwise, except that we have to be a bit careful to get \mathcal{U} to be of the right form. The actual definition is as follows. For $\sigma \in \omega^{<\omega}$, $\sigma \neq \emptyset$,

$$\sigma \subseteq \mathcal{U}(Y) \iff \exists n (\sigma \subseteq \sigma_{Y^p(n)} \ \& \ \forall m > n (Y^p(m) \downarrow \rightarrow \sigma \subseteq \sigma_{Y^p(m)})),$$

where $Y^p(m) \downarrow$ means that $|Y^p| > m$. It is not hard to see that if $\tau_0 \subseteq \mathcal{U}(Y)$ and $\tau_1 \subseteq \mathcal{U}(Y)$, then τ_0 and τ_1 must be compatible. We let $\mathcal{U}(Y)$ be the union of all σ such that $\sigma \subseteq \mathcal{U}(Y)$. We let the reader verify that \mathcal{U} is a Σ_2^0 -universal conciliatory function, as it is a standard computability theoretic argument.

\mathcal{U} has a particular property that will be quite important: the value of $\mathcal{U}(Y)$ does not depend on initial segments of Y , and only depends on the tail of Y .

Definition 2.14. A function $\mathcal{A} : \hat{\omega}^\omega \rightarrow \hat{\omega}^\omega$ is *initializable* if for every $\tau \in \hat{\omega}^{<\omega}$, there is a continuous function $\theta_\tau : \hat{\omega}^\omega \rightarrow [\tau]$ such that $\mathcal{A} \equiv_p \mathcal{A} \circ \theta_\tau$.

To see that our function \mathcal{U} is initializable, suppose 0 is the code for the empty string (i.e., $\sigma_0 = \emptyset$), then let $\theta_\tau(Y) = \tau \hat{\ } 0 \hat{\ } Y$. It is not hard to see that $\mathcal{U}(Y) = \mathcal{U}(\tau \hat{\ } 0 \hat{\ } Y)$.

We have proved the following proposition.

Proposition 2.15. *There is a Σ_2^0 -universal initializable conciliatory function.*

3. NESTED LABELED TREES

In this section we give formal definitions of ${}^\sqcup\text{Tree}^n(\mathcal{Q})$, \triangleleft , Σ_T , and the Σ_T -complete function Ω_T . We end the section by extending these ideas to all infinite Borel ranks.

3.1. Nested Trees. Let us first give some intuition for the connection between nested labeled trees and Borel functions. First consider the characteristic function χ_U of an open set $U \subseteq \omega^\omega$. Since the predicate $x \in U$ can be described by an existential formula, we have an approximation procedure which starts by guessing $\chi_U(x) = 0$ until $x \in U$ is witnessed, and then changes the guess to $\chi_U(x) = 1$ after seeing such a witness. We denote the collection of all such guessing procedures, namely the pointclass Σ_1^0 , by the term $\langle 0 \rangle \rightarrow \langle 1 \rangle$. We think of the term $\langle 0 \rangle \rightarrow \langle 1 \rangle$ as representing a tree with two nodes whose root is labeled by 0, and leaf is labeled by 1. Similarly, we use the tree $\langle 1 \rangle \rightarrow \langle 0 \rangle$ (with a root node labeled 1, and a leaf node labeled 0) to name the pointclass Π_1^0 , and we use trees of the form $\langle 0 \rangle \rightarrow \langle 1 \rangle \rightarrow \dots \rightarrow \langle 0 \rangle \rightarrow \langle 1 \rangle$ to name the finite levels of the Hausdorff-Kuratowski difference hierarchy.

To represent self-dual pointclasses such as Δ_1^0 , we will need to consider forests rather than trees. Given a clopen set $C \subseteq \omega^\omega$, one decides whether $\chi_C(x) = 0$ or $\chi_C(x) = 1$ at once and there is no change of mind afterwards. We represent this procedure by the term $\langle 0 \rangle \sqcup \langle 1 \rangle$, which is identified with a forest consisting of two roots labeled by 0 and 1, respectively. All levels of the Hausdorff-Kuratowski difference hierarchy (hence all Wadge degrees of Δ_2^0 subsets of ω^ω) are named by terms obtained from the operations \rightarrow and \sqcup (that is, well-founded $\{0, 1\}$ -labeled trees and their disjoint unions). For instance, a term of the form $I_n \rightarrow \bigsqcup_k I_k$, (where I_ℓ is the chain of the form $\langle 0 \rangle \rightarrow \langle 1 \rangle \rightarrow \dots \rightarrow \langle 0 \rangle \rightarrow \langle 1 \rangle$ of length ℓ) names the $(\omega + n)^{th}$ -level of the difference hierarchy. To represent Δ_2^0 3-partitions, Selivanov used forests labeled with $\{0, 1, 2\}$ instead. The idea is the same: A $\{0, 1, 2\}$ -labeled tree guides the mind changes allowed when defining a Δ_2^0 3-partitions; since the tree is well-founded, the guessing process eventually stops.

If we want to move on to Δ_3^0 functions, that is when we need to start nesting trees. For instance, the tree $\langle T \rangle$ consisting only of a root labeled by a tree T , is thought of as the *jump* of the pointclass named by T . Thus, $\langle \langle 0 \rangle \rightarrow \langle 1 \rangle \rangle$ is the jump of Σ_1^0 — namely Σ_2^0 . By using nesting of trees in this way, we will be able to climb up the Borel hierarchy.

3.1.1. Language and terms. All \mathcal{Q} -valued Borel functions of *finite* rank will be described using terms (identified with forests) in the language consisting of constant symbols (corresponding to elements in \mathcal{Q}), and three function symbols: \rightarrow (concatenation), \sqcup (disjoint union), and $\langle \cdot \rangle$ (labeling). To represent \mathcal{Q} -valued Borel functions of *infinite* rank, we will need to add symbols representing transfinite jump operations $\langle \cdot \rangle^{\omega^\alpha}$.

We formally describe the collections $\text{Tree}(\mathcal{Q})$ and ${}^\sqcup\text{Tree}(\mathcal{Q})$ of countable well-founded \mathcal{Q} -labeled trees and their countable disjoint unions (i.e., forests) in the following inductive manner:

- (1) If $T \in \text{Tree}(\mathcal{Q})$, then $T \in {}^\sqcup\text{Tree}(\mathcal{Q})$.
- (2) For each $q \in \mathcal{Q}$, the term $\langle q \rangle$ is in $\text{Tree}(\mathcal{Q})$. It represents the tree with only one node labeled q .
- (3) For any countable collection $\{T_i\}_{i \in \omega}$ in $\text{Tree}(\mathcal{Q})$, the term $\sqcup_i T_i$ is in ${}^\sqcup\text{Tree}(\mathcal{Q})$. Terms of the form $\sqcup_i T_i$ will be called *\sqcup -type terms*, and represent forests obtained as the disjoint union of trees T_i .
- (4) For any $q \in \mathcal{Q}$ and \sqcup -type term $T \in {}^\sqcup\text{Tree}(\mathcal{Q})$, the term $\langle q \rangle \rightarrow T$ is in $\text{Tree}(\mathcal{Q})$. It represents the tree obtained by joining to a root labeled q all the components of the forest T .

Note that $\text{Tree}(\mathcal{Q})$ consist of the non- \sqcup -type terms in ${}^\sqcup\text{Tree}(\mathcal{Q})$. Then, define $\text{Tree}^0(\mathcal{Q}) = \mathcal{Q}$, $\text{Tree}^{n+1}(\mathcal{Q}) = \text{Tree}(\text{Tree}^n(\mathcal{Q}))$, and ${}^\sqcup\text{Tree}^{n+1}(\mathcal{Q}) = {}^\sqcup\text{Tree}(\text{Tree}^n(\mathcal{Q}))$.

The way they are defined, $\text{Tree}^m(\mathcal{Q})$ and $\text{Tree}^n(\mathcal{Q})$ are disjoint whenever $m < n$. However, we will later see that every tree in $\text{Tree}^m(\mathcal{Q})$ is equivalent to one in $\text{Tree}^n(\mathcal{Q})$ (Observation 3.2).

3.1.2. Quasi-ordering nested trees. In this section, we introduce a quasi-order \preceq on ${}^\sqcup\text{Tree}^{<\omega}(\mathcal{Q})$, which we will show is isomorphic to the Wadge quasi-ordering of \mathcal{Q} -valued functions of finite Borel rank. To simplify our notation, we always identify $\langle T \rangle$ with $\langle T \rangle \rightarrow \sqcup_i \mathbf{O}$, where \mathbf{O} is the empty forest, which we think of as an imaginary least element with respect to the quasi-order \preceq , that is, $\mathbf{O} \preceq T$ for any $T \in {}^\sqcup\text{Tree}^n(\mathcal{Q})$.

Definition 3.1. We inductively define a quasi-order \trianglelefteq on $\bigcup_n \text{Tree}^n(\mathcal{Q})$ as follows, where the symbols p and q range over \mathcal{Q} , and U, V, S , and T range over $\bigcup_n \text{Tree}^n(\mathcal{Q})$:

$$\begin{aligned} p \trianglelefteq q &\iff p \leq_{\mathcal{Q}} q, \\ \langle U \rangle \trianglelefteq \langle V \rangle &\iff U \trianglelefteq V, \end{aligned}$$

and if S and T are of the form $\langle U \rangle \rightarrow \sqcup_i S_i$ and $\langle V \rangle \rightarrow \sqcup_j T_j$, respectively, then

$$S \trianglelefteq T \iff \begin{cases} \text{either } U \trianglelefteq V & \text{and } (\forall i) S_i \trianglelefteq T, \\ \text{or } U \not\trianglelefteq V & \text{and } (\exists j) S \trianglelefteq T_j. \end{cases}$$

This pre-ordering induces an equivalence as usual: let $S \equiv T$ if $S \trianglelefteq T$ and $T \trianglelefteq S$. For $p \in \mathcal{Q}$, we let $p \equiv \langle p \rangle \equiv \langle \langle p \rangle \rangle \equiv \dots$, allowing us to compare trees of different levels.

Finally, \trianglelefteq is uniquely extended to a quasi-order on $\bigcup_n {}^{\cup}\text{Tree}^n(\mathcal{Q})$ by interpreting \sqcup as a countable supremum operation:

$$\sqcup_i S_i \trianglelefteq \sqcup_j T_j \iff (\forall i)(\exists j) S_i \trianglelefteq T_j.$$

Observation 3.2. For every $T \in {}^{\cup}\text{Tree}^{\leq n}(\mathcal{Q})$, there is $S \in {}^{\cup}\text{Tree}^n(\mathcal{Q})$ such that $S \equiv T$.

Proof. Assume that $m \leq n$ and $T \in {}^{\cup}\text{Tree}^m(\mathcal{Q})$. Then, consider the term $\iota(T) = T[\langle q \rangle^{n-m}/q]_{q \in \mathcal{Q}}$ obtained by substituting all occurrences of $q \in \mathcal{Q}$ by $\langle q \rangle^{n-m}$, where $\langle q \rangle^0 = q$ and $\langle q \rangle^{k+1} = \langle \langle q \rangle^k \rangle$. Note that $\iota(T) \in {}^{\cup}\text{Tree}^n(\mathcal{Q})$, and it is clear that $T \equiv \iota(T)$. \square

Observation 3.3. Consider $S, T \in {}^{\cup}\text{Tree}^n(\mathcal{Q})$, and use \subseteq to denote the ordering among the nodes of S and T , the roots being the \subseteq -least elements. It is not hard to see that $S \trianglelefteq T$ if and only if there exists a map $f: S \rightarrow T$ which is order preserving, in the sense for σ, τ nodes in S , $\sigma \subseteq \tau \Rightarrow f(\sigma) \subseteq f(\tau)$, and $\leq_{\mathcal{Q}}$ -increasing in the sense that the label(σ) \trianglelefteq label($f(\sigma)$) for every $\sigma \in S$. Such f does not need to be one-to-one.

Theorem 3.4 (Laver [Lav78]). *For $n \in \omega$, if \mathcal{Q} is better-quasi-ordered, then so is ${}^{\cup}\text{Tree}^{\leq n}(\mathcal{Q})$.*

Laver showed that if \mathcal{Q} is a bqo, so is $\text{Tree}(\mathcal{Q})$ for an even stronger notion of reducibility.

3.2. The associated pointclasses. As we mentioned before, each forest $T \in {}^{\cup}\text{Tree}^n(\mathcal{Q})$ defines a pointclass Σ_T . For instance, if $\mathcal{Q} = 2$, then

$$\Sigma_{\langle 0 \rangle \sqcup \langle 1 \rangle} = \mathbf{\Delta}_1^0, \quad \Sigma_{\langle 0 \rangle \rightarrow \langle 1 \rangle} = \mathbf{\Sigma}_1^0, \quad \Sigma_{\langle 1 \rangle \rightarrow \langle 0 \rangle} = \mathbf{\Pi}_1^0, \quad \Sigma_{\langle \langle 0 \rangle \rightarrow \langle 1 \rangle \rangle} = \mathbf{\Sigma}_2^0, \quad \text{and so on.}$$

The following observations will simplify our definitions.

Observation 3.5. Let \mathcal{F} be a nonempty closed subset of ω^ω . Then, for every function $\mathcal{A}: \mathcal{F} \rightarrow \mathcal{Q}$ there is a function $\widehat{\mathcal{A}}: \omega^\omega \rightarrow \mathcal{Q}$ which is Wadge equivalent to \mathcal{A} .

Proof. By zero-dimensionality of ω^ω , there is a retraction $\rho_{\mathcal{F}}: \omega^\omega \rightarrow \mathcal{F}$ (that is, $\rho_{\mathcal{F}}$ is continuous and $\rho_{\mathcal{F}} \upharpoonright \mathcal{F}$ is identity). Define $\widehat{\mathcal{A}} = \mathcal{A} \circ \rho_{\mathcal{F}}$. Then, we have $\widehat{\mathcal{A}} \leq_w \mathcal{A}$ via $\rho_{\mathcal{F}}$, and $\mathcal{A} \leq_w \widehat{\mathcal{A}}$ via the identity map.

The definition of this retraction is quite standard: Let $T \subseteq \omega^{<\omega}$ be a tree without dead end such that $\mathcal{F} = [T]$. We define $\rho_{\mathcal{F}}: \omega^{<\omega} \rightarrow T$ by induction: $\rho_{\mathcal{F}}(\sigma \hat{\ } n) = \rho_{\mathcal{F}}(\sigma) \hat{\ } m$ where $m \in \omega$ is the closest to n such that $\rho_{\mathcal{F}}(\sigma) \hat{\ } m \in T$. (By closest we mean such that $|m + \frac{1}{3} - n|$ is least, for instance.) We then extend $\rho_{\mathcal{F}}$ to ω^ω to \mathcal{F} by continuity. \square

Observation 3.6. Let \mathcal{V} be a nonempty open subset of ω^ω . Then, for every function $\mathcal{A}: \mathcal{V} \rightarrow \mathcal{Q}$ there is a function $\widehat{\mathcal{A}}: \omega^\omega \rightarrow \mathcal{Q}$ which is Wadge equivalent to \mathcal{A} .

Proof. Let $V = \{\tau_0, \tau_1, \dots\} \subseteq \omega^{<\omega}$ be a generator of \mathcal{V} . That is, V is so that $\{[\tau] : \tau \in V\}$ is a partition on \mathcal{V} in clopen sets. Then the bijection $n \cap X \mapsto \tau_n \cap X: \omega^\omega \rightarrow \mathcal{V}$ induces a function $\widehat{\mathcal{A}}: \omega^\omega \rightarrow \mathcal{Q}$ Wadge equivalent to \mathcal{A} \square

From now on, whenever we encounter a \mathcal{Q} -valued function whose domain is an either open or closed subsets of ω^ω , we identify it with the corresponding function of domain ω^ω .

Definition 3.7. For each $T \in \bigcup_n \text{Tree}^n(\mathcal{Q})$, we inductively define the class Σ_T of \mathcal{Q} -valued functions on ω^ω as follows:

- (1) Σ_q consists only of the constant function $X \mapsto q: \omega^\omega \rightarrow \mathcal{Q}$.
- (2) If T is of the form $\sqcup_i S_i$, then $\mathcal{A} \in \Sigma_T$ if and only if there are clopen partition $(\mathcal{C}_i)_{i \in \omega}$ of ω^ω such that $\mathcal{A} \upharpoonright \mathcal{C}_i \in \Sigma_{S_i}$ for each $i \in \omega$.
- (3) $\mathcal{A} \in \Sigma_{T \rightarrow S}$ if and only if there is an open set $\mathcal{V} \subseteq \omega^\omega$ such that $\mathcal{A} \upharpoonright (\omega^\omega \setminus \mathcal{V})$ is in Σ_T and $\mathcal{A} \upharpoonright \mathcal{V}$ is in Σ_S .
- (4) $\mathcal{A} \in \Sigma_{\langle T \rangle}$ if and only if there is a Σ_2^0 -measurable function $\mathcal{D}: \omega^\omega \rightarrow \omega^\omega$ and a Σ_T -function $\mathcal{B}: \omega^\omega \rightarrow \mathcal{Q}$ such that $\mathcal{A} = \mathcal{B} \circ \mathcal{D}$.

We say that a function $\mathcal{A}: \omega^\omega \rightarrow \mathcal{Q}$ is Σ_T -complete if $\mathcal{A} \in \Sigma_T$ and every Σ_T -function \mathcal{B} is Wadge reducible to \mathcal{A} .

Observation 3.8. Let $T \in \text{Tree}^n(\mathcal{Q})$ be a forest, and $\theta: \omega^\omega \rightarrow \omega^\omega$ be a continuous function. If $\mathcal{A}: \omega^\omega \rightarrow \mathcal{Q}$ is in Σ_T , then so is $\mathcal{A} \circ \theta$. This can be easily shown by induction on T as a term.

To prove Proposition 1.6 for functions of finite Borel rank, we check measurability of Σ_T -functions. We denote by Δ_ξ^0 the set of all Δ_ξ^0 -measurable functions.

Lemma 3.9. *Let S, T be terms and ξ be a countable ordinal.*

- (1) $\Sigma_T, \Sigma_S \subseteq \Delta_{2+\xi}^0$ implies $\Sigma_{T \rightarrow S} \subseteq \Delta_{2+\xi}^0$.
- (2) $\Sigma_T \subseteq \Delta_{1+\xi}^0$ implies $\Sigma_{\langle T \rangle} \subseteq \Delta_{2+\xi}^0$.

Proof. To see (1), let \mathcal{A} be a $\Sigma_{T \rightarrow S}$ -function. Then, there is an open set \mathcal{V} such that $\mathcal{A} \upharpoonright (\omega^\omega \setminus \mathcal{V})$ can be extended to a total Σ_T -function \mathcal{A}_0 , and $\mathcal{A} \upharpoonright \mathcal{V}$ can be extended to a total Σ_S -function \mathcal{A}_1 as in Observations 3.5 and 3.6. Thus, for any $q \in \mathcal{Q}$, we have

$$\begin{aligned} \mathcal{A}^{-1}[q] &= (\mathcal{A}_0^{-1}[q] \setminus \mathcal{V}) \cup (\mathcal{A}_1^{-1}[q] \cap \mathcal{V}) \\ \mathcal{A}^{-1}[q]^c &= (\mathcal{A}_0^{-1}[q]^c \setminus \mathcal{V}) \cup (\mathcal{A}_1^{-1}[q]^c \cap \mathcal{V}) \end{aligned}$$

This gives $\Sigma_{2+\xi}^0$ -definitions of $\mathcal{A}^{-1}[q]$ and $\mathcal{A}^{-1}[q]^c$ since \mathcal{A}_0 and \mathcal{A}_1 are $\Delta_{2+\xi}^0$ -measurable, and \mathcal{V} is open. Consequently, \mathcal{A} is $\Delta_{2+\xi}^0$ -measurable.

For (2), note that the composition $\mathcal{B} \circ \mathcal{D}$ of a $\Delta_{1+\xi}^0$ -measurable function \mathcal{B} and a $\Sigma_{1+\eta}^0$ -measurable function \mathcal{D} is always $\Delta_{1+\eta+\xi}^0$ -measurable. This is because, one can see that if $\mathcal{S} \subseteq \omega^\omega$ is $\Sigma_{1+\zeta}^0$ then $\mathcal{D}^{-1}[\mathcal{S}]$ is $\Sigma_{1+\zeta+\xi}^0$ by induction on Borel rank. Now, every $\mathcal{A} \in \Sigma_{\langle T \rangle}$ is given by the composition of the Σ_T -function \mathcal{B} and the Σ_2^0 -measurable function \mathcal{D} . This implies that $\mathcal{B} \circ \mathcal{D}$ is $\Delta_{2+\xi}^0$ -measurable since \mathcal{B} is $\Delta_{1+\xi}^0$ -measurable by our assumption. \square

In particular, if $T \in {}^{\cup}\text{Tree}^n(\mathcal{Q})$, then every Σ_T -function is Δ_{n+1}^0 -measurable. As a consequence, this verifies Proposition 1.6 for functions of finite Borel rank.

3.3. Σ_T -complete functions. For $\mathcal{Q} = 2$, Duparc [Dup01] defined a complete sets $\Omega_\nu \subseteq \omega^{<\omega}$ for the different levels of the 2-Wadge hierarchy. For $\mathcal{Q} = k \in \omega$, Selivanov [Sel07] defined complete Δ_2^0 functions $\mu_T: \omega^\omega \rightarrow k$ for each forest $T \in {}^{\cup}\text{Tree}(k)$ based on the similar ideas. In this section we extend Duparc and Selivanov's definition to all bqos \mathcal{Q} and nested forests $T \in {}^{\cup}\text{Tree}^n(\mathcal{Q})$, and later on throughout the Borel hierarchy.

The complete functions we will define are conciliatory; see Section 2.5.

3.3.1. Difference hierarchy and mind-change operation. The Hausdorff-Kuratowski difference hierarchy (and the Ershov hierarchy) can be understood using the notion of a *mind-change*. That is, a subset \mathcal{A} of ω^ω is in the n^{th} -level of the difference hierarchy if and only if the characteristic function of \mathcal{A} is approximated by a continuous function with n mind-changes.

We want to define an operation $\mathcal{A} \rightarrow \mathcal{B}$ for $\mathcal{A}, \mathcal{B}: \hat{\omega}^\omega \rightarrow \mathcal{Q}$ which represents a function that could act as \mathcal{A} , but at any time could change its mind and act as \mathcal{B} . To make it easier to describe such a process, we introduce notations representing this kind of approximation procedure. Suppose we first want to output a sequence and after ℓ steps, after having defined a sequence $Y \in \omega^\ell$, we change our mind and we want to output a new sequence $Z \in \omega^{<\omega}$. We will encode this by the following real:

$$Y \rightarrow Z := \langle 2Y(0), 2Y(1), \dots, 2Y(\ell - 1), 2Z(0) + 1, Z(1), Z(2), \dots \rangle.$$

We want to define this procedure on $\hat{\omega}^\omega$ as follows, where we will require that the first entry of the second sequence Z is not **pass**, that is, $Z \in \omega \times \hat{\omega}^\omega$.

Notation 3.10. Given $Y \in \hat{\omega}^\ell$ of length $\ell \in \omega \cup \{\omega\}$ and $Z \in \omega \times \hat{\omega}^\omega$, we define $Y \rightarrow Z$ as follows:

$$Y \rightarrow Z(n) = \begin{cases} 2Y(n), & \text{if } n < \ell \text{ and } Y(n) \neq \text{pass}, \\ \text{pass}, & \text{if } n < \ell \text{ and } Y(n) = \text{pass}, \\ 2Z(0) + 1, & \text{if } n = \ell, \\ Z(n - \ell + 1), & \text{if } n > \ell. \end{cases}$$

Observation 3.11. The map $(Y, Z) \mapsto Y \rightarrow Z$ admits a conciliatory inverse. Indeed, there uniquely exist conciliatory continuous functions π_0 and π_1 such that for any $X \in \hat{\omega}^\omega$,

$$X = \pi_0(X) \rightarrow \pi_1(X).$$

We hereafter fix such functions π_0, π_1 . Note that $\pi_1(X)^{\text{p}}$ is nonempty if and only if X has changed his mind at some point. In other words, if $\pi_1(X)^{\text{p}}$ is empty, then the sequence given by X is $\pi_0(X)$, and if $\pi_1(X)^{\text{p}}$ is nonempty, X has already deleted the former sequence $\pi_0(X)$, and now proposes $\pi_1(X)$.

Notation 3.12 (see also [Dup01, Definition 6] for $\mathcal{Q} = 2$). Let \mathcal{A} and \mathcal{B} be functions whose domains are subsets of $\hat{\omega}^\omega$. We define a function $\mathcal{A} \rightarrow \mathcal{B}: \hat{\omega}^\omega \rightarrow \mathcal{Q}$ as follows:

$$(\mathcal{A} \rightarrow \mathcal{B})(X) = \begin{cases} \mathcal{A}(\pi_0(X)) & \text{if } \pi_1(X) \text{ is empty,} \\ \mathcal{B}(\pi_1(X)) & \text{otherwise.} \end{cases}$$

It is easy to check that the operation \rightarrow can also be seen as an operation on the Wadge degrees:

Observation 3.13. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ be functions whose domains are subsets of $\hat{\omega}^\omega$. If $\mathcal{A} \leq_w \mathcal{C}$ and $\mathcal{B} \leq_w \mathcal{D}$ then $\mathcal{A} \rightarrow \mathcal{B} \leq_w \mathcal{C} \rightarrow \mathcal{D}$.

3.3.2. Σ_T -complete functions. We now inductively assign a function Ω_T to each forest $T \in {}^\omega\text{Tree}^n(\mathcal{Q})$, and we will show that Ω_T is Σ_T -complete. Recall that T is a tree if and only if the outermost function symbol is not the disjoint union \sqcup , and thus, Ω_T is defined by the construction in (1), (3), or (4) of Definition 3.7. If T is a tree, Ω_T will be a conciliatory function from $\hat{\omega}^\omega$ to \mathcal{Q} . If T is not a tree, Ω_T will be a function from $\omega \times \hat{\omega}^\omega$ to \mathcal{Q} , which is almost conciliatory, that is, $X^p = Y^p$ implies $\Omega_T(n \hat{\cap} X) = \Omega_T(n \hat{\cap} Y)$ for any $n \in \omega$. (Think of almost conciliatory functions as having domain $\omega^{\leq \omega} \setminus \{\emptyset\}$.)

Definition 3.14. Let $T \in {}^\omega\text{Tree}^n(\mathcal{Q})$. We inductively define Ω_T as follows:

- (1) Suppose that T is of the form $\langle q \rangle$ for some $q \in \mathcal{Q}$. Then define $\Omega_{\langle q \rangle}: \hat{\omega}^\omega \rightarrow \mathcal{Q}$ as the constant function $X \mapsto q$, that is,

$$(\forall X \in \hat{\omega}^\omega) \quad \Omega_{\langle q \rangle}(X) = q.$$

We sometimes abbreviate $\Omega_{\langle q \rangle}$ to Ω_q .

- (2) Suppose that T is of the form $\bigsqcup_n T_n$, where each T_n is a tree. Then define $\Omega_T: \omega \times \hat{\omega}^\omega \rightarrow \mathcal{Q}$ as follows:

$$\Omega_{\bigsqcup_n T_n}(X) = \bigoplus_{n \in \omega} \Omega_{T_n}(X)$$

- (3) Suppose that T is of the form $\langle S \rangle \rightarrow F$, where S is the label on the root of T (thus $S \in \text{Tree}^{n-1}(\mathcal{Q})$), and F is a forest. Then,

$$\Omega_{\langle S \rangle \rightarrow F} = \Omega_{\langle S \rangle} \rightarrow \Omega_F.$$

- (4) Suppose that T is of the form $\langle S \rangle$ for some tree S . Then define $\Omega_T: \hat{\omega}^\omega \rightarrow \mathcal{Q}$ as follows:

$$\Omega_{\langle S \rangle} = \Omega_S \circ \mathcal{U},$$

where \mathcal{U} is a fixed Σ_2^0 -universal initializable conciliatory function as in Proposition 2.15.

Observation 3.15. If $T \in \text{Tree}^n(\mathcal{Q})$, Ω_T is conciliatory, and if T is a \sqcup -type term, Ω_T is almost conciliatory. The proof is an easy induction on the term T .

Observation 3.16. For every $T \in {}^\omega\text{Tree}^n(\mathcal{Q})$, the function Ω_T is in Σ_T .

Proof. This is obvious if T is constructed from (1), (2) or (4). Thus, it suffices to show that $\Omega_{T \rightarrow S} \in \Sigma_{T \rightarrow S}$. Recall that every $X \in \hat{\omega}^\omega$ is of the form $\pi_0(X) \rightarrow \pi_1(X)$ by Observation 3.11. Let \mathcal{V} be an open set consisting of all sequences X such that $\pi_1(X)$ is nonempty (which indicates that X has changed his mind at some point). It is clear that $\Omega_{T \rightarrow S} \upharpoonright (\omega^\omega \setminus \mathcal{V}) = \Omega_T \circ \pi_0 \upharpoonright (\omega^\omega \setminus \mathcal{V})$, and $\Omega_{T \rightarrow S} \upharpoonright \mathcal{V} = \Omega_S \circ \pi_1 \upharpoonright \mathcal{V}$. By induction hypothesis and by Observation 3.8, the former function is in Σ_T and the latter function is in Σ_S . This concludes that $\Omega_{T \rightarrow S} \in \Sigma_{T \rightarrow S}$. \square

Lemma 3.17. For every $T \in {}^\omega\text{Tree}^n(\mathcal{Q})$, the function Ω_T is Σ_T -complete.

Proof. First assume that T is of the form $\sqcup_i T_i$, and let \mathcal{A} be a Σ_T -function. Then there is a clopen partition $(\mathcal{C}_i)_{i \in \omega}$ such that $\mathcal{A} \upharpoonright \mathcal{C}_i$ is in Σ_{T_i} for any $i \in \omega$. By induction hypothesis, we have a continuous function θ_i witnessing $\mathcal{A} \upharpoonright \mathcal{C}_i \leq_w \Omega_{S_i}$ for every $i \in \omega$. Thus, to see $\mathcal{A} \leq_w \Omega_T$, given $X \in \omega^\omega$ one can computably find $i_X \in \omega$ such that $X \in \mathcal{C}_{i_X}$, and then we have $\mathcal{A}(X) \leq_Q \Omega_T(i_X \hat{\ } \theta_{i_X}(X))$.

Next, let \mathcal{A} be a function in $\Sigma_{T \rightarrow S}$. Then, there is an open set \mathcal{V} such that $\mathcal{A} \upharpoonright \mathcal{V}$ is in Σ_S and $\mathcal{A} \upharpoonright (\omega^\omega \setminus \mathcal{V})$ is in Σ_T . Recall that the former condition means that there is a generator V of \mathcal{V} such that $\mathcal{A} \upharpoonright [\sigma] \in \Sigma_S$ for any $\sigma \in V$. By induction hypothesis, we have continuous functions θ witnessing $\mathcal{A} \upharpoonright (\omega^\omega \setminus \mathcal{V}) \leq_w \Omega_T$, and γ_σ witnessing $\mathcal{A} \upharpoonright [\sigma] \leq_w \Omega_S$ for every $\sigma \in V$. To see $\mathcal{A} \leq_w \Omega_{T \rightarrow S}$, given $X \in \omega^\omega$, we first follow θ until we see $X \upharpoonright s \in V$ for some s (if ever). If we see $X \upharpoonright s \in V$, then we change our mind (that is, delete the former sequence $\theta(X \upharpoonright s - 1)$), and now follow $\gamma_{X \upharpoonright s}$. Recall that, in the latter case, this process is coded as $\theta(X \upharpoonright s - 1) \rightarrow \gamma_{X \upharpoonright s}(X)$. This witnesses $\mathcal{A} \leq_w \Omega_{T \rightarrow S}$.

Let \mathcal{A} be a $\Sigma_{\langle T \rangle}$ -function. Then, there are Σ_2^0 -function \mathcal{D} and a Σ_T -function \mathcal{B} such that $\mathcal{A} = \mathcal{B} \circ \mathcal{D}$. By induction hypothesis, we have a continuous function $\theta: \omega^\omega \rightarrow \hat{\omega}^\omega$ witnessing $\mathcal{B} \leq_w \Omega_T$. Thus, $\mathcal{A}(X) \leq_Q \Omega_T \circ \theta \circ \mathcal{D}(X)$. Since $\theta \circ \mathcal{D}$ is Σ_2^0 , and \mathcal{U} is universal, there is a continuous function $\Psi: \omega^\omega \rightarrow \hat{\omega}^\omega$ such that $\theta \circ \mathcal{D} \equiv_p \mathcal{U} \circ \Psi(X)$. Then, since Ω_T is conciliatory by Observation 3.15,

$$\mathcal{A}(X) = \mathcal{B}(\mathcal{D}(X)) \leq_Q \Omega_T(\theta \circ \mathcal{D}(X)) = \Omega_T(\mathcal{U} \circ \Psi(X)) = \Omega_{\langle T \rangle}(\Psi(X)).$$

Consequently, the continuous function Ψ witnesses $\mathcal{A} \leq_w \Omega_{\langle T \rangle}$. \square

Lemma 3.18. *For $S, T \in {}^\omega \text{Tree}^n(\mathcal{Q})$, if $S \trianglelefteq T$, then $\Omega_S \leq_w \Omega_T$.*

Proof. We show the assertion by induction on the definition of \trianglelefteq in 3.1. First, it is clear that $\Omega_p \leq_w \Omega_q$ if and only if $p \trianglelefteq q$. Next suppose that we have $\langle U \rangle \trianglelefteq \langle V \rangle$, which is equivalent to $U \trianglelefteq V$ by definition. By the induction hypothesis, we have a \mathcal{Q} -Wadge reduction $\theta: \hat{\omega}^\omega \rightarrow \hat{\omega}^\omega$ witnessing $\Omega_U \leq_w \Omega_V$. By the Σ_2^0 universality of \mathcal{U} , there exists a continuous $\Phi: \hat{\omega}^\omega \rightarrow \hat{\omega}^\omega$ such that $\theta \circ \mathcal{U} \equiv_p \mathcal{U} \circ \Phi$. Since Ω_V is conciliatory by Observation 3.15, we have that for every $X \in \hat{\omega}$,

$$\Omega_{\langle U \rangle}(X) = \Omega_U(\mathcal{U}(X)) \leq_Q \Omega_V(\theta(\mathcal{U}(X))) = \Omega_V(\mathcal{U}(\Phi(X))) = \Omega_{\langle V \rangle}(\Phi(X)).$$

Thus Φ witnesses that $\Omega_{\langle U \rangle} \leq_w \Omega_{\langle V \rangle}$.

Now, consider $S = \langle U \rangle \rightarrow \bigsqcup_i S_i$ and $T = \langle V \rangle \rightarrow \bigsqcup_j T_j$. First consider the case that $\langle U \rangle \trianglelefteq \langle V \rangle$. In this case, by the definition of \trianglelefteq under the assumption that $\langle U \rangle \trianglelefteq \langle V \rangle$, we have that $S_i \trianglelefteq T$ for any $i \in \omega$. By the induction hypothesis, $\Omega_{S_i} \leq_w \Omega_T$ via a continuous function θ_i for any i , and $\Omega_{\langle U \rangle} \leq_w \Omega_{\langle V \rangle}$ via τ . The idea to define the reduction is as follows: Given $X \in \omega^\omega$, while X does not change its mind, use the reduction $\tau: \Omega_{\langle U \rangle} \leq_w \Omega_{\langle V \rangle}$. If X changes its mind and moves into some S_k , then we need to use the reduction $\theta_k: \Omega_{S_k} \leq_w \Omega_T$. Since we have already taken some steps within the domain of $\langle V \rangle$, we need to use the initializability of $\Omega_{\langle V \rangle}$ to start over with the reduction to Ω_T .

More formally: By the initializability of $\Omega_{\langle V \rangle}$ (see Definition 2.14), for any σ , there is a continuous function η_σ witnessing $\Omega_{\langle V \rangle} \leq_w \Omega_{\langle V \rangle} \upharpoonright [\sigma]$. We can then extend this map to a Wadge reduction $\Omega_T \leq_w \Omega_T \upharpoonright [\sigma \rightarrow \emptyset]$, where $\sigma \rightarrow \emptyset$ represents the string in the domain of Ω_T for which we haven't changed our mind yet, and we are still computing $\Omega_{\langle V \rangle}$. For each σ , let $\hat{\eta}_\sigma$ be such that $\Omega_T(Y) \leq_Q \Omega_T(\sigma \rightarrow \hat{\eta}_\sigma(Y))$ witnessing such reduction. Given X , it is of

the form $\pi_0(X) \rightarrow \pi_1(X)$ by Observation 3.11. If $\pi_1(X)$ is empty (that is, X never changes his mind), then note that $\Omega_S(X) = \Omega_{\langle U \rangle}(\pi_0(X))$. In this case, return $\tau(\pi_0(X)) \rightarrow \emptyset$. Let X be a sequence such that $\pi_1(X)$ is nonempty (that is, X has changed his mind at some point). Put $k = \pi_1(X)(0)$. Note that $\Omega_S(X) = \Omega_{\sqcup_i S_i}(\pi_1(X)) = \Omega_{S_k}(\pi_1(X)^-)$, where $\pi_1(X) = k \frown \pi_1(X)^-$. Then change our guess to $\hat{\eta}_{\tau(\pi_0(X))} \circ \theta_k(\pi_1(X)^-)$. We thus get

$$\begin{aligned} \Omega_S(X) &= \Omega_{S_k}(\pi_1(X)^-) \leq_{\mathcal{Q}} \\ &\quad \Omega_T(\theta_k(\pi_1(X)^-)) \leq_{\mathcal{Q}} \Omega_T(\tau(\pi_0(X)) \rightarrow \hat{\eta}_{\tau(\pi_0(X))}(\theta_k(\pi_1(X)^-))). \end{aligned}$$

Putting these two cases together, we have a Wadge reduction from Ω_S to Ω_T .

We now assume that $\langle U \rangle \not\leq \langle V \rangle$. In this case, by definition, $S \leq T$ if and only if $S \leq T_j$ for some $j \in \omega$. By induction hypothesis $\Omega_S \leq_w \Omega_{T_j}$ for some j . Clearly, this condition implies $\Omega_S \leq_w \Omega_T$. \square

We will prove the reverse direction, that $\Omega_T \leq_w \Omega_S \Rightarrow T \leq S$, in Subsection 4.3. We need to wait until then, because we need the jump inversion operator for the proof.

3.4. Infinite Borel ranks. We now extend our ideas from Section 3.1 to infinite Borel ranks. The reader who is only interested in Borel functions of finite rank can skip Section 3.4.

3.4.1. Language and terms (infinite Borel rank). Given a set \mathcal{Q} , let $\mathcal{L}(\mathcal{Q})$ be the language consisting of constant symbols q for each $q \in \mathcal{Q}$, an ω -ary function symbol \sqcup , a two-ary function symbol \rightarrow , and a unary function symbol $\langle \cdot \rangle^{\omega^\alpha}$ for every countable ordinal $\alpha < \omega_1$.

We define ${}^{\sqcup}\text{Tree}^{\omega^\alpha}(\mathcal{Q})$ as the set of all $\mathcal{L}(\mathcal{Q})$ -terms of rank below ω^α as follows:

Definition 3.19 (Terms of Rank below ω^α). We inductively define the sets $\text{Tree}^{\omega^\alpha}(\mathcal{Q})$ and ${}^{\sqcup}\text{Tree}^{\omega^\alpha}(\mathcal{Q})$ consisting of $\mathcal{L}(\mathcal{Q})$ -terms as follows:

- (1) If $\beta \leq \alpha$ and $T \in \text{Tree}^{\omega^\beta}(\mathcal{Q})$ then $T \in \text{Tree}^{\omega^\alpha}(\mathcal{Q})$ and $T \in {}^{\sqcup}\text{Tree}^{\omega^\alpha}(\mathcal{Q})$.
- (2) If $q \in \mathcal{Q}$ then $q \in \text{Tree}^1(\mathcal{Q})$ (where note that $\omega^0 = 1$), and call it $\langle \cdot \rangle$ -type.
- (3) If $\beta < \alpha$ and $T \in \text{Tree}^{\omega^\beta}(\mathcal{Q})$ then $\langle T \rangle^{\omega^\alpha} \in \text{Tree}^{\omega^\alpha}(\mathcal{Q})$, and call it $\langle \cdot \rangle$ -type.
- (4) If $T_i \in \text{Tree}^{\omega^\alpha}(\mathcal{Q})$ for every $i \in \omega$, then $\sqcup_i T_i \in {}^{\sqcup}\text{Tree}^{\omega^\alpha}(\mathcal{Q})$, and call it \sqcup -type.
- (5) For any $\langle \cdot \rangle$ -type term $T \in \text{Tree}^{\omega^\alpha}(\mathcal{Q})$ and \sqcup -type term $S \in {}^{\sqcup}\text{Tree}^{\omega^\alpha}(\mathcal{Q})$, the term $T \rightarrow S$ is in $\text{Tree}^{\omega^\alpha}(\mathcal{Q})$.

We define $\text{Tree}^{\omega^1}(\mathcal{Q}) = \bigcup_{\alpha} \text{Tree}^{\omega^\alpha}(\mathcal{Q})$ and ${}^{\sqcup}\text{Tree}^{\omega^1}(\mathcal{Q}) = \bigcup_{\alpha} {}^{\sqcup}\text{Tree}^{\omega^\alpha}(\mathcal{Q})$.

For instance, ${}^{\sqcup}\text{Tree}^{\omega}(\mathcal{Q})$ can be viewed as the closure of \mathcal{Q} of the operations $\langle \cdot \rangle$, \sqcup , and \rightarrow . Notice that this is far larger than $\bigcup_n {}^{\sqcup}\text{Tree}^n(\mathcal{Q})$ (even with respect to \leq) because a term in ${}^{\sqcup}\text{Tree}^{\omega}(\mathcal{Q})$ can contain unbounded applications of the labeling function $\langle \cdot \rangle$, e.g., $\sqcup_n \langle 0 \rightarrow 1 \rangle^n$. This reflects the fact that the pointclass Δ_{ω}^0 is strictly larger than $\bigcup_{n < \omega} \Delta_n^0$. On the other hand, the function $\langle \cdot \rangle^{\omega}$ would take out of ${}^{\sqcup}\text{Tree}^{\omega}(\mathcal{Q})$, reflecting that the conciliatory Σ_{ω}^0 universal function is not Δ_{ω}^0 .

For a set \mathcal{R} of $\mathcal{L}(\mathcal{Q})$ -terms and an ordinal α , we define:

$$\langle \mathcal{R} \rangle^{\omega^\alpha} = \{ \langle T \rangle^{\omega^\alpha} : T \in \mathcal{R} \}.$$

We then inductively define the set $\text{Tree}^{\omega^\alpha \cdot k}(\mathcal{Q}) \subseteq \text{Tree}^{\omega^{\alpha+1}}(\mathcal{Q})$ of $\mathcal{L}(\mathcal{Q})$ -terms (of rank $< \omega^\alpha \cdot k$) as follows:

$$\text{Tree}^{\omega^{\alpha \cdot 1}}(\mathcal{Q}) = \text{Tree}^{\omega^\alpha}(\mathcal{Q}), \quad \text{Tree}^{\omega^\alpha \cdot k+1}(\mathcal{Q}) = \text{Tree}^{\omega^\alpha}(\langle \text{Tree}^{\omega^\alpha \cdot k}(\mathcal{Q}) \rangle^{\omega^\alpha}).$$

In general, recall that every countable ordinal ξ can be written as $\omega^\alpha + \beta$ for some $\beta < \omega^{\alpha+1}$. Then we define $\text{Tree}^\xi(\mathcal{Q})$ and $\sqcup \text{Tree}^\xi(\mathcal{Q})$ as follows:

$$\text{Tree}^{\omega^\alpha + \beta}(\mathcal{Q}) = \text{Tree}^{\omega^\alpha}(\langle \text{Tree}^\beta(\mathcal{Q}) \rangle^{\omega^\alpha}), \quad \sqcup \text{Tree}^{\omega^\alpha + \beta}(\mathcal{Q}) = \sqcup \text{Tree}^{\omega^\alpha}(\langle \text{Tree}^\beta(\mathcal{Q}) \rangle^{\omega^\alpha}).$$

3.4.2. Quasi-ordering nested trees (infinite Borel rank). In this section, we extend the domain of the quasi-order \preceq to $\sqcup \text{Tree}^{\omega^1}(\mathcal{Q})$. As in Section 3.1.2, we first inductively define a quasi-order \preceq on $\text{Tree}^{\omega^1}(\mathcal{Q})$, and then, \preceq is uniquely extended to a quasi-order on $\sqcup \text{Tree}^{\omega^1}(\mathcal{Q})$ by interpreting \sqcup as a countable supremum operation. Recall the convention from Section 3.1.2 that we always identify $p \in \mathcal{Q}$ with $\langle p \rangle$, and $\langle T \rangle$ with $\langle T \rangle \rightarrow \sqcup_i \mathbf{O}$, where \mathbf{O} is the empty forest, viewed as an imaginary least element w.r.t. the quasi-order \preceq , that is, $\mathbf{O} \preceq T$ for any $T \in \text{Tree}^{\omega^1}(\mathcal{Q})$.

Definition 3.20. We inductively define a quasi-order \preceq on $\sqcup_n \text{Tree}^{\omega^1}(\mathcal{Q})$ as follows, where the symbols p and q range over \mathcal{Q} , and U, V, S , and T range over range over $\text{Tree}^{\omega^1}(\mathcal{Q})$:

$$p \preceq q \iff p \leq_{\mathcal{Q}} q,$$

$$\langle U \rangle^{\omega^\alpha} \preceq \langle V \rangle^{\omega^\beta} \iff \begin{cases} U \preceq V & \text{if } \alpha = \beta, \\ \langle U \rangle^{\omega^\alpha} \preceq V & \text{if } \alpha > \beta, \\ U \preceq \langle V \rangle^{\omega^\beta} & \text{if } \alpha < \beta. \end{cases}$$

and if S and T are of the form $\langle U \rangle^{\omega^\alpha} \rightarrow \sqcup_i S_i$ and $\langle V \rangle^{\omega^\beta} \rightarrow \sqcup_j T_j$, respectively, then

$$S \preceq T \iff \begin{cases} \text{either } \langle U \rangle^{\omega^\alpha} \preceq \langle V \rangle^{\omega^\beta} & \text{and } (\forall i) S_i \preceq T, \\ \text{or } \langle U \rangle^{\omega^\alpha} \not\preceq \langle V \rangle^{\omega^\beta} & \text{and } (\exists j) S \preceq T_j. \end{cases}$$

We now assign a class Σ_T to each forest $T \in \sqcup \text{Tree}^{\omega^1}(\mathcal{Q})$ as follows:

Definition 3.21. For $\alpha > 0$, $\mathcal{A} \in \Sigma_{\langle T \rangle^{\omega^\alpha}}$ if and only if there is a $\Sigma_{1+\omega^\alpha}^0$ -measurable function $\mathcal{D}: \omega^\omega \rightarrow \omega^\omega$ and a Σ_T -function $\mathcal{B}: \omega^\omega \rightarrow \mathcal{Q}$ such that $\mathcal{A} = \mathcal{B} \circ \mathcal{D}$. The other cases are as in Definition 3.7.

We check measurability of Σ_T -functions.

Lemma 3.22. *Let T be an $\mathcal{L}(\mathcal{Q})$ -term and ξ be a countable ordinal. Then, $\Sigma_T \subseteq \Delta_{1+\xi}^0$ implies $\Sigma_{\langle T \rangle^{\omega^\alpha}} \subseteq \Delta_{1+\omega^\alpha+\xi}^0$.*

Proof. One can use a similar argument as in Lemma 3.9 (2). □

By combining Lemmas 3.9 and 3.22, we obtain the direction from (2) to (1) in Proposition 1.8 for infinite Borel rank, that is, that if $T \in \sqcup \text{Tree}^\xi(\mathcal{Q})$, then $\Sigma_T \subseteq \Delta_{1+\xi}^0$.

3.4.3. Σ_T -complete functions (infinite Borel rank). Now we introduce a Σ_T -complete function Ω_T for each forest $T \in {}^\omega\text{Tree}^{\omega_1}(\mathcal{Q})$. To achieve this, we need a universal function at transfinite Borel ranks. Again, recall that every $\Sigma_{1+\xi}^0$ -measurable function is coded by a real (for instance, we can use Fact 6.2).

Definition 3.23. We say that $\mathcal{U}_\xi: \hat{\omega}^\omega \rightarrow \hat{\omega}^\omega$ is Σ_ξ^0 -universal if it is Σ_ξ^0 -measurable, and for every conciliatory Σ_ξ^0 -measurable function $\Psi: \hat{\omega}^\omega \rightarrow \hat{\omega}^\omega$, there is a continuous function θ such that $\Psi = \mathcal{U}_\xi \circ \theta$.

To show the existence of a Σ_T -complete function, we need to extend Proposition 2.15 to infinite Borel ranks.

Proposition 3.24. For any countable ordinal α , there is a $\Sigma_{1+\omega^\alpha}^0$ -universal initializable conciliatory function.

We prove this proposition in Section 6.

We now introduce Ω_T for each term $T \in {}^\omega\text{Tree}^{\omega_1}(\mathcal{Q})$. It suffices to describe how to define $\Omega_{\langle T \rangle \omega^\alpha}$, as the rest is as in Definition 3.14.

Definition 3.25 (Complete Function). Let α be a countable ordinal, and let T be a tree in $\text{Tree}^{\omega_1}(\mathcal{Q})$. Then we define the conciliatory function $\Omega_{\langle T \rangle \omega^\alpha}: \omega^{\leq \omega} \rightarrow \mathcal{Q}$ as follows:

$$\Omega_{\langle T \rangle \omega^\alpha} = \Omega_T \circ \mathcal{U}_{\omega^\alpha},$$

where $\mathcal{U}_{\omega^\alpha}$ is a fixed $\Sigma_{1+\omega^\alpha}^0$ -universal initializable conciliatory function as in Proposition 3.24.

It is not hard to prove the transfinite versions of 3.15, 3.16, 3.17, and 3.18 namely that Ω_T is Σ_T -complete, and that if $S \trianglelefteq T$, then $\Omega_S \leq_w \Omega_T$.

4. THE JUMP OPERATOR AND ITS INVERSION

The goal of this section is to define an inverse of the operation $\mathcal{B} \mapsto \mathcal{B} \circ \mathcal{U}$. (Recall that $\Omega_{\langle T \rangle}$ was defined as $\Omega_T \circ \mathcal{U}$.) This operation will be denoted by \mathcal{A}^\sharp , and we will prove that $(\mathcal{A} \circ \mathcal{U})^\sharp \equiv_w \mathcal{A}$ for every \mathcal{Q} -valued function \mathcal{A} . Furthermore, we will get that $(\mathcal{A}^\sharp) \circ \mathcal{U} \equiv_w \mathcal{A}$ if we also assume \mathcal{A} is initializable. The key technical notions are the *Turing jump operator via true stages* from computability theory and an uniform version of the Friedberg jump inversion theorem. The use of this jump operator is one of the aspects of our proof that makes it easier than Duparc's work for $\mathcal{Q} = 2$.

4.1. **Turing jump operator.** The Turing jump operator is one of the most basic notions in computability theory. For our proof, we need a version of this operator with nicer properties than the standard jump operator $X \mapsto X'$. The jump operator *via true stages* $\mathcal{J}: \omega^\omega \rightarrow \omega^\omega$ introduced by Marcone and Montalbán [MM11, Mon14] is exactly what we need. Marcone and Montalbán also defined its approximation on finite strings $J: \omega^{<\omega} \rightarrow \omega^{<\omega}$. Putting these together, what they defined was a Σ_2^0 conciliatory function $\mathcal{J}: \omega^{\leq \omega} \rightarrow \omega^{\leq \omega}$. The properties we need are the following:

- (1) (Σ_2^0 -universality from the right.) For every Σ_2^0 operator $G: \omega^\omega \rightarrow \hat{\omega}^\omega$, there is a computable $\theta: \mathcal{J}[\omega^\omega] \rightarrow \hat{\omega}^\omega$ such that $G \equiv_p \theta \circ \mathcal{J}$ (recall the definition of \equiv_p in Section 2.6). Furthermore, if G is Σ_2^0 relative to an oracle C , then we can still find θ computable so that, for every $X \in \omega^\omega$, $G(X)^p = \theta(\mathcal{J}(C \oplus X))^p$.

- (2) The image ω^ω under \mathcal{J} , namely $\mathcal{J}[\omega^\omega]$, is a closed subset of ω^ω . Furthermore, \mathcal{J} is one-to-one, and its inverse $\mathcal{J}^{-1}: \mathcal{J}[\omega^\omega] \rightarrow \omega^\omega$ is continuous.
- (3) (Denseness of forcing.) For every string $\gamma \in \omega^{<\omega}$, there is a string $\sigma \supseteq \gamma$, $\sigma \in \omega^{<\omega}$ which forces the jump in the following sense: We say that $\sigma \in \omega^{<\omega}$ *forces the jump*, if for every $\tau \supseteq \sigma$, $\mathcal{J}(\tau) \supseteq \mathcal{J}(\sigma)$.

For (1), note that if the range of G is contained in ω^ω , one can find $\theta: \mathcal{J}[\omega^\omega] \rightarrow \omega^\omega$ such that $G = \theta \circ \mathcal{J}$. The relativized version also holds. These properties are immediate from the definitions in Marcone and Montalbán [MM11, Mon14]. The last property (3) requires a minute of thought, though it is quite standard.

Notice that for the usual Turing jump operator $X \mapsto X'$, the image is not closed. Another advantage of \mathcal{J} is that its finite approximation can be easily iterated, allowing us to keep the denseness of forcing when we consider transfinite iterates of the jump.

We use \mathcal{J}^C to denote the operator $X \mapsto \mathcal{J}(C \oplus X)$. It satisfy the same properties of \mathcal{J} we mentioned above. Let $\mathcal{J}^{n,C}$ be the n -th iterate of the jump operator relative to C , that is, put $\mathcal{J}^{1,C} = \mathcal{J}^C$ and $\mathcal{J}^{n+1,C} = \mathcal{J}^C \circ \mathcal{J}^{n,C}$. We also use the symbol $\mathcal{J}^{-1,C}$ to denote $(\mathcal{J}^C)^{-1}$. The Σ_2^0 universality of \mathcal{J} can be iterated through the finite levels of the Borel hierarchy.

Fact 4.1. *For every Σ_{n+1}^0 -measurable function $\mathcal{A}: \omega^\omega \rightarrow \hat{\omega}^\omega$, there are $C \in \omega^\omega$ and a computable $\Phi: \omega^\omega \rightarrow \hat{\omega}^\omega$ such that $\mathcal{A} \equiv_p \Phi \circ \mathcal{J}^{n,C}$.*

Duparc [Dup01, Definition 25] introduced an operation $\not\sim$ on subsets of ω^ω . We extend Duparc's operation $\not\sim$ to \mathcal{Q} -valued functions, but our definition is quite different from Duparc's, which is rather hard to understand.

Definition 4.2. For any $\mathcal{A}: \omega^\omega \rightarrow \mathcal{Q}$ and any oracle $Z \in \omega^\omega$ we introduce the Z -jump inversion of \mathcal{A} , $\mathcal{A}^{\not\sim Z}: \mathcal{J}^Z[\omega^\omega] \rightarrow \mathcal{Q}$, as follows:

$$\mathcal{A}^{\not\sim Z}(Y) = \mathcal{A}(\mathcal{J}^{-1,Z}(Y)),$$

Note that the domain of $\mathcal{A}^{\not\sim Z}$ is the range of \mathcal{J}^Z , which is closed as mentioned above, and therefore one can think of $\mathcal{A}^{\not\sim Z}$ as a function on ω^ω by Observation 3.5, where we composed $\mathcal{A}^{\not\sim Z}$ with a continuous retraction $\eta: \omega^\omega \rightarrow \mathcal{J}^Z[\omega^\omega]$.

Remark 4.3. We also apply the operator $\not\sim$ to a conciliatory function $\mathcal{A}: \hat{\omega}^\omega \rightarrow \mathcal{Q}$. In this case, via a computable homeomorphism $I: \hat{\omega}^\omega \rightarrow \omega^\omega$, we identify \mathcal{A} with $\mathcal{A} \circ I^{-1}: \omega^\omega \rightarrow \mathcal{Q}$. Clearly, \mathcal{A} is Wadge equivalent to $\mathcal{A} \circ I^{-1}$. Then, if the domain of \mathcal{A} is $\hat{\omega}^\omega$, the actual definition of $\mathcal{A}^{\not\sim Z}$ is $\mathcal{A} \circ I^{-1} \circ \mathcal{J}^{-1,Z}$, and the domain of $\mathcal{A}^{\not\sim Z}$ is $\mathcal{J}^Z[\omega^\omega]$. Note that $\mathcal{A} = \mathcal{A}^{\not\sim Z} \circ \mathcal{J}^Z \circ I$. We should be careful that $\mathcal{A}^{\not\sim}$ is not necessarily conciliatory even if \mathcal{A} is conciliatory.

Observation 4.4. Let \mathcal{A} be any partial \mathcal{Q} -valued function, and $Y, Z \in \omega^\omega$ be oracles. Then

$$Y \leq_T Z \quad \Rightarrow \quad \mathcal{A}^{\not\sim Y} \geq_w \mathcal{A}^{\not\sim Z}.$$

Proof. The map $X \mapsto \mathcal{J}^Y(X)$ is Σ_2^0 relative to Y , and hence in particular Σ_2^0 relative to Z . Therefore, there is a computable function Φ so that $\Phi \circ \mathcal{J}^Z = \mathcal{J}^Y$. This witnesses that $\mathcal{A}^{\not\sim Z} \leq_w \mathcal{A}^{\not\sim Y}$ as follows: For $\mathcal{J}^Z(X) \in \mathcal{J}^Z[\omega^\omega]$,

$$\mathcal{A}^{\not\sim Z}(\mathcal{J}^Z(X)) = \mathcal{A}(X) = \mathcal{A}^{\not\sim Y}(\mathcal{J}^Y(X)) = \mathcal{A}^{\not\sim Y}(\Phi(\mathcal{J}^Z(X))). \quad \square$$

Recall that if \mathcal{Q} is bqo, then so are the Wadge degrees of \mathcal{Q} -valued functions (Theorem 2.3). In particular, there is no infinite decreasing chain of the Wadge degrees. Thus, Observation 4.4 implies that, for any sufficiently powerful oracle $Z \in \omega^\omega$, we have $\mathcal{A}^{\not\sim Z} \leq_w \mathcal{A}^{\not\sim Y}$ for any other oracle $Y \in \omega^\omega$.

Notation 4.5. If \mathcal{A} is a \mathcal{Q} -valued function for a bqo \mathcal{Q} , we hereafter use the notation $\mathcal{A}^{\not\sim}$ to denote a representative of the minimum one among Wadge degrees of $\{\mathcal{A}^{\not\sim Z} : Z \in \omega^\omega\}$, that is,

$$\mathcal{A}^{\not\sim} \equiv_w \mathcal{A}^{\not\sim Z} \text{ for some } Z \in \omega^\omega, \text{ and } \mathcal{A}^{\not\sim} \leq_w \mathcal{A}^{\not\sim Y} \text{ for all } Y \in \omega^\omega.$$

Here are some basic properties of the jump inversion operator. In particular, (2) of the next lemma shows that $\not\sim$ is well-defined on \mathcal{Q} -Wadge degrees.

Lemma 4.6. *For any $\mathcal{A}, \mathcal{B}: \omega^\omega \rightarrow \mathcal{Q}$, the following holds.*

- (1) *If \mathcal{A} is Σ_{n+1}^0 -measurable, then $\mathcal{A}^{\not\sim}$ is Σ_n^0 -measurable.*
- (2) *If $\mathcal{A} \leq_w \mathcal{B}$, then $\mathcal{A}^{\not\sim} \leq_w \mathcal{B}^{\not\sim}$.*
- (3) *If either $\mathcal{A}^{\not\sim}$ or $\mathcal{B}^{\not\sim}$ is non-self-dual, then $\mathcal{A}^{\not\sim} \leq_w \mathcal{B}^{\not\sim}$ implies $\mathcal{A} \leq_w \mathcal{B}$.*

Proof. (1) Since \mathcal{A} is Σ_{n+1}^0 , there is a Δ_0^0 formula φ in the language of second-order arithmetic and a $Z \in \omega^\omega$ such that

$$\mathcal{A}(X) = q \iff (\exists a_0)(\forall a_1) \dots (\check{Q}a_{n-1})(Qa_n) \varphi(q, a_0, \dots, a_n, X, Z),$$

where Q is the existential quantifier if n is even; and Q is the universal quantifier if n is odd, and \check{Q} is the other way around. Clearly, there is a Δ_0^0 formula ψ such that

$$\mathcal{A}(X) = q \iff (\exists a_0)(\forall a_1) \dots (\check{Q}a_{n-1}) \psi(q, a_0, \dots, a_{n-1}, J^Z(X)).$$

Consequently, $\mathcal{A}^{\not\sim Z}$ is Σ_n^0 . The same argument works for any $C \geq_T C$, and thus $\mathcal{A}^{\not\sim}$ is Σ_n^0 too.

(2) Assume that $\mathcal{A} \leq_w \mathcal{B}$ via a continuous function θ . Then θ is C -computable for some oracle C . Let D be such that $\mathcal{A}^{\not\sim} \equiv_w \mathcal{A}^{\not\sim D}$ and $\mathcal{B}^{\not\sim} \equiv_w \mathcal{B}^{\not\sim D}$. Then, by the universality of \mathcal{J}^D from the right, there is a continuous function θ' such that $\theta' \circ \mathcal{J}^D = \mathcal{J}^D \circ \theta$ since $D \geq_T C$. This θ' witnesses that $\mathcal{A}^{\not\sim} \leq_w \mathcal{B}^{\not\sim}$ since for any $Y \in \mathcal{J}^D[\omega^\omega]$, and $X = \mathcal{J}^{-1,D}(Y)$,

$$\begin{aligned} \mathcal{A}^{\not\sim D}(Y) &= \mathcal{A}(X) \leq_{\mathcal{Q}} \\ &\mathcal{B}(\theta(X)) = \mathcal{B}^{\not\sim D}(\mathcal{J}^D(\theta(X))) = \mathcal{B}^{\not\sim D}(\theta'(\mathcal{J}^D(X))) = \mathcal{B}^{\not\sim D}(\theta'(Y)). \end{aligned}$$

(3) If $\mathcal{A} \not\leq_w \mathcal{B}$, then Player I has a winning strategy in the game $G_w(\mathcal{A}, \mathcal{B})$, that is, there is a C -computable Lipschitz function θ such that $\mathcal{A}(\theta(X)) \not\leq_{\mathcal{Q}} \mathcal{B}(X)$. By the same argument as above, let D be such that $\mathcal{A}^{\not\sim} \equiv_w \mathcal{A}^{\not\sim D}$ and $\mathcal{B}^{\not\sim} \equiv_w \mathcal{B}^{\not\sim D}$, and then, there is a continuous function θ' such that $\mathcal{A}^{\not\sim D}(\theta'(Y)) \not\leq_{\mathcal{Q}} \mathcal{B}^{\not\sim D}(Y)$ for all $Y \in \mathcal{J}^D[\omega^\omega]$. However, if $\mathcal{A}^{\not\sim D} \leq_w \mathcal{B}^{\not\sim D}$ via a continuous function η , we would have the following:

$$\begin{aligned} \mathcal{A}^{\not\sim D}(Z) &\leq_{\mathcal{Q}} \mathcal{B}^{\not\sim D}(\eta(Z)) \not\leq_{\mathcal{Q}} \mathcal{A}^{\not\sim D}(\theta' \circ \eta(Z)), \\ \mathcal{B}^{\not\sim D}(Z) &\not\leq_{\mathcal{Q}} \mathcal{A}^{\not\sim D}(\theta'(Z)) \leq_{\mathcal{Q}} \mathcal{B}^{\not\sim D}(\eta \circ \theta'(Z)). \end{aligned}$$

Thus both $\mathcal{A}^{\not\sim D}$ and $\mathcal{B}^{\not\sim D}$ are self-dual, which contradicts our assumption since $\mathcal{A}^{\not\sim} \equiv_w \mathcal{A}^{\not\sim D}$ and $\mathcal{B}^{\not\sim} \equiv_w \mathcal{B}^{\not\sim D}$. \square

4.2. The operation $\not\sim$ inverts the jump. We now prove a key result which is that $\not\sim$ is the inverse of $\mathcal{B} \mapsto \mathcal{B} \circ \mathcal{U}$. This is somewhat an analogue of [Dup01, Propositions 29 and 30], which roughly says that the jump inversion operator $\not\sim$ bridges $\text{Tree}^n(\mathcal{Q})$ and $\text{Tree}^{n+1}(\mathcal{Q})$: namely that $\Omega_{\langle T \rangle}^{\not\sim} \equiv_w \Omega_T$.

Recall from Remark 4.3 that $\Omega_{\langle T \rangle}^{\not\sim} = \Omega_T \circ \mathcal{U} \circ I^{-1} \circ \mathcal{J}^{-1,C}$, where $I : \hat{\omega}^\omega \rightarrow \omega^\omega$ is a computable homeomorphism. The following lemma shows how \mathcal{U} and \mathcal{J}^C interact. It states that $\mathcal{U} \circ I^{-1} \circ \mathcal{J}^{-1,C}$ is Wadge-equivalent to the identity function.

Lemma 4.7. *The map $\mathcal{U} \circ I^{-1} \circ \mathcal{J}^{-1,C} : \mathcal{J}^C[\omega^\omega] \rightarrow \hat{\omega}^\omega$ is \equiv_p -equivalent to a continuous map, and the identity map on $\hat{\omega}^\omega$ continuously reduces to it, that is, there is a continuous $\theta : \hat{\omega}^\omega \rightarrow \mathcal{J}^C[\omega^\omega]$ such that $\mathcal{U} \circ I^{-1} \circ \mathcal{J}^{-1,C} \circ \theta \equiv_p \text{id}$.*

Proof. We first see that $\mathcal{U} \circ I^{-1} \circ \mathcal{J}^{-1,C}$ is continuous. Since $\mathcal{U} \circ I^{-1} : \omega^\omega \rightarrow \hat{\omega}^\omega$ is Σ_2^0 -measurable, there is a computable $\Phi : \omega^\omega \rightarrow \hat{\omega}^\omega$ such that $\mathcal{U} \circ I^{-1} \equiv_p \Phi \circ \mathcal{J}^C$. Then, $\mathcal{U} \circ I^{-1} \circ \mathcal{J}^{-1,C} \equiv_p \Phi$ on its domain.

To prove the other direction we need the Friedberg jump inversion theorem [Fri57]. The standard proof of the Friedberg jump inversion theorem (relative to C) gives a C' -computable function ψ such that for any X ,

$$X \oplus C' \equiv_T \psi(X) \oplus C' \equiv_T (\psi(X) \oplus C)' \equiv_T \mathcal{J}^C(\psi(X)).$$

Note that by identifying $\hat{\omega}^\omega$ and ω^ω via an effective homeomorphism, we can assume that the domain of ψ is $\hat{\omega}^\omega$ and the range of ψ is included in ω^ω since an effective homeomorphism preserves Turing degrees in a uniform manner. By carefully checking the standard proof of the Friedberg jump inversion theorem, one can see the following uniform version: There are C' -computable function $\psi : \hat{\omega}^\omega \rightarrow \omega^\omega$ such that, for every $X \in \hat{\omega}^\omega$, $\mathcal{J}^C(\psi(X))$ is uniformly Turing equivalent to $X \oplus C'$. That is, there is a computable operator $\theta_0 : \omega^\omega \rightarrow \hat{\omega}^\omega$ and a C' -computable operator $\theta_1 : \omega^\omega \rightarrow \hat{\omega}^\omega$ such that, for every $X \in \hat{\omega}^\omega$,

$$\theta_0 \circ \mathcal{J}^C \circ \psi(X) = X \quad \text{and} \quad \theta_1(X) = \mathcal{J}^C \circ \psi(X).$$

Then, on $\hat{\omega}^\omega$,

$$\begin{aligned} \text{id} &= \theta_0 \circ \mathcal{J}^C \circ \psi \\ &\equiv_p \mathcal{U} \circ \Phi_2 \circ \psi && \text{where } \Phi_2 \text{ comes from the universality of } \mathcal{U} \\ &= \mathcal{U} \circ I^{-1} \circ \mathcal{J}^{-1,C} \circ \mathcal{J}^C \circ I \circ \Phi_2 \circ \psi \\ &= \mathcal{U} \circ I^{-1} \circ \mathcal{J}^{-1,C} \circ \theta_2 \circ \mathcal{J}^C \circ \psi && \text{where } \theta_2 \text{ comes from the universality of } \mathcal{J}^C \\ &= \mathcal{U} \circ I^{-1} \circ \mathcal{J}^{-1,C} \circ \theta_2 \circ \theta_1 && \text{using that } \mathcal{J}^C \circ \psi = \theta_1. \end{aligned}$$

Note that the range of Φ_e is included in $\hat{\omega}^\omega$, and therefore, that of $\mathcal{J}^C \circ I \circ \Phi_2$ is in $\mathcal{J}^C[\omega^\omega]$. Hence, the range of $\theta_2 \circ \theta_1$ is included in $\mathcal{J}^C[\omega^\omega]$. Since θ_2 and θ_1 are continuous, $\theta = \theta_2 \circ \theta_1$ is the desired continuous function. \square

Corollary 4.8. *Let $\mathcal{A} : \hat{\omega}^\omega \rightarrow \mathcal{Q}$ be conciliatory. Then,*

$$(\mathcal{A} \circ \mathcal{U})^{\not\sim} \equiv_w \mathcal{A}.$$

In particular, if $T \in \text{Tree}^n(\mathcal{Q})$, then $\Omega_{\langle T \rangle}^{\not\sim} \equiv_w \Omega_T$.

Proof. All one needs to observe is that $(\mathcal{A} \circ \mathcal{U})^\not\sim = \mathcal{A} \circ \mathcal{U} \circ I^{-1} \circ \mathcal{J}^{-1, C}$. Then $(\mathcal{A} \circ \mathcal{U})^\not\sim \leq_w \mathcal{A}$ because $\mathcal{U} \circ \mathcal{J}^{-1, C}$ is continuous by Lemma 4.7 and \mathcal{A} is conciliatory. Moreover, we have $\mathcal{A} \leq_w (\mathcal{A} \circ \mathcal{U})^\not\sim$ via θ of the previous lemma.

For the second part just notice that $\Omega_{\langle T \rangle}^\not\sim = (\Omega_T \circ \mathcal{U})^\not\sim \equiv_w \Omega_T$. \square

The operation $\not\sim$ is not one-to-one on \mathcal{Q} -Wadge degrees. But it is if we restrict it to initializable degrees. We first need to prove the following lemma, which is where the denseness of forcing of \mathcal{J} is needed.

Definition 4.9. For a function $\mathcal{A}: \omega^\omega \rightarrow \mathcal{Q}$, we say that \mathcal{A} is *initializable* if for every $\sigma \in \omega^{<\omega}$, $\mathcal{A} \leq_w \mathcal{A} \upharpoonright [\sigma]$.

That is, \mathcal{A} is initializable if and only if $\mathcal{F}(\mathcal{A}) = \omega^\omega$. Recall the definition of \mathcal{F} from Section 2.4. Also recall from Proposition 2.6 that \mathcal{A} is non-self-dual if and only if $\mathcal{F}(\mathcal{A})$ is non-empty.

Notice that this definition matches Definition 2.14 for the case $\mathcal{Q} = \omega^\omega$ were all elements are $\leq_{\mathcal{Q}}$ -incomparable.

Lemma 4.10. *If \mathcal{A} is initializable, then $\mathcal{A}^\not\sim$ is non-self-dual.*

Proof. Let C be an oracle which computes Wadge reductions $\mathcal{A} \leq_w \mathcal{A} \upharpoonright [\tau]$ for all τ . We say that σ forces its jump relative to C if for any $\tau \supseteq \sigma$, $J^C(\tau) \supseteq J^C(\sigma)$ holds. We claim that for such σ , we have

$$\mathcal{A}^{\not\sim C} \leq_w \mathcal{A}^{\not\sim C} \upharpoonright [J^C(\sigma)].$$

To prove this, let θ be C -computable such that $X \mapsto \sigma \hat{\ } \theta(X)$ is a Wadge reduction $\mathcal{A} \leq_w \mathcal{A} \upharpoonright [\sigma]$. Then, using the universality of \mathcal{J}^C from the right, we have a computable function θ' such that $\theta' \circ \mathcal{J}^C(X) = \mathcal{J}^C(\sigma \hat{\ } \theta(X))$ for all $X \in \omega^\omega$. Since σ forces its jump relative to C , we have

$$\mathcal{J}^C(\sigma) \subseteq \mathcal{J}^C(\sigma \hat{\ } \theta(X)) = \theta'(\mathcal{J}^C(X)).$$

Thus θ' gives a Wadge reduction from $\mathcal{A}^{\not\sim C}$ to $\mathcal{A}^{\not\sim C} \upharpoonright [J^C(\sigma)]$ as follows: Given $Y \in \mathcal{J}^C[\omega^\omega]$ and $X = \mathcal{J}^{-1, C}(Y)$,

$$\mathcal{A}^{\not\sim C}(Y) = \mathcal{A}(X) \leq_{\mathcal{Q}}$$

$$\mathcal{A}(\sigma \hat{\ } \theta(X)) = \mathcal{A}^{\not\sim C}(\mathcal{J}^C(\sigma \hat{\ } \theta(X))) = \mathcal{A}(\theta'(\mathcal{J}^C(X))) = \mathcal{A}(\theta'(Y)).$$

Now, we say that $X \in \omega^\omega$ forces its jump relative to C if there are infinitely many $n \in \omega$ such that $X \upharpoonright n$ forces its jump relative to C . Using the density of forcing, one can easily construct such an X . Let $t_0 < t_1 < t_2 < \dots$ be such that $X \upharpoonright t_i$ forces its jump relative to C . We then get that $\mathcal{J}^C(X \upharpoonright t_i) \subseteq \mathcal{J}^C(X)$ for all i , and furthermore, every initial segment of $\mathcal{J}^C(X)$ is an initial segment of some $\mathcal{J}^C(X \upharpoonright t_i)$. It follows that for every initial segment $\tau \subseteq \mathcal{J}^C(X)$, $\mathcal{A}^{\not\sim C} \leq_w \mathcal{A}^{\not\sim C} \upharpoonright [\tau]$. Therefore, $\mathcal{J}^C(X) \in \mathcal{F}(\mathcal{A}^{\not\sim C})$ as desired. \square

Corollary 4.11. *If \mathcal{A} is conciliatory and initializable, then $\mathcal{A}^\not\sim \circ \mathcal{U} \equiv_w \mathcal{A}$.*

Proof. By Corollary 4.8 applied to $\mathcal{A}^\not\sim$,

$$(\mathcal{A}^\not\sim \circ \mathcal{U})^\not\sim \equiv_w \mathcal{A}^\not\sim.$$

By the previous lemma, $\mathcal{A}^\not\sim$ is non-self-dual. But then, by Lemma 4.6(3) applied to the equivalence above, $\mathcal{A}^\not\sim \circ \mathcal{U} \equiv_w \mathcal{A}$. \square

4.3. Preservation of ordering. By a quite straightforward inductive proof, we are now ready to show Proposition 1.7 for finite Borel rank. Here, by Lemma 3.17, it suffices to show that $\Omega_S \leq_w \Omega_T$ if and only if $S \trianglelefteq T$ for any $S, T \in \sqcup \text{Tree}^n(\mathcal{Q})$. Recall that we have already proved the right-to-left direction in Lemma 3.18.

Proof of Proposition 1.7. We show the assertion by induction on the terms S and T . Assume $\Omega_S \leq_w \Omega_T$.

First, it is clear that $\Omega_p \leq_w \Omega_q$ if and only if $p \trianglelefteq q$. Suppose now that $\Omega_{\langle U \rangle} \leq_w \Omega_{\langle V \rangle}$. By Corollary 4.8, $\Omega_{\langle U \rangle}^{\neq} \equiv_w \Omega_U$. We thus get $\Omega_U \leq_w \Omega_V$. Hence, by the inductive hypothesis, $U \trianglelefteq V$ and $\langle U \rangle \trianglelefteq \langle V \rangle$.

Now, consider $S = \langle U \rangle \rightarrow \bigsqcup_i S_i$ and $T = \langle V \rangle \rightarrow \bigsqcup_j T_j$. Let us first consider the case that $\langle U \rangle \trianglelefteq \langle V \rangle$. In this case, by the definition of \trianglelefteq , under the assumption that $\langle U \rangle \trianglelefteq \langle V \rangle$, $S \trianglelefteq T$ if and only if $S_i \trianglelefteq T$ for any $i \in \omega$. We get this from the induction hypothesis, as clearly $\Omega_S \leq_w \Omega_T$ implies $\Omega_{S_i} \leq_w \Omega_T$ for any i .

Let us now assume that $\langle U \rangle \not\trianglelefteq \langle V \rangle$. By induction hypothesis, we have $\Omega_{\langle U \rangle} \not\leq_w \Omega_{\langle V \rangle}$. In this case, by definition, $S \trianglelefteq T$ if and only if $S \trianglelefteq T_j$ for some $j \in \omega$. We thus need to show that $\Omega_S \leq_w \Omega_{T_j}$ for some j .

Assume that $\Omega_S \leq_w \Omega_{T_j}$ via a continuous function θ . There must be a sequence X such that $\pi_1(X)$ is empty, but $\pi_1 \circ \theta(X)$ is nonempty (that is, X never changes his mind, whereas $\theta(X)$ changes her mind at some point): This is because if not, $\pi_0 \circ \theta$ would give a reduction from $\Omega_{\langle U \rangle}$ to $\Omega_{\langle V \rangle}$, which contradicts our assumption. Now, let n be the point where $\theta(X \upharpoonright n)$ changes her mind, and let k be the first entry of $\pi_1 \circ \theta(X \upharpoonright n)$. Then $\pi_1 \circ \theta \upharpoonright [X \upharpoonright n]$ gives a reduction from $\Omega_S \upharpoonright [X \upharpoonright n]$ to T_k . We now need a reduction from Ω_S to $\Omega_S \upharpoonright [X \upharpoonright n]$.

Since $\Omega_{\langle U \rangle}$ is initializable, for any σ , there is a continuous function η_σ witnessing $\Omega_{\langle U \rangle} \leq_w \Omega_{\langle U \rangle} \upharpoonright [\sigma]$. It is not hard then, to use η_σ to build a reduction from Ω_S to $\Omega_S \upharpoonright [\sigma \rightarrow \emptyset]$. This concludes the proof since $\pi_1(X)$ is empty, and thus $X \upharpoonright n$ is of the form $\sigma \rightarrow \emptyset$. \square

We have just proved that the map $T \mapsto \Omega_T$ is an order-preserving embedding of $\sqcup \text{Tree}^n(\mathcal{Q})$ into the \mathcal{Q} -Wadge degrees of Δ_{n+1} -measurable functions. What is left to do is to show this embedding is onto.

4.4. Another construction of a Σ_2^0 -universal function. We end this section by giving a second construction of a conciliatory Σ_2^0 -universal function. The reason we prove this again is that the following proof can be easily extended to through the transfinite, once we define the transfinite jump operation in Section 6.1. Recall that our first construction of a conciliatory Σ_2^0 -universal function in Section 2.6 was direct and did not use \mathcal{J} .

Second proof of Proposition 2.15. Let $\{\Phi_e : e \in \omega\}$ be a computable enumeration of all computable operators $:\omega^\omega \rightarrow \omega^{\leq \omega}$. By the universality of \mathcal{J} from the right, we know that for every Σ_2^0 operator $G : \omega^\omega \rightarrow \omega^{\leq \omega}$, there is an $e \in \omega$ and a $C \in \omega^\omega$ such that $G = \Phi_e \circ \mathcal{J}^C$, where Φ_e is the e -th partial computable function. It is not hard to see that the map

$$e \hat{\ } C \oplus X \mapsto \Phi_e \circ \mathcal{J}^C(X)$$

is Σ_2^0 universal (from the left). However, we need \mathcal{U} to also be initializable, so we need to tweak this definition a bit.

For $Z \in \hat{\omega}^\omega$, let $Z^{+1}(n) = Z(n) + 1$ if $Z(n) \neq \text{pass}$, and $Z^{+1}(\text{pass}) = \text{pass}$. We define \mathcal{U} as follows

$$\mathcal{U}(Y) = \begin{cases} \Phi_e \circ \mathcal{J}^C(X) & \text{if } Y = \sigma \hat{\ } 0 \hat{\ } Z^{+1} \text{ and } Z = e \hat{\ } C \oplus X \\ \emptyset & \text{if } Y \text{ has infinitely many } 0\text{'s and is not of the form } \sigma \hat{\ } 0 \hat{\ } Z^{+1} \end{cases}$$

It is clear that \mathcal{U} is still Σ_2^0 universal. It is also easily seen to be Σ_2^0 itself, as deciding in which case we are and where to split σ and Z is Σ_2^0 , but then recovering e , C and X is computable. It is clearly initializable as $\mathcal{U}(X) = \mathcal{U}(\sigma \hat{\ } 0 \hat{\ } X)$ for every σ and X . \square

5. PROOF OF ONTONESS (FINITE BOREL RANK)

Recall that we divided Theorem 1.4 into Propositions 1.6, 1.7, 1.8 and 1.9, and the only one that is left to prove is the latter one. This whole section is dedicated to proving Proposition 1.9 for finite rank, that is, that given a Δ_{1+n}^0 -measurable function $\mathcal{A}: \omega^\omega \rightarrow \mathcal{Q}$, we need to show that there is $T \in \sqcup \text{Tree}^n(\mathcal{Q})$ such that $\mathcal{A} \equiv_w \Omega_T$. Furthermore, we will show that if \mathcal{A} is non-self-dual, then T will be a tree, while if T is self dual, T will be a \sqcup -type term.

The proof is by induction on \leq_w , which we know is well-founded (even bqo). We divide the proof in various cases depending on the properties of \mathcal{A} .

Case 1, Constant. If \mathcal{A} is Wadge equivalent to a constant function, then it is clearly equivalent to $\Omega_{\langle q \rangle}$ for some $q \in \mathcal{Q}$.

Case 2, Self Dual. If \mathcal{A} is self dual, we proved in Proposition 2.6 that $\mathcal{A} \equiv_w \bigoplus_{i \in \omega} \mathcal{A}_i$ where each \mathcal{A}_i is non-self-dual and $\mathcal{A}_i <_w \mathcal{A}$. By the induction hypothesis, there exists trees $T_i \in \text{Tree}^n(\mathcal{Q})$ such that $\mathcal{A}_i \equiv_w \Omega_{T_i}$. It then follows that $\mathcal{A} \equiv_w \Omega_{\sqcup_i T_i}$.

These covers the continuous case, as any continuous function a clopen sum of constant functions.

Case 3, Initializable. Suppose now that \mathcal{A} is initializable. We first claim that $\mathcal{A}^\zeta <_w \mathcal{A}$. That $\mathcal{A}^\zeta \leq_w \mathcal{A}$ follows from the fact that $\mathcal{J}^{-1,C}$ is continuous. Suppose n is the least such that \mathcal{A} is \mathcal{Q} -Wadge equivalent to a Δ_{n+1}^0 function. Since \mathcal{A}^ζ is Δ_n^0 by Lemma 4.6, it is not \mathcal{Q} -Wadge equivalent to \mathcal{A} .

Also recall from Lemma 4.10 that \mathcal{A}^ζ is non-self-dual. By the induction hypothesis, we then have that there is $T \in \text{Tree}^{n-1}(\mathcal{Q})$ such that $\mathcal{A}^\zeta \equiv_w \Omega_T$. Moreover, by Corollary 4.11, $\Omega_T \equiv_w \Omega_{\langle T \rangle}^\zeta$, and thus $\mathcal{A}^\zeta \equiv_w \Omega_{\langle T \rangle}^\zeta$. Therefore, by Lemma 4.6 (3), we obtain $\mathcal{A} \equiv_w \Omega_{\langle T \rangle}$.

Observation 5.1. Let us comment on the case when the domain of \mathcal{A} is closed subset $\mathcal{F} \subseteq \omega^\omega$. In this case we say that \mathcal{A} is initializable if for every $\sigma \in \omega^\omega$ extendible in \mathcal{F} , $\mathcal{A} \leq_w \mathcal{A} \upharpoonright [\sigma]$. In Observation 3.5, we notice we could view such a map as a map defined on ω^ω by composing with a retraction $\rho_{\mathcal{F}}$. One can show that for the retraction defined there, the map we get is also initializable.

Case 4, Non-self-dual and not initializable. Suppose now that \mathcal{A} is non-self-dual and not Wadge equivalent to any initializable function.

Lemma 5.2. *If $\mathcal{F}(\mathcal{A}) \neq \emptyset$, then $\mathcal{A} \upharpoonright \mathcal{F}(\mathcal{A})$ is initializable.*

Furthermore, \mathcal{A} is Wadge equivalent to a initializable function, then $\mathcal{A} \equiv_w \mathcal{A} \upharpoonright \mathcal{F}(\mathcal{A})$.

Proof. Let

$$F = \{\sigma \in \omega^{<\omega} : \mathcal{A} \leq_w \mathcal{A} \upharpoonright [\sigma]\}.$$

Then F is a tree without dead ends whose paths are exactly $\mathcal{F}(\mathcal{A})$. (We will sometimes write $\sigma \in \mathcal{F}(\mathcal{A})$ to mean $\sigma \in F$.) We need to show that for each $\sigma \in F$, there is a continuous reduction $\mathcal{A} \upharpoonright \mathcal{F}(\mathcal{A}) \leq_w \mathcal{A} \upharpoonright \mathcal{F}(\mathcal{A}) \cap [\sigma]$. Since $\sigma \in F$, there is a continuous reduction $\theta: \mathcal{A} \leq_w \mathcal{A} \upharpoonright [\sigma]$. Think of θ as a function $:\omega^{<\omega} \rightarrow \omega^{<\omega}$. We claim that $\theta[F] \subseteq F$, hence obtaining a reduction $\mathcal{A} \upharpoonright \mathcal{F}(\mathcal{A}) \leq_w \mathcal{A} \upharpoonright \mathcal{F}(\mathcal{A}) \cap [\sigma]$ as wanted. Suppose not, and that for some $\tau \in F$, $\theta(\tau) \notin F$. We then get a continuous reduction of $\mathcal{A} \upharpoonright [\tau]$ to $\mathcal{A} \upharpoonright [\theta(\tau)]$. But $\mathcal{A} \upharpoonright [\tau] \equiv_w \mathcal{A}$ while $\mathcal{A} \not\leq_w \mathcal{A} \upharpoonright [\theta(\tau)]$, getting the desired contradiction.

For the second part of the lemma, one needs to observe that if $\mathcal{A} \equiv_w \mathcal{B}$, then $\mathcal{A} \upharpoonright \mathcal{F}(\mathcal{A}) \equiv_w \mathcal{B} \upharpoonright \mathcal{F}(\mathcal{B})$. The reason is that if θ is a reduction $\mathcal{A} \leq_w \mathcal{B}$, then for every $\sigma \in \mathcal{F}(\mathcal{A})$, $\theta(\sigma)$ must be in $\mathcal{F}(\mathcal{B})$ as $\mathcal{B} \leq_w \mathcal{A} \leq_w \mathcal{A} \upharpoonright [\sigma] \leq_w \mathcal{B} \upharpoonright [\theta(\sigma)]$. Now, if \mathcal{B} is initializable, $\mathcal{F}(\mathcal{B}) = \omega^\omega$, and hence $\mathcal{A} \upharpoonright \mathcal{F}(\mathcal{A}) \equiv_w \mathcal{B} \upharpoonright \mathcal{F}(\mathcal{B}) = \mathcal{B} \equiv_w \mathcal{A}$. \square

Let V be the set of all minimal strings leaving from $\mathcal{F}(\mathcal{A})$, and then let $\{\sigma_n\}_{n \in \omega}$ be an enumeration of V . Then consider $\mathcal{B} = \mathcal{A} \upharpoonright \mathcal{F}(\mathcal{A})$ and $\mathcal{C} = \bigoplus_n (\mathcal{A} \upharpoonright [\sigma_n])$. One can easily check that \mathcal{B} and \mathcal{C} are Δ_{1+n}^0 -measurable whenever \mathcal{A} is.

We first see that $\mathcal{B}, \mathcal{C} <_w \mathcal{A}$. It is easy to check that $\mathcal{B}, \mathcal{C} \leq_w \mathcal{A}$. By Lemma 5.2, \mathcal{B} is initializable, and therefore not Wadge equivalent to \mathcal{A} . To see $\mathcal{C} <_w \mathcal{A}$, we note that $\mathcal{A} \upharpoonright [\sigma] <_w \mathcal{A}$ for all $\sigma \in V$. Therefore, if $\mathcal{C} \equiv_w \mathcal{A}$ then it would imply that \mathcal{A} is σ -join-reducible, which contradicts our assumption that \mathcal{A} is non-self-dual by Proposition 2.6. Thus, we get that $\mathcal{B}, \mathcal{C} <_w \mathcal{A}$.

By the induction hypothesis, we get $U \in \text{Tree}^{n-1}(\mathcal{Q})$ and $S \in {}^\omega\text{Tree}^n(\mathcal{Q})$ such that $\mathcal{B} \equiv_w \Omega_{\langle U \rangle}$ and $\mathcal{C} \equiv_w \Omega_S$.

Finally, we claim that $\mathcal{A} \equiv_w \mathcal{B} \rightarrow \mathcal{C}$. This would give us that $\mathcal{A} \equiv_w \Omega_{\langle U \rangle \rightarrow S}$. It is straightforward to see that $\mathcal{A} \leq_w \mathcal{B} \rightarrow \mathcal{C}$. To see $\mathcal{B} \rightarrow \mathcal{C} \leq_w \mathcal{A}$, construct a continuous reduction as follows: Take $X \in \omega^\omega$, which we write as $\pi_0(X) \rightarrow \pi_1(X)$. While we are reading $\pi_0(X)$ we stay inside $\mathcal{F}(\mathcal{A})$ using the continuous embedding $\rho_B: \mathcal{B} \rightarrow \mathcal{A} \upharpoonright \mathcal{F}(\mathcal{A})$. If we ever change our mind, $\mathcal{B} \rightarrow \mathcal{C}(X) = \mathcal{C}(\pi_1(X))$. We still have that $\rho_B(\pi_0(X))$ is a finite string in $\mathcal{F}(\mathcal{A})$, and we so can use the reductions $\mathcal{C} \leq_w \mathcal{A} \leq_w \mathcal{A} \upharpoonright [\rho_B(\pi_0(X))]$.

More formally: For each $\tau \in \mathcal{F}(\mathcal{A})$, let η_τ be a Wadge reduction $\mathcal{A} \leq_w \mathcal{A} \upharpoonright \tau$. Let $\theta_C: \mathcal{C} \rightarrow \mathcal{A}$. Then

$$X \mapsto (\rho_B(\pi_0(X)) \frown \eta_{\rho_B(\pi_0(X))}(\theta_C(\pi_1(X))))$$

is a reduction $\mathcal{B} \rightarrow \mathcal{C} \leq_w \mathcal{A}$.

6. INFINITE BOREL RANK

We now start to deal with functions of infinite Borel rank.

6.1. Transfinite jump operator. We now need to consider α -th Turing jump for transfinite α . We again use the machinery developed in [MM11, Mon14]. There, the ω -th Turing jump is defined by taking the first bit of each of the finite jumps:

$$\mathcal{J}^\omega(X) = \langle \mathcal{J}(X)(0), \mathcal{J}^2(X)(0), \mathcal{J}^3(X)(0), \dots \rangle.$$

Because of the way these operators are defined in [MM11, Mon14] taking one bit from each jump is enough to code the whole sequence $\{\mathcal{J}^n(X) : n \in \omega\}$. The reason is that

$\mathcal{J}^n(X)(0)$ codes a least two bits of $J^{n-1}(X)$, and at least three bits of $J^{n-2}(X), \dots$, and at least n bits of $\mathcal{J}(X)$. See [MM11, Mon14] for more details.

The definition of the ω^α -th jump is similar, taking one bit from a sequence of jumps that converges to ω^α . To make this precise, we need to assign to each countable ordinal α a *fundamental sequence* $(\alpha[n])_{n \in \omega}$ which is a non-decreasing sequence with $\alpha = \sup_n \alpha[n] + 1$. When $\alpha = \beta + 1$, we just define $\alpha[n] = \beta$, and when α is a limit, $(\alpha[n])_{n \in \omega}$ is any increasing sequences towards α . Notice that in either case $\omega^\alpha = \sum_n \omega^{\alpha[n]}$. In [MM11, Mon14], they needed fundamental sequences with particular properties, but we do not need to get into that now.

Something we need in this paper but that was not necessary in [MM11, Mon14] is *relative* transfinite Turing jump operators. If all we needed to do is relativize the whole construction, that would not be any harder. The problem is that we need to consider transfinite jumps, which are built by iterating jump operators that use different oracles. For instance, If we deal with the ω -th Turing jump, we shall consider a sequence $\mathcal{C} = (C_n)_{n \in \omega}$ of oracles to compute the different values $\mathcal{J}^{n, C_n}(X)(0)$. We call such a sequence an ω -*oracle*.

In general, we define an ω^α -oracle as a sequence $\mathcal{C} = (C_n)_{n \in \omega}$ of $\omega^{\alpha[n]}$ -oracles. To define the notion of a ξ -oracle for ordinals that are not of the form ω^α , note that every ordinal ξ can be written as $\xi = \omega^\alpha + \beta$ for some $\beta < \omega^{\alpha+1}$ (consider the Cantor normal form). Then we define a ξ -oracle as a pair $\mathcal{C} = (C_0, C_1)$ of an ω^α -oracle and a β -oracle. Note that for each countable ordinal ξ , there is a well-founded tree $\Lambda_\xi \subseteq \omega^{<\omega}$ such that each ξ -oracle \mathcal{C} can be thought of as a Λ_ξ -indexed collection $(C_\sigma)_{\sigma \in \Lambda_\xi}$. Given ξ -oracles \mathcal{C} and \mathcal{D} , we write $\mathcal{C} \leq_T \mathcal{D}$ if $C_\sigma \leq_T D_\sigma$ uniformly holds for any $\sigma \in \Lambda_\xi$.

To introduce the transfinite jump operator, for an ω^α -oracle $\mathcal{C} = (C_n)_{n \in \omega}$, we use the following abbreviations:

$$\begin{aligned} \mathcal{J}_{[0,n]}^{\omega^\alpha, \mathcal{C}} &= \mathcal{J}^{\omega^{\alpha[n-1]}, C_{n-1}} \circ \mathcal{J}^{\omega^{\alpha[n-2]}, C_{n-2}} \circ \dots \circ \mathcal{J}^{\omega^{\alpha[0]}, C_0}, \\ \mathcal{J}_{[0,n]}^{\omega^\alpha, \mathcal{C}} &= \mathcal{J}^{\omega^{\alpha[n-1]}, C_{n-1}} \circ \mathcal{J}^{\omega^{\alpha[n-2]}, C_{n-2}} \circ \dots \circ \mathcal{J}^{\omega^{\alpha[0]}, C_0}. \end{aligned}$$

Definition 6.1 (Montalbán [Mon14]). For any countable ordinal α and ω^α -oracle \mathcal{C} , define the ω^α -th jump operation $\mathcal{J}^{\omega^\alpha, \mathcal{C}}$ and its approximation $J^{\omega^\alpha, \mathcal{C}}$ as follows:

$$\begin{aligned} \mathcal{J}^{\omega^\alpha, \mathcal{C}}(Z)(n) &= \mathcal{J}_{[0, n+1]}^{\omega^\alpha, \mathcal{C}}(Z)(0), \\ J^{\omega^\alpha, \mathcal{C}}(\sigma)(n) &= J_{[0, n+1]}^{\omega^\alpha, \mathcal{C}}(\sigma)(0). \end{aligned}$$

If a countable ordinal ξ is of the form $\omega^\alpha + \beta$ for some $\beta < \omega^{\alpha+1}$, for a ξ -oracle $\mathcal{C} = (C_0, C_1)$, we define $\mathcal{J}^{\xi, \mathcal{C}} = \mathcal{J}^{\beta, C_1} \circ \mathcal{J}^{\omega^\alpha, C_0}$ and $J^{\xi, \mathcal{C}} = J^{\beta, C_1} \circ J^{\omega^\alpha, C_0}$. We also use $\mathcal{J}^{-\xi, \mathcal{C}}$ to denote $(\mathcal{J}^{\xi, \mathcal{C}})^{-1}$.

The property mentioned in the first paragraph in Section 6.1 implies the following well-known fact which connects the transfinite jump operation and the Borel hierarchy.

Fact 6.2 (Universality from the right). *If $\mathcal{A}: \omega^\omega \rightarrow \hat{\omega}^\omega$ is $\Sigma_{1+\xi}^0$ -measurable relative to \mathcal{C} , then there is a computable function θ such that $\mathcal{A} \equiv_p \theta \circ \mathcal{J}^{\xi, \mathcal{C}}$.*

(Recall that we write $\mathcal{A} \equiv_p \mathcal{B}$ if $\mathcal{A}(X)^p = \mathcal{B}(X)^p$ for all $X \in \mathcal{X}$.) By using Fact 6.2, it is not hard to construct a $\Sigma_{1+\xi}^0$ -universal initializable conciliatory function (Proposition 3.24) by a similar argument as in Section 4.4.

Proof of Proposition 3.24. By $\Sigma_{1+\xi}^0$ -universality of \mathcal{J}^ξ from the right (Fact 6.2), it is obvious that $e \hat{\ } C \oplus X \mapsto \Phi_e \circ \mathcal{J}^{\xi, C}(X)$ is $\Sigma_{1+\xi}^0$ -universal from the left. As in the proof of Proposition 2.15 (see Section 4.4), by modifying this function, one can obtain a $\Sigma_{1+\xi}^0$ -universal initializable conciliatory function. \square

We now introduce the transfinite version of the jump inversion operator.

Definition 6.3. For any $\mathcal{A}: \omega^\omega \rightarrow \mathcal{Q}$ and any oracle \mathcal{C} we introduce the ω^α -th jump inversion of \mathcal{A} , $\mathcal{A}^{\not\sim \omega^\alpha, \mathcal{C}}: \mathcal{J}^{\omega^\alpha, \mathcal{C}}[\omega^\omega] \rightarrow \mathcal{Q}$ as follows:

$$\mathcal{A}^{\not\sim \omega^\alpha, \mathcal{C}}(X) = \mathcal{A} \circ \mathcal{J}^{-\omega^\alpha, \mathcal{C}}(X).$$

As before, the domain of $\mathcal{A}^{\not\sim \omega^\alpha, \mathcal{C}}$ is closed. If the domain of \mathcal{A} is $\hat{\omega}^\omega$, then $\mathcal{A}^{\not\sim \omega^\alpha, \mathcal{C}}$ is defined by $\mathcal{A} \circ I^{-1} \circ \mathcal{J}^{-\omega^\alpha, \mathcal{C}}$, where $I: \hat{\omega}^\omega \rightarrow \omega^\omega$ is a homeomorphism. The following transfinite version of Observation 4.4 is also straightforward.

Observation 6.4. Let \mathcal{A} be any partial \mathcal{Q} -valued function, and Y and Z be ξ -oracles. If $Z \geq_T Y$, then $\mathcal{A}^{\not\sim \xi, Z} \leq_w \mathcal{A}^{\not\sim \xi, Y}$.

Thus, there is \mathcal{C} such that $\mathcal{A}^{\not\sim \omega^\alpha, \mathcal{C}} \leq_w \mathcal{A}^{\not\sim \omega^\alpha, \mathcal{D}}$ for any \mathcal{D} by well-foundedness of the Wadge degrees of \mathcal{Q} -valued functions (Theorem 2.3). Thus we use $\mathcal{A}^{\not\sim \omega^\alpha}$ to denote such $\mathcal{A}^{\not\sim \omega^\alpha, \mathcal{C}}$. By Observation 6.4, \mathcal{C} can be chosen as a constant sequence, that is, $C_\sigma = C$ for any $\sigma \in \Lambda_{\omega^\alpha}$. If \mathcal{C} is a constant sequence consisting of $C \in \omega^\omega$, we simply write it as C instead of \mathcal{C} . We now see the transfinite version of Lemma 4.6.

Lemma 6.5. *For any $\mathcal{A}, \mathcal{B}: \omega^\omega \rightarrow \mathcal{Q}$, the following holds.*

- (1) *If \mathcal{A} is $\Sigma_{1+\omega^\alpha+\beta}^0$ -measurable, then $\mathcal{A}^{\not\sim \omega^\alpha}$ is $\Sigma_{1+\beta}^0$ -measurable.*
- (2) *If $\mathcal{A} \leq_w \mathcal{B}$ then $\mathcal{A}^{\not\sim \omega^\alpha} \leq_w \mathcal{B}^{\not\sim \omega^\alpha}$.*
- (3) *If either $\mathcal{A}^{\not\sim \omega^\alpha}$ or $\mathcal{B}^{\not\sim \omega^\alpha}$ is non-self-dual, then $\mathcal{A}^{\not\sim \omega^\alpha} \leq_w \mathcal{B}^{\not\sim \omega^\alpha}$ implies $\mathcal{A} \leq_w \mathcal{B}$.*

Proof. For (1), let C be such that $\mathcal{A}^{\not\sim \omega^\alpha} \equiv_w \mathcal{A}^{\not\sim \omega^\alpha, C}$. If \mathcal{A} is $\Delta_{1+\omega^\alpha+\beta}^0$ -measurable, then there is $D \geq_T C$ such that $\mathcal{A} = \Phi_e \circ \mathcal{J}^{\beta, D} \circ \mathcal{J}^{\omega^\alpha, D}$ by Fact 6.2. Then we also have $\mathcal{A}^{\not\sim \omega^\alpha} \equiv_w \mathcal{A}^{\not\sim \omega^\alpha, D}$, and moreover, $\mathcal{A}^{\not\sim \omega^\alpha, D} = \Phi_e \circ \mathcal{J}^{\beta, D}$, which is clearly $\Delta_{1+\beta}^0$ -measurable. It is straightforward to show (2) and (3) by using the same argument as in the proof of Lemma 4.6. \square

To get the transfinite version of Lemma 4.7, we use the transfinite version of the Friedberg jump inversion theorem [Mac77]: There exists a $C^{(\omega^\alpha)}$ -computable function ψ such that

$$X \oplus C^{(\omega^\alpha)} \equiv_T \psi(X) \oplus C^{(\omega^\alpha)} \equiv_T (\psi(X) \oplus C)^{(\omega^\alpha)} \equiv_T \mathcal{J}^{\omega^\alpha, C}(\psi(X))$$

holds uniformly in X . Here, as usual, $Z^{(\xi)}$ denotes the ξ -th Turing jump of Z . As in the proof of Lemma 4.7, in particular, there are a computable operator $\theta_0: \omega^\omega \rightarrow \hat{\omega}^\omega$ and a $C^{(\omega^\alpha)}$ -computable operator $\theta_1: \omega^\omega \rightarrow \hat{\omega}^\omega$ such that for every $X \in \hat{\omega}^\omega$,

$$\theta_0 \circ \mathcal{J}^{\omega^\alpha, C} \circ \psi(X) = X \quad \text{and} \quad \theta_1(X) = \mathcal{J}^{\omega^\alpha, C} \circ \psi(X).$$

Corollary 6.6. $\Omega_{\langle T \rangle^{\omega^\alpha}}^{\not\sim \omega^\alpha} \equiv_w \Omega_T$.

Proof. As in the proofs of Lemma 4.7 and Corollary 4.8, it follows from the above formula. \square

6.2. Generalization of initializability.

Definition 6.7. For a countable ordinal α , we say that \mathcal{A} is α -stable if \mathcal{A} is Wadge equivalent to an initializable function, and $\mathcal{A}^{\not\sim\omega^\beta} \equiv_w \mathcal{A}$ holds for any $\beta < \alpha$.

By Lemma 6.5 (2), if \mathcal{B} is Wadge equivalent to \mathcal{A} and if \mathcal{A} is α -stable, then so is \mathcal{B} .

Lemma 6.8. For any $T \in \text{Tree}^{\omega^1}(\mathcal{Q})$ and countable ordinal α , $\Omega_{\langle T \rangle^{\omega^\alpha}}$ is α -stable.

Proof. First note that, for any countable ordinal α , $\Omega_{\langle T \rangle^{\omega^\alpha}}$ is initializable since $\Omega_{\langle T \rangle^{\omega^\alpha}} = \Omega_T \circ \mathcal{U}_{\omega^\alpha}$ for an initializable function $\mathcal{U}_{\omega^\alpha}$. Recall that $I: \hat{\omega}^\omega \rightarrow \omega^\omega$ is a computable homeomorphism. Since $I \circ \mathcal{U}_{\omega^\alpha} \circ I^{-1}$ is $\Sigma_{1+\omega^\alpha}^0$ -measurable, by universality of $\mathcal{J}^{\omega^\alpha}$ from the right (Fact 6.2), $\mathcal{U}_{\omega^\alpha} = I^{-1} \circ \Phi \circ \mathcal{J}^{\omega^\alpha, C} \circ I$ for some Φ and C . Fix $\beta < \alpha$. We show that $\Omega_{\langle T \rangle^{\omega^\alpha}}$ is Wadge reducible to $\Omega_{\langle T \rangle^{\omega^\alpha}}^{\not\sim\omega^\beta, C}$. Let $\psi: \omega^\omega \rightarrow \omega^\omega$ be the ω^β -th jump inversion map relative to C , that is, there are a computable operator $\theta_0: \omega^\omega \rightarrow \hat{\omega}^\omega$ and a $C^{(\omega^\beta)}$ -computable operator $\theta_1: \omega^\omega \rightarrow \hat{\omega}^\omega$ such that for every $X \in \omega^\omega$,

$$\theta_0 \circ \mathcal{J}^{\omega^\beta, C} \circ \psi(X) = X \quad \text{and} \quad \theta_1(X) = \mathcal{J}^{\omega^\beta, C} \circ \psi(X).$$

Then, $\mathcal{J}^{\omega^\alpha, C}(Z) = \mathcal{J}^{\omega^\alpha, C} \circ \theta_0 \circ \mathcal{J}^{\omega^\beta, C} \circ \psi(Z)$. Since $\beta < \alpha$, the map $\mathcal{J}^{\omega^\alpha, C} \circ \theta_0 \circ \mathcal{J}^{\omega^\beta, C}$ is still $\Sigma_{1+\omega^\alpha}^0$ relative to C . By universality of $\mathcal{J}^{\omega^\alpha}$ from the right (Fact 6.2), there is a computable function θ_2 such that $\theta_2 \circ \mathcal{J}^{\omega^\alpha} = \mathcal{J}^{\omega^\alpha, C} \circ \theta_0 \circ \mathcal{J}^{\omega^\beta, C}$. Hence,

$$\mathcal{J}^{\omega^\alpha, C} = \theta_2 \circ \mathcal{J}^{\omega^\alpha, C} \circ \psi.$$

Then,

$$\mathcal{U}_{\omega^\alpha} = I^{-1} \circ \Phi \circ \mathcal{J}^{\omega^\alpha, C} \circ I = I^{-1} \circ \Phi \circ \theta_2 \circ \mathcal{J}^{\omega^\alpha, C} \circ \psi \circ I.$$

Clearly, $I^{-1} \circ \Phi \circ \theta_2 \circ \mathcal{J}^{\omega^\alpha, C}$ (hence, its restriction up to ω^ω) is $\Sigma_{1+\alpha}^0$ relative to C , and therefore, by universality of $\mathcal{U}_{\omega^\alpha}$ from the left, there is a continuous function $\theta_3: \omega^\omega \rightarrow \hat{\omega}^\omega$ such that

$$\mathcal{U}_{\omega^\alpha} = \mathcal{U}_{\omega^\alpha} \circ \theta_3 \circ \psi \circ I.$$

By our explicit construction of $\mathcal{U}_{\omega^\alpha}$ and $C^{(\omega^\beta)}$ -computability of ψ , one can assume that θ_3 is $C^{(\omega^\beta)}$ -computable. By universality of $\mathcal{J}^{\omega^\beta}$ from the right, there is a computable function θ_4 such that $\mathcal{J}^{\omega^\beta} \circ I \circ \theta_3 = \theta_4 \circ \mathcal{J}^{\omega^\beta}$, and therefore,

$$\mathcal{J}^{\omega^\beta} \circ I \circ \theta_3 \circ \psi \circ I = \theta_4 \circ \mathcal{J}^{\omega^\beta} \circ \psi \circ I = \theta_4 \circ \theta_1 \circ I.$$

Then, by combining the above formulas, we get

$$\begin{aligned} \Omega_{\langle T \rangle^{\omega^\alpha}} &= \Omega_T \circ \mathcal{U}_{\omega^\alpha} = \Omega_T \circ \mathcal{U}_{\omega^\alpha} \circ \theta_3 \circ \psi \circ I = \Omega_{\langle T \rangle^{\omega^\alpha}} \circ \theta_3 \circ \psi \circ I \\ &= \Omega_{\langle T \rangle^{\omega^\alpha}}^{\not\sim\omega^\beta, C} \circ \mathcal{J}^{\omega^\beta, C} \circ I \circ \theta_3 \circ \psi \circ I = \Omega_{\langle T \rangle^{\omega^\alpha}}^{\not\sim\omega^\beta, C} \circ \theta_4 \circ \theta_1 \circ I. \end{aligned}$$

Consequently, the continuous function $\theta_4 \circ \theta_1 \circ I$ gives a Wadge reduction from $\Omega_{\langle T \rangle^{\omega^\alpha}}$ to $\Omega_{\langle T \rangle^{\omega^\alpha}}^{\not\sim\omega^\beta, C}$. \square

The purpose of this section is to prove the following transfinite version of Lemma 4.10.

Lemma 6.9. If \mathcal{A} is α -stable, then $\mathcal{A}^{\not\sim\omega^\alpha}$ is σ -join-irreducible.

Before proving Lemma 6.9 we first check that this Lemma immediately implies our main theorems.

Proof of Proposition 1.7 from Lemma 6.9. We only show that $\Omega_{\langle U \rangle^{\omega^\alpha}} \leq_w \Omega_{\langle V \rangle^{\omega^\beta}}$ if and only if $\langle U \rangle^{\omega^\alpha} \leq \langle V \rangle^{\omega^\beta}$. For the other cases, we can use a similar argument as in the proof of Proposition 1.7 for finite Borel rank. To verify the above equivalence, by Lemma 6.6, we have $\Omega_{\langle U \rangle^{\omega^\alpha}}^{\not\sim} \equiv_w \Omega_U$. Since $\Omega_{\langle U \rangle^{\omega^\alpha}}$ is α -stable by Lemma 6.8, $\Omega_{\langle U \rangle^{\omega^\alpha}}^{\not\sim}$ is σ -join-irreducible by Lemma 6.9, and thus non-self-dual by Proposition 2.6. Thus, if $\alpha = \beta$, by Lemma 6.5,

$$\Omega_{\langle U \rangle^{\omega^\alpha}} \leq_w \Omega_{\langle V \rangle^{\omega^\beta}} \iff \Omega_U \equiv_w \Omega_{\langle U \rangle^{\omega^\alpha}}^{\not\sim} \leq_w \Omega_{\langle V \rangle^{\omega^\beta}}^{\not\sim} \equiv_w \Omega_V.$$

This ensures the desired assertion by induction hypothesis. If $\alpha > \beta$, then, since $\Omega_{\langle U \rangle^{\omega^\alpha}}$ is α -stable by Lemma 6.8, we have $\Omega_{\langle U \rangle^{\omega^\alpha}}^{\not\sim} \equiv_w \Omega_{\langle U \rangle^{\omega^\alpha}}$, and therefore,

$$\Omega_{\langle U \rangle^{\omega^\alpha}} \leq_w \Omega_{\langle V \rangle^{\omega^\beta}} \iff \Omega_{\langle U \rangle^{\omega^\alpha}} \equiv_w \Omega_{\langle U \rangle^{\omega^\alpha}}^{\not\sim} \leq_w \Omega_{\langle V \rangle^{\omega^\beta}}^{\not\sim} \equiv_w \Omega_V.$$

This ensures the desired assertion by induction hypothesis. The same argument works in the case $\alpha < \beta$. \square

Proof of Proposition 1.9 from Lemma 6.9. Fix a Δ_ξ^0 -measurable function \mathcal{A} . Let δ be the smallest ordinal such that \mathcal{A} is not $(\gamma + 1)$ -stable for some $\gamma < \delta$. If $\delta = 0$, then δ is not Wadge-initializable, and we can use the same argument as in the proof of Proposition 1.9 for functions of finite Borel rank.

Suppose that $\delta > 0$. Then, note that δ must be a successor ordinal, say $\delta = \alpha + 1$, and thus \mathcal{A} is α -stable. Let β be a unique ordinal that $\xi = \omega^\alpha + \beta$. By Lemma 6.5 (1), $\mathcal{A}^{\not\sim \omega^\alpha}$ is Δ_β^0 -measurable. Moreover, by minimality of α , we have $\mathcal{A}^{\not\sim \omega^\alpha} <_w \mathcal{A}$ and $\mathcal{A}^{\not\sim \omega^\alpha}$ is non-self-dual by Lemma 6.9 and Proposition 2.6. By induction hypothesis, $\mathcal{A}^{\not\sim \omega^\alpha} \equiv_w \Omega_T$ for some tree $T \in \text{Tree}^\beta(\mathcal{Q})$. Then we have $\mathcal{A}^{\not\sim \omega^\alpha} \equiv_w \Omega_{\langle T \rangle^{\omega^\alpha}}^{\not\sim}$ by Lemma 6.6. Note that Ω_T is σ -join-irreducible by Observations 3.15 and 2.8 since T is a tree, and therefore non-self-dual by Proposition 2.6. Therefore we get $\mathcal{A} \equiv_w \Omega_{\langle T \rangle^{\omega^\alpha}}$ by Lemma 6.5 (3). Claim that

$$\langle T \rangle^{\omega^\alpha} \in \langle \text{Tree}^\beta(\mathcal{Q}) \rangle^{\omega^\alpha} \subseteq \text{Tree}^\xi(\mathcal{Q}).$$

It is clear if $\beta < \omega^{\alpha+1}$ by definition. If $\beta \geq \omega^{\alpha+1}$, then we must have $\beta = \xi$. Recall that ξ is of the form $\omega^\gamma + \delta$ for some $\gamma < \omega_1$ and $\delta < \omega^{\gamma+1}$. We then have $\alpha < \gamma$. Thus, if $T \in \text{Tree}^\xi(\mathcal{Q}) = \text{Tree}^\beta(\mathcal{Q})$, then $\langle T \rangle^{\omega^\alpha} \in \text{Tree}^\xi(\mathcal{Q})$. This concludes the proof. \square

6.3. Proof of Lemma 6.9. It remains to show Lemma 6.9. The statement of this lemma resembles Lemma 4.10, that is, it looks like a transfinite version of Lemma 4.10. Nevertheless, our proof requires a very different argument. The notation for the proof get a bit more complicated than in Lemma 4.10, as one needs to keep track of ω -iterations of the jump. However, it still is much simpler than Duparc's [Dup] proof for $\mathcal{Q} = 2$.

6.3.1. Proof of Lemma 6.9 (for $\alpha \leq 1$). Throughout this subsection, for notational simplicity, we always assume that $\mathcal{B}^{\not\sim} \equiv_w \mathcal{B}^{\not\sim 0}$ for any function \mathcal{B} . We will deal with nonempty oracles in the next Section 6.3.2.

We first consider the base case $\alpha = 0$. Recall that \mathcal{A} is 0-stable if and only if it is Wadge equivalent to an initializable function. Lemma 4.10 states that if \mathcal{A} is 0-stable, then $\mathcal{A}^{\not\sim}$ is σ -join-irreducible. In the proof of Lemma 4.10 we showed that if \mathcal{B} is actually

initializable $\sigma \in \omega^{<\omega}$ forces its jump, then $\mathcal{B}^\omega \leq_w \mathcal{B}^\omega \upharpoonright J(\sigma)$, and that if $X \in \omega^\omega$ forces its jump, then $\mathcal{J}(X) \in \mathcal{F}(\mathcal{B}^\omega)$. Since every string can be extended to one that forces its jump, the set of $X \in \omega^\omega$ which force their jump is dense. We thus get that the set of X such that $\mathcal{J}(X) \in \mathcal{F}(\mathcal{B}^\omega)$ is dense. In the case when \mathcal{A} is not actually initializable, but Wadge equivalent to an initializable, recall from Lemma 5.2 that $\mathcal{A} \equiv_w \mathcal{A} \upharpoonright \mathcal{F}(\mathcal{A})$. (Recall Observation 3.5 for how to deal with functions whose domain is a closed set as if their domain was all of ω^ω .)

Then, the proof of Lemma 4.10 actually implies that

- (1) if \mathcal{A} is 0-stable, then $\{X : \mathcal{J}(X) \in \mathcal{F}((\mathcal{A} \upharpoonright \mathcal{F}(\mathcal{A}))^\omega)\}$ is dense in $\mathcal{F}(\mathcal{A})$.

In the rest of this section, we give a proof of Lemma 6.9 for $\alpha = 1$, that is, if \mathcal{A} is 1-stable, then \mathcal{A}^{ω} is σ -join-irreducible. By definition, \mathcal{A} is 1-stable if and only if \mathcal{A} is Wadge equivalent to an initializable function and $\mathcal{A}^\omega \equiv_w \mathcal{A}$. The latter condition is equivalent to that $\mathcal{A}^{\omega^n} \equiv_w \mathcal{A}$ for any $n \in \omega$. Given a 1-stable function \mathcal{A} , we inductively define a \mathcal{Q} -valued function \mathcal{A}_n by

$$\mathcal{A}_0 = \mathcal{A} \quad \text{and} \quad \mathcal{A}_{n+1} = (\mathcal{A}_n \upharpoonright \mathcal{F}(\mathcal{A}_n))^\omega.$$

Observation 6.10. If \mathcal{A} is 1-stable, then $\mathcal{A} \equiv_w \mathcal{A}_n$ for any $n \in \omega$.

Proof. Recall that by Lemma 5.2, a function \mathcal{B} is Wadge equivalent to an initializable function if and only if $\mathcal{B} \equiv_w \mathcal{B} \upharpoonright \mathcal{F}(\mathcal{B})$. Fix n , and inductively assume that $\mathcal{A} \equiv_w \mathcal{A}_n$. Therefore, we have $\mathcal{A} \upharpoonright \mathcal{F}(\mathcal{A}) \equiv_w \mathcal{A}_n \upharpoonright \mathcal{F}(\mathcal{A}_n)$. Since \mathcal{A} is 1-stable, in particular, \mathcal{A} is Wadge equivalent to an initializable function. Then, $\mathcal{A} \equiv_w \mathcal{A} \upharpoonright \mathcal{F}(\mathcal{A})$, and therefore, $\mathcal{A}_n \upharpoonright \mathcal{F}(\mathcal{A}_n)$ is 1-stable. Thus, $\mathcal{A}_{n+1} \equiv_w \mathcal{A}_n \upharpoonright \mathcal{F}(\mathcal{A}_n)$, and therefore $\mathcal{A}_{n+1} \equiv_w \mathcal{A}$. \square

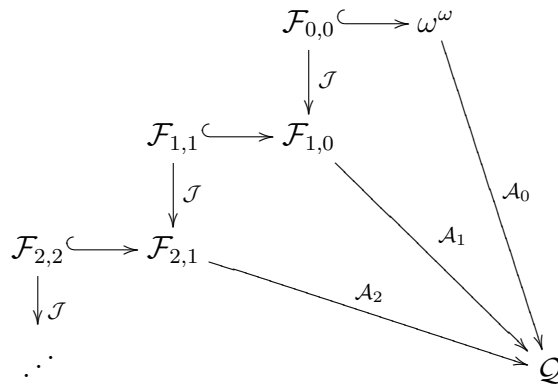
In particular, \mathcal{A}_n is initializable for any $n \in \omega$. Thus, the property (1) implies that

- (2) if \mathcal{A} is 1-stable, then $\mathcal{J}^{-1}[\mathcal{F}(\mathcal{A}_{n+1})^\omega]$ is dense in $\mathcal{F}(\mathcal{A}_n)$ for any $n \in \omega$.

For notational convenience, we define

$$\mathcal{F}_{n,n} = \mathcal{F}(\mathcal{A}_n) \quad \text{and} \quad \mathcal{F}_{n+1,n} = \text{dom}(\mathcal{A}_{n+1}) = \mathcal{J}[\mathcal{F}_{n,n}].$$

Note that $\mathcal{F}_{n+1,n}$ is the domain of $\mathcal{A}_{n+1,n+1}$, and $\mathcal{F}_{n,n}$ and $\mathcal{F}_{n+1,n}$ are closed sets (since \mathcal{J}^{-1} is continuous). In the following diagram, the arrow \hookrightarrow indicates the inclusion map.



We also define $\mathcal{F}_{0,1} = \mathcal{J}^{-1}[\mathcal{F}_{1,1}]$ which is included in $\mathcal{F}_{0,0}$, and in general $\mathcal{F}_{n,n+1} = \mathcal{J}^{-1}[\mathcal{F}_{n+1,n+1}]$, which is not necessarily closed. By using these notations, the property

(2) can be rephrased as: If \mathcal{A} is 1-stable, then $\mathcal{F}_{n,n+1}$ is dense in $\mathcal{F}_{n,n}$. Therefore, we have the following commutative diagram.

$$\begin{array}{ccccccc}
 & & \mathcal{F}_{n,n+1} & \xrightarrow{\text{dense}} & \mathcal{F}_{n,n} & \hookrightarrow & \mathcal{F}_{n,n-1} & \xrightarrow{\mathcal{A}_n} & \mathcal{Q} \\
 & & \uparrow \mathcal{J}^{-1} & & \downarrow \mathcal{J} & & & \nearrow \mathcal{A}_{n+1} & \\
 \mathcal{F}_{n+1,n+2} & \xrightarrow{\text{dense}} & \mathcal{F}_{n+1,n+1} & \hookrightarrow & \mathcal{F}_{n+1,n} & & & &
 \end{array}$$

Generally, for $m < n$, we define $\mathcal{F}_{m,n} = \mathcal{J}^{-1}[\mathcal{F}_{m+1,n}]$. In particular, $\mathcal{F}_{0,n} = \mathcal{J}^{-n}[\mathcal{F}_{n,n}]$. This gives a decreasing sequence $(\mathcal{F}_{0,n})_{n \in \omega}$. Define $\mathcal{F}_{0,\omega} = \bigcap_{n \in \omega} \mathcal{F}_{0,n}$. In other words,

$$\mathcal{F}_{0,\omega} = \{X \in \omega^\omega : (\forall n \in \omega) \mathcal{J}^n(X) \in \mathcal{F}_{n,n}\}.$$

Then we define $\mathcal{F}_{\omega,\omega} = \mathcal{J}^\omega[\mathcal{F}_{0,\omega}]$.

$$\begin{array}{ccccccc}
 \mathcal{F}_{\omega,0} & \hookrightarrow & \dots & \xrightarrow{\text{dense}} & \mathcal{F}_{2,0} & \xrightarrow{\text{dense}} & \mathcal{F}_{1,0} & \xrightarrow{\text{dense}} & \mathcal{F}_{0,0} & \hookrightarrow & \omega^\omega \\
 & & & & \uparrow \mathcal{J}^{-1} & & \uparrow \mathcal{J}^{-1} & & \downarrow \mathcal{J} & & \searrow \mathcal{A}_0 \\
 & & \dots & \xrightarrow{\text{dense}} & \mathcal{F}_{2,1} & \xrightarrow{\text{dense}} & \mathcal{F}_{1,1} & \hookrightarrow & \mathcal{F}_{1,0} & & \searrow \mathcal{A}_1 \\
 & & \dots & \xrightarrow{\text{dense}} & \mathcal{F}_{2,2} & \hookrightarrow & \mathcal{F}_{2,1} & & & & \searrow \mathcal{A}_2 \\
 & & \dots & & \downarrow \mathcal{J} & & & & & & \searrow \mathcal{A}^{\omega} \\
 & & \dots & & \dots & & & & & & \searrow \mathcal{A}^{\omega} \\
 \mathcal{F}_{\omega,\omega} & \hookrightarrow & \mathcal{F}(\mathcal{A}^{\omega}) & & & & & & & & \mathcal{Q}
 \end{array}$$

Hereafter, we identify the closed set $\mathcal{F}_{n,n}$ with the pruned tree whose infinite paths are exactly the elements of $\mathcal{F}_{n,n}$.

We devote the rest of this section to prove the following claim:

(3) If \mathcal{A} is 1-stable, then $\{X : \mathcal{J}^\omega(X) \in \mathcal{F}(\mathcal{A}^{\omega})\}$ is dense in $\mathcal{F}(\mathcal{A})$.

(Recall that $\mathcal{F}(\mathcal{A}) = \mathcal{F}_{0,0}$.) Clearly, the claim (3) entails Lemma 6.9 for $\alpha = 1$ as it implies that $\mathcal{F}(\mathcal{A}^{\omega}) \neq \emptyset$. Our strategy has two steps: First to prove that $\mathcal{F}_{0,\omega}$ is dense in $\mathcal{F}(\mathcal{A})$. Second to prove that $(\mathcal{J}^\omega)^{-1}[\mathcal{F}(\mathcal{A}^{\omega})] \supseteq \mathcal{F}_{0,\omega}$ by showing that $\mathcal{F}_{\omega,\omega} \subseteq \mathcal{F}(\mathcal{A}^{\omega})$ and using that $(\mathcal{J}^\omega)^{-1}[\mathcal{F}_{\omega,\omega}] = \mathcal{F}_{0,\omega}$.

Lemma 6.11. *If \mathcal{A} is 1-stable, then $\mathcal{F}_{0,\omega}$ is dense in $\mathcal{F}_{0,0}$.*

Proof. Fix $\sigma \in \mathcal{F}_{0,0}$ and put $\sigma_0 = \sigma$. We will construct a sequence $(\sigma_n)_{n \in \omega}$ of finite strings such that $\sigma_n \in \mathcal{F}_{n,n}$, and

$$(\mathcal{J}^{n-m})^{-1}(\sigma_n) \subseteq (\mathcal{J}^{n-m+1})^{-1}(\sigma_{n+1}) \in \mathcal{F}_{m,m}$$

for any $m \leq n$. Then we will define $X := \bigcup_n (\mathcal{J}^n)^{-1}(\sigma_n)$ and ensure that $X \in \mathcal{F}_{0,\omega}$, that is, $\mathcal{J}^n(X) \in \mathcal{F}_{n,n}$. Given n , inductively assume that $\sigma_n \in \mathcal{F}_{n,n}$. Now, by the

property (2), $\mathcal{F}_{n,n+1}$ is dense in $\mathcal{F}_{n,n}$ for any $n \in \omega$. Since $\mathcal{F}_{n,n+1} = \mathcal{J}^{-1}[\mathcal{F}_{n+1,n+1}]$, there is $Y \in \mathcal{F}_{n+1,n+1}$ such that

$$\sigma_n \subset \mathcal{J}^{-1}(Y) \in \mathcal{F}_{n,n}.$$

Since \mathcal{J}^{-1} is continuous, we can find an initial segment $\sigma_{n+1} \subset Y$ such that $\sigma_n \subseteq \mathcal{J}^{-1}(\sigma_{n+1})$. Clearly $\sigma_{n+1} \in \mathcal{F}_{n+1,n+1}$. For every $m \leq n$, by continuity of $(\mathcal{J}^{n-m})^{-1}$, we also have

$$(\mathcal{J}^{n-m})^{-1}(\sigma_n) \subseteq (\mathcal{J}^{n-m})^{-1} \circ \mathcal{J}^{-1}(\sigma_{n+1}) = (\mathcal{J}^{n-m+1})^{-1}(\sigma_{n+1})$$

and $(\mathcal{J}^{n-m})^{-1}(\sigma_n)$ is extendible in $(\mathcal{J}^{n-m})^{-1}[\mathcal{F}_{n,n}] = \mathcal{F}_{m,n} \subseteq \mathcal{F}_{m,m}$, that is, $(\mathcal{J}^{n-m})^{-1}(\sigma_n) \in \mathcal{F}_{m,m}$ as wanted.

For $X = \bigcup_n (\mathcal{J}^n)^{-1}(\sigma_n)$, we claim that $\mathcal{J}^m(X) = Y_m := \bigcup_{n \geq m} (\mathcal{J}^{n-m})^{-1}(\sigma_n)$. This is because we have

$$(\mathcal{J}^m)^{-1}(Y_m) = \bigcup_n (\mathcal{J}^m)^{-1} \circ (\mathcal{J}^{n-m})^{-1}(\sigma_n) = \bigcup_n (\mathcal{J}^n)^{-1}(\sigma_n) = X.$$

The first equality is due to continuity of $(\mathcal{J}^m)^{-1}$ and the property that $((\mathcal{J}^{n-m})^{-1}(\sigma_n))_{n \geq m}$ is increasing. Therefore $\mathcal{J}^m(X) = Y_m$. Since $(\mathcal{J}^{n-m})^{-1}(\sigma_n) \in \mathcal{F}_{m,m}$, and $\mathcal{F}_{m,m}$ is closed, we have $\mathcal{J}^m(X) \in \mathcal{F}_{m,m}$ for all $m \in \omega$, and therefore $\sigma \subseteq X \in \mathcal{F}_{0,\omega}$. This shows that $\mathcal{F}_{0,\omega}$ is dense in $\mathcal{F}_{0,0}$. \square

Lemma 6.12. *If \mathcal{A} is 1-stable, then $\mathcal{F}_{\omega,\omega} \subseteq \mathcal{F}(\mathcal{A}^{\omega})$.*

Proof. Take $Z \in \mathcal{F}_{\omega,\omega}$; We want to show that $Z \in \mathcal{F}(\mathcal{A}^{\omega})$. Given $n \in \omega$, we will define a continuous function $\eta : \mathcal{J}^\omega[\omega^\omega] \rightarrow \mathcal{J}^\omega[\omega^\omega] \cap [\mathcal{J}^\omega(X) \upharpoonright n]$ witnessing that $\mathcal{A}^{\omega} \leq_w \mathcal{A}^{\omega} \upharpoonright [Z \upharpoonright n]$.

Let $X \in \omega^\omega = \mathcal{J}^{-\omega}(Z) \in \mathcal{F}_{\omega,0}$. Note that $\mathcal{J}^\omega(X) \upharpoonright n = \langle \mathcal{J}(X) \upharpoonright 1, \dots, \mathcal{J}^n(X) \upharpoonright 1 \rangle$. Recall that $\mathcal{J}^\omega(X) \in \mathcal{F}_{\omega,\omega}$ if and only if $\mathcal{J}^n(X) \in \mathcal{F}_{n,n} = \mathcal{F}(\mathcal{A}_n)$ for all $n \in \omega$. Therefore, there is a continuous reduction

$$\theta_n : \mathcal{A}^{\omega} \leq_w \mathcal{A}_n \upharpoonright [\mathcal{J}^n(X) \upharpoonright 1]$$

since $\mathcal{A}^{\omega} \leq_w \mathcal{A} \leq_w \mathcal{A}_n \leq_w \mathcal{A}_n \upharpoonright [\mathcal{J}^n(X) \upharpoonright 1]$, using Observation 6.10. The objective now is to define a continuous function η so that it mimics θ_n when the input is the an ω -jump instead of an n -jump. That is, we want η so that

$$\eta(\mathcal{J}^\omega(Y))(k) = \mathcal{J}^k(\mathcal{J}^{-n} \circ \theta_n \circ \mathcal{J}^n(Y)) \upharpoonright 1.$$

Since $\mathcal{J}^k(\mathcal{J}^{-n} \circ \theta_n \circ \mathcal{J}^n(Y))$ is clearly uniformly computable from $\mathcal{J}^\omega(Y)$, it is not hard to define such function η . In other words, we have that $\eta(\mathcal{J}^\omega(Y)) = \mathcal{J}^\omega(\mathcal{J}^{-n} \circ \theta_n \circ \mathcal{J}^n(Y))$. Consequently,

$$\begin{aligned} \mathcal{A}^{\omega}(\mathcal{J}^\omega(Y)) &= \mathcal{A}(Y) = \mathcal{A}^{\omega}(\mathcal{J}^n(Y)) \leq_Q \mathcal{A}^{\omega}(\theta_n \circ \mathcal{J}^n(Y)) \\ &= \mathcal{A}(\mathcal{J}^{-n} \circ \theta_n \circ \mathcal{J}^n(Y)) = \mathcal{A}^{\omega}(\mathcal{J}^\omega(\mathcal{J}^{-n} \circ \theta_n \circ \mathcal{J}^n(Y))) = \mathcal{A}^{\omega}(\eta(\mathcal{J}^\omega(Y))). \end{aligned}$$

Since $\eta(\mathcal{J}^\omega(Y))$ extends $\mathcal{J}^\omega(X) \upharpoonright n$, this witnesses that $\mathcal{A}^{\omega} \leq_w \mathcal{A}^{\omega} \upharpoonright [\mathcal{J}^\omega(X) \upharpoonright n]$. \square

This concludes the proof of Lemma 6.9 for $\alpha = 1$.

6.3.2. *Proof of Lemma 6.9 (for general α).* In this section, we describe the proof of Lemma 6.9 for general α , which will be almost no different from the proof for $\alpha = 1$. This section is just for the sake of completeness. We also explicitly describe how to deal with oracles.

Now, fix a countable ordinal α . By induction, we assume that we have already shown the following claim for any $\beta < \alpha$, if \mathcal{A} is β -stable, then for any oracle D , there is an ω^β -oracle $C \geq_T D$ such that

$$(4) \quad \{X : \mathcal{J}^{\omega^\beta, C}(X) \in \mathcal{F}((\mathcal{A} \upharpoonright \mathcal{F}(\mathcal{A}))^{\mathcal{J}^{\omega^\beta, C}})\} \text{ is dense in } \mathcal{F}(\mathcal{A}).$$

We now fix an α -stable function \mathcal{A} . We will define oracles $(C_n)_{n \in \omega}$. Then, for notational simplicity, we will use the following notations:

$$\mathcal{J}_n = \mathcal{J}^{\omega^{\alpha[n]}, C_n}, \quad \mathcal{B}^{\mathcal{J}_n} = \mathcal{B}^{\mathcal{J}^{\omega^{\alpha[n]}, C_n}}.$$

As in the previous section, we inductively define a \mathcal{Q} -valued function \mathcal{A}_n and a closed set $\mathcal{F}_{n,n}$ as follows:

$$\begin{aligned} \mathcal{A}_0 &= \mathcal{A}, & \mathcal{F}_{0,0} &= \mathcal{F}(\mathcal{A}). \\ \mathcal{A}_{n+1} &= (\mathcal{A}_n \upharpoonright \mathcal{F}_{n,n})^{\mathcal{J}_n}, & \mathcal{F}_{n+1,n+1} &= \mathcal{F}(\mathcal{A}_{n+1}). \end{aligned}$$

To define \mathcal{A}_{n+1} , we need to specify oracles $(C_m)_{m \leq n}$. Before defining these oracles, we introduce several notations. We define

$$\begin{aligned} \mathcal{J}_{[m,n]} &= \mathcal{J}_{n-1} \circ \mathcal{J}_{n-2} \circ \cdots \circ \mathcal{J}_{m+1} \circ \mathcal{J}_m, \\ \mathcal{B}_{[m,n]}^{\mathcal{J}} &= ((\dots (\mathcal{B}^{\mathcal{J}_m})^{\mathcal{J}_{m+1}} \dots)^{\mathcal{J}_{n-2}})^{\mathcal{J}_{n-1}}. \end{aligned}$$

Note that the sequences defined in the previous section satisfy $\mathcal{A}_n = \mathcal{A}^{\mathcal{J}_n} \upharpoonright \mathcal{F}_{n,n-1}$. Now, in our new definition, $\mathcal{A}^{\mathcal{J}_n}$ is replaced with $\mathcal{A}_{[0,n]}^{\mathcal{J}}$, that is,

$$\mathcal{A}_n = \mathcal{A}_{[0,n]}^{\mathcal{J}} \upharpoonright \mathcal{F}_{n,n-1}, \text{ where } \mathcal{F}_{n,n-1} = \mathcal{J}[\mathcal{F}_{n-1,n-1}].$$

We now start to define a sequence $(C_n)_{n \in \omega}$ of oracles. Let C_{-1} be an oracle such that $\mathcal{A}^{\mathcal{J}^{\omega^\alpha}} \equiv_w \mathcal{A}^{\mathcal{J}^{\omega^\alpha, C_{-1}}}$. Define $\mathcal{A}_0 = \mathcal{A}$, and assume that $(C_m)_{m < n}$ are defined, and $\mathcal{A} \equiv_w \mathcal{A}_n$ as in Observation 6.10. In particular, \mathcal{A}_n is α -stable. Then, by induction hypothesis (4), and initializability of \mathcal{A}_n , there is an oracle $C \geq_T C_{n-1}$ such that

- (a) $(\mathcal{A}_n \upharpoonright \mathcal{F}_{n,n})^{\mathcal{J}^{\omega^{\beta[n]}, C}} \equiv_w (\mathcal{A}_n \upharpoonright \mathcal{F}_{n,n})^{\mathcal{J}^{\omega^{\beta[n]}}$.
- (b) $(\mathcal{J}^{\omega^{\beta[n]}, C})^{-1}[\mathcal{F}((\mathcal{A}_n \upharpoonright \mathcal{F}_{n,n})^{\mathcal{J}^{\omega^{\beta[n]}, C}})]$ is dense in $\mathcal{F}_{n,n}$.
- (c) For any $\sigma \in \mathcal{F}_{n,n}$, there is a C -computable Wadge-reduction $\mathcal{A}_{[0,n]}^{\mathcal{J}} \leq_w \mathcal{A}_n \upharpoonright \mathcal{F}_{n,n} \cap [\sigma]$ (recall our proof of Lemma 6.12).

Define $C_n = C$ for such C , and then define $\mathcal{C} = (C_n)_{n \in \omega}$. We also define $(\mathcal{F}_{m,n})_{m,n \in \omega}$ as in the previous section. Then, for instance, the above condition (b) can be rephrased

as: $\mathcal{F}_{n,n+1}$ is dense in $\mathcal{F}_{n,n}$. We then get the following commutative diagram:

$$\begin{array}{ccccccc}
 \dots & \hookrightarrow & \mathcal{F}_{n,n+1} & \xrightarrow{\text{dense}} & \mathcal{F}_{n,n} & \hookrightarrow & \mathcal{F}_{n,n-1} & \hookrightarrow & \dots & \hookrightarrow & \mathcal{J}_{[0,n]}[\omega^\omega] & \xrightarrow{\mathcal{A}_{[0,n]}^\omega} & \omega^\omega \\
 & & \mathcal{J}_n^{-1} \uparrow & & \downarrow \mathcal{J}_n & & \downarrow \mathcal{J}_n & & & & \downarrow \mathcal{J}_n & \nearrow \mathcal{A}_{[0,n+1]}^\omega & \uparrow \\
 \dots & \xrightarrow{\text{dense}} & \mathcal{F}_{n+1,n+1} & \hookrightarrow & \mathcal{F}_{n+1,n} & \hookrightarrow & \mathcal{F}_{n+1,n-1} & \hookrightarrow & \dots & \hookrightarrow & \mathcal{J}_{[0,n+1]}[\omega^\omega] & & \\
 & & & & & & & & & & & & \uparrow \\
 & & & & & & & & & & & & \mathcal{A}_{n+1}
 \end{array}$$

Define $\mathcal{F}_{0,\omega} = \bigcap_{n \in \omega} \mathcal{F}_{0,n}$. In other words,

$$\mathcal{F}_{0,\omega} = \{X \in \omega^\omega : (\forall n \in \omega) \mathcal{J}^{[0,n]}(X) \in \mathcal{F}_{n,n}\}.$$

Then we define $\mathcal{F}_{\omega,\omega} = \mathcal{J}^\omega[\mathcal{F}_{0,\omega}]$. As in the previous section, we will show the following claim:

$$(5) \quad (\mathcal{J}^{\omega^\alpha, \mathcal{C}})^{-1}[\mathcal{F}(\mathcal{A}^{\omega^\alpha, \mathcal{C}})] \text{ is dense in } \mathcal{F}(\mathcal{A}).$$

The claim (5) entails that $\mathcal{F}(\mathcal{A}^{\omega^\alpha, \mathcal{C}})$ is nonempty, and therefore $\mathcal{A}^{\omega^\alpha, \mathcal{C}}$ is σ -join-irreducible by Proposition 2.6. Here, since $\mathcal{C} \geq_T C_{-1}$, we have that $\mathcal{A}^{\omega^\alpha, \mathcal{C}} \equiv_w \mathcal{A}^{\omega^\alpha}$. Therefore, the claim (5) implies that $\mathcal{A}^{\omega^\alpha}$ is σ -join-irreducible as desired. Hence, it suffices to show the claim (5) to prove Lemma 6.9. We will use almost the same strategy as in the previous section.

Lemma 6.13. $\mathcal{F}_{0,\omega}$ is dense in $\mathcal{F}_{0,0}$.

Proof. Fix $\sigma \in \mathcal{F}_{0,0}$ and put $\sigma_0 = \sigma$. We will construct a sequence $(\sigma_n)_{n \in \omega}$ of finite strings such that $\sigma_n \in \mathcal{F}_{n,n}$, and

$$\mathcal{J}_{[m,n]}^{-1}(\sigma_n) \subseteq \mathcal{J}_{[m,n+1]}^{-1}(\sigma_{n+1}) \in \mathcal{F}_{m,m}$$

for any $m \leq n$. Then we will define $X := \bigcup_n \mathcal{J}_{[0,n]}^{-1}(\sigma_n)$ and ensure that $X \in \mathcal{F}_{0,\omega}$, that is, $\mathcal{J}_{[0,n]}(X) \in \mathcal{F}_{n,n}$. Given n , inductively assume that $\sigma_n \in \mathcal{F}_{n,n}$. Now, by the property (b), $\mathcal{F}_{n,n+1}$ is dense in $\mathcal{F}_{n,n}$ for any $n \in \omega$. Since $\mathcal{F}_{n,n+1} = \mathcal{J}_n^{-1}[\mathcal{F}_{n+1,n+1}]$, there is $Y \in \mathcal{F}_{n+1,n+1}$ such that

$$\sigma_n \subset \mathcal{J}_n^{-1}(Y) \in \mathcal{F}_{n,n}.$$

Since \mathcal{J}_n^{-1} is continuous, we can find an initial segment $\sigma_{n+1} \subset Y$ such that $\sigma_n \subseteq \mathcal{J}_n^{-1}(\sigma_{n+1})$. Clearly $\sigma_{n+1} \in \mathcal{F}_{n+1,n+1}$. For every $m \leq n$, by continuity of $\mathcal{J}_{[m,n]}^{-1}$, we also have

$$\mathcal{J}_{[m,n]}^{-1}(\sigma_n) \subseteq \mathcal{J}_{[m,n]}^{-1} \circ \mathcal{J}_n^{-1}(\sigma_{n+1}) = \mathcal{J}_{[m,n+1]}^{-1}(\sigma_{n+1})$$

and $\mathcal{J}_{[m,n]}^{-1}(\sigma_n)$ is extendible in $\mathcal{J}_{[m,n]}^{-1}[\mathcal{F}_{n,n}] = \mathcal{F}_{m,n} \subseteq \mathcal{F}_{m,m}$, that is, $\mathcal{J}_{[m,n]}^{-1}(\sigma_n) \in \mathcal{F}_{m,m}$ as wanted.

For $X = \bigcup_n \mathcal{J}_{[0,n]}^{-1}(\sigma_n)$, we claim that $\mathcal{J}_{[0,m]}(X) = Y_m := \bigcup_{n \geq m} \mathcal{J}_{[m,n]}^{-1}(\sigma_n)$. This is because we have

$$\mathcal{J}_{[0,m]}^{-1}(Y_m) = \bigcup_{n \geq m} \mathcal{J}_{[0,m]}^{-1} \circ \mathcal{J}_{[m,n]}^{-1}(\sigma_n) = \bigcup_n \mathcal{J}_{[0,n]}^{-1}(\sigma_n) = X.$$

The first equality is due to continuity of $\mathcal{J}_{[0,m]}^{-1}$ and the property that $(\mathcal{J}_{[m,n]}^{-1}(\sigma_n))_{n \geq m}$ is increasing. Therefore $\mathcal{J}_{[0,m]}(X) = Y_m$. Since $\mathcal{J}_{[m,n]}^{-1}(\sigma_n) \in \mathcal{F}_{m,m}$, and $\mathcal{F}_{m,m}$ is closed,

we have $\mathcal{J}_{[0,m]}(X) \in \mathcal{F}_{m,m}$ for all $m \in \omega$, and therefore $\sigma \subset X \in \mathcal{F}_{0,\omega}$. This shows that $\mathcal{F}_{0,\omega}$ is dense in $\mathcal{F}_{0,0}$. \square

For notational simplicity, we use the following notations:

$$\mathcal{J}_\omega = \mathcal{J}^{\omega^\alpha, \mathcal{C}}, \quad \mathcal{A}_\omega^\zeta = \mathcal{A}^{\zeta^{\omega^\alpha, \mathcal{C}}}.$$

Lemma 6.14. *If \mathcal{A} is α -stable, then $\mathcal{F}_{\omega,\omega} \subseteq \mathcal{F}(\mathcal{A}_\omega^\zeta)$.*

Proof. Fix $X \in \omega^\omega$ such that $\mathcal{J}_\omega(X) \in \mathcal{F}_{\omega,\omega}$. Given $n \in \omega$, we will define a continuous function $\eta : \mathcal{J}_\omega[\omega^\omega] \rightarrow \mathcal{J}_\omega[\omega^\omega] \cap [\mathcal{J}_\omega(X) \upharpoonright n]$ witnessing that $\mathcal{A}_\omega^\zeta \leq_w \mathcal{A}_\omega^\zeta \upharpoonright [\mathcal{J}_\omega(X) \upharpoonright n]$. Note that $\mathcal{J}_\omega(X) \upharpoonright n = \langle \mathcal{J}_{[0,1]}(X) \upharpoonright 1, \dots, \mathcal{J}_{[0,n]}(X) \upharpoonright 1 \rangle$. Note also that $\mathcal{J}_\omega(X) \in \mathcal{F}_{\omega,\omega}$ if and only if $\mathcal{J}_{[0,n]}(X) \in \mathcal{F}_{n,n}$ for all $n \in \omega$. By the condition (c), Player II has a C_n -computable Wadge reduction $\theta_n : \mathcal{A}_{[0,n]}^\zeta \leq_w \mathcal{A}_n \upharpoonright [\mathcal{J}_{[0,n]}(X) \upharpoonright 1]$. We let η be a continuous function such that for any k ,

$$\eta(\mathcal{J}_\omega(Y))(k) = \mathcal{J}_{[0,k]}(\mathcal{J}_{[0,n]}^{-1} \circ \theta_n \circ \mathcal{J}_{[0,n]}(Y)) \upharpoonright 1.$$

In other words, $\eta(\mathcal{J}_\omega(Y)) = \mathcal{J}_\omega(\mathcal{J}_{[0,n]}^{-1} \circ \theta_n \circ \mathcal{J}_{[0,n]}(Y))$. Consequently,

$$\begin{aligned} \mathcal{A}_\omega^\zeta(\mathcal{J}_\omega(Y)) &= \mathcal{A}(Y) = \mathcal{A}_{[0,n]}^\zeta(\mathcal{J}_{[0,n]}(Y)) \leq_{\mathcal{Q}} \mathcal{A}_{[0,n]}^\zeta(\theta_n \circ \mathcal{J}_{[0,n]}(Y)) \\ &= \mathcal{A}(\mathcal{J}_{[0,n]}^{-1} \circ \theta_n \circ \mathcal{J}_{[0,n]}(Y)) = \mathcal{A}_\omega^\zeta(\mathcal{J}_\omega(\mathcal{J}_{[0,n]}^{-1} \circ \theta_n \circ \mathcal{J}_{[0,n]}(Y))) = \mathcal{A}_\omega^\zeta(\eta(\mathcal{J}_\omega(Y))). \end{aligned}$$

Since $\eta(\mathcal{J}_\omega(Y))$ extends $\mathcal{J}_\omega(X) \upharpoonright n$, this witnesses that $\mathcal{A}_\omega^\zeta \leq_w \mathcal{A}_\omega^\zeta \upharpoonright [\mathcal{J}_\omega(X) \upharpoonright n]$. \square

This concludes the proof of Lemma 6.9.

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