ON THE STRUCTURE OF THE WADGE DEGREES OF BQO-VALUED BOREL FUNCTIONS

TAKAYUKI KIHARA AND ANTONIO MONTALBÁN

Abstract. In this article, we give a full description of the Wadge degrees of Borel functions from ω^ω to a better-quasi-ordering $\mathcal{Q}$. More precisely, for any countable ordinal ξ, we show that the Wadge degrees of $\mathbf{\Delta}^0_{1+\xi}$-measurable functions $\omega^\omega \to \mathcal{Q}$ can be represented by countable joins of the ξ-th transfinite nests of $\mathcal{Q}$-labeled well-founded trees.

Contents

[0] Saved: April 14, 2018 Last modified by T.
Compiled: April 14, 2018

The first-named author was partially supported by JSPS KAKENHI grant 17H06738, 15H03634, and the JSPS Core-to-Core Program (A. Advanced Research Networks).

The second-named author was partially supported by NSF grant DMS-0901169 and the Packard Fellowship.

1. Introduction

In his doctorate thesis [Wad83], Wadge proposed a notion of reducibility between sets of reals that is not only natural, but also surprisingly well behaved, as opposed to most computability theoretic reducibilities which have a rather messy structure.

Definition 1.1 (Wadge [Wad83])**.** Given $\mathcal{A}, \mathcal{B} \subseteq \omega^\omega$, we say that $\mathcal{A}$ is *Wadge reducible* to $\mathcal{B}$, and write $\mathcal{A} \leq_w \mathcal{B}$, if there is a continuous function $f\colon \omega^\omega \to \omega^\omega$ such that $X \in \mathcal{A} \iff f(X) \in \mathcal{B}$ for all $X \in \omega^\omega$.

The relation $\leq_w$ is a pre-ordering, and, as usual, it induces an equivalence $\equiv_w$ and a degree structure. Wadge showed the Wadge degrees are semi-linearly-ordered in the sense that all anti-chains have size at most 2. Then, Martin and Monk showed they are well-founded. (This is all assuming Γ-determinacy when dealing with sets in a pointclass Γ.) Furthermore, each Wadge degree is in a sense *natural*, and can be assigned a *name* using an ordinal less than Θ and a symbol from $\{\Delta, \Sigma, \Pi\}$ ([VW78]; see also the Cabal volume [KLS12]), a name from which we can understand the nature of that Wadge degree. Based on this perspective, Duparc [Dup01, Dup] gave an explicit description of each Borel Wadge degree of a subset of ω^ω.

The Wadge degrees were later extended in various directions. We can encapsulate all those extensions within the following framework:

Definition 1.2. Let $(\mathcal{Q}; \leq_\mathcal{Q})$ be a partial ordering. For $\mathcal{Q}$-valued functions $\mathcal{A}, \mathcal{B}\colon \omega^\omega \to \mathcal{Q}$, we say that $\mathcal{A}$ *is* $\mathcal{Q}$*-Wadge reducible to* $\mathcal{B}$ (written $\mathcal{A} \leq_w \mathcal{B}$) if there is a continuous function $\theta\colon \omega^\omega \to \omega^\omega$ such that

$$(\forall X \in \omega^\omega)\ \mathcal{A}(X) \leq_\mathcal{Q} \mathcal{B}(\theta(X)).$$

The original Wadge degrees are the case $\mathcal{Q} = 2$ in the definition above, coding sets by their characteristic functions $\omega^\omega \to 2$ and viewing 2 as the partial ordering with two incomparable elements 0 and 1.

The first extension already considered by Wadge [Wad83, Section 1.E], was to *partial* functions $\omega^\omega \to \{0, 1\}$, or equivalently, total functions $\omega^\omega \to \{\bot, 0, 1\}$, where $\bot$ is thought of as being below both 0 and 1, which are incomparable with each other. The degree structure we obtain is also semi-well-ordered, but slightly different than the structure of the Wadge degrees. These degrees are connected to recent work of Day, Downey, and Westrick [DDW17], as observed by Kihara [Kih17].

Shortly after, Steel studied the Wadge degrees of ordinal-valued functions with domain ω^ω, and showed they are well-ordered (see [Dup03, Theorem 1]). Later, Steel, van Engelen and Miller [vEMS87] employed bqo theory to unify these results, and showed that if $\mathcal{Q}$ is better-quasi-ordered (bqo, see Definition 2.1), then so is the poset of the Wadge degrees of $\mathcal{Q}$-valued Borel functions. We delay the definition of better-quasi-ordering until Definition 2.1, and for now let us just say that better-quasi-orderings are well-founded, have no infinite antichains, and have very good closure properties. van Engelen, Miller and Steel's results are even more surprising than Wadge–Martin–Monk's semi-well-orderness of the 2-Wadge degrees: Naturally defined better-quasi-orders usually have a well-behaved, easy-to-visualize structure.

For a bqo $\mathcal{Q}$, the $\mathcal{Q}$-Wadge degrees are recently found to play an important role in computability theory. In the context of uniform Martin's conjecture, the authors

[KM] showed that there is a natural isomorphism between the structure of $\mathcal{Q}$-Wadge degrees and that of the "natural" many-one degrees of $\mathcal{Q}$-valued problems. Hence, exploring $\mathcal{Q}$-Wadge degrees is the same thing as exploring natural $\mathcal{Q}$-many-one degrees[1]. The objective of this paper is to describe the structure of the $\mathcal{Q}$-Wadge degrees by showing that it is isomorphic to another partial ordering that is easier to visualize and understand.

In the last decade, Selivanov [Sel07, Sel11] started studying the case of *k-partitions*, that is, the case when $\mathcal{Q} = k$, the poset with k incomparable elements for finite k. Selivanov [Sel07] gave a full description of the Wadge degrees of $\mathbf{\Delta}^0_2$ k-partitions, naming each such degree by a k-labeled well-founded forest, in a way that the name describes the nature of the k-Wadge degree. What he does is essentially a generalization of the Hausdroff-Kuratowski hierarchy from $k = 2$ to larger k's, where the structure becomes much richer. More precisely, for a set $\mathcal{Q}$, let $\mathrm{Tree}(\mathcal{Q})$ be the set of all $\mathcal{Q}$-labeled well-founded countable trees, and let ${}^{\sqcup}\mathrm{Tree}(\mathcal{Q})$ be the set of all $\mathcal{Q}$-labeled well-founded countable forests. Note that every such forest F can be thought of as a collection (or a disjoint union) of countably many $\mathcal{Q}$-labeled well-founded countable trees. Hertling introduced a quasi-order $\trianglelefteq$ on ${}^{\sqcup}\mathrm{Tree}(k)$, given by $S \trianglelefteq T$ if there is a homomorphism from S to T which preserves inclusion of strings ($\subseteq$) and preserves labels (as defined in Section 3.1.1).

Theorem 1.3 (Selivanov [Sel07])**.** *Let $k \in \omega$. The Wadge degree structure of the $\mathbf{\Delta}^0_2$-measurable k-valued functions is isomorphic to the quotient order of $({}^{\sqcup}\mathrm{Tree}(k); \trianglelefteq)$ on well-founded k-labeled forests.*

We have recently learned that Selivanov has extended his result to the class of Δ^0_3 k-partitions, using forests labeled with labeled trees [Sel17a, Sel17b]. His techniques are very different from ours.

The objective of this paper is to give a description of the Wadge degrees of Borel functions $\omega^\omega \to \mathcal{Q}$, where $\mathcal{Q}$ is any better-quasi-ordering (bqo), generalizing Selivanov's results from Δ^0_3 to all Borel functions and from finite k to all bqos $\mathcal{Q}$.

To name the Wadge degrees of $\mathbf{\Delta}^0_{n+1}$-measurable $\mathcal{Q}$-valued functions, we will use trees labeled by trees labeled by trees ... labeled by $\mathcal{Q}$. That is, we will define $\mathrm{Tree}^n(\mathcal{Q})$ as $\mathrm{Tree}(\mathrm{Tree}(\cdots\mathrm{Tree}(\mathcal{Q})\cdots))$ iterated n times, then define ${}^{\sqcup}\mathrm{Tree}^n(\mathcal{Q})$ as the disjoint unions of these trees (see Section 3.1.2). We think of each forest $T \in {}^{\sqcup}\mathrm{Tree}^n(\mathcal{Q})$ as a process of mind-changes which captures a natural class Σ_T of $\mathbf{\Delta}^0_{n+1}$-measurable functions. Based on this viewpoint, we will then define a quasi-order $\trianglelefteq$ on ${}^{\sqcup}\mathrm{Tree}^n(\mathcal{Q})$ that matches Wadge reducibility on the classes of functions described by these forests.

Theorem 1.4. *Let $\mathcal{Q}$ be a bqo. Then, the Wadge degree structure of the $\mathbf{\Delta}^0_{n+1}$-measurable $\mathcal{Q}$-valued functions is isomorphic to the quotient order of $({}^{\sqcup}\mathrm{Tree}^n(\mathcal{Q}), \trianglelefteq)$.*

Note that the restriction of our quasi-order $\trianglelefteq$ to ${}^{\sqcup}\mathrm{Tree}^n(\mathcal{Q})$ is essentially equivalent to the homomorphic quasi-order which has been studied in, e.g. [Sel07, Sel17a, Sel17b]. Using the terminology of the homomorphic quasi-order, [Sel17a, Sel17b] has proposed an

[1]Selivanov [Sel83, Sel95] has also looked at a notion of naturalness (which is, a priori, entirely different from [KM]) to connect hierarchies of many-one degrees and Wadge degrees, although his naturalness has nothing to do with Martin's conjecture (and, in particular, with "natural" solutions to Post's problem).

idea of a strategy for proving Theorem 1.4 for $\mathcal{Q} = k$ (where the arXiv version of [Sel17a] has appeared in 2014, which includes a few words mentioning his idea); however, the proposed ideas are again completely different from ours. Moreover, as seen in Section 3.4, the idea of homomorphic quasi-order does not work when we move to infinite Borel ranks, while our quasi-ordering $\trianglelefteq$ naturally extends to infinitary versions. To extend Theorem 1.4 through the Borel hierarchy, we will introduce the ξ-th iterated version ${}^{\sqcup}\mathrm{Tree}^{\xi}(\mathcal{Q})$ for each countable ordinal ξ, and show the following transfinite version:

Theorem 1.5. *Let $\mathcal{Q}$ be a bqo, and ξ be a countable ordinal. Then, the Wadge degree structure of the $\mathbf{\Delta}^0_{1+\xi}$-measurable $\mathcal{Q}$-valued functions is isomorphic to the quotient order of $({}^{\sqcup}\mathrm{Tree}^{\xi}(\mathcal{Q}), \trianglelefteq)$.*

We deal with functions of finite Borel rank and prove Theorem 1.4 in Sections 3–5. We will then extend those ideas to infinite Borel rank and prove Theorem 1.5 in Section 6.

The main steps for the proof are as follows. First, we need to formally define ${}^{\sqcup}\mathrm{Tree}^{\xi}(\mathcal{Q})$ and the ordering $\trianglelefteq$. Then, as suggested above, in Section 3.2, we will assign a pointclass Σ_T of $\mathcal{Q}$-valued functions to each forest $T \in {}^{\sqcup}\mathrm{Tree}^{\xi}(\mathcal{Q})$. For instance, $\Sigma_{\langle 0\rangle\to\langle 1\rangle}$ is the class of characteristic functions of $\mathbf{\Sigma}^0_1$ sets, and $\Sigma_{\langle 0\rangle\to\langle 1\rangle\to\langle 0\rangle}$ is the class of characteristic functions of sets which are differences of two open sets.

Proposition 1.6. *For every $T \in {}^{\sqcup}\mathrm{Tree}^{\xi}(\mathcal{Q})$, every function in Σ_T is $\mathbf{\Delta}^0_{1+\xi}$-measurable.*

These pointclasses will match the ordering $\trianglelefteq$ on forests in the following sense:

Proposition 1.7. *For $S, T \in {}^{\sqcup}\mathrm{Tree}^{\xi}(\mathcal{Q})$, $S \trianglelefteq T$ if and only if every Σ_S function is Wadge reducible to some Σ_T function.*

For a pointclass Σ_T, we will define a Σ_T-complete function Ω_T, that is, Ω_T is in Σ_T and any other function in Σ_T is Wadge reducible to Ω_T.

Proposition 1.8. *For each $T \in {}^{\sqcup}\mathrm{Tree}^{\xi}(\mathcal{Q})$, there is a Σ_T-complete function Ω_T.*

We can then restate Proposition 1.7 as $S \trianglelefteq T \iff \Omega_S \leq_w \Omega_T$. This gives us an embedding of ${}^{\sqcup}\mathrm{Tree}^{\xi}(\mathcal{Q})$ into the $\Delta_{1+\xi}$ $\mathcal{Q}$-Wadge degrees. The last step is to show that this embedding is onto.

Proposition 1.9. *Every $\mathbf{\Delta}^0_{1+\xi}$-measurable function $\mathcal{A} : \omega^\omega \to \mathcal{Q}$ is Wadge equivalent to a Σ_T-complete function for some $T \in {}^{\sqcup}\mathrm{Tree}^{\xi}(\mathcal{Q})$.*

Moreover, if $\mathcal{A}$ is non-self-dual (Definition 2.5), then T can be chosen from $\mathrm{Tree}^{\xi}(\mathcal{Q})$.

2. The $\mathcal{Q}$-Wadge degrees

Let us start by describing what we knew about the structure of the $\mathcal{Q}$-Wadge degrees.

2.1. **The Borel hierarchy of functions.** We should be careful here, as there are several different definitions of the Borel hierarchy (specifically, at limit ranks). We adopt the following definition: For $\alpha > 0$, a set $\mathcal{S} \subseteq \omega^\omega$ is $\mathbf{\Sigma}^0_\alpha$ if $\mathcal{S}$ can be written as $\mathcal{S} = \bigcup_{n\in\omega} \mathcal{S}_n$ where each $\mathcal{S}_n$ is $\mathbf{\Pi}^0_{\beta_n}$ for some $\beta_n < \alpha$. Then, we define $\mathbf{\Pi}^0_\alpha$ and $\mathbf{\Delta}^0_\alpha$ in the usual manner. For a countable ordinal ξ, and a topological space $\mathcal{X}$, a function $\mathcal{A}\colon \omega^\omega \to \mathcal{X}$ is $\mathbf{\Sigma}^0_\xi$*-measurable* if $\mathcal{A}^{-1}[U]$ is $\mathbf{\Sigma}^0_\xi$ for each open set $U \subseteq \mathcal{X}$. In particular: $\mathcal{A}\colon \omega^\omega \to \omega^\omega$ is $\mathbf{\Sigma}^0_\xi$-measurable if $\mathcal{A}^{-1}[\sigma]$ is $\mathbf{\Sigma}^0_\xi$ for each $\sigma \in \omega^{<\omega}$, where $[\sigma] = \{X \in \omega^\omega : \sigma \subseteq X\}$; $\mathcal{A}\colon \omega^\omega \to \mathcal{Q}$ is $\mathbf{\Sigma}^0_\xi$-measurable if it is with respect to the discrete topology on $\mathcal{Q}$. If $\mathcal{Q}$ is a discrete space, the class of total $\mathbf{\Sigma}^0_\xi$-measurable functions $\omega^\omega \to \mathcal{Q}$ is the same as that of $\mathbf{\Delta}^0_\xi$-measurable functions. Note that the range of a Borel function from ω^ω to a discrete space is countable; otherwise ZFC would prove the existence of $2^{\aleph_1}$ many pairwise different Borel subsets of ω^ω. Thus, $\mathcal{A} : \omega^\omega \to \mathcal{Q}$ is $\mathbf{\Delta}^0_\xi$-measurable if and only if the range of $\mathcal{A}$ is countable, and $\mathcal{A}^{-1}[\{q\}]$ is $\mathbf{\Delta}^0_\xi$ for any $q \in \mathcal{Q}$. Since we will be dealing with Borel functions $\mathcal{A}\colon \omega^\omega \to \mathcal{Q}$, they will always have countable range. We can thus assume from the rest of the paper that $\mathcal{Q}$ is actually countable, even though all the results will extend to uncountable $\mathcal{Q}$ for functions with countable range.

Continuous functions are exactly the $\mathbf{\Sigma}^0_1$-measurable functions. For each continuous function G there is a partial computable operator $\Phi_e\colon \omega^{\leq\omega} \to \omega^{\leq\omega}$ and an oracle $C \in \omega^\omega$ such that $G(X) = \Phi_e(C \oplus X)$ for all $X \in \omega^\omega$. Also, we will often identify a continuous function $\omega^\omega \to \omega^\omega$ with its corresponding approximation function $\omega^{\leq\omega} \to \omega^{\leq\omega}$.

For functions $\mathcal{A}, \mathcal{B}\colon \omega^\omega \to \omega^\omega$, we say that $\mathcal{A}$ is *Wadge reducible* to $\mathcal{B}$ if there is a continuous function θ such that $\mathcal{A} = \mathcal{B} \circ \theta$. Note that this matches Definition 1.2 if we think of $\mathcal{Q}$ as ω^ω where every two reals are incomparable under $\leq_Q$.

2.2. **Wadge degrees and games.** Wadge [Wad83, Theorem B8] introduced a perfect-information, infinite, two-player game, known as the *Wadge game*, which can be used to define Wadge reducibility. For $\mathcal{Q}$-valued functions $\mathcal{A}, \mathcal{B}\colon \omega^\omega \to \mathcal{Q}$, here is the $\mathcal{Q}$-valued version $G_w(\mathcal{A}, \mathcal{B})$ of the Wadge game: At the n-th round of the game, Player I chooses $x_n \in \omega$ and II chooses $y_n \in \omega \cup \{\mathsf{pass}\}$ (where $\mathsf{pass} \notin \omega$). Eventually Players I and II produce infinite sequences $X = (x_n)_{n\in\omega}$ and $Y = (y_n)_{n\in\omega}$, respectively. Let Y^{p} denote the result dropping all passes from Y. We say that *Player II wins the game* $G_w(\mathcal{A}, \mathcal{B})$ if

$$Y^{\mathsf{p}} \text{ is an infinite sequence, and } \mathcal{A}(X) \leq_{\mathcal{Q}} \mathcal{B}(Y^{\mathsf{p}}).$$

As in Wadge [Wad83, Theorem B8], one can easily check that $\mathcal{A} \leq_w \mathcal{B}$ holds if and only if Player II has a winning strategy for the game $G_w(\mathcal{A}, \mathcal{B})$. We will often identify a winning strategy with a continuous function generated by it.

2.3. **Better quasi orderings.** We will not use the precise definition of bqo — we will add it for completeness. What we will use is Theorem 2.3 below that guarantees that the Wedge degree are well-founded when Q is a bqo.

To define bqos, we need to introduce some notation. Let $[\omega]^\omega$ be the set of all strictly increasing sequences on ω, whose topology is inherited from ω^ω. We also assume that a quasi-order $\mathcal{Q}$ is equipped with the discrete topology. Given $X \in [\omega]^\omega$, let X^- denote the result of dropping the first entry from X (or equivalently, $X^- = X \setminus \{\min X\}$, if we think of $X \in [\omega]^\omega$ as an infinite subset of ω).

Definition 2.1 (Nash-Williams [NW65])**.** A quasi-order $\mathcal{Q}$ is a *better-quasi-order* (abbreviated as bqo) if, for any continuous function $f\colon [\omega]^\omega \to \mathcal{Q}$, there is $X \in [\omega]^\omega$ such that $f(X) \leq_{\mathcal{Q}} f(X^-)$.

The formulation of the definition above is due to Simpson [Sim85]. He also shows that one can use Borel functions f in the definition and obtain the same notion.

Example 2.2. For a natural number k, the discrete order $\mathcal{Q} = (k; =)$, denoted by k, is a bqo. More generally, every finite partial ordering is a bqo.

Every bqo is also a *well-quasi-order* (often abbreviated as wqo), that is, is well-founded and has no infinite antichain. Bqo's were introduced by Nash-Williams to prove wqo results, as bqo's have better closure properties than wqo's under infinitary operations. For instance, Laver [Lav78] showed that if $\mathcal{Q}$ is a bqo, then so are $\mathrm{Tree}(\mathcal{Q})$ ordered by $\trianglelefteq$, and the class of scattered Q-labeled linear orderings ordered by $\leq_Q$-preserving embeddability. The most relevant such result for us is the following:

Theorem 2.3 (van Engelen–Miller–Steel [vEMS87, Theorem 3.2])**.** *If $\mathcal{Q}$ is a bqo, then the Wadge degrees of $\mathcal{Q}$-valued Borel functions on ω^ω form a bqo too.*

2.4. **Self-duality and join-reducibility.** Two important notions when trying to understand the notion of the $\mathcal{Q}$-Wadge degrees is that of σ-join-reducibility and self-duality.

Definition 2.4. We say that a $\mathcal{Q}$-Wadge degree $\mathbf{a}$ is *σ-join-reducible* if $\mathbf{a}$ is the least upper bound of a countable collection $(\mathbf{b}_i)_{i\in\omega}$ of $\mathcal{Q}$-Wadge degrees such that $\mathbf{b}_i <_w \mathbf{a}$. Otherwise, we say that $\mathbf{a}$ is *σ-join-irreducible.*

Definition 2.5 (Louveau and Saint-Raymond [LSR90])**.** We say that a function $\mathcal{A}\colon \omega^\omega \to \mathcal{Q}$ is *self-dual* if there is a continuous function $\theta\colon \omega^\omega \to \omega^\omega$ such that $\mathcal{A}(\theta(X)) \not\leq_{\mathcal{Q}} \mathcal{A}(X)$ for all $X \in \omega^\omega$.

For example, in the case $\mathcal{Q} = 2$, the Δ Wadge degrees are the self-dual ones, and the Σ's and the Π's are not. Also, each Δ degree of successor rank is the least upper bound of the Σ degree and the Π degree immediately below it.

Before stating the equivalence of these two notions, the following definition gives us a useful tool to study the Wadge degree of a function $\mathcal{A}\colon \omega^\omega \to \mathcal{Q}$. For $\sigma \in \omega^{<\omega}$, define the function $\mathcal{A}\restriction[\sigma]$ by $(\mathcal{A}\restriction[\sigma])(X) = \mathcal{A}(\sigma^\frown X)$ for any $X \in \omega^\omega$ (see also Observations 3.5 and 3.6), where $\sigma^\frown X$ is the concatenation of σ and X. Notice that for each $\sigma \in \omega^{<\omega}$, $\mathcal{A}\restriction[\sigma] \leq_w \mathcal{A}$ by essentially the identity operation. For some of these σ we will have $\mathcal{A}\restriction[\sigma] \equiv_w \mathcal{A}$ and for some $\mathcal{A}\restriction[\sigma] <_w \mathcal{A}$. Define

$$\mathcal{F}(\mathcal{A}) = \{X : (\forall n)\ \mathcal{A}\restriction[X\restriction n] \equiv_w \mathcal{A}\}.$$

One more definition, given $\mathcal{A}_n\colon \omega^\omega \to \mathcal{Q}$, $\bigoplus_{n\in\omega} \mathcal{A}_n$ is defined by

$$(\bigoplus_{n\in\omega} \mathcal{A}_n)(n^\frown X) = \mathcal{A}_n(X).$$

Proposition 2.6. *Let $\mathcal{Q}$ be a bqo and $\mathcal{A}\colon \omega^\omega \to \mathcal{Q}$ a Borel function. The following are equivalent:*

(1) *$\mathcal{A}$ is σ-join-reducible.*
(2) *$\mathcal{A} \equiv_w \bigoplus_{n\in\omega} \mathcal{A}_n$, for some $\mathcal{A}_n$ which are σ-join-irreducible and $\mathcal{A}_n <_w \mathcal{A}$.*
(3) *$\mathcal{F}(\mathcal{A})$ is empty.*
(4) *$\mathcal{A}$ is self-dual.*

Proof. The equivalence between (1) and (4) was proved by Block [Blo14, Proposition 3.5.4], and is a generalization of Steel–van Wesep's theorem [VW78] from $\mathcal{Q} = 2$ to general $\mathcal{Q}$.

Let us prove (3)⇒(1). Suppose $\mathcal{F}(\mathcal{A})$ is empty, and let V be the set of minimal strings in $\omega^{<\omega}$ such that $\mathcal{A}\restriction[\sigma] <_w \mathcal{A}$. Then $\{[\sigma] : \sigma \in V\}$ is a clopen partition of ω^ω. It is not hard to see that $\mathcal{A} \equiv_w \bigoplus_{\sigma\in V} \mathcal{A}\restriction[\sigma]$, and hence that $\mathcal{A}$ is σ-join-reducible.

For the direction (1)⇒(2), suppose that $\mathcal{A}$ is σ-join-reducible, and that its Wadge degree is the least upper bound of $\mathcal{B}_i$, for $i \in \omega$, with $\mathcal{B}_i <_w \mathcal{A}$. Since $\mathcal{B}_j \leq_w \bigoplus_{i\in\omega} \mathcal{B}_i$ for all $j \in \omega$, we get that $\mathcal{A} \leq_w \bigoplus_{i\in\omega} \mathcal{B}_i$. Furthermore, since $\mathcal{Q}$-Wadge degrees are bqo, we can use transfinite induction and assume that each $\mathcal{B}_i$ is either σ-join-irreducible or a sum of σ-join-irreducibles. We would then get that $\mathcal{A}$ is itself equivalent to a sum of σ-join-irreducibles.

For (2)⇒(3), let θ witness that $\mathcal{A} \leq_w \bigoplus_{i\in\omega} \mathcal{A}_i$. For each $X \in \omega^\omega$, there exists n such that $\theta(X\restriction n)$ is non-empty. If i is the first entry of $\theta(X\restriction n)^{\mathsf{p}}$, we then get that θ witnesses that $\mathcal{A}\restriction[X\restriction n] \leq_w \mathcal{A}_i <_w \mathcal{A}$. It follows that $X \notin \mathcal{F}(\mathcal{A})$ and hence that $\mathcal{F}(\mathcal{A})$ is empty. □

Note that the equivalence between (1)–(3) is the $\mathcal{Q}$-version of the standard fact in the Wadge degree theory, cf. [Dup01].

2.5. **Conciliatory functions.** There is another way of characterizing non-self-dual functions, and it is using conciliatory functions. Essentially, these are functions whose domain is $\omega^{\leq\omega}$ instead of just ω^ω. For a Borel function $\mathcal{A}\colon \omega^\omega \to \mathcal{Q}$, it will follow from our results that $\mathcal{A}$ is non-self-dual if and only if it can be extended to a function $\tilde{\mathcal{A}}\colon \omega^{\leq\omega} \to \mathcal{Q}$ that is Wadge equivalent to $\mathcal{A}$ (in the sense that we describe below). This was proved by Duparc [Dup01] for $\mathcal{Q} = 2$ — he actually introduced the notion of a conciliatory set. We generalize the notion of a conciliatory set in the $\mathcal{Q}$-valued setting and prove this result as a consequence of Proposition 1.9 and Observation 3.15.

To be able to deal with Wadge reducibility and with complexity pointclasses, we will use the following representation of conciliatory functions. Fix a symbol 'pass' and define

$$\hat{\omega} = \omega \cup \{\mathsf{pass}\}.$$

Given $X \in \hat{\omega}^\omega$, we use the notation $X^{\mathsf{p}} \in \omega^{\leq\omega}$ to denote the string obtained by removing all pass's from X (see also the definition of the Wadge game; Section 2.2).

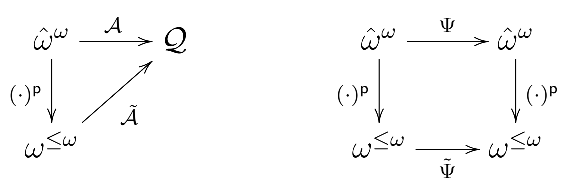
Definition 2.7. A function $\mathcal{A}\colon \hat{\omega}^\omega \to \mathcal{Q}$ is *conciliatory* if

$$(\forall X, Y \in \hat{\omega}^\omega)\ [X^{\mathsf{p}} = Y^{\mathsf{p}} \implies \mathcal{A}(X) = \mathcal{A}(Y)].$$

A function $\Psi\colon \hat{\omega}^\omega \to \hat{\omega}^\omega$ is *conciliatory* if

$$(\forall X, Y \in \hat{\omega}^\omega)\ [X^{\mathsf{p}} = Y^{\mathsf{p}} \implies \Psi(X)^{\mathsf{p}} = \Psi(Y)^{\mathsf{p}}].$$

In other words, there are $\tilde{A} : \omega^{\leq\omega} \to \mathcal{Q}$ and $\tilde{\Psi} : \omega^{\leq\omega} \to \mathcal{Q}$ such that the following diagrams commute:



Conciliatory functions are in one-to-one correspondence with functions $\omega^{\leq\omega} \to \mathcal{Q}$ and $\omega^{\leq\omega} \to \omega^{\leq\omega}$ respectively. However, when we think of their Wadge degrees and of their complexity, it is better to think of them as maps defined on $\hat{\omega}^\omega$. The obvious topology to give to $\hat{\omega}^\omega$ is the product topology of the discrete space $\hat{\omega}$, which is homeomorphic to ω^ω (just because there is a bijection between $\hat{\omega}$ and ω). We will thus treat $\hat{\omega}^\omega$ exactly as we treat ω^ω when we define complexity classes of sets and functions. For instance, a Wadge reduction between conciliatory functions $\mathcal{A}\colon \hat{\omega}^\omega \to \mathcal{Q}$ and $\mathcal{B}\colon \hat{\omega}^\omega \to \mathcal{Q}$, would be a continuous function $\theta\colon \hat{\omega}^\omega \to \hat{\omega}^\omega$ which is not necessarily conciliatory. Thus, this function θ is not necessarily well-defined as a function on $\omega^{\leq\omega}$.

Via the identification between $\hat{\omega}^\omega$ and ω^ω, conciliatory functions are just a special class of functions on ω^ω. Then, for instance, we can transform a conciliatory function $\mathcal{A}\colon \hat{\omega}^\omega \to \mathcal{Q}$ into a function $\underline{\mathcal{A}}\colon \omega^\omega \to \mathcal{Q}$ which is Wadge equivalent to $\mathcal{A}$. Thus, the conciliatory Wadge degrees are just a subset of the standard Wadge degrees of functions on ω^ω. However, they will be very useful to us when we define the Σ_T-complete functions Ω_T.

Observation 2.8. Every conciliatory function is σ-join-irreducible.

Proof. If $\mathcal{A}$ is conciliatory, it is easy to see that $\mathsf{pass}^\omega \in \mathcal{F}(\mathcal{A})$, where pass^ω is the infinite sequence consisting only of pass. Thus, by Proposition 2.6, $\mathcal{A}$ is σ-join-irreducible. □

It is the converse direction of this observation that is hard to prove.

The following lemmas and observations will help us gain some intuition on conciliatory functions, even though they will not be used in the rest of the paper.

Observation 2.9. Every partial computable operator Φ_e can be viewed as a conciliatory function. Essentially, it just outputs passes while it is waiting either for a new value of the oracle, or a new computation to converge. By the same reason, every continuous function $\omega^\omega \to \omega^\omega$ can be extended to a conciliatory function as we mentioned above.

Lemma 2.10. *A function $G\colon \omega^{\leq\omega} \to \omega^{\leq\omega}$ can be represented as a continuous conciliatory function $\hat{\omega}^\omega \to \hat{\omega}^\omega$ if and only if $\sigma \subseteq \tau$ implies $G(\sigma) \subseteq G(\tau)$ for every $\sigma, \tau \in \omega^{<\omega}$, and $G(X) = \bigcup_n G(X \upharpoonright n)$ for every $X \in \omega^\omega$.*

Sketch of the proof. For the left-to-right implication, suppose $\hat{G}$ is a continuous conciliatory function such that $\hat{G}(X)^{\mathsf{p}} = G(X^{\mathsf{p}})$ for all $X \in \hat{\omega}^\omega$. Suppose $\tau = \sigma^\frown\gamma$. Every initial segment of $G(\sigma)$ must be an initial segment of $G(\tau)$ because every initial segment of $G(\sigma)^{\mathsf{p}}$ is contained in $\hat{G}(\sigma^\frown\mathsf{pass}\ \mathsf{pass}\cdots\mathsf{pass})^{\mathsf{p}}$ for some number of passes. Then,

$$G(\tau) = \hat{G}(\sigma^\frown\mathsf{pass}\ \mathsf{pass}\cdots\mathsf{pass}^\frown\gamma^\frown\mathsf{pass}^\omega)^{\mathsf{p}}.$$

It follows that $G(\sigma) \subseteq G(\tau)$. By the same argument, if $\sigma \subseteq X$, then $G(\sigma) \subseteq G(X)$. We leave the remaining details to the reader. □

One can show that a function $G\colon \omega^{\leq\omega} \to \mathcal{Q}$ can be represented as a continuous conciliatory function $\hat{G} : \hat{\omega}^\omega \to \mathcal{Q}$ if and only if it is constant. (Just think of $\mathcal{Q}$ as ω, being the first entry of the output of a function as in the lemma.) The case of Σ^0_2 functions gets more interesting.

Lemma 2.11. *A function $G\colon \omega^{\leq\omega} \to \omega^{\leq\omega}$ can be represented as a $\mathbf{\Sigma}^0_2$ conciliatory function if and only if for every $X \in \omega^\omega$, $G(X)$ is the pointwise limit of $G(X \upharpoonright m)$ in the following sense: for every $\sigma \in \omega^{<\omega}$,*

$$\sigma \subseteq G(X) \iff \exists n \forall m > n\ (\sigma \subseteq G(X \upharpoonright m)).$$

In particular, a function $G\colon \omega^{\leq\omega} \to \mathcal{Q}$ is $\mathbf{\Sigma}^0_2$ conciliatory if and only if $G(X) = \lim_n G(X \upharpoonright n)$ for every $X \in \omega^\omega$.

Sketch of the proof. For the left-to-right implication, suppose $\hat{G}$ is a $\mathbf{\Sigma}^0_2$ conciliatory function such that $\hat{G}(X)^{\mathsf{p}} = G(X^{\mathsf{p}})$ for all $X \in \hat{\omega}^\omega$. By definition, the predicate $\tau \subseteq \hat{G}(X)$ is Σ^0_2-definable with parameters. For $\sigma \in \omega^{<\omega}$ and $X \in \omega^\omega$, note that the predicate $\sigma \subset \hat{G}(X)^{\mathsf{p}}$ is equivalent to the existence of $\tau \in \hat{\omega}^{<\omega}$ such that $\tau^{\mathsf{p}} = \sigma$ and $\tau \subseteq \hat{G}(X)$. The latter condition is also Σ^0_2-definable with parameters. Thus, there is a Δ^0_1 predicate R with parameters such that

$$\sigma \subseteq \hat{G}(X)^{\mathsf{p}} \iff \exists n \forall m > n \; R(\sigma, n, m, X \upharpoonright m)$$

for $\sigma \in \omega^{<\omega}$ and $X \in \hat{\omega}^\omega$. Suppose, toward a contradiction that $X \in \omega^\omega$ and $\sigma \subseteq G(X)$, but there exists $k_0 < k_1 < \cdots$ such that $\sigma \not\subseteq G(X \upharpoonright k_n)$. We will then define $Y \in \hat{\omega}^\omega$ with $Y^{\mathsf{p}} = X$ such that $\sigma \not\subseteq \hat{G}(Y) = G(X)$. We define Y by finite approximations $Y_0 \subseteq Y_1 \subseteq Y_2 \cdots$ so that ${Y_n}^{\mathsf{p}} = X \upharpoonright k_n$. At each stage n, since $\sigma \not\subseteq G(X \upharpoonright k_n)$, there is an m_n such that $\neg R(\sigma, n, m_n, Y_n{}^\frown \mathsf{pass}^k)$, where k is so that $|Y_n{}^\frown \mathsf{pass}^k| = m_n$. Define Y_{n+1} to be $Y_n{}^\frown \mathsf{pass}^k{}^\frown X \upharpoonright [k_n + 1, k_{n+1}]$, so that ${Y_{n+1}}^{\mathsf{p}} = X \upharpoonright k_{n+1}$. We then have $\forall n \neg R(\sigma, n, m_n, Y \upharpoonright m_n)$, and hence that $\sigma \not\subseteq \hat{G}(Y)$.

We leave the converse direction to the reader. It is a standard argument in computability theory. □

The following lemma is also quite standard. It is just a uniform version of the limit lemma.

Lemma 2.12. *Every partial $\mathbf{\Sigma}^0_2$ function $G\colon \omega^\omega \to \omega^\omega$ can be extended to a $\mathbf{\Sigma}^0_2$ conciliatory function $\hat{G}\colon \hat{\omega}^\omega \to \hat{\omega}^\omega$, so that $\hat{G}(X) = G(X)$ for all $X \in \omega^\omega$.*

2.6. **Universal $\mathbf{\Sigma}^0_2$ conciliatory functions.** First, let us observe that there is no universal total $\mathbf{\Sigma}^0_2$-measurable function on ω^ω, as it would be $\mathbf{\Delta}^0_2$, and there is no greatest $\mathbf{\Delta}^0_2$ Wadge degree. This is the main reason we need to deal with conciliatory functions in this paper. Hereafter, for functions $\mathcal{A}, \mathcal{B} : \mathcal{X} \to \hat{\omega}^\omega$ for $\mathcal{X} \in \{\omega^\omega, \hat{\omega}^\omega\}$, we write $\mathcal{A} \equiv_{\mathsf{p}} \mathcal{B}$ if $\mathcal{A}(X)^{\mathsf{p}} = \mathcal{B}(X)^{\mathsf{p}}$ for all $X \in \mathcal{X}$.

Definition 2.13. Let $\mathcal{U}\colon \hat{\omega}^\omega \to \hat{\omega}^\omega$ be a conciliatory function. We say that $\mathcal{U}$ is $\mathbf{\Sigma}^0_2$-*universal* if it is $\mathbf{\Sigma}^0_2$-measurable, and for every $\mathbf{\Sigma}^0_2$-measurable conciliatory function $G\colon \hat{\omega}^\omega \to \hat{\omega}^\omega$, there exists a continuous function $\theta\colon \hat{\omega}^\omega \to \hat{\omega}^\omega$ such that $G \equiv_{\mathsf{p}} \mathcal{U} \circ \theta$.

Let us define a $\mathbf{\Sigma}^0_2$-universal function $\mathcal{U}$. Let $\{\sigma_n : n \in \omega\}$ be an effective enumeration of $\omega^{<\omega}$. Think of an input Y to $\mathcal{U}$ as a code for a sequence of strings $\sigma_{Y(0)}, \sigma_{Y(1)}, \sigma_{Y(2)}, \dots$ and $\mathcal{U}(Y)$ as the pointwise limit of these strings. That is, we would like to define $\mathcal{U}(Y)(j) = \lim_{i\to\infty} \sigma_{Y(i)}(j)$ if the limit exists, and let it be undefined otherwise, except that we have to be a bit careful to get $\mathcal{U}$ to be of the right form. The actual definition is as follows. For $\sigma \in \omega^{<\omega}$, $\sigma \neq \emptyset$,

$$\sigma \subseteq \mathcal{U}(Y) \iff \exists n \left(\sigma \subseteq \sigma_{Y^{\mathsf{p}}(n)} \;\&\; \forall m > n \left(Y^{\mathsf{p}}(m)\downarrow \to \sigma \subseteq \sigma_{Y^{\mathsf{p}}(m)}\right)\right),$$

where $Y^{\mathsf{p}}(m)\downarrow$ means that $|Y^{\mathsf{p}}| > m$. It is not hard to see that if $\tau_0 \subseteq \mathcal{U}(Y)$ and $\tau_1 \subseteq \mathcal{U}(Y)$, then τ_0 and τ_1 must be compatible. We let $\mathcal{U}(Y)$ be the union of all σ such that $\sigma \subseteq \mathcal{U}(Y)$. We let the reader verify that $\mathcal{U}$ is a $\mathbf{\Sigma}^0_2$-universal conciliatory function, as it is a standard computability theoretic argument.

$\mathcal{U}$ has a particular property that will be quite important: the value of $\mathcal{U}(Y)$ does not depend on initial segments of Y, and only depends on the tail of Y.

Definition 2.14. A function $\mathcal{A}\colon \hat{\omega}^\omega \to \hat{\omega}^\omega$ is *initializable* if for every $\tau \in \hat{\omega}^{<\omega}$, there is a continuous function $\theta_\tau\colon \hat{\omega}^\omega \to [\tau]$ such that $\mathcal{A} \equiv_{\mathsf{p}} \mathcal{A} \circ \theta_\tau$.

Essentially the same notion has also been studied by e.g. [Dup01]. To see that our function $\mathcal{U}$ is initializable, suppose 0 is the code for the empty string (i.e., $\sigma_0 = \emptyset$), then let $\theta_\tau(Y) = \tau^\frown 0^\frown Y$. It is not hard to see that $\mathcal{U}(Y) = \mathcal{U}(\tau^\frown 0^\frown Y)$.

We have proved the following proposition.

Proposition 2.15. *There is an initializable $\mathbf{\Sigma}^0_2$-universal conciliatory function.*

3. Nested labeled trees

In this section we give formal definitions of ${}^{\sqcup}\mathrm{Tree}^n(\mathcal{Q})$, $\trianglelefteq$, Σ_T, and the Σ_T-complete function Ω_T. We end the section by extending these ideas to all infinite Borel ranks.

3.1. Nested Trees.

Let us first give some intuition for the connection between nested labeled trees and Borel functions. First consider the characteristic function χ_U of an open set $U \subseteq \omega^\omega$. Since the predicate $x \in U$ can be described by an existential formula, we have an approximation procedure which starts by guessing $\chi_U(x) = 0$ until $x \in U$ is witnessed, and then changes the guess to $\chi_U(x) = 1$ after seeing such a witness. We denote the collection of all such guessing procedures, namely the pointclass $\mathbf{\Sigma}^0_1$, by the term $\langle 0\rangle^\to\langle 1\rangle$. We think of the term $\langle 0\rangle^\to\langle 1\rangle$ as representing a tree with two nodes whose root is labeled by 0, and leaf is labeled by 1. Similarly, we use the tree $\langle 1\rangle^\to\langle 0\rangle$ (with a root note labeled 1, and a leaf node labeled 0) to name the pointclass $\mathbf{\Pi}^0_1$, and we use trees of the form $\langle 0\rangle^\to\langle 1\rangle^\to \dots {}^\to\langle 0\rangle^\to\langle 1\rangle$ to name the finite levels of the Hausdorff-Kuratowski difference hierarchy.

To represent self-dual pointclasses such as $\mathbf{\Delta}^0_1$, we will need to consider forests rather than trees. Given a clopen set $C \subseteq \omega^\omega$, one decides whether $\chi_C(x) = 0$ or $\chi_C(x) = 1$ at once and there is no change of mind afterwords. We represent this procedure by the term $\langle 0\rangle \sqcup \langle 1\rangle$, which is identified with a forest consisting of two roots labeled by 0 and 1, respectively. All levels of the Hausdorff-Kuratowski difference hierarchy (hence all Wadge degrees of $\mathbf{\Delta}^0_2$ subsets of ω^ω) are named by terms obtained from the operations ${}^\to$ and $\sqcup$ (that is, well-founded $\{0,1\}$-labeled trees and their disjoint unions). For instance, a term of the form $I_n{}^\to \bigsqcup_k I_k$, (where I_ℓ is the chain of the form $\langle 0\rangle^\to\langle 1\rangle^\to \dots {}^\to\langle 0\rangle^\to\langle 1\rangle$ of length ℓ) names the $(\omega+n)^{th}$-level of the difference hierarchy. To represent Δ^0_2 3-partitions, Selivanov used forests labeled with $\{0,1,2\}$ instead. The idea is the same: A $\{0,1,2\}$-labeled tree guides the mind changes allowed when defining a Δ^0_2 3-partitions; since the tree is well-founded, the guessing process eventually stops.

If we want to move on to Δ^0_3 functions, that is when we need to start nesting trees. For instance, the tree $\langle T\rangle$ consisting only of a root labeled by a tree T, is thought of as the *jump* of the pointclass named by T. Thus, $\langle\langle 0\rangle^\to\langle 1\rangle\rangle$ is the jump of $\mathbf{\Sigma}^0_1$ — namely $\mathbf{\Sigma}^0_2$. By using nesting of trees in this way, we will be able to climb up the Borel hierarchy.

3.1.1. *Homomorphic quasi-order.* For a quasi-ordered set $\mathcal{Q}$, it is now easy to guess that every $\mathcal{Q}$-valued $\mathbf{\Delta}^0_2$-measurable functions can be described as a countable $\mathcal{Q}$-labeled well-founded forest. There is a known way of comparing the complexity of $\mathcal{Q}$-labeled forests. Formally, a *$\mathcal{Q}$-labeled forest* is a tuple $(F, \preceq_F, \lambda_F)$ of a forest $(F, \preceq_F)$ and a labeling function $\lambda_F : F \to \mathcal{Q}$. A *homomorphism* h between $\mathcal{Q}$-labeled forests S and T is a order-preserving $\leq_{\mathcal{Q}}$-increasing maps, that is, for any $\sigma, \tau \in S$, $\sigma \preceq_S \tau$ implies $h(\sigma) \preceq_T h(\tau)$, and $\lambda_S(\sigma) \leq_{\mathcal{Q}} \lambda_T(h(\sigma))$. Write $S \trianglelefteq_{\mathcal{Q}} T$ if there is a homomorphism from S to T.

Let $\mathrm{Tree}_*(\mathcal{Q})$ be the collection of all countable $\mathcal{Q}$-labeled well-founded trees, and ${}^{\sqcup}\mathrm{Tree}_*(\mathcal{Q})$ be the collection of all countable forests written as a disjoint union of trees in $\mathrm{Tree}_*(\mathcal{Q})$. Selivanov [Sel07, Sel17a, Sel17b] showed that $({}^{\sqcup}\mathrm{Tree}_*(k), \trianglelefteq_k)$ characterizes the Wadge degrees of $\mathbf{\Delta}^0_2$ k-partitions, and that $({}^{\sqcup}\mathrm{Tree}_*(\mathrm{Tree}_*(k)), \trianglelefteq_{\mathrm{Tree}_*(k)})$ characterizes $\mathbf{\Delta}^0_3$ k-partitions. As mentioned in [Sel17a, Sel17b], it is natural to consider iterated nesting of labeled forests and homomorphic quasi-order. However, we will see that this approach does not extend to infinite Borel ranks (see Section 3.4). For the sake of uniformity, we will directly define a quasi-order $\trianglelefteq$ on the nested forests, which enable us to compare functions of different Borel ranks, and is extendible directly to infinite Borel ranks (but does not differ from the homomorphic quasi-order if restricted to forests of nesting depth n for a fixed $n \in \omega$). In order to define $\trianglelefteq$, we will represent trees and forests as terms.

3.1.2. *Language and terms.* All $\mathcal{Q}$-valued Borel functions of *finite* rank will be described using terms (identified with forests) in the language consisting of constant symbols (corresponding to elements in $\mathcal{Q}$), and three function symbols: ${}^{\to}$ (concatenation), $\sqcup$ (disjoint union), and $\langle\cdot\rangle$ (labeling). To represent $\mathcal{Q}$-valued Borel functions of *infinite* rank, we will need to add symbols representing transfinite jump operations $\langle\cdot\rangle^{\omega^\alpha}$.

We formally describe the collections $\mathrm{Tree}(\mathcal{Q})$ and ${}^{\sqcup}\mathrm{Tree}(\mathcal{Q})$ of countable well-founded $\mathcal{Q}$-labeled trees and their countable disjoint unions (i.e., forests) in the following inductive manner:

(1) If $T \in \mathrm{Tree}(\mathcal{Q})$, then $T \in {}^{\sqcup}\mathrm{Tree}(\mathcal{Q})$.
(2) For each $q \in \mathcal{Q}$, the term $\langle q\rangle$ is in $\mathrm{Tree}(\mathcal{Q})$. It represents the tree with only one node labeled q.
(3) For any countable collection $\{T_i\}_{i\in I}$ in $\mathrm{Tree}(\mathcal{Q})$, where $\emptyset \neq I \subseteq \omega$, the term $\sqcup_i T_i$ is in ${}^{\sqcup}\mathrm{Tree}(\mathcal{Q})$. Terms of the form $\sqcup_i T_i$ will be called *$\sqcup$-type terms*, and represent forests obtained as the disjoint union of trees T_i.
(4) For any $q \in \mathcal{Q}$ and $\sqcup$-type term $T \in {}^{\sqcup}\mathrm{Tree}(\mathcal{Q})$, the term $\langle q\rangle^{\to} T$ is in $\mathrm{Tree}(\mathcal{Q})$. It represents the tree obtained by joining to a root labeled q all the components of the forest T.

Note that $\mathrm{Tree}(\mathcal{Q})$ consist of the non-$\sqcup$-type terms in ${}^{\sqcup}\mathrm{Tree}(\mathcal{Q})$. Then, define $\mathrm{Tree}^0(\mathcal{Q}) = \mathcal{Q}$, $\mathrm{Tree}^{n+1}(\mathcal{Q}) = \mathrm{Tree}(\mathrm{Tree}^n(\mathcal{Q}))$, and ${}^{\sqcup}\mathrm{Tree}^{n+1}(\mathcal{Q}) = {}^{\sqcup}\mathrm{Tree}(\mathrm{Tree}^n(\mathcal{Q}))$.

The way they are defined, $\mathrm{Tree}^m(\mathcal{Q})$ and $\mathrm{Tree}^n(\mathcal{Q})$ are disjoint whenever $m < n$. However, we will later see that every tree is $\mathrm{Tree}^m(\mathcal{Q})$ is equivalent to one in $\mathrm{Tree}^n(\mathcal{Q})$ (Observation 3.2).

Note that $\mathrm{Tree}(\mathcal{Q})$ and ${}^{\sqcup}\mathrm{Tree}(\mathcal{Q})$ corresponds to the $\mathcal{Q}$-labeled trees and forests in the sense of Section 3.1.1. For instance, the term $\langle q\rangle^{\to} \sqcup_{i\in\omega} \langle p_i\rangle$ represents an infinitely branching tree of height 2 whose root is labeled by q, and whose ith immediate successor is labeled by p_i.

3.1.3. *Quasi-ordering nested trees.* In this section, we introduce a quasi-order $\trianglelefteq$ on ${}^{\sqcup}\mathrm{Tree}^{<\omega}(\mathcal{Q})$, which we will show is isomorphic to the Wadge quasi-ordering of $\mathcal{Q}$-valued functions of finite Borel rank. To simplify our notation, we always identify $\langle T\rangle$ with $\langle T\rangle^{\to}\sqcup_i\mathbf{O}$, where $\mathbf{O}$ is the empty forest, which we think of as an imaginary least element with respect to the quasi-order $\trianglelefteq$, that is, $\mathbf{O}\trianglelefteq T$ for any $T\in{}^{\sqcup}\mathrm{Tree}^n(\mathcal{Q})$. By permitting the use of $\mathbf{O}$, one can also assume that an index set I in (3) is always ω.

Definition 3.1. We inductively define a quasi-order $\trianglelefteq$ on $\bigcup_n\mathrm{Tree}^n(\mathcal{Q})$ as follows, where the symbols p and q range over $\mathcal{Q}$, and U, V, S, and T range over $\bigcup_n\mathrm{Tree}^n(\mathcal{Q})$:

$$p\trianglelefteq q\iff p\leq_{\mathcal{Q}} q,$$
$$\langle U\rangle\trianglelefteq\langle V\rangle\iff U\trianglelefteq V,$$

and if S and T are of the form $\langle U\rangle^{\to}\sqcup_i S_i$ and $\langle V\rangle^{\to}\sqcup_j T_j$, respectively, then

$$S\trianglelefteq T\iff\begin{cases}\text{either } U\trianglelefteq V & \text{and } (\forall i)\ S_i\trianglelefteq T,\\ \text{or} \quad\ \ U\not\trianglelefteq V & \text{and } (\exists j)\ S\trianglelefteq T_j.\end{cases}$$

This pre-ordering induces an equivalence as usual: let $S\equiv T$ if $S\trianglelefteq T$ and $T\trianglelefteq S$. For $p\in\mathcal{Q}$, we let $p\equiv\langle p\rangle\equiv\langle\langle p\rangle\rangle\equiv\cdots$, allowing us to compare trees of different levels. However, note that $T\not\equiv\langle T\rangle$ when $T\notin\mathcal{Q}$.

Finally, $\trianglelefteq$ is uniquely extended to a quasi-order on $\bigcup_n{}^{\sqcup}\mathrm{Tree}^n(\mathcal{Q})$ by interpreting $\sqcup$ as a countable supremum operation:

$$\sqcup_i S_i\trianglelefteq\sqcup_j T_j\iff(\forall i)(\exists j)\ S_i\trianglelefteq T_j.$$

Observation 3.2. For every $T\in{}^{\sqcup}\mathrm{Tree}^{\leq n}(\mathcal{Q})$, there is $S\in{}^{\sqcup}\mathrm{Tree}^n(\mathcal{Q})$ such that $S\equiv T$.

Proof. By induction on the rank of the trees. Assume that $m\leq n$ and $T\in{}^{\sqcup}\mathrm{Tree}^m(\mathcal{Q})$. Then, consider the term $\iota(T)=T[\langle q\rangle^{n-m}/q]_{q\in\mathcal{Q}}$ obtained by substituting all occurrences of $q\in\mathcal{Q}$ by $\langle q\rangle^{n-m}$, where $\langle q\rangle^0=q$ and $\langle q\rangle^{k+1}=\langle\langle q\rangle^k\rangle$. Note that $\iota(T)\in{}^{\sqcup}\mathrm{Tree}^n(\mathcal{Q})$, and it is clear that $T\equiv\iota(T)$. $\square$

Observation 3.3. If we identify a term in ${}^{\sqcup}\mathrm{Tree}^n(\mathcal{Q})$ with the corresponding $\mathrm{Tree}^{n-1}(\mathcal{Q})$-labeled forest, then it is not hard to see by induction on the rank of the trees that the quasi-order $\trianglelefteq$ restricted to ${}^{\sqcup}\mathrm{Tree}^n(\mathcal{Q})$ is exactly the same as the homomorphic quasi-order in Section 3.1.1.

Theorem 3.4 (Laver [Lav78])**.** *For $n\in\omega$, if $\mathcal{Q}$ is better-quasi-ordered, then so is ${}^{\sqcup}\mathrm{Tree}^{\leq n}(\mathcal{Q})$.*

This is a consequence of Laver's results, together with closure properties of the class of bqos. Laver showed that if Q is a bqo, so is $\mathrm{Tree}(\mathcal{Q})$ for an even stronger notion of reducibility.

3.2. **The associated pointclasses.** As we mentioned before, each forest $T\in{}^{\sqcup}\mathrm{Tree}^n(\mathcal{Q})$ defines a pointclass Σ_T. For instance, if $\mathcal{Q}=2$, then

$$\Sigma_{\langle 0\rangle\sqcup\langle 1\rangle}=\mathbf{\Delta}^0_1,\quad\Sigma_{\langle 0\rangle\to\langle 1\rangle}=\mathbf{\Sigma}^0_1,\quad\Sigma_{\langle 1\rangle\to\langle 0\rangle}=\mathbf{\Pi}^0_1,\quad\Sigma_{\langle\langle 0\rangle\to\langle 1\rangle\rangle}=\mathbf{\Sigma}^0_2,\quad\text{and so on.}$$

The following observations will simplify our definitions.

Observation 3.5. Let $\mathcal{F}$ be a nonempty closed subset of ω^ω. Then, for every function $\mathcal{A}\colon\mathcal{F}\to\mathcal{Q}$ there is a function $\widehat{\mathcal{A}}\colon\omega^\omega\to\mathcal{Q}$ which is Wadge equivalent to $\mathcal{A}$.

Proof. By zero-dimensionality of ω^ω, there is a retraction $\rho_{\mathcal{F}}\colon \omega^\omega \to \mathcal{F}$ (that is, $\rho_{\mathcal{F}}$ is continuous and $\rho_{\mathcal{F}} \restriction \mathcal{F}$ is identity). Define $\widehat{\mathcal{A}} = \mathcal{A} \circ \rho_{\mathcal{F}}$. Then, we have $\widehat{\mathcal{A}} \leq_w \mathcal{A}$ via $\rho_{\mathcal{F}}$, and $\mathcal{A} \leq_w \widehat{A}$ via the identity map.

The definition of this retraction is quite standard: Let $T \subseteq \omega^{<\omega}$ be a tree without dead ends such that $\mathcal{F} = [T]$. We define $\rho_{\mathcal{F}}\colon \omega^{<\omega} \to T$ by induction: $\rho_{\mathcal{F}}(\sigma^\frown n) = \rho_{\mathcal{F}}(\sigma)^\frown m$ where $m \in \omega$ is the closest to n such that $\rho_{\mathcal{F}}(\sigma)^\frown m \in T$. (By closest we mean such that $|m + \frac{1}{3} - n|$ is least, for instance.) We then extend $\rho_{\mathcal{F}}$ to ω^ω to $\mathcal{F}$ by continuity. □

Observation 3.6. Let $\mathcal{V}$ be a nonempty open subset of ω^ω. Then, for every function $\mathcal{A}\colon \mathcal{V} \to \mathcal{Q}$ there is a function $\widehat{\mathcal{A}}\colon \omega^\omega \to \mathcal{Q}$ which is Wadge equivalent to $\mathcal{A}$.

Proof. Let $V = \{\tau_0, \tau_1, ...\} \subseteq \omega^{<\omega}$ be a generator of $\mathcal{V}$. That is, V is so that $\{[\tau] : \tau \in V\}$ is a partition on $\mathcal{V}$ in clopen sets. Then the bijection $n^\frown X \mapsto {\tau_n}^\frown X : \omega^\omega \to \mathcal{V}$ induces a function $\widehat{\mathcal{A}}\colon \omega^\omega \to \mathcal{Q}$ Wadge equivalent to $\mathcal{A}$ □

From now on, whenever we encounter a $\mathcal{Q}$-valued function whose domain is an either open or closed subset of ω^ω, we identify it with the corresponding function of domain ω^ω.

Definition 3.7. For each $T \in \bigcup_n {}^{\sqcup}\mathrm{Tree}^n(\mathcal{Q})$, we inductively define the class Σ_T of $\mathcal{Q}$-valued functions on ω^ω as follows:

1. Σ_q consists only of the constant function $X \mapsto q\colon \omega^\omega \to \mathcal{Q}$.
2. If T is of the form $\sqcup_i S_i$, then $\mathcal{A} \in \Sigma_T$ if and only if there is a clopen partition $(\mathcal{C}_i)_{i\in\omega}$ of ω^ω such that $\mathcal{A} \restriction \mathcal{C}_i \in \Sigma_{S_i}$ for each $i \in \omega$.
3. $\mathcal{A} \in \Sigma_{T\to S}$ if and only if there is an open set $\mathcal{V} \subseteq \omega^\omega$ such that $\mathcal{A} \restriction (\omega^\omega \setminus \mathcal{V})$ is in Σ_T and $\mathcal{A} \restriction \mathcal{V}$ is in Σ_S.
4. $\mathcal{A} \in \Sigma_{\langle T\rangle}$ if and only if there is a $\mathbf{\Sigma}^0_2$-measurable function $\mathcal{D}\colon \omega^\omega \to \omega^\omega$ and a Σ_T-function $\mathcal{B}\colon \omega^\omega \to \mathcal{Q}$ such that $\mathcal{A} = \mathcal{B} \circ \mathcal{D}$.

We say that a function $\mathcal{A}\colon \omega^\omega \to \mathcal{Q}$ is Σ_T*-complete* if $\mathcal{A} \in \Sigma_T$ and every Σ_T-function $\mathcal{B}$ is Wadge reducible to $\mathcal{A}$.

Observation 3.8. Let $T \in {}^{\sqcup}\mathrm{Tree}^n(\mathcal{Q})$ be a forest, and $\theta\colon \omega^\omega \to \omega^\omega$ be a continuous function. If $\mathcal{A}\colon \omega^\omega \to \mathcal{Q}$ is in Σ_T, then so is $\mathcal{A} \circ \theta$. This can be easily shown by induction on T as a term.

To prove Proposition 1.6 for functions of finite Borel rank, we check measurability of Σ_T-functions. We denote by $\mathbf{\Delta}^0_\xi$ the set of all $\mathbf{\Delta}^0_\xi$-measurable functions.

Lemma 3.9. *Let S, T be terms and ξ be a countable ordinal.*

1. $\Sigma_T, \Sigma_S \subseteq \mathbf{\Delta}^0_{2+\xi}$ *implies* $\Sigma_{T\to S} \subseteq \mathbf{\Delta}^0_{2+\xi}$.
2. $\Sigma_T \subseteq \mathbf{\Delta}^0_{1+\xi}$ *implies* $\Sigma_{\langle T\rangle} \subseteq \mathbf{\Delta}^0_{2+\xi}$.

Proof. To see (1), let $\mathcal{A}$ be a $\Sigma_{T\to S}$-function. Then, there is an open set $\mathcal{V}$ such that $\mathcal{A} \restriction (\omega^\omega \setminus \mathcal{V})$ can be extended to a total Σ_T-function $\mathcal{A}_0$, and $\mathcal{A} \restriction \mathcal{V}$ can be extended to a total Σ_S-function $\mathcal{A}_1$ as in Observations 3.5 and 3.6. Thus, for any $q \in \mathcal{Q}$, we have

$$\begin{aligned} \mathcal{A}^{-1}[q] &= (\mathcal{A}_0^{-1}[q] \setminus \mathcal{V}) \cup (\mathcal{A}_1^{-1}[q] \cap \mathcal{V}) \\ \mathcal{A}^{-1}[q]^c &= (\mathcal{A}_0^{-1}[q]^c \setminus \mathcal{V}) \cup (\mathcal{A}_1^{-1}[q]^c \cap \mathcal{V}) \end{aligned}$$

This gives $\mathbf{\Sigma}^0_{2+\xi}$-definitions of $\mathcal{A}^{-1}[q]$ and $\mathcal{A}^{-1}[q]^c$ since $\mathcal{A}_0$ and $\mathcal{A}_1$ are $\mathbf{\Delta}^0_{2+\xi}$-measurable, and $\mathcal{V}$ is open. Consequently, $\mathcal{A}$ is $\mathbf{\Delta}^0_{2+\xi}$-measurable.

For (2), note that the composition $\mathcal{B}\circ\mathcal{D}$ of a $\mathbf{\Delta}^0_{1+\xi}$-measurable function $\mathcal{B}$ and a $\mathbf{\Sigma}^0_{1+\eta}$-measurable function $\mathcal{D}$ is always $\mathbf{\Delta}^0_{1+\eta+\xi}$-measurable. This is because, one can see that if $\mathcal{S}\subseteq\omega^\omega$ is $\mathbf{\Sigma}^0_{1+\xi}$ then $\mathcal{D}^{-1}[\mathcal{S}]$ is $\mathbf{\Sigma}^0_{1+\eta+\xi}$ by induction on Borel rank. Now, every $\mathcal{A}\in\Sigma_{\langle T\rangle}$ is given by the composition of the Σ_T-function $\mathcal{B}$ and the $\mathbf{\Sigma}^0_2$-measurable function $\mathcal{D}$. This implies that $\mathcal{B}\circ\mathcal{D}$ is $\mathbf{\Delta}^0_{2+\xi}$-measurable since $\mathcal{B}$ is $\mathbf{\Delta}^0_{1+\xi}$-measurable by our assumption. □

In particular, if $T\in{}^{\sqcup}\mathrm{Tree}^n(\mathcal{Q})$, then every Σ_T-function is $\mathbf{\Delta}^0_{n+1}$-measurable. As a consequence, this verifies Proposition 1.6 for functions of finite Borel rank.

3.3. **Σ_T-complete functions.** For $\mathcal{Q}=2$, Duparc [Dup01] defined complete sets $\Omega_\nu\subseteq\omega^{\leq\omega}$ for the different levels of the 2-Wadge hierarchy. For $Q=k\in\omega$, Selivanov [Sel07] defined complete Δ^0_2 functions $\mu_T\colon\omega^\omega\to k$ for each forest $T\in{}^{\sqcup}\mathrm{Tree}(k)$ based on similar ideas. In this section we extend Duparc and Selivanov's definition to all bqos $\mathcal{Q}$ and nested forests $T\in{}^{\sqcup}\mathrm{Tree}^n(\mathcal{Q})$, and later on throughout the Borel hierarchy.

The complete functions we will define are conciliatory; see Section 2.5.

3.3.1. *Difference hierarchy and mind-change operation.* The Hausdorff-Kuratowski difference hierarchy (and the Ershov hierarchy) can be understood using the notion of a *mind-change*. That is, a subset $\mathcal{A}$ of ω^ω is in the n^{th}-level of the difference hierarchy if and only if the characteristic function of $\mathcal{A}$ is approximated by a continuous function with n mind-changes.

We want to define an operation $\mathcal{A}^\to\mathcal{B}$ for $\mathcal{A},\mathcal{B}\colon\hat{\omega}^\omega\to\mathcal{Q}$ which represents a function that could act as $\mathcal{A}$, but at any time could change its mind and act as $\mathcal{B}$. To make it easier to describe such a process, we introduce notations representing this kind of approximation procedure. Suppose we first want to output a sequence and after ℓ steps, after having defined a sequence $Y\in\omega^\ell$, we change our mind and we want to output a new sequence $Z\in\omega^{\leq\omega}$. We will encode this by the following real:

$$Y^\to Z:=\langle 2Y(0),2Y(1),\dots,2Y(\ell-1),2Z(0)+1,Z(1),Z(2),\dots\rangle.$$

We want to define this procedure on $\hat{\omega}^\omega$ as follows, where we will require that the first entry of the second sequence Z is not pass, that is, $Z\in\omega\times\hat{\omega}^\omega$.

Notation 3.10. Given $Y\in\hat{\omega}^\ell$ of length $\ell\in\omega\cup\{\omega\}$ and $Z\in\omega\times\hat{\omega}^\omega$, we define $Y^\to Z$ as follows:

$$Y^\to Z(n)=\begin{cases}2Y(n), & \text{if } n<\ell \text{ and } Y(n)\neq\mathsf{pass},\\ \mathsf{pass}, & \text{if } n<\ell \text{ and } Y(n)=\mathsf{pass},\\ 2Z(0)+1, & \text{if } n=\ell,\\ Z(n-\ell+1), & \text{if } n>\ell.\end{cases}$$

Observation 3.11. The map $(Y,Z)\mapsto Y^\to Z$ admits a conciliatory inverse. Indeed, there uniquely exist conciliatory continuous functions π_0 and π_1 such that for any $X\in\hat{\omega}^\omega$,

$$X=\pi_0(X)^\to\pi_1(X).$$

We hereafter fix such functions π_0, π_1. Note that $\pi_1(X)^{\mathsf{p}}$ is nonempty if and only if X has changed his mind at some point. In other words, if $\pi_1(X)^{\mathsf{p}}$ is empty, then the sequence given by X is $\pi_0(X)$, and if $\pi_1(X)^{\mathsf{p}}$ is nonempty, X has already deleted the former sequence $\pi_0(X)$, and now proposes $\pi_1(X)$.

Notation 3.12 (see also [Dup01, Definition 6] for $\mathcal{Q} = 2$). Let $\mathcal{A}$ and $\mathcal{B}$ be functions whose domains are subsets of $\hat{\omega}^\omega$. We define a function $\mathcal{A}^{\to}\mathcal{B} \colon \hat{\omega}^\omega \to \mathcal{Q}$ as follows:

$$(\mathcal{A}^{\to}\mathcal{B})(X) = \begin{cases} \mathcal{A}(\pi_0(X)) & \text{if } \pi_1(X) \text{ is empty,} \\ \mathcal{B}(\pi_1(X)) & \text{otherwise.} \end{cases}$$

It is easy to check that the operation $\to$ can also be seen as an operation on the Wadge degrees:

Observation 3.13. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ be functions whose domains are subsets of $\hat{\omega}^\omega$. If $\mathcal{A} \leq_w \mathcal{C}$ and $\mathcal{B} \leq_w \mathcal{D}$ then $\mathcal{A}^{\to}\mathcal{B} \leq_w \mathcal{C}^{\to}\mathcal{D}$.

3.3.2. Σ_T*-complete functions.* We now inductively assign a function Ω_T to each forest $T \in {}^{\sqcup}\mathrm{Tree}^n(\mathcal{Q})$, and we will show that Ω_T is Σ_T-complete. Recall that T is a tree if and only if the outermost function symbol is not the disjoint union $\sqcup$, and thus, Ω_T is defined by the construction in (1), (3), or (4) of Definition 3.7. If T is a tree, Ω_T will be a conciliatory function from $\hat{\omega}^\omega$ to $\mathcal{Q}$. If T is not a tree, Ω_T will be a function from $\omega \times \hat{\omega}^\omega$ to $\mathcal{Q}$, which is *almost conciliatory*, that is, $X^{\mathsf{p}} = Y^{\mathsf{p}}$ implies $\Omega_T(n^\frown X) = \Omega_T(n^\frown Y)$ for any $n \in \omega$. (Think of almost conciliatory functions as having domain $\omega^{\leq\omega} \setminus \{\emptyset\}$.)

Definition 3.14. Let $T \in {}^{\sqcup}\mathrm{Tree}^n(\mathcal{Q})$. We inductively define Ω_T as follows:

(1) Suppose that T is of the form $\langle q \rangle$ for some $q \in \mathcal{Q}$. Then define $\Omega_{\langle q \rangle} \colon \hat{\omega}^\omega \to \mathcal{Q}$ as the constant function $X \mapsto q$, that is,
$$(\forall X \in \hat{\omega}^\omega) \quad \Omega_{\langle q \rangle}(X) = q.$$
We sometimes abbreviate $\Omega_{\langle q \rangle}$ to Ω_q.

(2) Suppose that T is of the form $\bigsqcup_n T_n$, where each T_n is a tree. Then define $\Omega_T \colon \omega \times \hat{\omega}^\omega \to \mathcal{Q}$ as follows:
$$\Omega_{\bigsqcup_n T_n}(X) = \bigoplus_{n \in \omega} \Omega_{T_n}(X)$$

(3) Suppose that T is of the form $\langle S \rangle^{\to} F$, where S is the label on the root of T (thus $S \in \mathrm{Tree}^{n-1}(\mathcal{Q})$), and F is a forest. Then,
$$\Omega_{\langle S \rangle^{\to} F} = \Omega_{\langle S \rangle}{}^{\to}\Omega_F.$$

(4) Suppose that T is of the form $\langle S \rangle$ for some tree S. Then define $\Omega_T \colon \hat{\omega}^\omega \to \mathcal{Q}$ as follows:
$$\Omega_{\langle S \rangle} = \Omega_S \circ \mathcal{U},$$
where $\mathcal{U}$ is a fixed $\mathbf{\Sigma}^0_2$-universal initializable conciliatory function as in Proposition 2.15.

Observation 3.15. If $T \in \mathrm{Tree}^n(\mathcal{Q})$, then Ω_T is conciliatory. If T is a $\sqcup$-type term, Ω_T is almost conciliatory. The proof is an easy induction on the term T.

Observation 3.16. For every $T \in {}^{\sqcup}\mathrm{Tree}^n(\mathcal{Q})$, the function Ω_T is in Σ_T.

Proof. This is obvious if T is constructed from (1), (2) or (4). Thus, it suffices to show that $\Omega_{T\to S} \in \Sigma_{T\to S}$. Recall that every $X \in \hat{\omega}^\omega$ is of the form $\pi_0(X)^\to\pi_1(X)$ by Observation 3.11. Let $\mathcal{V}$ be an open set consisting of all sequences X such that $\pi_1(X)$ is nonempty (which indicates that X has changed his mind at some point). It is clear that $\Omega_{T\to S}\restriction(\omega^\omega \setminus \mathcal{V}) = \Omega_T \circ \pi_0 \restriction(\omega^\omega \setminus \mathcal{V})$, and $\Omega_{T\to S}\restriction\mathcal{V} = \Omega_S \circ \pi_1 \restriction \mathcal{V}$. By induction hypothesis and by Observation 3.8, the former function is in Σ_T and the latter function is in Σ_S. This concludes that $\Omega_{T\to S} \in \Sigma_{T\to S}$. □

Lemma 3.17. *For every $T \in {}^\sqcup\mathrm{Tree}^n(\mathcal{Q})$, the function Ω_T is Σ_T-complete.*

Proof. First assume that T is of the form $\sqcup_i T_i$, and let $\mathcal{A}$ be a Σ_T-function. Then there is a clopen partition $(\mathcal{C}_i)_{i\in\omega}$ such that $\mathcal{A}\restriction\mathcal{C}_i$ is in Σ_{T_i} for any $i \in \omega$. By induction hypothesis, we have a continuous function θ_i witnessing $\mathcal{A}\restriction\mathcal{C}_i \leq_w \Omega_{T_i}$ for every $i \in \omega$. Thus, to see $\mathcal{A} \leq_w \Omega_T$, given $X \in \omega^\omega$ one can continuously find $i_X \in \omega$ such that $X \in \mathcal{C}_{i_X}$, and then we have $\mathcal{A}(X) \leq_\mathcal{Q} \Omega_T(i_X{}^\frown\theta_{i_X}(X))$.

Next, let $\mathcal{A}$ be a function in $\Sigma_{T\to S}$. Then, there is an open set $\mathcal{V}$ such that $\mathcal{A}\restriction\mathcal{V}$ is in Σ_S and $\mathcal{A}\restriction(\omega^\omega \setminus \mathcal{V})$ is in Σ_T. Recall that the former condition means that there is a generator V of $\mathcal{V}$ such that $\mathcal{A}\restriction[\sigma] \in \Sigma_S$ for any $\sigma \in V$. By induction hypothesis, we have continuous functions θ witnessing $\mathcal{A}\restriction(\omega^\omega \setminus \mathcal{V}) \leq_w \Omega_T$, and γ_σ witnessing $\mathcal{A}\restriction[\sigma] \leq_w \Omega_S$ for every $\sigma \in V$. To see $\mathcal{A} \leq_w \Omega_{T\to S}$, given $X \in \omega^\omega$, we first follow θ until we see $X \restriction s \in V$ for some s (if ever). If we see $X\restriction s \in V$, then we change our mind (that is, delete the former sequence $\theta(X \restriction s-1)$), and now follow $\gamma_{X\restriction s}$. Recall that, in the latter case, this process is coded as $\theta(X \restriction s-1)^\to\gamma_{X\restriction s}(X)$. This witnesses $\mathcal{A} \leq_w \Omega_{T\to S}$.

Let $\mathcal{A}$ be a $\Sigma_{\langle T\rangle}$-function. Then, there are a $\boldsymbol{\Sigma}^0_2$-measurable function $\mathcal{D}$ and a Σ_T-function $\mathcal{B}$ such that $\mathcal{A} = \mathcal{B}\circ\mathcal{D}$. By induction hypothesis, we have a continuous function $\theta\colon \omega^\omega \to \hat{\omega}^\omega$ witnessing $\mathcal{B} \leq_w \Omega_T$. Thus, $\mathcal{A}(X) \leq_\mathcal{Q} \Omega_T \circ\theta\circ\mathcal{D}(X)$. Since $\theta\circ\mathcal{D}$ is $\boldsymbol{\Sigma}^0_2$, and $\mathcal{U}$ is universal, there is a continuous function $\Psi\colon \omega^\omega \to \hat{\omega}^\omega$ such that $\theta\circ\mathcal{D} \equiv_\mathsf{p} \mathcal{U}\circ\Psi(X)$. Then, since Ω_T is conciliatory by Observation 3.15,

$$\mathcal{A}(X) = \mathcal{B}(\mathcal{D}(X)) \leq_\mathcal{Q} \Omega_T(\theta\circ\mathcal{D}(X)) = \Omega_T(\mathcal{U}\circ\Psi(X)) = \Omega_{\langle T\rangle}(\Psi(X)).$$

Consequently, the continuous function Ψ witnesses $\mathcal{A} \leq_w \Omega_{\langle T\rangle}$. □

Lemma 3.18. *For $S, T \in {}^\sqcup\mathrm{Tree}^n(\mathcal{Q})$, if $S \trianglelefteq T$, then $\Omega_S \leq_w \Omega_T$.*

Proof. We show the assertion by induction on the definition of $\trianglelefteq$ in Definition 3.1. First, it is clear that $\Omega_p \leq_w \Omega_q$ if and only if $p \trianglelefteq q$. Next suppose that we have $\langle U\rangle \trianglelefteq \langle V\rangle$ (where U and V are not of $\sqcup$-type), which is equivalent to $U \trianglelefteq V$ by definition. By the induction hypothesis, we have a $\mathcal{Q}$-Wadge reduction $\theta\colon \hat{\omega}^\omega \to \hat{\omega}^\omega$ witnessing $\Omega_U \leq_w \Omega_V$. By the $\boldsymbol{\Sigma^0_2}$ universality of $\mathcal{U}$, there exists a continuous $\Phi\colon \hat{\omega}^\omega \to \hat{\omega}^\omega$ such that $\theta\circ\mathcal{U} \equiv_\mathsf{p} \mathcal{U}\circ\Phi$. Since Ω_V is conciliatory by Observation 3.15, we have that for every $X \in \hat{\omega}$,

$$\Omega_{\langle U\rangle}(X) = \Omega_U(\mathcal{U}(X)) \leq_\mathcal{Q} \Omega_V(\theta(\mathcal{U}(X))) = \Omega_V(\mathcal{U}(\Phi(X))) = \Omega_{\langle V\rangle}(\Phi(X)).$$

Thus Φ witnesses that $\Omega_{\langle U\rangle} \leq_w \Omega_{\langle V\rangle}$.

Now, consider $S = \langle U\rangle^\to \bigsqcup_i S_i$ and $T = \langle V\rangle^\to \bigsqcup_j T_j$. First consider the case that $\langle U\rangle \trianglelefteq \langle V\rangle$. In this case, by the definition of $\trianglelefteq$ under the assumption that $\langle U\rangle \trianglelefteq \langle V\rangle$, we have that $S_i \trianglelefteq T$ for any $i \in \omega$. By the induction hypothesis, $\Omega_{S_i} \leq_w \Omega_T$ via a continuous function θ_i for any i, and $\Omega_{\langle U\rangle} \leq_w \Omega_{\langle V\rangle}$ via τ. The idea to define the reduction is as follows: Given $X \in \omega^\omega$, while X does not change its mind, use the reduction $\tau\colon \Omega_{\langle U\rangle} \leq_w \Omega_{\langle V\rangle}$. If X changes its mind and moves into some S_k, then we need to use the reduction $\theta_k\colon \Omega_{S_k} \leq_w \Omega_T$. Since we have already taken some steps within the domain of $\langle V\rangle$, we need to use the initializability of $\Omega_{\langle V\rangle}$ to start over with the reduction to Ω_T.

More formally: By the initializability of $\Omega_{\langle V\rangle}$ (see Definition 2.14), for any σ, there is a continuous function η_σ witnessing $\Omega_{\langle V\rangle} \leq_w \Omega_{\langle V\rangle}\upharpoonright[\sigma]$. We can then extend this map to a Wadge reduction $\Omega_T \leq_w \Omega_T\upharpoonright[\sigma^\rightarrow\emptyset]$, where $\sigma^\rightarrow\emptyset$ represents the string in the domain of Ω_T for which we haven't changed our mind yet, and we are still computing $\Omega_{\langle V\rangle}$. For each σ, let $\hat\eta_\sigma$ be such that $\Omega_T(Y) \leq_{\mathcal{Q}} \Omega_T(\sigma^\rightarrow\hat\eta_\sigma(Y))$, witnessing such reduction. A given X can be written as $\pi_0(X)^\rightarrow\pi_1(X)$ by Observation 3.11. If $\pi_1(X)$ is empty (that is, X never changes his mind), then note that $\Omega_S(X) = \Omega_{\langle U\rangle}(\pi_0(X))$. In this case, return $\tau(\pi_0(X))^\rightarrow\emptyset$. Let X be a sequence such that $\pi_1(X)$ is nonempty (that is, X has changed his mind at some point). Put $k = \pi_1(X)(0)$. Note that $\Omega_S(X) = \Omega_{\bigsqcup_i S_i}(\pi_1(X)) = \Omega_{S_k}(\pi_1(X)^-)$, where $\pi_1(X) = k^\frown\pi_1(X)^-$. Then change our guess to $\hat\eta_{\tau(\pi_0(X))}\circ\theta_k(\pi_1(X)^-)$. We thus get

$$\Omega_S(X) = \Omega_{S_k}(\pi_1(X)^-) \leq_{\mathcal{Q}} \Omega_T(\theta_k(\pi_1(X)^-)) \leq_{\mathcal{Q}} \Omega_T(\tau(\pi_0(X)) \;^\rightarrow\; \hat\eta_{\tau(\pi_0(X))}(\theta_k(\pi_1(X)^-))).$$

Putting these two cases together, we have a Wadge reduction from Ω_S to Ω_T.

We now assume that $\langle U\rangle \not\trianglelefteq \langle V\rangle$. In this case, by definition, $S \trianglelefteq T$ if and only if $S \trianglelefteq T_j$ for some $j \in \omega$. By induction hypothesis $\Omega_S \leq_w \Omega_{T_j}$ for some j. Clearly, this condition implies $\Omega_S \leq_w \Omega_T$. □

We will prove the reverse direction, that $\Omega_T \leq_w \Omega_S \Rightarrow T \trianglelefteq S$, in Subsection 4.3. We need to wait until then, because we need the jump inversion operator for the proof.

3.4. **Infinite Borel ranks.** We now extend our ideas from Section 3.1 to infinite Borel ranks. The reader who is only interested in Borel functions of finite rank can skip Section 3.4.

We first describe a naive idea using homomorphic quasi-order in Section 3.1.1, and we will see that the naive approach does not work. For a countable ordinal ξ, one may think that it is natural to define the trees $\mathrm{Tree}^\xi_*(\mathcal{Q})$ of the ξth nesting rank as $\mathrm{Tree}_*(\bigcup_{\eta<\xi}\mathrm{Tree}^\eta_*(\mathcal{Q}))$. Then it is straightforward to extend Definition 3.14 to define Ω_T for $T \in \mathrm{Tree}^\xi_*(\mathcal{Q})$. However, it is not hard to see that Ω_T is still $\mathbf{\Delta}^0_\omega$-measurable whatever ξ is.

This failed approach corresponds to a known fact for $\mathcal{Q} = 2$: It is known that the Wadge rank of a $\mathbf{\Sigma}^0_n$-complete set is $\omega_1 \uparrow\uparrow n$ (i.e., the nth level of the super-exponential hierarchy of base ω_1) and thus smaller than the (ω_1+1)st epsilon number ε_{ω_1+1} (which is the same as the first fixed point of the exponential of base ω_1). One may guess that the height of the Wadge degrees of $\mathbf{\Delta}^0_\omega$ sets is ε_{ω_1+1}; however, it is wrong. As calculated by Wadge [Wad83], the correct height of $\mathbf{\Delta}^0_\omega$ sets is the $(\omega_1\cdot 2)$nd epsilon number $\varepsilon_{\omega_1\cdot 2}$ (which is the same as the ω_1th fixed point of the exponential of base ω_1). This observation reveals the existence of a long hierarchy of $\mathbf{\Delta}^0_\omega$ sets which are not of finite Borel rank, and one can now see that $(\mathrm{Tree}^\xi_*(2))_{\omega\leq\xi<\omega_1}$ only describes such sets, i.e., the sets of Wadge rank between ε_{ω_1+1} and $\varepsilon_{\omega_1\cdot 2}$.

On the other hand, our quasi-ordering $\trianglelefteq$ in Section 3.1.3 is significantly different from the homomorphic quasi-order when considering the nesting of unbounded depth. For a fixed n, ${}^{\sqcup}\mathrm{Tree}^{\leq n}(\mathrm{Tree}(\mathcal{Q}))$ is isomorphic to ${}^{\sqcup}\mathrm{Tree}^{n+1}(\mathcal{Q})$ (which is also isomorphic to the quotient order under the homomorphic quasi-ordering on ${}^{\sqcup}\mathrm{Tree}^{n+1}_{*}(\mathcal{Q})$), and under a natural isomorphism $\iota\colon {}^{\sqcup}\mathrm{Tree}^{\leq n}(\mathrm{Tree}(\mathcal{Q})) \simeq {}^{\sqcup}\mathrm{Tree}^{n+1}(\mathcal{Q})$, each tree $T \in \mathrm{Tree}(\mathcal{Q})$ is interpreted as $\iota(T) = \langle T\rangle^n$. However, if we move to the nesting of unbounded depth, we observe that ${}^{\sqcup}\mathrm{Tree}^{<\omega}(\mathrm{Tree}(\mathcal{Q}))$ and ${}^{\sqcup}\mathrm{Tree}^{<\omega}(\mathcal{Q})$ are not necessarily isomorphic: An intended interpretation of $T \in \mathrm{Tree}(\mathcal{Q})$ in the former ordering is more complex than $\langle T\rangle^n$ for any $n \in \omega$, which does not live in the latter ordering.

For instance, observe that if $\mathcal{Q} = 2$ and $T = \langle 0\rangle^{\to}\langle 1\rangle$ (which is in $\mathrm{Tree}(\mathcal{Q})$), then $T \not\equiv \langle T\rangle$ in ${}^{\sqcup}\mathrm{Tree}^{<\omega}(\mathcal{Q})$ since T and $\langle T\rangle$ represent $\mathbf{\Sigma}^0_1$ and $\mathbf{\Sigma}^0_2$ respectively, but $T \equiv \langle T\rangle$ in ${}^{\sqcup}\mathrm{Tree}^{<\omega}(\mathrm{Tree}(\mathcal{Q}))$. Note that this strange phenomenon is simply caused by abuse of notation (that is, we should have distinguished two labeling functions for inner trees $\mathrm{Tree}(\mathcal{Q})$ and outer forests ${}^{\sqcup}\mathrm{Tree}^{<\omega}(\cdot)$).

To avoid the notational confusion, let $\langle\cdot\rangle^*$ be a new function symbol, and for a quasi-ordering $\mathcal{P}$, define the new quasi-ordering $\langle\mathcal{P}\rangle^*$ as follows:

$$\langle\mathcal{P}\rangle^* = \{\langle p\rangle^* : p \in \mathcal{P}\} \text{ ordered by } \langle p\rangle^* \leq \langle q\rangle^* \iff p \leq_{\mathcal{P}} q.$$

Trivially, $\mathcal{P} \simeq \langle\mathcal{P}\rangle^*$, and thus ${}^{\sqcup}\mathrm{Tree}^{<\omega}(\mathrm{Tree}(\mathcal{Q})) \simeq {}^{\sqcup}\mathrm{Tree}^{<\omega}(\langle\mathrm{Tree}(\mathcal{Q})\rangle^*)$. This indicates that $T \equiv \langle T\rangle$ in the former ordering should be understood as $\langle T\rangle^* \equiv \langle\langle T\rangle^*\rangle$. The latter equivalence is not very surprising. (Note that the rank of $\langle\langle 0\rangle^{\to}\langle 1\rangle\rangle^*$ in ${}^{\sqcup}\mathrm{Tree}^{<\omega}(\langle\mathrm{Tree}(2)\rangle^*)$ is ε_{ω_1+1}.)

Now let ${}^{\sqcup}\mathrm{Tree}^{\omega}(\mathcal{Q})$ be the set of all terms in the language introduced in Section 3.1.2. It is straightforward to extend the domain of our quasi-ordering $\trianglelefteq$ in Definition 3.1 to ${}^{\sqcup}\mathrm{Tree}^{\omega}(\mathcal{Q})$. Later we will show that the $\mathcal{Q}$-valued $\mathbf{\Delta}^0_\omega$-functions are exactly those represented by the terms in ${}^{\sqcup}\mathrm{Tree}^{\omega}(\mathcal{Q})$. Then we will let ${}^{\sqcup}\mathrm{Tree}^{\omega+1}(\mathcal{Q})$ be isomorphic to ${}^{\sqcup}\mathrm{Tree}^{\omega}(\mathrm{Tree}(\mathcal{Q}))$, but to avoid the notational confusion, we will introduce a new symbol $\langle\cdot\rangle^\omega$, and define ${}^{\sqcup}\mathrm{Tree}^{\omega+1}(\mathcal{Q}) = {}^{\sqcup}\mathrm{Tree}^{\omega}(\langle\mathrm{Tree}(\mathcal{Q})\rangle^\omega)$. Similarly, for instance, ${}^{\sqcup}\mathrm{Tree}^{\omega\cdot 3+6}(\mathcal{Q})$ will be defined as follows:

$$\begin{aligned}{}^{\sqcup}\mathrm{Tree}^{\omega\cdot 3+6}(\mathcal{Q}) &= {}^{\sqcup}\mathrm{Tree}^{\omega}(\langle\mathrm{Tree}^{\omega}(\langle\mathrm{Tree}^{\omega}(\langle\mathrm{Tree}^{6}(\mathcal{Q})\rangle^\omega)\rangle^\omega)\rangle^\omega)\\ &\simeq {}^{\sqcup}\mathrm{Tree}^{\omega}(\mathrm{Tree}^{\omega}(\mathrm{Tree}^{\omega}(\mathrm{Tree}^{6}(\mathcal{Q})))).\end{aligned}$$

In this way, by using $\mathcal{Q}$, ${}^{\to}$, $\sqcup$, $\langle\cdot\rangle$, and $\langle\cdot\rangle^\omega$, one can define ${}^{\sqcup}\mathrm{Tree}^{\xi}(\mathcal{Q})$ for all $\xi \leq \omega^2$. Then, ${}^{\sqcup}\mathrm{Tree}^{\omega^2+1}(\mathcal{Q}) = {}^{\sqcup}\mathrm{Tree}^{\omega^2}(\langle\mathrm{Tree}(\mathcal{Q})\rangle^{\omega^2}) \simeq {}^{\sqcup}\mathrm{Tree}^{\omega^2}(\mathrm{Tree}(\mathcal{Q}))$, where $\langle\cdot\rangle^{\omega^2}$ is a new symbol. To introduce ${}^{\sqcup}\mathrm{Tree}^{\xi}(\mathcal{Q})$ for all $\xi < \omega_1$ we need a function symbol $\langle\cdot\rangle^{\omega^\alpha}$ for each $\alpha < \omega_1$ as described below.

3.4.1. *Language and terms (infinite Borel rank).* Given a set $\mathcal{Q}$, let $\mathcal{L}(\mathcal{Q})$ be the language consisting of constant symbols q for each $q \in \mathcal{Q}$, an ω-ary function symbol $\sqcup$, a two-ary function symbol ${}^{\to}$, and a unary function symbol $\langle\cdot\rangle^{\omega^\alpha}$ for every countable ordinal $\alpha < \omega_1$.

We define ${}^{\sqcup}\mathrm{Tree}^{\omega^\alpha}(\mathcal{Q})$ as the set of all $\mathcal{L}(\mathcal{Q})$-terms of rank below ω^α as follows:

Definition 3.19 (Terms of Rank below ω^α)**.** We inductively define the sets $\mathrm{Tree}^{\omega^\alpha}(\mathcal{Q})$ and ${}^{\sqcup}\mathrm{Tree}^{\omega^\alpha}(\mathcal{Q})$ consisting of $\mathcal{L}(\mathcal{Q})$-terms as follows:

(1) If $\beta \leq \alpha$ and $T \in \mathrm{Tree}^{\omega^\beta}(\mathcal{Q})$ then $T \in \mathrm{Tree}^{\omega^\alpha}(\mathcal{Q})$ and $T \in {}^{\sqcup}\mathrm{Tree}^{\omega^\alpha}(\mathcal{Q})$.

(2) If $q \in \mathcal{Q}$ then $\langle q\rangle \in \mathrm{Tree}^1(\mathcal{Q})$ (where note that $\omega^0 = 1$), and call it $\langle\rangle$-*type*.
(3) If $\beta < \alpha$ and $T \in \mathrm{Tree}^{\omega^\alpha}(\mathcal{Q})$ then $\langle T\rangle^{\omega^\beta} \in \mathrm{Tree}^{\omega^\alpha}(\mathcal{Q})$, and call it $\langle\rangle$-*type*.
(4) If $T_i \in \mathrm{Tree}^{\omega^\alpha}(\mathcal{Q})$ for every $i \in \omega$, then $\sqcup_i T_i \in {}^\sqcup\mathrm{Tree}^{\omega^\alpha}(\mathcal{Q})$, and call it $\sqcup$-*type*.
(5) For any $\langle\rangle$-type term $T \in \mathrm{Tree}^{\omega^\alpha}(\mathcal{Q})$ and $\sqcup$-type term $S \in {}^\sqcup\mathrm{Tree}^{\omega^\alpha}(\mathcal{Q})$, the term $T^\to S$ is in $\mathrm{Tree}^{\omega^\alpha}(\mathcal{Q})$.

We define $\mathrm{Tree}^{\omega_1}(\mathcal{Q}) = \bigcup_\alpha \mathrm{Tree}^{\omega^\alpha}(\mathcal{Q})$ and ${}^\sqcup\mathrm{Tree}^{\omega_1}(\mathcal{Q}) = \bigcup_\alpha {}^\sqcup\mathrm{Tree}^{\omega^\alpha}(\mathcal{Q})$.

For instance, ${}^\sqcup\mathrm{Tree}^{\omega}(\mathcal{Q})$ can be viewed as the closure of $\mathcal{Q}$ under the operations $\langle\cdot\rangle$, $\sqcup$, and $^\to$. Notice that this is far larger than $\bigcup_n {}^\sqcup\mathrm{Tree}^{n}(\mathcal{Q})$ (even with respect to $\trianglelefteq$) because a term in ${}^\sqcup\mathrm{Tree}^{\omega}(\mathcal{Q})$ can contain unbounded applications of the labeling function $\langle\cdot\rangle$, e.g., $\sqcup_n\langle 0^\to 1\rangle^n$. This reflects the fact that the pointclass $\mathbf{\Delta}^0_\omega$ is strictly larger than $\bigcup_{n<\omega}\mathbf{\Delta}^0_n$. On the other hand, the function $\langle\cdot\rangle^\omega$ would take us out of ${}^\sqcup\mathrm{Tree}^{\omega}(\mathcal{Q})$, reflecting that the conciliatory Σ^0_ω universal function is not $\mathbf{\Delta}^0_\omega$.

For a set $\mathcal{R}$ of $\mathcal{L}(\mathcal{Q})$-terms and an ordinal α, we define:

$$\langle\mathcal{R}\rangle^{\omega^\alpha} = \{\langle T\rangle^{\omega^\alpha} : T \in \mathcal{R}\}.$$

We then inductively define the set $\mathrm{Tree}^{\omega^\alpha\cdot k}(\mathcal{Q}) \subseteq \mathrm{Tree}^{\omega^{\alpha+1}}(\mathcal{Q})$ of $\mathcal{L}(\mathcal{Q})$-terms (of rank $< \omega^\alpha \cdot k$ for $1 \leq k < \omega$) as follows:

$$\mathrm{Tree}^{\omega^\alpha\cdot 1}(\mathcal{Q}) = \mathrm{Tree}^{\omega^\alpha}(\mathcal{Q}), \qquad \mathrm{Tree}^{\omega^\alpha\cdot(k+1)}(\mathcal{Q}) = \mathrm{Tree}^{\omega^\alpha}(\langle\mathrm{Tree}^{\omega^\alpha\cdot k}(\mathcal{Q})\rangle^{\omega^\alpha}).$$

In general, recall that every countable ordinal ξ can be written as $\omega^\alpha + \beta$ for some $\beta < \omega^{\alpha+1}$. Then we define $\mathrm{Tree}^\xi(\mathcal{Q})$ and ${}^\sqcup\mathrm{Tree}^\xi(\mathcal{Q})$ as follows:

$$\mathrm{Tree}^{\omega^\alpha+\beta}(\mathcal{Q}) = \mathrm{Tree}^{\omega^\alpha}(\langle\mathrm{Tree}^{\beta}(\mathcal{Q})\rangle^{\omega^\alpha}), \qquad {}^\sqcup\mathrm{Tree}^{\omega^\alpha+\beta}(\mathcal{Q}) = {}^\sqcup\mathrm{Tree}^{\omega^\alpha}(\langle\mathrm{Tree}^{\beta}(\mathcal{Q})\rangle^{\omega^\alpha}).$$

Note that one can also decompose ξ as $\xi = \omega^\alpha \cdot k + \gamma$ for $k < \omega$ and $\gamma < \omega^\alpha$ to define $\mathrm{Tree}^\xi(\mathcal{Q})$ in a straightforward manner, but it gives the same definition as above.

3.4.2. *Quasi-ordering nested trees (infinite Borel rank).* In this section, we extend the domain of the quasi-order $\trianglelefteq$ to ${}^\sqcup\mathrm{Tree}^{\omega_1}(\mathcal{Q})$. As in Section 3.1.3, we first inductively define a quasi-order $\trianglelefteq$ on $\mathrm{Tree}^{\omega_1}(\mathcal{Q})$, and then, $\trianglelefteq$ is uniquely extended to a quasi-order on ${}^\sqcup\mathrm{Tree}^{\omega_1}(\mathcal{Q})$ by interpreting $\sqcup$ as a countable supremum operation. Recall the convention from Section 3.1.3 that we always identify $p \in \mathcal{Q}$ with $\langle p\rangle$, and $\langle T\rangle$ with $\langle T\rangle^\to \sqcup_i \mathbf{O}$, where $\mathbf{O}$ is the empty forest, viewed as an imaginary least element w.r.t. the quasi-order $\trianglelefteq$, that is, $\mathbf{O} \trianglelefteq T$ for any $T \in \mathrm{Tree}^{\omega_1}(\mathcal{Q})$. We also identify $p \in \mathcal{Q}$ with $\langle p\rangle^{\omega^\alpha}$ for any $\alpha < \omega_1$.

Definition 3.20. We inductively define a quasi-order $\trianglelefteq$ on $\bigcup_n \mathrm{Tree}^{\omega_1}(\mathcal{Q})$ as follows, where the symbols p and q range over $\mathcal{Q}$, and U, V, S, and T range over range over $\mathrm{Tree}^{\omega_1}(\mathcal{Q})$:

$$p \trianglelefteq q \iff p \leq_\mathcal{Q} q,$$

$$\langle U\rangle^{\omega^\alpha} \trianglelefteq \langle V\rangle^{\omega^\beta} \iff \begin{cases} U \trianglelefteq V & \text{if } \alpha = \beta, \\ \langle U\rangle^{\omega^\alpha} \trianglelefteq V & \text{if } \alpha > \beta, \\ U \trianglelefteq \langle V\rangle^{\omega^\beta} & \text{if } \alpha < \beta. \end{cases}$$

and if S and T are of the form $\langle U\rangle^{\omega^\alpha\to}\sqcup_i S_i$ and $\langle V\rangle^{\omega^\beta\to}\sqcup_j T_j$, respectively, then

$$S\trianglelefteq T\iff\begin{cases}\text{either } \langle U\rangle^{\omega^\alpha}\trianglelefteq\langle V\rangle^{\omega^\beta} & \text{and } (\forall i)\ S_i\trianglelefteq T,\\ \text{or } \quad\ \ \langle U\rangle^{\omega^\alpha}\not\trianglelefteq\langle V\rangle^{\omega^\beta} & \text{and } (\exists j)\ S\trianglelefteq T_j.\end{cases}$$

We now assign a class Σ_T to each forest $T\in{}^{\sqcup}\mathrm{Tree}^{\omega_1}(\mathcal{Q})$ as follows:

Definition 3.21. For $\alpha>0$, $\mathcal{A}\in\Sigma_{\langle T\rangle^{\omega^\alpha}}$ if and only if there is a $\mathbf{\Sigma}^0_{1+\omega^\alpha}$-measurable function $\mathcal{D}\colon\omega^\omega\to\omega^\omega$ and a Σ_T-function $\mathcal{B}\colon\omega^\omega\to\mathcal{Q}$ such that $\mathcal{A}=\mathcal{B}\circ\mathcal{D}$. The other cases are as in Definition 3.7.

We check measurability of Σ_T-functions.

Lemma 3.22. *Let T be an $\mathcal{L}(\mathcal{Q})$-term and ξ be a countable ordinal. Then, $\Sigma_T\subseteq\mathbf{\Delta}^0_{1+\xi}$ implies $\Sigma_{\langle T\rangle^{\omega^\alpha}}\subseteq\mathbf{\Delta}^0_{1+\omega^\alpha+\xi}$.*

Proof. One can use a similar argument as in Lemma 3.9 (2). □

By combining Lemmas 3.9 and 3.22, we obtain the direction from (2) to (1) in Proposition 1.8 for infinite Borel rank, that is, that if $T\in{}^{\sqcup}\mathrm{Tree}^{\xi}(\mathcal{Q})$, then $\Sigma_T\subseteq\mathbf{\Delta}^0_{1+\xi}$.

3.4.3. *Σ_T-complete functions (infinite Borel rank).* Now we introduce a Σ_T-complete function Ω_T for each forest $T\in{}^{\sqcup}\mathrm{Tree}^{\omega_1}(\mathcal{Q})$. To achieve this, we need a universal function at transfinite Borel ranks. Again, recall that every $\mathbf{\Sigma}^0_{1+\xi}$-measurable function is coded by a real (for instance, we can use Fact 6.2 below).

Definition 3.23. We say that $\mathcal{U}_\xi\colon\hat{\omega}^\omega\to\hat{\omega}^\omega$ is *$\mathbf{\Sigma}^0_\xi$-universal* if it is $\mathbf{\Sigma}^0_\xi$-measurable, and for every conciliatory $\mathbf{\Sigma}^0_\xi$-measurable function $\Psi\colon\hat{\omega}^\omega\to\hat{\omega}^\omega$, there is a continuous function θ such that $\Psi=\mathcal{U}_\xi\circ\theta$.

To show the existence of a Σ_T-complete function, we need to extend Proposition 2.15 to infinite Borel ranks.

Proposition 3.24. *For any countable ordinal α, there is an initializable $\mathbf{\Sigma}^0_{1+\omega^\alpha}$-universal conciliatory function.*

We prove this proposition in Section 6.

We now introduce Ω_T for each term $T\in{}^{\sqcup}\mathrm{Tree}^{\omega_1}(\mathcal{Q})$. It suffices to describe how to define $\Omega_{\langle T\rangle^{\omega^\alpha}}$, as the rest is as in Definition 3.14.

Definition 3.25 (Complete Function)**.** Let α be a countable ordinal, and let T be a tree in $\mathrm{Tree}^{\omega_1}(\mathcal{Q})$. Then we define the conciliatory function $\Omega_{\langle T\rangle^{\omega^\alpha}}:\omega^{\leq\omega}\to\mathcal{Q}$ as follows:

$$\Omega_{\langle T\rangle^{\omega^\alpha}}=\Omega_T\circ\mathcal{U}_{\omega^\alpha},$$

where $\mathcal{U}_{\omega^\alpha}$ is a fixed initializable $\mathbf{\Sigma}^0_{1+\omega^\alpha}$-universal conciliatory function as in Proposition 3.24.

It is not hard to prove the transfinite versions of 3.15, 3.16, 3.17, and 3.18 namely that Ω_T is Σ_T-complete, and that if $S\trianglelefteq T$, then $\Omega_S\leq_w\Omega_T$.

4. The jump operator and its inversion

The goal of this section is to define an inverse of the operation $\mathcal{B} \mapsto \mathcal{B} \circ \mathcal{U}$. (Recall that $\Omega_{\langle T \rangle}$ was defined as $\Omega_T \circ \mathcal{U}$.) This operation will be denoted by $\mathcal{A}^{\not\to}$, and we will prove that $(\mathcal{A} \circ \mathcal{U})^{\not\to} \equiv_w \mathcal{A}$ for every $\mathcal{Q}$-valued function $\mathcal{A}$. Furthermore, we will get that $(A^{\not\to}) \circ \mathcal{U} \equiv_w \mathcal{A}$ if we also assume $\mathcal{A}$ is initializable. The key technical notions are the *Turing jump operator via true stages* from computability theory and a uniform version of the Friedberg jump inversion theorem. The use of this jump operator is one of the aspects of our proof that makes it easier than Duparc's work for $\mathcal{Q} = 2$.

4.1. **Turing jump operator.** The Turing jump operator is one of the most basic notions in computability theory. For our proof, we need a version of this operator with nicer properties than the standard jump operator $X \mapsto X'$. The jump operator *via true stages* $\mathcal{J} : \omega^\omega \to \omega^\omega$ introduced by Marcone and Montalbán [MM11, Mon14] is exactly what we need. Marcone and Montalbán also defined its approximation on finite strings $J : \omega^{<\omega} \to \omega^{<\omega}$. Putting these together, what they defined was a Σ^0_2 conciliatory function $\mathcal{J} : \omega^{\leq\omega} \to \omega^{\leq\omega}$. The properties we need are the following:

(1) (Σ^0_2-universality from the right.) For every Σ^0_2 operator $G : \omega^\omega \to \hat{\omega}^\omega$, there is a computable $\theta : \mathcal{J}[\omega^\omega] \to \hat{\omega}^\omega$ such that $G \equiv_{\mathsf{p}} \theta \circ \mathcal{J}$ (recall the definition of $\equiv_{\mathsf{p}}$ in Section 2.6). Furthermore, if G is Σ^0_2 relative to an oracle C, then we can still find θ computable so that, for every $X \in \omega^\omega$, $G(X)^{\mathsf{p}} = \theta(\mathcal{J}(C \oplus X))^{\mathsf{p}}$.
(2) The image ω^ω under $\mathcal{J}$, namely $\mathcal{J}[\omega^\omega]$, is a closed subset of ω^ω. Furthermore, $\mathcal{J}$ is one-to-one, and its inverse $\mathcal{J}^{-1} : \mathcal{J}[\omega^\omega] \to \omega^\omega$ is continuous.
(3) (Denseness of forcing.) For every string $\gamma \in \omega^{<\omega}$, there is a string $\sigma \supseteq \gamma, \sigma \in \omega^{<\omega}$ which forces the jump in the following sense: We say that $\sigma \in \omega^{<\omega}$ *forces the jump*, if for every $\tau \supseteq \sigma$, $\mathcal{J}(\tau) \supseteq \mathcal{J}(\sigma)$.

For (1), note that if the range of G is contained in ω^ω, one can find $\theta : \mathcal{J}[\omega^\omega] \to \omega^\omega$ such that $G = \theta \circ \mathcal{J}$. The relativized version also holds. These properties are immediate from the definitions in Marcone and Montalbán [MM11, Mon14]. The last property (3) requires a minute of thought, though it is quite standard.

Notice that for the usual Turing jump operator $X \mapsto X'$, the image is not closed. Another advantage of $\mathcal{J}$ is that its finite approximation can be easily iterated, allowing us to keep the denseness of forcing when we consider transfinite iterates of the jump.

We use $\mathcal{J}^C$ to denote the operator $X \mapsto \mathcal{J}(C \oplus X)$. It satisfies the same properties of $\mathcal{J}$ we mentioned above. Let $\mathcal{J}^{n,C}$ be the n-th iterate of the jump operator relative to C, that is, put $\mathcal{J}^{1,C} = \mathcal{J}^C$ and $\mathcal{J}^{n+1,C} = \mathcal{J}^C \circ \mathcal{J}^{n,C}$. We also use the symbol $\mathcal{J}^{-1,C}$ to denote $(\mathcal{J}^C)^{-1}$. The Σ^0_2 universality of $\mathcal{J}$ can be iterated through the finite levels of the Borel hierarchy.

Fact 4.1. *For every $\mathbf{\Sigma}^0_{n+1}$-measurable function $\mathcal{A} : \omega^\omega \to \hat{\omega}^\omega$, there are $C \in \omega^\omega$ and a computable $\Phi : \omega^\omega \to \hat{\omega}^\omega$ such that $\mathcal{A} \equiv_{\mathsf{p}} \Phi \circ \mathcal{J}^{n,C}$.*

Duparc [Dup01, Definition 25] introduced an operation $\not\to$ on subsets of ω^ω. We extend Duparc's operation $\not\to$ to $\mathcal{Q}$-valued functions, but our definition is quite different from Duparc's, which is rather hard to understand.

Definition 4.2. For any $\mathcal{A}\colon \omega^\omega \to \mathcal{Q}$ and any oracle $Z \in \omega^\omega$ we introduce the *Z-jump inversion* of $\mathcal{A}$, $\mathcal{A}^{\not\nearrow Z}\colon \mathcal{J}^Z[\omega^\omega] \to \mathcal{Q}$, as follows:

$$\mathcal{A}^{\not\nearrow Z}(Y) = \mathcal{A}(\mathcal{J}^{-1,Z}(Y)),$$

Note that the domain of $\mathcal{A}^{\not\nearrow Z}$ is the range of $\mathcal{J}^Z$, which is closed as mentioned above, and therefore one can think of $\mathcal{A}^{\not\nearrow Z}$ as a function on ω^ω by Observation 3.5, where we composed $\mathcal{A}^{\not\nearrow Z}$ with a continuous retraction $\eta\colon \omega^\omega \to \mathcal{J}^Z[\omega^\omega]$.

Remark 4.3. We also apply the operator $\not\nearrow$ to a conciliatory function $\mathcal{A} : \hat{\omega}^\omega \to \mathcal{Q}$. In this case, via a computable homeomorphism $I : \hat{\omega}^\omega \to \omega^\omega$, we identify $\mathcal{A}$ with $\mathcal{A} \circ I^{-1} : \omega^\omega \to \mathcal{Q}$. Clearly, $\mathcal{A}$ is Wadge equivalent to $\mathcal{A} \circ I^{-1}$. Then, if the domain of $\mathcal{A}$ is $\hat{\omega}^\omega$, the actual definition of $\mathcal{A}^{\not\nearrow Z}$ is $\mathcal{A} \circ I^{-1} \circ \mathcal{J}^{-1,Z}$, and the domain of $\mathcal{A}^{\not\nearrow Z}$ is $\mathcal{J}^Z[\omega^\omega]$. Note that $\mathcal{A} = \mathcal{A}^{\not\nearrow Z} \circ \mathcal{J}^Z \circ I$. We should be careful as the domain of $\mathcal{A}^{\not\nearrow Z}$ is a subset of ω^ω even if $\mathcal{A}$ has the domain $\hat{\omega}^\omega$.

Observation 4.4. Let $\mathcal{A}$ be any partial $\mathcal{Q}$-valued function, and $Y, Z \in \omega^\omega$ be oracles. Then

$$Y \leq_T Z \quad \Rightarrow \quad \mathcal{A}^{\not\nearrow Y} \geq_w \mathcal{A}^{\not\nearrow Z}.$$

Proof. The map $X \mapsto \mathcal{J}^Y(X)$ is Σ^0_2 relative to Y, and hence in particular Σ^0_2 relative to Z. Therefore, there is a computable function Φ so that $\Phi \circ \mathcal{J}^Z = \mathcal{J}^Y$. This witnesses that $\mathcal{A}^{\not\nearrow Z} \leq_w \mathcal{A}^{\not\nearrow Y}$ as follows: For $\mathcal{J}^Z(X) \in \mathcal{J}^Z[\omega^\omega]$,

$$\mathcal{A}^{\not\nearrow Z}(\mathcal{J}^Z(X)) = \mathcal{A}(X) = \mathcal{A}^{\not\nearrow Y}(\mathcal{J}^Y(X)) = \mathcal{A}^{\not\nearrow Y}(\Phi(\mathcal{J}^Z(X))). \qquad \square$$

Recall that if $\mathcal{Q}$ is bqo, then so are the Wadge degrees of $\mathcal{Q}$-valued functions (Theorem 2.3). In particular, there is no infinite decreasing chain of the Wadge degrees. Thus, Observation 4.4 implies that, for any sufficiently powerful oracle $Z \in \omega^\omega$, we have $\mathcal{A}^{\not\nearrow Z} \leq_w \mathcal{A}^{\not\nearrow Y}$ for any other oracle $Y \in \omega^\omega$.

Notation 4.5. If $\mathcal{A}$ is a $\mathcal{Q}$-valued function for a bqo $\mathcal{Q}$, we hereafter use the notation $\mathcal{A}^{\not\nearrow}$ to denote a representative of the minimum one among Wadge degrees of $\{\mathcal{A}^{\not\nearrow Z} : Z \in \omega^\omega\}$, that is,

$$\mathcal{A}^{\not\nearrow} \equiv_w \mathcal{A}^{\not\nearrow Z} \text{ for some } Z \in \omega^\omega, \text{ and } \mathcal{A}^{\not\nearrow} \leq_w \mathcal{A}^{\not\nearrow Y} \text{ for all } Y \in \omega^\omega.$$

Here are some basic properties of the jump inversion operator. In particular, (2) of the next lemma shows that $\not\nearrow$ is well-defined on $\mathcal{Q}$-Wadge degrees.

Lemma 4.6. *For any $\mathcal{A}, \mathcal{B}\colon \omega^\omega \to \mathcal{Q}$, the following holds.*

1. *If $\mathcal{A}$ is $\mathbf{\Sigma}^0_{n+1}$-measurable, then $\mathcal{A}^{\not\nearrow}$ is $\mathbf{\Sigma}^0_n$-measurable.*
2. *If $\mathcal{A} \leq_w \mathcal{B}$, then $\mathcal{A}^{\not\nearrow} \leq_w \mathcal{B}^{\not\nearrow}$.*
3. *If either $\mathcal{A}^{\not\nearrow}$ or $\mathcal{B}^{\not\nearrow}$ is non-self-dual, then $\mathcal{A}^{\not\nearrow} \leq_w \mathcal{B}^{\not\nearrow}$ implies $\mathcal{A} \leq_w \mathcal{B}$.*

Proof. (1) Since $\mathcal{A}$ is $\mathbf{\Sigma}^0_{n+1}$, there is a Δ^0_0 formula φ in the language of second-order arithmetic and a $Z \in \omega^\omega$ such that

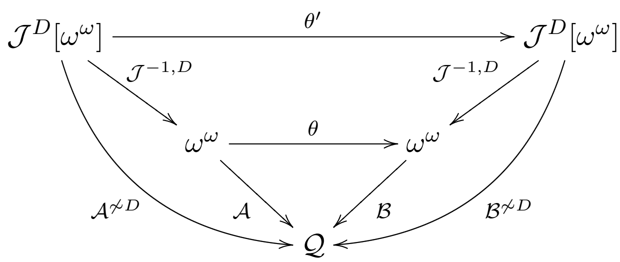
$$\mathcal{A}(X) = q \iff (\exists a_0)(\forall a_1)\dots(\check{\mathsf{Q}} a_{n-1})(\mathsf{Q} a_n)\ \varphi(q, a_0, \dots, a_n, X, Z),$$

where Q is the existential quantifier if n is even; and Q is the universal quantifier if n is odd, and $\check{\mathsf{Q}}$ is the other way around. Clearly, there is a Δ^0_0 formula ψ such that

$$\mathcal{A}(X) = q \iff (\exists a_0)(\forall a_1)\dots(\check{\mathsf{Q}} a_{n-1})\ \psi(q, a_0, \dots, a_{n-1}, J^Z(X)).$$

Consequently, $\mathcal{A}^{\not\sim Z}$ is $\mathbf{\Sigma}^0_n$. The same argument works for any $C \geq_T Z$, and thus $\mathcal{A}^{\not\sim}$ is $\mathbf{\Sigma}^0_n$ too.

(2) Assume that $\mathcal{A} \leq_w \mathcal{B}$ via a continuous function θ. Then θ is C-computable for some oracle C. Let D be such that $\mathcal{A}^{\not\sim} \equiv_w \mathcal{A}^{\not\sim D}$ and $\mathcal{B}^{\not\sim} \equiv_w \mathcal{B}^{\not\sim D}$. Then, by the universality of $\mathcal{J}^D$ from the right, there is a continuous function θ' such that $\theta' \circ \mathcal{J}^D = \mathcal{J}^D \circ \theta$ since $D \geq_T C$. As seen in the following diagram, this θ' witnesses that $\mathcal{A}^{\not\sim} \leq_w \mathcal{B}^{\not\sim}$.



More formally: For any $Y \in \mathcal{J}^D[\omega^\omega]$, and $X = \mathcal{J}^{-1,D}(Y)$,

$$\mathcal{A}^{\not\sim D}(Y) = \mathcal{A}(X) \leq_{\mathcal{Q}} \mathcal{B}(\theta(X)) = \mathcal{B}^{\not\sim D}(\mathcal{J}^D(\theta(X))) = \mathcal{B}^{\not\sim D}(\theta'(\mathcal{J}^D(X))) = \mathcal{B}^{\not\sim D}(\theta'(Y)).$$

(3) If $\mathcal{A} \not\leq_w \mathcal{B}$, then Player I has a winning strategy in the game $G_w(\mathcal{A},\mathcal{B})$, that is, there is a C-computable Lipschitz function θ such that $\mathcal{A}(\theta(X)) \not\leq_{\mathcal{Q}} \mathcal{B}(X)$. By the same argument as above, let D be such that $\mathcal{A}^{\not\sim} \equiv_w \mathcal{A}^{\not\sim D}$ and $\mathcal{B}^{\not\sim} \equiv_w \mathcal{B}^{\not\sim D}$, and then, there is a continuous function θ' such that $\mathcal{A}^{\not\sim D}(\theta'(Y)) \not\leq_{\mathcal{Q}} \mathcal{B}^{\not\sim D}(Y)$ for all $Y \in \mathcal{J}^D[\omega^\omega]$. However, if $\mathcal{A}^{\not\sim D} \leq_w \mathcal{B}^{\not\sim D}$ via a continuous function η, we would have the following:

$$\mathcal{A}^{\not\sim D}(Z) \leq_{\mathcal{Q}} \mathcal{B}^{\not\sim D}(\eta(Z)) \not\geq_{\mathcal{Q}} \mathcal{A}^{\not\sim D}(\theta' \circ \eta(Z)),$$
$$\mathcal{B}^{\not\sim D}(Z) \not\geq_{\mathcal{Q}} \mathcal{A}^{\not\sim D}(\theta'(Z)) \leq_{\mathcal{Q}} \mathcal{B}^{\not\sim D}(\eta \circ \theta'(Z)).$$

Thus both $\mathcal{A}^{\not\sim D}$ and $\mathcal{B}^{\not\sim D}$ are self-dual, which contradicts our assumption since $\mathcal{A}^{\not\sim} \equiv_w \mathcal{A}^{\not\sim D}$ and $\mathcal{B}^{\not\sim} \equiv_w \mathcal{B}^{\not\sim D}$. □

4.2. **The operation $\not\sim$ inverts the jump.** We now prove a key result which is that $\not\sim$ is the inverse of $\mathcal{B} \mapsto \mathcal{B} \circ \mathcal{U}$. This is somewhat an analogue of [Dup01, Propositions 29 and 30], which roughly says that the jump inversion operator $\not\sim$ bridges $\mathrm{Tree}^n(\mathcal{Q})$ and $\mathrm{Tree}^{n+1}(\mathcal{Q})$: namely that $\Omega^{\not\sim}_{\langle T\rangle} \equiv_w \Omega_T$.

Recall from Remark 4.3 that $\Omega^{\not\sim}_{\langle T\rangle} = \Omega_T \circ \mathcal{U} \circ I^{-1} \circ \mathcal{J}^{-1,C}$, where $I : \hat{\omega}^\omega \to \omega^\omega$ is a computable homeomorphism. The proof of the following lemma shows how $\mathcal{U}$ and $\mathcal{J}^C$ interact: $\mathcal{U} \circ I^{-1} \circ \mathcal{J}^{-1,C}$ is Wadge-equivalent to the identity function.

Lemma 4.7. *Let $\mathcal{A}\colon \hat{\omega}^\omega \to \mathcal{Q}$ be conciliatory. Then,*

$$(\mathcal{A} \circ \mathcal{U})^{\not\sim} \equiv_w \mathcal{A}.$$

In particular, if $T \in \mathrm{Tree}^n(\mathcal{Q})$, then $\Omega^{\not\sim}_{\langle T\rangle} \equiv_w \Omega_T$.

Proof. First observe that $(\mathcal{A} \circ \mathcal{U})^{\not\sim} = \mathcal{A} \circ \mathcal{U} \circ I^{-1} \circ \mathcal{J}^{-1,C}$ for some oracle C. Since $\mathcal{U} \circ I^{-1} : \omega^\omega \to \hat{\omega}^\omega$ is Σ^0_2-measurable, there is a computable $\Phi\colon \omega^\omega \to \hat{\omega}^\omega$ such that $\mathcal{U} \circ I^{-1} \equiv_{\mathsf{p}} \Phi \circ \mathcal{J}^C$. Then, $\mathcal{U} \circ I^{-1} \circ \mathcal{J}^{-1,C} \equiv_{\mathsf{p}} \Phi$ on its domain. Then $(\mathcal{A} \circ \mathcal{U})^{\not\sim} \leq_w \mathcal{A}$ via Φ because $\mathcal{A}$ is conciliatory.

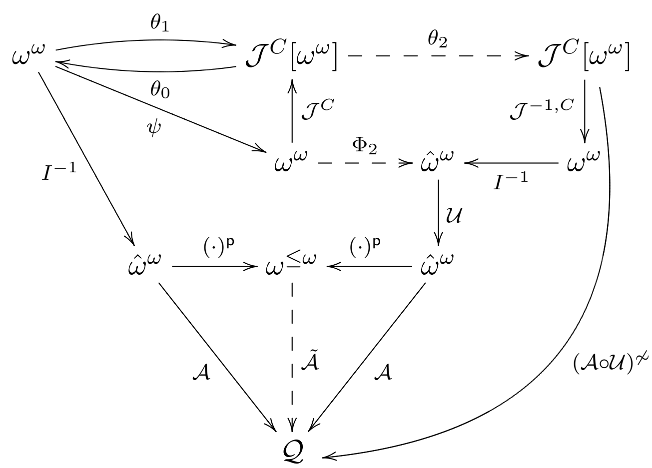
To prove the other direction we need the Friedberg jump inversion theorem [Fri57]. The standard proof of the Friedberg jump inversion theorem (relative to C) gives a C'-computable function ψ such that for any X,

$$X \oplus C' \equiv_T \psi(X) \oplus C' \equiv_T (\psi(X) \oplus C)' \equiv_T \mathcal{J}^C(\psi(X)).$$

By carefully checking the standard proof of the Friedberg jump inversion theorem, one can see the following uniform version: There is a C'-computable function $\psi\colon \omega^\omega \to \omega^\omega$ such that, for every $X \in \omega^\omega$, $\mathcal{J}^C(\psi(X))$ is uniformly Turing equivalent to $X \oplus C'$. That is, there is a computable operator $\theta_0\colon \omega^\omega \to \omega^\omega$ and a C'-computable operator $\theta_1\colon \omega^\omega \to \omega^\omega$ such that, for every $X \in \omega^\omega$,

$$\theta_0 \circ \mathcal{J}^C \circ \psi(X) = X \quad \text{and} \quad \theta_1(X) = \mathcal{J}^C \circ \psi(X).$$

Then, by universality of $\mathcal{U}$ we get a C-computable function Φ_2 such that $I^{-1} \circ \theta_0 \circ \mathcal{J}^C \equiv_{\mathsf{p}} \mathcal{U} \circ \Phi_2$, i.e., $(\cdot)^{\mathsf{p}} \circ I^{-1} \circ \theta_0 \circ \mathcal{J}^C = (\cdot)^{\mathsf{p}} \circ \mathcal{U} \circ \Phi_2$. Moreover, by universality of $\mathcal{J}$, we also get a computable function θ_2 such that $\mathcal{J}^C \circ I \circ \Phi_2 = \theta_2 \circ \mathcal{J}^C$. We now obtain the following commutative diagram:



This commutative diagram shows that $\mathcal{A} \leq_w (\mathcal{A} \circ \mathcal{U})^{\not\to}$ is witnessed by the continuous function $\theta_2 \circ \theta_1 \circ I$.

For the second part just notice that $\Omega_{\langle T \rangle}^{\not\to} = (\Omega_T \circ \mathcal{U})^{\not\to} \equiv_w \Omega_T$. □

The operation $\not\to$ is not one-to-one on $\mathcal{Q}$-Wadge degrees. But it is if we restrict it to initializable degrees. We first need to prove the following lemma, which is where the denseness of forcing of $\mathcal{J}$ is needed.

Definition 4.8. For a function $\mathcal{A}\colon \omega^\omega \to \mathcal{Q}$, we say that $\mathcal{A}$ is *initializable* if for every $\sigma \in \omega^{<\omega}$, $\mathcal{A} \leq_w \mathcal{A} \upharpoonright [\sigma]$.

That is, $\mathcal{A}$ is initializable if and only if $\mathcal{F}(\mathcal{A}) = \omega^\omega$. Recall the definition of $\mathcal{F}$ from Section 2.4. Also recall from Proposition 2.6 that $\mathcal{A}$ is non-self-dual if and only if $\mathcal{F}(\mathcal{A})$ is non-empty.

Notice that this definition matches Definition 2.14 for the case $\mathcal{Q} = \omega^\omega$ were all elements are $\leq_Q$-incomparable.

Lemma 4.9. *If $\mathcal{A}$ is initializable, then $\mathcal{A}^{\not\to}$ is non-self-dual.*

Proof. Let C be an oracle which computes Wadge reductions $\mathcal{A} \leq_w \mathcal{A}\upharpoonright[\tau]$ for all τ. We say that σ *forces its jump relative to* C if for any $\tau \supseteq \sigma$, $J^C(\tau) \supseteq J^C(\sigma)$ holds. We claim that for such σ, we have

$$\mathcal{A}^{\not\leftarrow C} \leq_w \mathcal{A}^{\not\leftarrow C}\upharpoonright[J^C(\sigma)].$$

To prove this, let θ be C-computable such that $X \mapsto \sigma^\frown\theta(X)$ is a Wadge reduction $\mathcal{A} \leq_w \mathcal{A}\upharpoonright[\sigma]$. Then, using the universality of $\mathcal{J}^C$ from the right, we have a computable function θ' such that $\theta' \circ \mathcal{J}^C(X) = \mathcal{J}^C(\sigma^\frown\theta(X))$ for all $X \in \omega^\omega$. Since σ forces its jump relative to C, we have

$$\mathcal{J}^C(\sigma) \subseteq \mathcal{J}^C(\sigma^\frown\theta(X)) = \theta'(\mathcal{J}^C(X)).$$

Thus θ' gives a Wadge reduction from $\mathcal{A}^{\not\leftarrow C}$ to $\mathcal{A}^{\not\leftarrow C}\upharpoonright[J^C(\sigma)]$ as follows: Given $Y \in \mathcal{J}^C[\omega^\omega]$ and $X = \mathcal{J}^{-1,C}(Y)$,

$$\mathcal{A}^{\not\leftarrow C}(Y) = \mathcal{A}(X) \leq_{\mathcal{Q}} \mathcal{A}(\sigma^\frown\theta(X)) = \mathcal{A}^{\not\leftarrow C}(\mathcal{J}^C(\sigma^\frown\theta(X))) = \mathcal{A}(\theta'(\mathcal{J}^C(X))) = \mathcal{A}(\theta'(Y)).$$

Now, we say that $X \in \omega^\omega$ *forces its jump relative to* C if there are infinitely many $n \in \omega$ such that $X\upharpoonright n$ forces its jump relative to C. Using the density of forcing, one can easily construct such an X. Let $t_0 < t_1 < t_2 < \cdots$ be such that $X\upharpoonright t_i$ forces its jump relative to C. We then get that $\mathcal{J}^C(X\upharpoonright t_i) \subseteq \mathcal{J}^C(X)$ for all i, and furthermore, every initial segment of $\mathcal{J}^C(X)$ is an initial segment of some $\mathcal{J}^C(X\upharpoonright t_i)$. It follows that for every initial segment $\tau \subseteq \mathcal{J}^C(X)$, $\mathcal{A}^{\not\leftarrow C} \leq_w \mathcal{A}^{\not\leftarrow C}\upharpoonright[\tau]$. Therefore, $\mathcal{J}^C(X) \in \mathcal{F}(\mathcal{A}^{\not\leftarrow C})$ as desired. □

4.3. **Preservation of ordering.** By a quite straightforward inductive proof, we are now ready to show Proposition 1.7 for finite Borel rank. Here, by Lemma 3.17, it suffices to show that $\Omega_S \leq_w \Omega_T$ if and only if $S \trianglelefteq T$ for any $S, T \in {}^{\sqcup}\mathrm{Tree}^n(\mathcal{Q})$. Recall that we have already proved the right-to-left direction in Lemma 3.18.

Proof of Proposition 1.7. We show the assertion by induction on the terms S and T. Assume $\Omega_S \leq_w \Omega_T$.

First, it is clear that $\Omega_p \leq_w \Omega_q$ if and only if $p \trianglelefteq q$. Suppose now that $\Omega_{\langle U\rangle} \leq_w \Omega_{\langle V\rangle}$. By Lemma 4.7, $\Omega^{\not\leftarrow}_{\langle U\rangle} \equiv_w \Omega_U$ and similarly for V. We thus get $\Omega_U \leq_w \Omega_V$ by Lemma 4.6 (2). Hence, by the inductive hypothesis, $U \trianglelefteq V$ and $\langle U\rangle \trianglelefteq \langle V\rangle$.

Now, consider $S = \langle U\rangle^{\to} \bigsqcup_i S_i$ and $T = \langle V\rangle^{\to} \bigsqcup_j T_j$. Let us first consider the case that $\langle U\rangle \trianglelefteq \langle V\rangle$. In this case, by the definition of $\trianglelefteq$, under the assumption that $\langle U\rangle \trianglelefteq \langle V\rangle$, $S \trianglelefteq T$ if and only if $S_i \trianglelefteq T$ for any $i \in \omega$. We get this from the induction hypothesis, as clearly $\Omega_S \leq_w \Omega_T$ implies $\Omega_{S_i} \leq_w \Omega_T$ for any i.

Let us now assume that $\langle U\rangle \not\trianglelefteq \langle V\rangle$. By induction hypothesis, we have $\Omega_{\langle U\rangle} \not\leq_w \Omega_{\langle V\rangle}$. In this case, by definition, $S \trianglelefteq T$ if and only if $S \trianglelefteq T_j$ for some $j \in \omega$. We thus need to show that $\Omega_S \leq_w \Omega_{T_j}$ for some j.

Assume that $\Omega_S \leq_w \Omega_T$ via a continuous function θ. There must be a sequence X such that $\pi_1(X)$ is empty, but $\pi_1 \circ \theta(X)$ is nonempty (that is, X never changes his mind, whereas $\theta(X)$ changes her mind at some point): This is because if not, $\pi_0 \circ \theta$ would give a reduction from $\Omega_{\langle U\rangle}$ to $\Omega_{\langle V\rangle}$, which contradicts our assumption. Now, let n be the point where $\theta(X\upharpoonright n)$ changes her mind, and let k be the first entry of $\pi_1 \circ \theta(X\upharpoonright n)$.

Then $\pi_1 \circ \theta \restriction [X \restriction n]$ gives a reduction from $\Omega_S \restriction [X \restriction n]$ to T_k. We now need a reduction from Ω_S to $\Omega_S \restriction [X \restriction n]$.

Since $\Omega_{\langle U \rangle}$ is initializable, for any σ, there is a continuous function η_σ witnessing $\Omega_{\langle U \rangle} \leq_w \Omega_{\langle U \rangle} \restriction [\sigma]$. It is not hard then, to use η_σ to build a reduction from Ω_S to $\Omega_S \restriction [\sigma^\to \emptyset]$. This concludes the proof since $\pi_1(X)$ is empty, and thus $X \restriction n$ is of the form $\sigma^\to \emptyset$. □

We have just proved that the map $T \mapsto \Omega_T$ is an order-preserving embedding of ${}^\sqcup\mathrm{Tree}^n(\mathcal{Q})$ into the $\mathcal{Q}$-Wadge degrees of Δ_{n+1}-measurable functions. What is left to do is to show this embedding is onto.

4.4. **Another construction of a $\mathbf{\Sigma}^0_2$-universal function.** We end this section by giving a second construction of an initializable $\mathbf{\Sigma}^0_2$-universal conciliatory function. The reason we prove this again is that the following proof can be easily extended through the transfinite, once we define the transfinite jump operation in Section 6.1. Recall that our first construction of a conciliatory $\mathbf{\Sigma}^0_2$-universal function in Section 2.6 was direct and did not use $\mathcal{J}$.

Second proof of Proposition 2.15. Let $\{\Phi_e : e \in \omega\}$ be a computable enumeration of all computable operators : $\omega^\omega \to \omega^{\leq\omega}$. By the universality of $\mathcal{J}$ from the right, we know that for every $\mathbf{\Sigma^0_2}$ operator $G\colon \omega^\omega \to \omega^{\leq\omega}$, there is an $e \in \omega$ and a $C \in \omega^\omega$ such that $G = \Phi_e \circ \mathcal{J}^C$, where Φ_e is the e-th partial computable function. It is not hard to see that the map

$$e^\frown C \oplus X \mapsto \Phi_e \circ \mathcal{J}^C(X)$$

is $\mathbf{\Sigma^0_2}$ universal (from the left). However, we need $\mathcal{U}$ to also be initializable, so we need to tweak this definition a bit.

For $Z \in \hat{\omega}^\omega$, let $Z^{+1}(n) = Z(n) + 1$ if $Z(n) \neq \mathsf{pass}$, and $Z^{+1}(\mathsf{pass}) = \mathsf{pass}$. We define $\mathcal{U}$ as follows

$$\mathcal{U}(Y) = \begin{cases} \Phi_e \circ \mathcal{J}^C(X) & \text{if } Y = \sigma^\frown 0^\frown Z^{+1} \text{ and } Z = e^\frown C \oplus X \\ \emptyset & \text{if } Y \text{ has infinitely many 0's and is not of the form } \sigma^\frown 0^\frown Z^{+1} \end{cases}$$

It is clear that $\mathcal{U}$ is still $\mathbf{\Sigma^0_2}$ universal. It is also easily seen to be Σ^0_2 itself, as deciding in which case we are and where to split σ and Z is Σ^0_2, but then recovering e, C and X is computable. It is clearly initializable as $\mathcal{U}(X) = \mathcal{U}(\sigma^\frown 0^\frown X)$ for every σ and X. □

5. Proof of ontoness (finite Borel rank)

Recall that we divided Theorem 1.4 into Propositions 1.6, 1.7, 1.8 and 1.9, and the only one that is left to prove is the latter one. This whole section is dedicated to proving Proposition 1.9 for finite rank, that is, that given a $\mathbf{\Delta}^0_{1+n}$-measurable function $\mathcal{A}\colon \omega^\omega \to \mathcal{Q}$, we need to show that there is $T \in {}^\sqcup\mathrm{Tree}^n(\mathcal{Q})$ such that $\mathcal{A} \equiv_w \Omega_T$. Furthermore, we will show that if $\mathcal{A}$ is non-self-dual, then T will be a tree, while if T is self dual, T will be a $\sqcup$-type term.

The proof is by induction on $\leq_w$, which we know is well-founded (even bqo). We divide the proof in various cases depending on the properties of $\mathcal{A}$.

Case 1, *Constant.* If $\mathcal{A}$ is Wadge equivalent to a constant function, then it is clearly equivalent to $\Omega_{\langle q \rangle}$ for some $q \in \mathcal{Q}$.

Case 2, *Self Dual.* If $\mathcal{A}$ is self dual, we proved in Proposition 2.6 that $\mathcal{A} \equiv_w \bigoplus_{i\in\omega} \mathcal{A}_i$ where each $\mathcal{A}_i$ is non-self-dual and $\mathcal{A}_i <_w \mathcal{A}$. By the induction hypothesis, there exists trees $T_i \in \mathrm{Tree}^n(\mathcal{Q})$ such that $\mathcal{A}_i \equiv_w \Omega_{T_i}$. It then follows that $\mathcal{A} \equiv_w \Omega_{\sqcup_i T_i}$.

These covers the continuous case, as any continuous function is a clopen sum of constant functions.

Case 3, *Initializable.* Suppose now that $\mathcal{A}$ is initializable. We first claim that $\mathcal{A}^{\not\sim} <_w \mathcal{A}$. That $\mathcal{A}^{\not\sim} \leq_w \mathcal{A}$ follows from the fact that $\mathcal{J}^{-1,C}$ is continuous. Suppose n is the least such that $\mathcal{A}$ is $\mathcal{Q}$-Wadge equivalent to a Δ^0_{n+1} function. Since $\mathcal{A}^{\not\sim}$ is Δ^0_n by Lemma 4.6, it is not $\mathcal{Q}$-Wadge equivalent to $\mathcal{A}$.

Also recall from Lemma 4.9 that $\mathcal{A}^{\not\sim}$ is non-self-dual. By the induction hypothesis, we then have that there is $T \in \mathrm{Tree}^{n-1}(\mathcal{Q})$ such that $\mathcal{A}^{\not\sim} \equiv_w \Omega_T$. Moreover, by Lemma 4.7 applied to Ω_T, we get $\Omega_T \equiv_w \Omega^{\not\sim}_{\langle T\rangle}$, and thus $\mathcal{A}^{\not\sim} \equiv_w \Omega^{\not\sim}_{\langle T\rangle}$. Therefore, by Lemma 4.6 (3), we obtain $\mathcal{A} \equiv_w \Omega_{\langle T\rangle}$.

Observation 5.1. Let us comment on the case when the domain of $\mathcal{A}$ is closed subset $\mathcal{F} \subseteq \omega^\omega$. In this case we say that $\mathcal{A}$ is initializable if for every $\sigma \in \omega^\omega$ extendible in $\mathcal{F}$, $\mathcal{A} \leq_w \mathcal{A}\restriction[\sigma]$. In Observation 3.5, we notice we could view such a map as a map defined on ω^ω by composing with a retraction $\rho_{\mathcal{F}}$. One can show that for the retraction defined there, the map we get is also initializable.

Case 4, *Non-self-dual and not initializable.* Suppose now that $\mathcal{A}$ is non-self-dual and not Wadge equivalent to any initializable function. The following is the $\mathcal{Q}$-version of the basic property of initializability.

Lemma 5.2. *If $\mathcal{F}(\mathcal{A}) \neq \emptyset$, then $\mathcal{A}\restriction\mathcal{F}(\mathcal{A})$ is initializable.*

Furthermore, if $\mathcal{A}$ is Wadge equivalent to an initializable function, then $\mathcal{A} \equiv_w \mathcal{A}\restriction\mathcal{F}(A)$.

Proof. Let
$$F = \{\sigma \in \omega^{<\omega} : \mathcal{A} \leq_w \mathcal{A}\restriction[\sigma]\}.$$
Then F is a tree without dead ends whose paths are exactly $\mathcal{F}(\mathcal{A})$. (We will sometimes write $\sigma \in \mathcal{F}(\mathcal{A})$ to mean $\sigma \in F$.) We need to show that for each $\sigma \in F$, there is a continuous reduction $\mathcal{A}\restriction\mathcal{F}(\mathcal{A}) \leq_w \mathcal{A}\restriction\mathcal{F}(\mathcal{A}) \cap [\sigma]$. Since $\sigma \in F$, there is a continuous reduction $\theta\colon \mathcal{A} \leq_w \mathcal{A}\restriction[\sigma]$. Think of θ as a function $: \omega^{<\omega} \to \omega^{<\omega}$. We claim that $\theta[F] \subseteq F$, hence obtaining a reduction $\mathcal{A}\restriction\mathcal{F}(\mathcal{A}) \leq_w \mathcal{A}\restriction\mathcal{F}(\mathcal{A}) \cap [\sigma]$ as wanted. Suppose not, and that for some $\tau \in F$, $\theta(\tau) \notin F$. We then get a continuous reduction of $\mathcal{A}\restriction[\tau]$ to $\mathcal{A}\restriction[\theta(\tau)]$. But $\mathcal{A}\restriction[\tau] \equiv_w \mathcal{A}$ while $\mathcal{A} \not\leq_w \mathcal{A}\restriction[\theta(\tau)]$, getting the desired contradiction.

For the second part of the lemma, one needs to observe that if $\mathcal{A} \equiv_w \mathcal{B}$, then $\mathcal{A}\restriction\mathcal{F}(\mathcal{A}) \equiv_w \mathcal{B}\restriction\mathcal{F}(\mathcal{B})$. The reason is that if θ is a reduction $\mathcal{A} \leq_w \mathcal{B}$, then for every $\sigma \in \mathcal{F}(\mathcal{A})$, $\theta(\sigma)$ must be in $\mathcal{F}(\mathcal{B})$ as $\mathcal{B} \leq_w \mathcal{A} \leq_w \mathcal{A}\restriction[\sigma] \leq_w \mathcal{B}\restriction[\theta(\sigma)]$. Now, if $\mathcal{B}$ is initializable, $\mathcal{F}(\mathcal{B}) = \omega^\omega$, and hence $\mathcal{A}\restriction\mathcal{F}(\mathcal{A}) \equiv_w \mathcal{B}\restriction\mathcal{F}(\mathcal{B}) = \mathcal{B} \equiv_w \mathcal{A}$. □

Let V be the set of all minimal strings leaving from $\mathcal{F}(\mathcal{A})$, and then let $\{\sigma_n\}_{n\in\omega}$ be an enumeration of V. Then consider $\mathcal{B} = \mathcal{A}\restriction\mathcal{F}(\mathcal{A})$ and $\mathcal{C} = \bigoplus_n(\mathcal{A}\restriction[\sigma_n])$. One can easily check that $\mathcal{B}$ and $\mathcal{C}$ are $\mathbf{\Delta}^0_{1+n}$-measurable whenever $\mathcal{A}$ is.

We first see that $\mathcal{B}, \mathcal{C} <_w \mathcal{A}$. It is easy to check that $\mathcal{B}, \mathcal{C} \leq_w \mathcal{A}$. By Lemma 5.2, $\mathcal{B}$ is initializable, and therefore not Wadge equivalent to $\mathcal{A}$. To see $\mathcal{C} <_w \mathcal{A}$, we note that $\mathcal{A}\restriction[\sigma] <_w \mathcal{A}$ for all $\sigma \in V$. Therefore, if $\mathcal{C} \equiv_w \mathcal{A}$ then it would imply that $\mathcal{A}$ is σ-join-reducible, which contradicts our assumption that $\mathcal{A}$ is non-self-dual by Proposition 2.6. Thus, we get that $\mathcal{B}, \mathcal{C} <_w \mathcal{A}$.

By the induction hypothesis, we get $U \in \mathrm{Tree}^{n-1}(\mathcal{Q})$ and $S \in {}^{\sqcup}\mathrm{Tree}^{n}(\mathcal{Q})$ such that $\mathcal{B} \equiv_w \Omega_{\langle U\rangle}$ and $\mathcal{C} \equiv_w \Omega_S$.

Finally, we claim that $\mathcal{A} \equiv_w \mathcal{B}^{\to}\mathcal{C}$. This would give us that $\mathcal{A} \equiv_w \Omega_{\langle U\rangle\to S}$. It is straightforward to see that $\mathcal{A} \leq_w \mathcal{B}^{\to}\mathcal{C}$. To see $\mathcal{B}^{\to}\mathcal{C} \leq_w \mathcal{A}$, construct a continuous reduction as follows: Take $X \in \omega^\omega$, which we write as $\pi_0(X)^{\to}\pi_1(X)$. While we are reading $\pi_0(X)$ we stay inside $\mathcal{F}(\mathcal{A})$ using the continuous retraction $\rho_{\mathcal{F}(\mathcal{A})}$. If we ever change our mind, $\mathcal{B}^{\to}\mathcal{C}(X) = \mathcal{C}(\pi_1(X))$. We still have that $\rho_{\mathcal{F}(\mathcal{A})}(\pi_0(X))$ is a finite string which is extendible in $\mathcal{F}(\mathcal{A})$, and we so can use the reductions $\mathcal{C} \leq_w \mathcal{A} \leq_w \mathcal{A}\restriction[\rho_{\mathcal{F}(\mathcal{A})}(\pi_0(X))]$.

More formally: For each $\tau \in \mathcal{F}(\mathcal{A})$, let η_τ be a Wadge reduction $\mathcal{A} \leq_w \mathcal{A}\restriction\tau$. Let $\theta_C\colon \mathcal{C} \to \mathcal{A}$. Then

$$X \mapsto \left(\rho_{\mathcal{F}(\mathcal{A})}(\pi_0(X))^\frown\eta_{\rho_{\mathcal{F}(\mathcal{A})}(\pi_0(X))}(\theta_C(\pi_1(X)))\right)$$

is a reduction $\mathcal{B}^{\to}\mathcal{C} \leq_w \mathcal{A}$.

6. Infinite Borel rank

We now start to deal with functions of infinite Borel rank.

6.1. **Transfinite jump operator.** We now need to consider α-th Turing jump for transfinite α. We again use the machinery developed in [MM11, Mon14]. There, the ω-th Turing jump is defined by taking the first bit of each of the finite jumps:

$$\mathcal{J}^\omega(X) = \langle \mathcal{J}(X)(0), \mathcal{J}^2(X)(0), \mathcal{J}^3(X)(0), \dots\rangle.$$

Because of the way these operators are defined in [MM11, Mon14] taking one bit from each jump is enough to code the whole sequence $\{\mathcal{J}^n(X) : n \in \omega\}$. The reason is that $\mathcal{J}^n(X)(0)$ codes at least two bits of $\mathcal{J}^{n-1}(X)$, and at least three bits of $\mathcal{J}^{n-2}(X)$,..., and at least n bits of $\mathcal{J}(X)$. See [MM11, Mon14] for more details.

The definition of the ω^α-th jump is similar, taking one bit from a sequence of jumps that converges to ω^α. To make this precise, we need to assign to each countable ordinal α a *fundamental sequence* $(\alpha[n])_{n\in\omega}$ which is a non-decreasing sequence with $\alpha = \sup_n(\alpha[n]+1)$. When $\alpha = \beta+1$, we just define $\alpha[n] = \beta$, and when α is a limit, $(\alpha[n])_{n\in\omega}$ is any increasing sequences with limit α. Notice that in either case $\omega^\alpha = \sum_n \omega^{\alpha[n]}$. The constructions in [MM11, Mon14] required fundamental sequences with particular properties, but we do not need to get into that now.

Something we need in this paper but that was not necessary in [MM11, Mon14] is *relative* transfinite Turing jump operators. If all we needed to do was relativize the whole construction, that would not be any harder. The problem is that we need to consider transfinite jumps, which are built by iterating jump operators that use different oracles. For instance, If we deal with the ω-th Turing jump, we shall consider a sequence $\mathcal{C} = (C_n)_{n\in\omega}$ of oracles to compute the different values $\mathcal{J}^{n,C_n}(X)(0)$. We call such a sequence an *ω-oracle*.

In general, we define an ω^α-oracle as a sequence $\mathcal{C} = (C_n)_{n\in\omega}$ of $\omega^{\alpha[n]}$-oracles. To define the notion of a ξ-oracle for ordinals that are not of the form ω^α, note that every ordinal ξ can be written as $\xi = \omega^\alpha + \beta$ for some unique α and $\beta < \omega^{\alpha+1}$ (consider the Cantor normal form). Then we define a ξ-oracle as a pair $\mathcal{C} = (C_0, C_1)$ of an ω^α-oracle and a β-oracle. Note that for each countable ordinal ξ, there is a well-founded tree $\Lambda_\xi \subseteq \omega^{<\omega}$ such that each ξ-oracle $\mathcal{C}$ can be thought of as a Λ_ξ-indexed collection $(C_\sigma)_{\sigma\in\Lambda_\xi}$. Given ξ-oracles $\mathcal{C}$ and $\mathcal{D}$, we write $\mathcal{C} \leq_T \mathcal{D}$ if If $C_\sigma \leq_T D_\sigma$ uniformly holds for any $\sigma \in \Lambda_\xi$.

To introduce the transfinite jump operator, for an ω^α-oracle $\mathcal{C} = (C_n)_{n\in\omega}$, we use the following abbreviations:

$$\mathcal{J}^{\omega^\alpha,\mathcal{C}}_{[0,n)} = \mathcal{J}^{\omega^{\alpha[n-1]},C_{n-1}} \circ \mathcal{J}^{\omega^{\alpha[n-2]},C_{n-2}} \circ \cdots \circ \mathcal{J}^{\omega^{\alpha[0]},C_0},$$
$$J^{\omega^\alpha,\mathcal{C}}_{[0,n)} = J^{\omega^{\alpha[n-1]},C_{n-1}} \circ J^{\omega^{\alpha[n-2]},C_{n-2}} \circ \cdots \circ J^{\omega^{\alpha[0]},C_0}.$$

Definition 6.1 (Montalbán [Mon14]). For any countable ordinal α and ω^α-oracle $\mathcal{C}$, define the ω^α-th jump operation $\mathcal{J}^{\omega^\alpha,\mathcal{C}}$ and its approximation $J^{\omega^\alpha,\mathcal{C}}$ as follows:

$$\mathcal{J}^{\omega^\alpha,\mathcal{C}}(Z)(n) = \mathcal{J}^{\omega^\alpha,\mathcal{C}}_{[0,n+1)}(Z)(0),$$
$$J^{\omega^\alpha,\mathcal{C}}(\sigma)(n) = J^{\omega^\alpha,\mathcal{C}}_{[0,n+1)}(\sigma)(0).$$

If a countable ordinal ξ is of the form $\omega^\alpha + \beta$ for some $\beta < \omega^{\alpha+1}$, for a ξ-oracle $\mathcal{C} = (C_0, C_1)$, we define $\mathcal{J}^{\xi,\mathcal{C}} = \mathcal{J}^{\beta,C_1} \circ \mathcal{J}^{\omega^\alpha,C_0}$ and $J^{\xi,\mathcal{C}} = J^{\beta,C_1} \circ J^{\omega^\alpha,C_0}$. We also use $\mathcal{J}^{-\xi,\mathcal{C}}$ to denote $(\mathcal{J}^{\xi,\mathcal{C}})^{-1}$.

The property mentioned in the first paragraph in Section 6.1, $\mathcal{J}^{\xi,C}(Z)$ is Turing equivalent to the usual ξth Turing jump of $C \oplus Z$ (which computes all $\Delta^0_{1+\xi}(C \oplus Z)$ sets), and the equivalence is uniform. This implies the following well-known fact which connects the transfinite jump operation and the Borel hierarchy.

Fact 6.2 (Universality from the right). *If $\mathcal{A}\colon \omega^\omega \to \hat{\omega}^\omega$ is $\mathbf{\Sigma}^0_{1+\xi}$-measurable relative to C, then there is a computable function θ such that $\mathcal{A} \equiv_{\mathsf{p}} \theta \circ \mathcal{J}^{\xi,C}$.*

This is because the ξ-th Turing jump of $X \oplus C$ can computably figure out the value of $\mathcal{A}(X)$.

(Recall that we write $\mathcal{A} \equiv_{\mathsf{p}} \mathcal{B}$ if $\mathcal{A}(X)^{\mathsf{p}} = \mathcal{B}(X)^{\mathsf{p}}$ for all $X \in \mathcal{X}$.) By using Fact 6.2, it is not hard to construct a $\mathbf{\Sigma}^0_{1+\xi}$-universal initializable conciliatory function (Proposition 3.24) by a similar argument as in Section 4.4.

Proof of Proposition 3.24. By $\Sigma^0_{1+\xi}$-universality of $\mathcal{J}^\xi$ from the right (Fact 6.2), it is obvious that $e^\frown C \oplus X \mapsto \Phi_e \circ \mathcal{J}^{\xi,C}(X)$ is $\mathbf{\Sigma}^0_{1+\xi}$-universal from the left. As in the proof of Proposition 2.15 (see Section 4.4), by modifying this function, one can obtain a $\mathbf{\Sigma}^0_{1+\xi}$-universal initializable conciliatory function. □

We now introduce the transfinite version of the jump inversion operator.

Definition 6.3. For any $\mathcal{A}\colon \omega^\omega \to \mathcal{Q}$ and any oracle $\mathcal{C}$ we introduce the ω^α*-th jump inversion* of $\mathcal{A}$, $\mathcal{A}^{\not\nearrow\omega^\alpha,\mathcal{C}}\colon \mathcal{J}^{\omega^\alpha,\mathcal{C}}[\omega^\omega] \to \mathcal{Q}$ as follows:

$$\mathcal{A}^{\not\nearrow\omega^\alpha,\mathcal{C}}(X) = \mathcal{A} \circ \mathcal{J}^{-\omega^\alpha,\mathcal{C}}(X).$$

As before, the domain of $\mathcal{A}^{\not\nearrow\omega^\alpha,\mathcal{C}}$ is closed. If the domain of $\mathcal{A}$ is $\hat{\omega}^\omega$, then $\mathcal{A}^{\not\nearrow\omega^\alpha,\mathcal{C}}$ is defined by $\mathcal{A} \circ I^{-1} \circ \mathcal{J}^{-\omega^\alpha,\mathcal{C}}$, where $I : \hat{\omega}^\omega \to \omega^\omega$ is a homeomorphism. The following transfinite version of Observation 4.4 is also straightforward.

Observation 6.4. Let $\mathcal{A}$ be any partial $\mathcal{Q}$-valued function, and Y and Z be ξ-oracles. If $Z \geq_T Y$, then $\mathcal{A}^{\not\nearrow\xi,Z} \leq_w \mathcal{A}^{\not\nearrow\xi,Y}$.

Thus, there is $\mathcal{C}$ such that $\mathcal{A}^{\not\sim\omega^\alpha,\mathcal{C}} \leq_w \mathcal{A}^{\not\sim\omega^\alpha,\mathcal{D}}$ for any $\mathcal{D}$ by well-foundedness of the Wadge degrees of $\mathcal{Q}$-valued functions (Theorem 2.3). Thus we use $\mathcal{A}^{\not\sim\omega^\alpha}$ to denote such $\mathcal{A}^{\not\sim\omega^\alpha,\mathcal{C}}$. By Observation 6.4, $\mathcal{C}$ can be chosen as a constant sequence, that is, $C_\sigma = C$ for any $\sigma \in \Lambda_{\omega^\alpha}$. If $\mathcal{C}$ is a constant sequence consisting of $C \in \omega^\omega$, we simply write it as C instead of $\mathcal{C}$. We now see the transfinite version of Lemma 4.6.

Lemma 6.5. *For any $\mathcal{A}, \mathcal{B}\colon \omega^\omega \to \mathcal{Q}$, the following holds.*

1. *If $\mathcal{A}$ is $\mathbf{\Sigma}^0_{1+\omega^\alpha+\beta}$-measurable, then $\mathcal{A}^{\not\sim\omega^\alpha}$ is $\mathbf{\Sigma}^0_{1+\beta}$-measurable.*
2. *If $\mathcal{A} \leq_w \mathcal{B}$ then $\mathcal{A}^{\not\sim\omega^\alpha} \leq_w \mathcal{B}^{\not\sim\omega^\alpha}$.*
3. *If either $\mathcal{A}^{\not\sim\omega^\alpha}$ or $\mathcal{B}^{\not\sim\omega^\alpha}$ is non-self-dual, then $\mathcal{A}^{\not\sim\omega^\alpha} \leq_w \mathcal{B}^{\not\sim\omega^\alpha}$ implies $\mathcal{A} \leq_w \mathcal{B}$.*

Proof. For (1), let C be such that $\mathcal{A}^{\not\sim\omega^\alpha} \equiv_w \mathcal{A}^{\not\sim\omega^\alpha,C}$. If $\mathcal{A}$ is $\mathbf{\Delta}^0_{1+\omega^\alpha+\beta}$-measurable, then there is $D \geq_T C$ such that $\mathcal{A} = \Phi_e \circ \mathcal{J}^{\beta,D} \circ \mathcal{J}^{\omega^\alpha,D}$ by Fact 6.2. Then we also have $\mathcal{A}^{\not\sim\omega^\alpha} \equiv_w \mathcal{A}^{\not\sim\omega^\alpha,D}$, and moreover, $\mathcal{A}^{\not\sim\omega^\alpha,D} = \Phi_e \circ \mathcal{J}^{\beta,D}$, which is clearly $\mathbf{\Delta}^0_{1+\beta}$-measurable. It is straightforward to show (2) and (3) by using the same argument as in the proof of Lemma 4.6. □

To get the transfinite version of Lemma 4.7, we use the transfinite version of the Friedberg jump inversion theorem [Mac77]: There exists a $C^{(\omega^\alpha)}$-computable function ψ such that

$$X \oplus C^{(\omega^\alpha)} \equiv_T \psi(X) \oplus C^{(\omega^\alpha)} \equiv_T (\psi(X) \oplus C)^{(\omega^\alpha)} \equiv_T \mathcal{J}^{\omega^\alpha,C}(\psi(X))$$

holds uniformly in X. Here, as usual, $Z^{(\xi)}$ denotes the ξ-th Turing jump of Z. As in the proof of Lemma 4.7, there are a computable operator $\theta_0 : \omega^\omega \to \hat{\omega}^\omega$ and a $C^{(\omega^\alpha)}$-computable operator $\theta_1 : \omega^\omega \to \hat{\omega}^\omega$ such that for every $X \in \hat{\omega}^\omega$,

$$\theta_0 \circ \mathcal{J}^{\omega^\alpha,C} \circ \psi(X) = X \quad \text{and} \quad \theta_1(X) = \mathcal{J}^{\omega^\alpha,C} \circ \psi(X).$$

Corollary 6.6. $\Omega^{\not\sim\omega^\alpha}_{\langle T\rangle^{\omega^\alpha}} \equiv_w \Omega_T$.

Proof. As in the proof of Lemma 4.7, it follows from the above formula. □

6.2. Generalization of initializability.

Definition 6.7. For a countable ordinal α, we say that $\mathcal{A}$ is *α-stable* if $\mathcal{A}$ is Wadge equivalent to an initializable function, and $\mathcal{A}^{\not\sim\omega^\beta} \equiv_w \mathcal{A}$ holds for any $\beta < \alpha$.

By Lemma 6.5 (2), if $\mathcal{B}$ is Wadge equivalent to $\mathcal{A}$ and if $\mathcal{A}$ is α-stable, then so is $\mathcal{B}$.

Lemma 6.8. *For any $T \in \mathrm{Tree}^{\omega_1}(\mathcal{Q})$ and countable ordinal α, $\Omega_{\langle T\rangle^{\omega^\alpha}}$ is α-stable.*

Proof. First note that, for any countable ordinal α, $\Omega_{\langle T\rangle^{\omega^\alpha}}$ is initializable since $\Omega_{\langle T\rangle^{\omega^\alpha}} = \Omega_T \circ \mathcal{U}_{\omega^\alpha}$ for an initializable function $\mathcal{U}_{\omega^\alpha}$. Fix $\beta < \alpha$. We show that $\Omega_{\langle T\rangle^{\omega^\alpha}}$ is Wadge reducible to $\Omega^{\not\sim\omega^\beta,C}_{\langle T\rangle^{\omega^\alpha}}$. Let $\psi : \omega^\omega \to \omega^\omega$ be the ω^β-th jump inversion map relative to C, that is, there are a computable operator $\theta_0 : \omega^\omega \to \omega^\omega$ and a $C^{(\omega^\beta)}$-computable operator $\theta_1 : \omega^\omega \to \omega^\omega$ such that for every $X \in \omega^\omega$,

$$\theta_0 \circ \mathcal{J}^{\omega^\beta,C} \circ \psi(X) = X \quad \text{and} \quad \theta_1(X) = \mathcal{J}^{\omega^\beta,C} \circ \psi(X).$$

Since $\beta < \alpha$, the map $\mathcal{J}^{\omega^\alpha,C} \circ \theta_0 \circ \mathcal{J}^{\omega^\beta,C}$ is still $\Sigma^0_{1+\omega^\alpha}$ relative to C. By universality of $\mathcal{J}^{\omega^\alpha}$ from the right (Fact 6.2), there is a computable function θ_2 such that

$$\theta_2 \circ \mathcal{J}^{\omega^\alpha} = \mathcal{J}^{\omega^\alpha,C} \circ \theta_0 \circ \mathcal{J}^{\omega^\beta,C}.$$

Then, recall that $I\colon \hat{\omega}^\omega \to \omega^\omega$ is a computable homeomorphism. Since $I \circ \mathcal{U}_{\omega^\alpha} \circ I^{-1}$ is $\mathbf{\Sigma}^0_{1+\omega^\alpha}$-measurable, by universality of $\mathcal{J}^{\omega^\alpha}$ from the right (Fact 6.2), there is Φ such that

$$I \circ \mathcal{U}_{\omega^\alpha} \circ I^{-1} = \Phi \circ \mathcal{J}^{\omega^\alpha, C}.$$

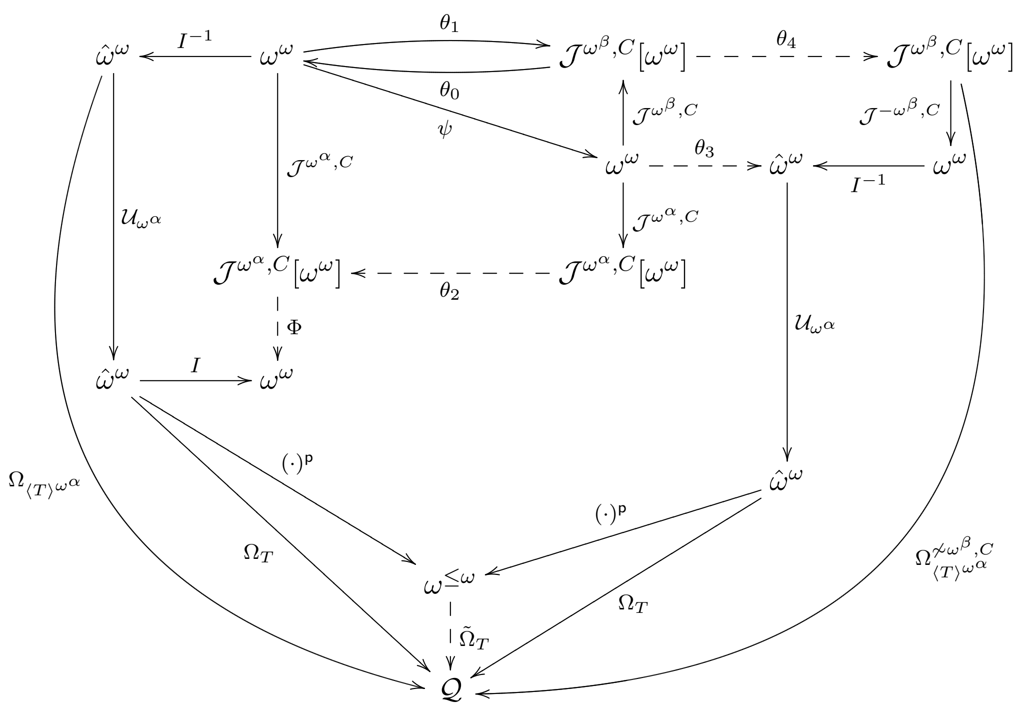
Clearly, $I^{-1} \circ \Phi \circ \theta_2 \circ \mathcal{J}^{\omega^\alpha, C}$ (hence, its restriction up to ω^ω) is $\Sigma^0_{1+\alpha}$ relative to C, and therefore, by universality of $\mathcal{U}_{\omega^\alpha}$ from the left, there is a continuous function $\theta_3 : \omega^\omega \to \hat{\omega}^\omega$ such that

$$\mathcal{U}_{\omega^\alpha} \circ \theta_3 \equiv_{\mathsf{p}} \mathcal{U}_{\omega^\alpha} \circ I^{-1} \circ \Phi \circ \theta_2 \circ \mathcal{J}^{\omega^\alpha, C}.$$

By our explicit construction of $\mathcal{U}_{\omega^\alpha}$ and $C^{(\omega^\beta)}$-computability of ψ, one can assume that θ_3 is $C^{(\omega^\beta)}$-computable. By universality of $\mathcal{J}^{\omega^\beta}$ from the right, there is a computable function θ_4 such that

$$\theta_4 \circ \mathcal{J}^{\omega^\beta, C} = \mathcal{J}^{\omega^\beta, C} \circ I \circ \theta_3.$$

We now obtain the following commutative diagram:



This commutative diagram shows that

$$\Omega_{\langle T \rangle^{\omega^\alpha}} \equiv_{\mathsf{p}} \Omega^{\not\sim \omega^\beta, C}_{\langle T \rangle^{\omega^\alpha}} \circ \theta_4 \circ \theta_1 \circ I.$$

Consequently, the continuous function $\theta_4 \circ \theta_1 \circ I$ gives a Wadge reduction from $\Omega_{\langle T \rangle^{\omega^\alpha}}$ to $\Omega^{\not\sim \omega^\beta, C}_{\langle T \rangle^{\omega^\alpha}}$. □

The purpose of this section is to prove the following transfinite version of Lemma 4.9.

Lemma 6.9. *If $\mathcal{A}$ is α-stable, then $\mathcal{A}^{\not\sim \omega^\alpha}$ is σ-join-irreducible.*

Before proving Lemma 6.9 we first check that this Lemma immediately implies our main theorems.

Proof of Proposition 1.7 from Lemma 6.9. We only show that $\Omega_{\langle U\rangle^{\omega^\alpha}} \leq_w \Omega_{\langle V\rangle^{\omega^\beta}}$ if and only if $\langle U\rangle^{\omega^\alpha} \trianglelefteq \langle V\rangle^{\omega^\beta}$. For the other cases, we can use a similar argument as in the proof of Proposition 1.7 for finite Borel rank. To verify the above equivalence, by Lemma 6.6, we have $\Omega^{\not\sim\omega^\alpha}_{\langle U\rangle^{\omega^\alpha}} \equiv_w \Omega_U$. Since $\Omega_{\langle U\rangle^{\omega^\alpha}}$ is α-stable by Lemma 6.8, $\Omega^{\not\sim\omega^\alpha}_{\langle U\rangle^{\omega^\alpha}}$ is σ-join-irreducible by Lemma 6.9, and thus non-self-dual by Proposition 2.6. Thus, if $\alpha=\beta$, by Lemma 6.5,

$$\Omega_{\langle U\rangle^{\omega^\alpha}} \leq_w \Omega_{\langle V\rangle^{\omega^\beta}} \iff \Omega_U \equiv_w \Omega^{\not\sim\omega^\alpha}_{\langle U\rangle^{\omega^\alpha}} \leq_w \Omega^{\not\sim\omega^\alpha}_{\langle V\rangle^{\omega^\beta}} \equiv_w \Omega_V.$$

This ensures the desired assertion by induction hypothesis. If $\alpha>\beta$, then, since $\Omega_{\langle U\rangle^{\omega^\alpha}}$ is α-stable by Lemma 6.8, we have $\Omega^{\not\sim\omega^\beta}_{\langle U\rangle^{\omega^\alpha}} \equiv_w \Omega_{\langle U\rangle^{\omega^\alpha}}$, and therefore,

$$\Omega_{\langle U\rangle^{\omega^\alpha}} \leq_w \Omega_{\langle V\rangle^{\omega^\beta}} \iff \Omega_{\langle U\rangle^{\omega^\alpha}} \equiv_w \Omega^{\not\sim\omega^\beta}_{\langle U\rangle^{\omega^\alpha}} \leq_w \Omega^{\not\sim\omega^\beta}_{\langle V\rangle^{\omega^\beta}} \equiv_w \Omega_V.$$

This ensures the desired assertion by induction hypothesis. The same argument works in the case $\alpha<\beta$. □

Proof of Proposition 1.9 from Lemma 6.9. Fix a $\mathbf{\Delta}^0_\xi$-measurable function $\mathcal{A}$. Let δ be the smallest ordinal such that $\mathcal{A}$ is not $(\gamma+1)$-stable for some $\gamma<\delta$. If $\delta=0$, then $\mathcal{A}$ is not Wadge-initializable, and we can use the same argument as in the proof in Section 5 of Proposition 1.9 for functions of finite Borel rank.

Suppose that $\delta>0$. Then, note that δ must be a successor ordinal, say $\delta=\alpha+1$, and thus $\mathcal{A}$ is α-stable. Let β be a unique ordinal that $\xi=\omega^\alpha+\beta$. By Lemma 6.5 (1), $\mathcal{A}^{\not\sim\omega^\alpha}$ is $\mathbf{\Delta}^0_\beta$-measurable. Moreover, by minimality of α, we have $\mathcal{A}^{\not\sim\omega^\alpha} <_w \mathcal{A}$ and $\mathcal{A}^{\not\sim\omega^\alpha}$ is non-self-dual by Lemma 6.9 and Proposition 2.6. By induction hypothesis, $\mathcal{A}^{\not\sim\omega^\alpha} \equiv_w \Omega_T$ for some tree $T\in\mathrm{Tree}^\beta(\mathcal{Q})$. Then we have $\mathcal{A}^{\not\sim\omega^\alpha} \equiv_w \Omega^{\not\sim\omega^\alpha}_{\langle T\rangle^{\omega^\alpha}}$ by Lemma 6.6. Note that Ω_T is σ-join-irreducible by Observations 3.15 and 2.8 since T is a tree, and therefore non-self-dual by Proposition 2.6. Therefore we get $\mathcal{A} \equiv_w \Omega_{\langle T\rangle^{\omega^\alpha}}$ by Lemma 6.5 (3). We claim that

$$\langle T\rangle^{\omega^\alpha} \in \langle\mathrm{Tree}^\beta(\mathcal{Q})\rangle^{\omega^\alpha} \subseteq \mathrm{Tree}^\xi(\mathcal{Q}).$$

It is clear if $\beta<\omega^{\alpha+1}$ by definition. If $\beta\geq\omega^{\alpha+1}$, then we must have $\beta=\xi$. Recall that ξ is of the form $\omega^\gamma+\delta$ for some $\gamma<\omega_1$ and $\delta<\omega^{\gamma+1}$. We then have $\alpha<\gamma$. Thus, if $T\in\mathrm{Tree}^\xi(\mathcal{Q})=\mathrm{Tree}^\beta(\mathcal{Q})$, then $\langle T\rangle^{\omega^\alpha}\in\mathrm{Tree}^\xi(\mathcal{Q})$. This concludes the proof. □

6.3. **Proof of Lemma 6.9.** It remains to show Lemma 6.9. The statement of this lemma resembles Lemma 4.9, that is, it looks like a transfinite version of Lemma 4.9. Nevertheless, our proof requires a very different argument. The notation for the proof get a bit more complicated than in Lemma 4.9, as one needs to keep track of ω-iterations of the jump. However, it still is much simpler than Duparc's [Dup] proof for $\mathcal{Q}=2$.

6.3.1. *Proof of Lemma 6.9 (for $\alpha\leq 1$).* Throughout this subsection, for notational simplicity, we always assume that $\mathcal{B}^{\not\sim} \equiv_w \mathcal{B}^{\not\sim\emptyset}$ for any function $\mathcal{B}$. We will deal with nonempty oracles in the next Section 6.3.2.

We first consider the base case $\alpha=0$. Recall that $\mathcal{A}$ is 0-stable if and only if it is Wadge equivalent to an initializable function. Lemma 4.9 states that if $\mathcal{A}$ is 0-stable, then $\mathcal{A}^{\not\sim}$ is σ-join-irreducible. In the proof of Lemma 4.9 we showed that if $\mathcal{B}$ is actually initializable and $\sigma\in\omega^{<\omega}$ forces its jump, then $\mathcal{B}^{\not\sim} \leq_w \mathcal{B}^{\not\sim}\upharpoonright J(\sigma)$, and that if $X \in \omega^\omega$ forces its jump, then $\mathcal{J}(X) \in \mathcal{F}(\mathcal{B}^{\not\curlywedge})$. Since every string can be extended to one that forces its jump, the set of $X \in \omega^\omega$ which force their jump is dense. We thus get that the set of X such that $\mathcal{J}(X) \in \mathcal{F}(\mathcal{B}^{\not\curlywedge})$ is dense. In the case when $\mathcal{A}$ is not actually initializable, but Wadge equivalent to an initializable, recall from Lemma 5.2 that $\mathcal{A} \equiv_w \mathcal{A} \restriction \mathcal{F}(\mathcal{A})$. (Recall Observation 3.5 for how to deal with functions whose domain is a closed set as if their domain was all of ω^ω.)

Then, the proof of Lemma 4.9 actually implies that

(1) if $\mathcal{A}$ is 0-stable, then $\{X : \mathcal{J}(X) \in \mathcal{F}((\mathcal{A} \restriction \mathcal{F}(\mathcal{A}))^{\not\curlywedge})\}$ is dense in $\mathcal{F}(\mathcal{A})$.

In the rest of this section, we give a proof of Lemma 6.9 for $\alpha = 1$, that is, if $\mathcal{A}$ is 1-stable, then $\mathcal{A}^{\not\curlywedge\omega}$ is σ-join-irreducible. By definition, $\mathcal{A}$ is 1-stable if and only if $\mathcal{A}$ is Wadge equivalent to an initializable function and $\mathcal{A}^{\not\curlywedge} \equiv_w \mathcal{A}$. The latter condition is equivalent to $\mathcal{A}^{\not\curlywedge n} \equiv_w \mathcal{A}$ for any natural number $n \geq 1$. Given a 1-stable function $\mathcal{A}$, we inductively define a $\mathcal{Q}$-valued function $\mathcal{A}_n$ by

$$\mathcal{A}_0 = \mathcal{A} \quad \text{and} \quad \mathcal{A}_{n+1} = (\mathcal{A}_n \restriction \mathcal{F}(\mathcal{A}_n))^{\not\curlywedge}.$$

Observation 6.10. If $\mathcal{A}$ is 1-stable, then $\mathcal{A} \equiv_w \mathcal{A}_n$ for any $n \in \omega$.

Proof. Recall that by Lemma 5.2, a function $\mathcal{B}$ is Wadge equivalent to an initalizable function if and only if $\mathcal{B} \equiv_w \mathcal{B} \restriction \mathcal{F}(\mathcal{B})$. Fix n, and inductively assume that $\mathcal{A} \equiv_w \mathcal{A}_n$. Therefore, we have $\mathcal{A} \restriction \mathcal{F}(\mathcal{A}) \equiv_w \mathcal{A}_n \restriction \mathcal{F}(\mathcal{A}_n)$. Since $\mathcal{A}$ is 1-stable, in particular, $\mathcal{A}$ is Wadge equivalent to an initializable function. Then, $\mathcal{A} \equiv_w \mathcal{A} \restriction \mathcal{F}(\mathcal{A})$, and therefore, $\mathcal{A}_n \restriction \mathcal{F}(\mathcal{A}_n)$ is 1-stable. Thus, $\mathcal{A}_{n+1} \equiv_w \mathcal{A}_n \restriction \mathcal{F}(\mathcal{A}_n)$, and therefore $\mathcal{A}_{n+1} \equiv_w \mathcal{A}$. □

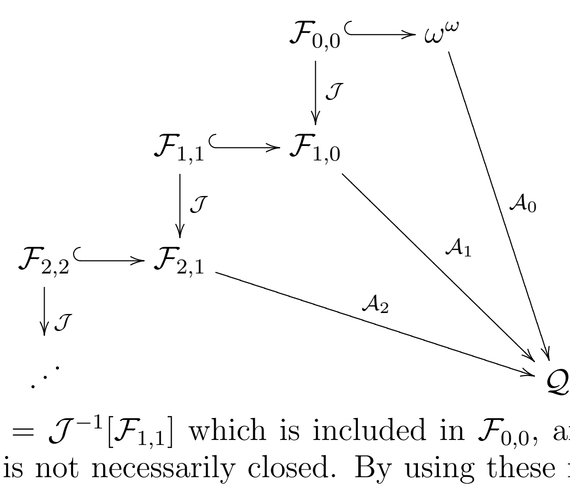
In particular, $\mathcal{A}_n$ is initializable for any $n \in \omega$. Thus, the property (1) implies that

(2) if $\mathcal{A}$ is 1-stable, then $\{X : \mathcal{J}(X) \in \mathcal{F}(\mathcal{A}_{n+1})\}$ is dense in $\mathcal{F}(\mathcal{A}_n)$ for any $n \in \omega$.

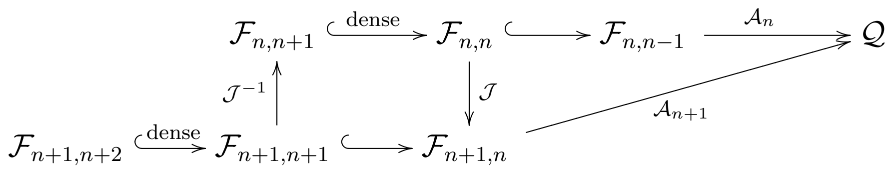
For notational convenience, we define

$$\mathcal{F}_{n,n} = \mathcal{F}(\mathcal{A}_n) \quad \text{and} \quad \mathcal{F}_{n+1,n} = dom(\mathcal{A}_{n+1}) = \mathcal{J}[\mathcal{F}_{n,n}].$$

Note that $\mathcal{F}_{n+1,n}$ is the domain of $\mathcal{A}_{n+1,n+1}$, and $\mathcal{F}_{n,n}$ and $\mathcal{F}_{n+1,n}$ are closed sets (since $\mathcal{J}^{-1}$ is continuous). In the following diagram, the arrow $\hookrightarrow$ indicates the inclusion map.



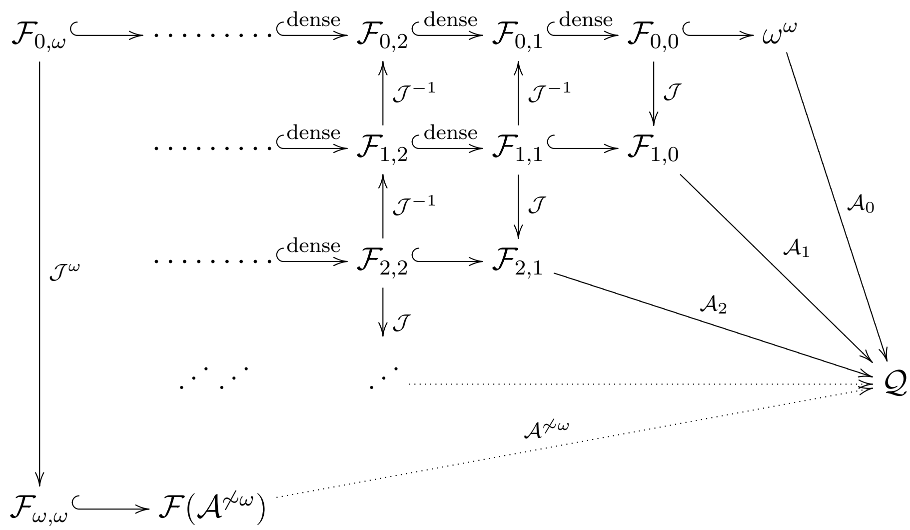
We also define $\mathcal{F}_{0,1} = \mathcal{J}^{-1}[\mathcal{F}_{1,1}]$ which is included in $\mathcal{F}_{0,0}$, and in general $\mathcal{F}_{n,n+1} = \mathcal{J}^{-1}[\mathcal{F}_{n+1,n+1}]$, which is not necessarily closed. By using these notations, the property (2) can be rephrased as: If $\mathcal{A}$ is 1-stable, then $\mathcal{F}_{n,n+1}$ is dense in $\mathcal{F}_{n,n}$. Therefore, we have the following commutative diagram.



Generally, for $m < n$, we define $\mathcal{F}_{m,n} = \mathcal{J}^{-1}[\mathcal{F}_{m+1,n}]$. In particular, $\mathcal{F}_{0,n} = \mathcal{J}^{-n}[\mathcal{F}_{n,n}]$. This gives a decreasing sequence $(\mathcal{F}_{0,n})_{n\in\omega}$. Define $\mathcal{F}_{0,\omega} = \bigcap_{n\in\omega} \mathcal{F}_{0,n}$. In other words,

$$\mathcal{F}_{0,\omega} = \{X \in \omega^\omega : (\forall n \in \omega)\ \mathcal{J}^n(X) \in \mathcal{F}_{n,n}\}.$$

Then we define $\mathcal{F}_{\omega,\omega} = \mathcal{J}^\omega[\mathcal{F}_{0,\omega}]$.



We devote the rest of this section to prove the following claim:

(3) If $\mathcal{A}$ is 1-stable, then $\{X : \mathcal{J}^\omega(X) \in \mathcal{F}(\mathcal{A}^{\not\sim\omega})\}$ is dense in $\mathcal{F}(\mathcal{A})$.

(Recall that $\mathcal{F}(\mathcal{A}) = \mathcal{F}_{0,0}$.) Clearly, the claim (3) entails Lemma 6.9 for $\alpha = 1$ as it implies that $\mathcal{F}(\mathcal{A}^{\not\sim\omega}) \neq \emptyset$. Our strategy has two steps: First is to prove that $\mathcal{F}_{0,\omega}$ is dense in $\mathcal{F}(\mathcal{A})$. Second to prove that $(\mathcal{J}^\omega)^{-1}[\mathcal{F}(\mathcal{A}^{\not\sim\omega})] \supseteq \mathcal{F}_{0,\omega}$ by showing that $\mathcal{F}_{\omega,\omega} \subseteq \mathcal{F}(\mathcal{A}^{\not\sim\omega})$ and using that $(\mathcal{J}^\omega)^{-1}[\mathcal{F}_{\omega,\omega}] = \mathcal{F}_{0,\omega}$.

Hereafter, we identify the closed set $\mathcal{F}_{n,n}$ with the pruned tree whose infinite paths are exactly the elements of $\mathcal{F}_{n,n}$.

Lemma 6.11. *If $\mathcal{A}$ is 1-stable, then $\mathcal{F}_{0,\omega}$ is dense in $\mathcal{F}_{0,0}$.*

Proof. Fix $\sigma \in \mathcal{F}_{0,0}$ and put $\sigma_0 = \sigma$. We will construct a sequence $(\sigma_n)_{n\in\omega}$ of finite strings such that $\sigma_n \in \mathcal{F}_{n,n}$, and

$$(\mathcal{J}^{n-m})^{-1}(\sigma_n) \subseteq (\mathcal{J}^{n-m+1})^{-1}(\sigma_{n+1}) \in \mathcal{F}_{m,m}$$

for any $m \leq n$. Then we will define $X := \bigcup_n (\mathcal{J}^n)^{-1}(\sigma_n)$ and ensure that $X \in \mathcal{F}_{0,\omega}$, that is, $\mathcal{J}^n(X) \in \mathcal{F}_{n,n}$ for all n. Given n, inductively assume that $\sigma_n \in \mathcal{F}_{n,n}$. Now, by the property (2), $\mathcal{F}_{n,n+1}$ is dense in $\mathcal{F}_{n,n}$ for any $n \in \omega$. Since $\mathcal{F}_{n,n+1} = \mathcal{J}^{-1}[\mathcal{F}_{n+1,n+1}]$, there is $Y \in \mathcal{F}_{n+1,n+1}$ such that

$$\sigma_n \subset \mathcal{J}^{-1}(Y) \in \mathcal{F}_{n,n}.$$

Since $\mathcal{J}^{-1}$ is continuous, we can find an initial segment $\sigma_{n+1} \subset Y$ such that $\sigma_n \subseteq \mathcal{J}^{-1}(\sigma_{n+1})$. Clearly $\sigma_{n+1} \in \mathcal{F}_{n+1,n+1}$. For every $m \leq n$, by continuity of $(\mathcal{J}^{n-m})^{-1}$, we also have

$$(\mathcal{J}^{n-m})^{-1}(\sigma_n) \subseteq (\mathcal{J}^{n-m})^{-1} \circ \mathcal{J}^{-1}(\sigma_{n+1}) = (\mathcal{J}^{n-m+1})^{-1}(\sigma_{n+1})$$

and $(\mathcal{J}^{n-m})^{-1}(\sigma_n)$ is extendible in $(\mathcal{J}^{n-m})^{-1}[\mathcal{F}_{n,n}] = \mathcal{F}_{m,n} \subseteq \mathcal{F}_{m,m}$, that is, $(\mathcal{J}^{n-m})^{-1}(\sigma_n) \in \mathcal{F}_{m,m}$ as wanted.

For $X = \bigcup_n (\mathcal{J}^n)^{-1}(\sigma_n)$, we claim that $\mathcal{J}^m(X) = Y_m := \bigcup_{n\geq m}(\mathcal{J}^{n-m})^{-1}(\sigma_n)$. This is because we have

$$(\mathcal{J}^m)^{-1}(Y_m) = \bigcup_n (\mathcal{J}^m)^{-1} \circ (\mathcal{J}^{n-m})^{-1}(\sigma_n) = \bigcup_n (\mathcal{J}^n)^{-1}(\sigma_n) = X.$$

The first equality is due to continuity of $(\mathcal{J}^m)^{-1}$ and the property that $((\mathcal{J}^{n-m})^{-1}(\sigma_n))_{n\geq m}$ is increasing. Therefore $\mathcal{J}^m(X) = Y_m$. Since $(\mathcal{J}^{n-m})^{-1}(\sigma_n) \in \mathcal{F}_{m,m}$, and $\mathcal{F}_{m,m}$ is closed, we have $\mathcal{J}^m(X) \in \mathcal{F}_{m,m}$ for all $m \in \omega$, and therefore $\sigma \subseteq X \in \mathcal{F}_{0,\omega}$. This shows that $\mathcal{F}_{0,\omega}$ is dense in $\mathcal{F}_{0,0}$. □

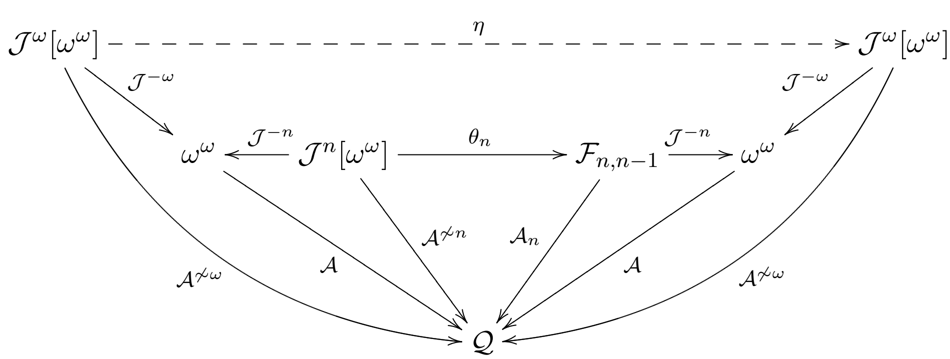
Lemma 6.12. *If $\mathcal{A}$ is 1-stable, then $\mathcal{F}_{\omega,\omega} \subseteq \mathcal{F}(\mathcal{A}^{\not\sim\omega})$.*

Proof. Take $Z \in \mathcal{F}_{\omega,\omega}$; we want to show that $Z \in \mathcal{F}(\mathcal{A}^{\not\sim\omega})$. Given $n \in \omega$, we will define a continuous function $\eta : \mathcal{J}^\omega[\omega^\omega] \to \mathcal{J}^\omega[\omega^\omega] \cap [\mathcal{J}^\omega(X)\restriction n]$ witnessing that $\mathcal{A}^{\not\sim\omega} \leq_w \mathcal{A}^{\not\sim\omega}\restriction[Z\restriction n]$.

Let $X = \mathcal{J}^{-\omega}(Z) \in \mathcal{F}_{\omega,0}$. Note that $\mathcal{J}^\omega(X)\restriction n = \langle \mathcal{J}(X)\restriction 1, \dots, \mathcal{J}^n(X)\restriction 1\rangle$. Recall that $\mathcal{J}^\omega(X) \in \mathcal{F}_{\omega,\omega}$ if and only if $\mathcal{J}^n(X) \in \mathcal{F}_{n,n} = \mathcal{F}(\mathcal{A}_n)$ for all $n \in \omega$. Therefore, there is a continuous reduction

$$\theta_n \colon \mathcal{A}^{\not\sim n} \leq_w \mathcal{A}_n\restriction[\mathcal{J}^n(X)\restriction 1]$$

since $\mathcal{A}^{\not\sim n} \leq_w \mathcal{A} \leq_w \mathcal{A}_n \leq_w \mathcal{A}_n\restriction[\mathcal{J}^n(X)\restriction 1]$, using Observation 6.10. We assume that θ_n is computable (as mentioned before, we will describe how to deal with oracles in the next Section 6.3.2). Since $\mathcal{J}^\omega \circ \mathcal{J}^{-n} \circ \theta_n \circ \mathcal{J}^n$ is clearly Σ^0_ω-measurable, by universality from the right (Observation 6.2), we have $\eta \circ \mathcal{J}^\omega = \mathcal{J}^\omega \circ \mathcal{J}^{-n} \circ \theta_n \circ \mathcal{J}^n$. Thus we get the following diagram:



This shows that $\mathcal{A}^{\not\sim\omega}(Z) \leq_{\mathcal{Q}} \mathcal{A}^{\not\sim\omega} \circ \eta(Z)$. More formally:

$$\begin{aligned}\mathcal{A}^{\not\sim\omega}(\mathcal{J}^\omega(Y)) &= \mathcal{A}(Y) = \mathcal{A}^{\not\sim n}(\mathcal{J}^n(Y)) \leq_{\mathcal{Q}} \mathcal{A}^{\not\sim n}(\theta_n \circ \mathcal{J}^n(Y)) \\ &= \mathcal{A}(\mathcal{J}^{-n} \circ \theta_n \circ \mathcal{J}^n(Y)) = \mathcal{A}^{\not\sim\omega}(\mathcal{J}^\omega(\mathcal{J}^{-n} \circ \theta_n \circ \mathcal{J}^n(Y))) = \mathcal{A}^{\not\sim\omega}(\eta(\mathcal{J}^\omega(Y))).\end{aligned}$$

Since $\theta(Z)$ extends $\mathcal{J}^n(X)\upharpoonright 1$ for any $Z \in \mathcal{J}^n[\omega^\omega]$, we have $\mathcal{J}^\omega(\mathcal{J}^{-n} \circ \theta(Z))(n) = \mathcal{J}^n(X)\upharpoonright 1$. Thus, $\mathcal{J}^\omega(\mathcal{J}^{-n} \circ \theta(Z))$ extends $\mathcal{J}^\omega(X)\upharpoonright n$ (see the first paragraph in Section 6.1), and so does $\eta(\mathcal{J}^\omega(Y))$ for any $Y \in \omega^\omega$. Hence, η witnesses that $\mathcal{A}^{\not\sim\omega} \leq_w \mathcal{A}^{\not\sim\omega}\upharpoonright[\mathcal{J}^\omega(X)\upharpoonright n]$. □

This concludes the proof of Lemma 6.9 for $\alpha = 1$.

6.3.2. *Proof of Lemma 6.9 (for general α).* In this section, we describe the proof of Lemma 6.9 for general α, which will be almost no different from the proof for $\alpha = 1$. This section is just for the sake of completeness. We also explicitly describe how to deal with oracles.

Now, fix a countable ordinal α. By induction, we assume that we have already shown the following claim for any $\beta < \alpha$: If $\mathcal{A}$ is β-stable, then for any oracle D, there is an ω^β-oracle $C \geq_T D$ such that

$$\{X : \mathcal{J}^{\omega^\beta,C}(X) \in \mathcal{F}((\mathcal{A}\upharpoonright\mathcal{F}(\mathcal{A}))^{\not\sim\omega^\beta,C})\} \text{ is dense in } \mathcal{F}(\mathcal{A}). \tag{4}$$

We now fix an α-stable function $\mathcal{A}$. We will define oracles $(C_n)_{n\in\omega}$. Then, for notational simplicity, we will use the following notations:

$$\mathcal{J}_n = \mathcal{J}^{\omega^{\alpha[n]},C_n}, \qquad \mathcal{B}^{\not\sim n} = \mathcal{B}^{\not\sim\omega^{\alpha[n]},C_n}.$$

As in the precious section, we inductively define a $\mathcal{Q}$-valued function $\mathcal{A}_n$ and a closed set $\mathcal{F}_{n,n}$ as follows:

$$\mathcal{A}_0 = \mathcal{A}, \qquad \mathcal{F}_{0,0} = \mathcal{F}(\mathcal{A}).$$
$$\mathcal{A}_{n+1} = (\mathcal{A}_n\upharpoonright\mathcal{F}_{n,n})^{\not\sim n}, \qquad \mathcal{F}_{n+1,n+1} = \mathcal{F}(\mathcal{A}_{n+1}).$$

To define $\mathcal{A}_{n+1}$, we need to specify oracles $(C_m)_{m\leq n}$. Before defining these oracles, we introduce several notations. We define

$$\mathcal{J}_{[m,n)} = \mathcal{J}_{n-1} \circ \mathcal{J}_{n-2} \circ \cdots \circ \mathcal{J}_{m+1} \circ \mathcal{J}_m,$$
$$\mathcal{B}^{\not\sim}_{[m,n)} = ((\dots(\mathcal{B}^{\not\sim m})^{\not\sim m+1}\dots)^{\not\sim n-2})^{\not\sim n-1}.$$

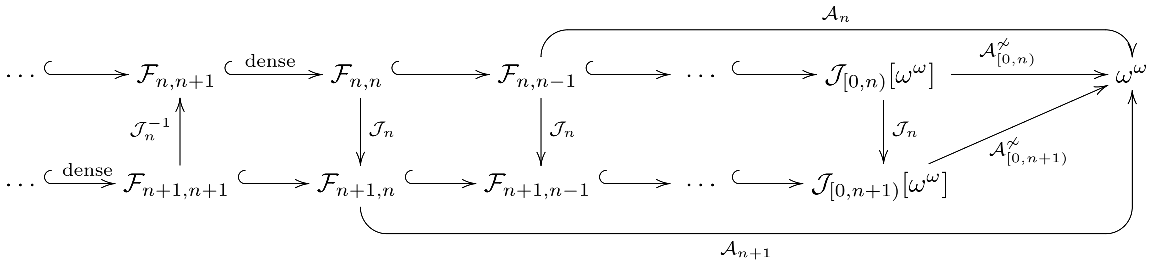
Note that the sequences defined in the previous section satisfy $\mathcal{A}_n = \mathcal{A}^{\not\sim n}\upharpoonright\mathcal{F}_{n,n-1}$. Now, in our new definition, $\mathcal{A}^{\not\sim n}$ is replaced with $\mathcal{A}^{\not\sim}_{[0,n)}$, that is,

$$\mathcal{A}_n = \mathcal{A}^{\not\sim}_{[0,n)}\upharpoonright\mathcal{F}_{n,n-1}, \text{ where } \mathcal{F}_{n,n-1} = \mathcal{J}[\mathcal{F}_{n-1,n-1}].$$

We now start to define a sequence $(C_n)_{n\in\omega}$ of oracles. Let C_{-1} be an oracle such that $\mathcal{A}^{\not\sim\omega^\alpha} \equiv_w \mathcal{A}^{\not\sim\omega^\alpha,C_{-1}}$. Define $\mathcal{A}_0 = \mathcal{A}$, and assume that $(C_m)_{m<n}$ are defined, and $\mathcal{A} \equiv_w \mathcal{A}_n$ as in Observation 6.10. In particular, $\mathcal{A}_n$ is α-stable. Then, by induction hypothesis (4), and initializability of $\mathcal{A}_n$, there is an oracle $C \geq_T C_{n-1}$ such that

(a) $(\mathcal{A}_n\upharpoonright\mathcal{F}_{n,n})^{\not\sim\omega^{\beta[n]},C} \equiv_w (\mathcal{A}_n\upharpoonright\mathcal{F}_{n,n})^{\not\sim\omega^{\beta[n]}}$.
(b) $(\mathcal{J}^{\omega^{\beta[n]},C})^{-1}[\mathcal{F}((\mathcal{A}_n\upharpoonright\mathcal{F}_{n,n})^{\not\sim\omega^{\beta[n]},C})]$ is dense in $\mathcal{F}_{n,n}$.
(c) For any $\sigma \in \mathcal{F}_{n,n}$, there is a C-computable Wadge-reduction $\mathcal{A}^{\not\sim}_{[0,n)} \leq_w \mathcal{A}_n\upharpoonright\mathcal{F}_{n,n}\cap[\sigma]$ (recall our proof of Lemma 6.12).

Define $C_n = C$ for such C, and then define $\mathcal{C} = (C_n)_{n\in\omega}$. We also define $(\mathcal{F}_{m,n})_{m,n\in\omega}$ as in the previous section. Then, for instance, the above condition (b) can be rephrased as: $\mathcal{F}_{n,n+1}$ is dense in $\mathcal{F}_{n,n}$. We then get the following commutative diagram:



Define $\mathcal{F}_{0,\omega} = \bigcap_{n\in\omega} \mathcal{F}_{0,n}$. In other words,

$$\mathcal{F}_{0,\omega} = \{X \in \omega^\omega : (\forall n \in \omega)\ \mathcal{J}^{[0,n)}(X) \in \mathcal{F}_{n,n}\}.$$

Then we define $\mathcal{F}_{\omega,\omega} = \mathcal{J}^\omega[\mathcal{F}_{0,\omega}]$. As in the previous section, we will show the following claim:

$$(\mathcal{J}^{\omega^\alpha,\mathcal{C}})^{-1}[\mathcal{F}(\mathcal{A}^{\not<\omega^\alpha,\mathcal{C}})] \text{ is dense in } \mathcal{F}(\mathcal{A}). \tag{5}$$

The claim (5) entails that $\mathcal{F}(\mathcal{A}^{\not<\omega^\alpha,\mathcal{C}})$ is nonempty, and therefore $\mathcal{A}^{\not<\omega^\alpha,\mathcal{C}}$ is σ-join-irreducible by Proposition 2.6. Here, since $\mathcal{C} \geq_T C_{-1}$, we have that $\mathcal{A}^{\not<\omega^\alpha,\mathcal{C}} \equiv_w \mathcal{A}^{\not<\omega^\alpha}$. Therefore, the claim (5) implies that $\mathcal{A}^{\not<\omega^\alpha}$ is σ-join-irreducible as desired. Hence, it suffices to show the claim (5) to prove Lemma 6.9. We will use almost the same strategy as in the previous section.

Lemma 6.13. *$\mathcal{F}_{0,\omega}$ is dense in $\mathcal{F}_{0,0}$.*

Proof. Fix $\sigma \in \mathcal{F}_{0,0}$ and put $\sigma_0 = \sigma$. We will construct a sequence $(\sigma_n)_{n\in\omega}$ of finite strings such that $\sigma_n \in \mathcal{F}_{n,n}$, and

$$\mathcal{J}^{-1}_{[m,n)}(\sigma_n) \subseteq \mathcal{J}^{-1}_{[m,n+1)}(\sigma_{n+1}) \in \mathcal{F}_{m,m}$$

for any $m \leq n$. Then we will define $X := \bigcup_n \mathcal{J}^{-1}_{[0,n)}(\sigma_n)$ and ensure that $X \in \mathcal{F}_{0,\omega}$, that is, $\mathcal{J}_{[0,n)}(X) \in \mathcal{F}_{n,n}$. Given n, inductively assume that $\sigma_n \in \mathcal{F}_{n,n}$. Now, by the property (b), $\mathcal{F}_{n,n+1}$ is dense in $\mathcal{F}_{n,n}$ for any $n \in \omega$. Since $\mathcal{F}_{n,n+1} = \mathcal{J}_n^{-1}[\mathcal{F}_{n+1,n+1}]$, there is $Y \in \mathcal{F}_{n+1,n+1}$ such that

$$\sigma_n \subset \mathcal{J}_n^{-1}(Y) \in \mathcal{F}_{n,n}.$$

Since $\mathcal{J}_n^{-1}$ is continuous, we can find an initial segment $\sigma_{n+1} \subset Y$ such that $\sigma_n \subseteq \mathcal{J}_n^{-1}(\sigma_{n+1})$. Clearly $\sigma_{n+1} \in \mathcal{F}_{n+1,n+1}$. For every $m \leq n$, by continuity of $\mathcal{J}^{-1}_{[m,n)}$, we also have

$$\mathcal{J}^{-1}_{[m,n)}(\sigma_n) \subseteq \mathcal{J}^{-1}_{[m,n)} \circ \mathcal{J}_n^{-1}(\sigma_{n+1}) = \mathcal{J}^{-1}_{[m,n+1)}(\sigma_{n+1})$$

and $\mathcal{J}^{-1}_{[m,n)}(\sigma_n)$ is extendible in $\mathcal{J}^{-1}_{[m,n)}[\mathcal{F}_{n,n}] = \mathcal{F}_{m,n} \subseteq \mathcal{F}_{m,m}$, that is, $\mathcal{J}^{-1}_{[m,n)}(\sigma_n) \in \mathcal{F}_{m,m}$ as wanted.

For $X = \bigcup_n \mathcal{J}^{-1}_{[0,n)}(\sigma_n)$, we claim that $\mathcal{J}_{[0,m)}(X) = Y_m := \bigcup_{n\geq m} \mathcal{J}^{-1}_{[m,n)}(\sigma_n)$. This is because we have

$$\mathcal{J}^{-1}_{[0,m)}(Y_m) = \bigcup_{n\geq m} \mathcal{J}^{-1}_{[0,m)} \circ \mathcal{J}^{-1}_{[m,n)}(\sigma_n) = \bigcup_n \mathcal{J}^{-1}_{[0,n)}(\sigma_n) = X.$$

The first equality is due to continuity of $\mathcal{J}^{-1}_{[0,m)}$ and the property that $(\mathcal{J}^{-1}_{[m,n)}(\sigma_n))_{n\geq m}$ is increasing. Therefore $\mathcal{J}_{[0,m)}(X) = Y_m$. Since $\mathcal{J}^{-1}_{[m,n)}(\sigma_n) \in \mathcal{F}_{m,m}$, and $\mathcal{F}_{m,m}$ is closed, we have $\mathcal{J}_{[0,m)}(X) \in \mathcal{F}_{m,m}$ for all $m \in \omega$, and therefore $\sigma \subset X \in \mathcal{F}_{0,\omega}$. This shows that $\mathcal{F}_{0,\omega}$ is dense in $\mathcal{F}_{0,0}$. □

For notational simplicity, we use the following notations:

$$\mathcal{J}_\omega = \mathcal{J}^{\omega^\alpha,\mathcal{C}}, \qquad \mathcal{A}^{\not\sim}_\omega = \mathcal{A}^{\not\sim\omega^\alpha,\mathcal{C}}.$$

Lemma 6.14. *If $\mathcal{A}$ is α-stable, then $\mathcal{F}_{\omega,\omega} \subseteq \mathcal{F}(\mathcal{A}^{\not\sim}_\omega)$.*

Proof. Fix $X \in \omega^\omega$ such that $\mathcal{J}_\omega(X) \in \mathcal{F}_{\omega,\omega}$. Given $n \in \omega$, we will define a continuous function $\eta : \mathcal{J}_\omega[\omega^\omega] \to \mathcal{J}_\omega[\omega^\omega] \cap [\mathcal{J}_\omega(X) \restriction n]$ witnessing that $\mathcal{A}^{\not\sim}_\omega \leq_w \mathcal{A}^{\not\sim}_\omega \restriction [\mathcal{J}_\omega(X) \restriction n]$. Note that $\mathcal{J}_\omega(X) \restriction n = \langle \mathcal{J}_{[0,1)}(X) \restriction 1, \dots, \mathcal{J}_{[0,n)}(X) \restriction 1\rangle$. Note also that $\mathcal{J}_\omega(X) \in \mathcal{F}_{\omega,\omega}$ if and only if $\mathcal{J}_{[0,n)}(X) \in \mathcal{F}_{n,n}$ for all $n \in \omega$. By the condition (c), there is a C_n-computable Wadge reduction $\theta_n \colon \mathcal{A}^{\not\sim}_{[0,n)} \leq_w \mathcal{A}_n \restriction [\mathcal{J}_{[0,n)}(X) \restriction 1]$. We let η be a continuous function such that for any k,

$$\eta(\mathcal{J}_\omega(Y))(k) = \mathcal{J}_{[0,k)}(\mathcal{J}^{-1}_{[0,n)} \circ \theta_n \circ \mathcal{J}_{[0,n)}(Y)) \restriction 1.$$

In other words, $\eta(\mathcal{J}_\omega(Y)) = \mathcal{J}_\omega(\mathcal{J}^{-1}_{[0,n)} \circ \theta_n \circ \mathcal{J}_{[0,n)}(Y))$. Consequently,

$$\begin{aligned}\mathcal{A}^{\not\sim}_\omega(\mathcal{J}_\omega(Y)) = \mathcal{A}(Y) = \mathcal{A}^{\not\sim}_{[0,n)}(\mathcal{J}_{[0,n)}(Y)) \leq_{\mathcal{Q}} \mathcal{A}^{\not\sim}_{[0,n)}(\theta_n \circ \mathcal{J}_{[0,n)}(Y))\\ = \mathcal{A}(\mathcal{J}^{-1}_{[0,n)} \circ \theta_n \circ \mathcal{J}_{[0,n)}(Y)) = \mathcal{A}^{\not\sim}_\omega(\mathcal{J}_\omega(\mathcal{J}^{-1}_{[0,n)} \circ \theta_n \circ \mathcal{J}_{[0,n)}(Y))) = \mathcal{A}^{\not\sim}_\omega(\eta(\mathcal{J}_\omega(Y))).\end{aligned}$$

Since $\eta(\mathcal{J}_\omega(Y))$ extends $\mathcal{J}_\omega(X) \restriction n$, this witnesses that $\mathcal{A}^{\not\sim}_\omega \leq_w \mathcal{A}^{\not\sim}_\omega \restriction [\mathcal{J}_\omega(X) \restriction n]$. □

This concludes the proof of Lemma 6.9.

References

[Blo14] Alexander C. Block. Operations on a Wadge-type hierarchy of ordinal-valued functions. Master's thesis, Universiteit van Amsterdam, 2014.

[DDW17] Adam Day, Rod Downey, and Linda B. Westrick. Reducibilities for discontinuous functions. unpublished, 2017.

[Dup] J. Duparc. Wadge hierarchy and Veblen hierarchy, part II: Borel sets of infinite rank. unpublished.

[Dup01] J. Duparc. Wadge hierarchy and Veblen hierarchy. I. Borel sets of finite rank. *J. Symbolic Logic*, 66(1):56–86, 2001.

[Dup03] J. Duparc. The Steel hierarchy of ordinal valued Borel mappings. *J. Symbolic Logic*, 68(1):187–234, 2003.

[Fri57] Richard Friedberg. A criterion for completeness of degrees of unsolvability. *J. Symb. Logic*, 22:159–160, 1957.

[Kih17] Takayuki Kihara. Ordinal ranks on the Baire and non-Baire class functions. unpublished, 2017.

[KLS12] Alexander S. Kechris, Benedikt Löwe, and John R. Steel, editors. *Wadge degrees and projective ordinals. The Cabal Seminar. Volume II*, volume 37 of *Lecture Notes in Logic*. Association for Symbolic Logic, La Jolla, CA; Cambridge University Press, Cambridge, 2012.

[KM] Takayuki Kihara and Antonio Montalbán. The uniform Martin's conjecture for many-one degrees. to appear in Trans. Amer. Math. Soc.

[Lav78] Richard Laver. Better-quasi-orderings and a class of trees. In *Studies in foundations and combinatorics*, volume 1 of *Adv. in Math. Suppl. Stud.*, pages 31–48. Academic Press, New York-London, 1978.

[LSR90] Alain Louveau and Jean Saint-Raymond. On the quasi-ordering of Borel linear orders under embeddability. *J. Symbolic Logic*, 55(2):537–560, 1990.

[Mac77] John M. Macintyre. Transfinite extensions of Friedberg's completeness criterion. *J. Symbolic Logic*, 42(1):1–10, 1977.

[MM11] Alberto Marcone and Antonio Montalbán. The Veblen functions for computability theorists. *Journal of Symbolic Logic*, 76(2):575–602, 2011.

[Mon14] Antonio Montalbán. Priority arguments via true stages. *J. Symb. Log.*, 79(4):1315–1335, 2014.

[MW85] Richard Mansfield and Galen Weitkamp. *Recursive aspects of descriptive set theory*. Oxford University Press, New York, 1985. with a chapter by Stephen Simpson.

[NW65] C. St. J. A. Nash-Williams. On well-quasi-ordering infinite trees. *Proc. Cambridge Philos. Soc.*, 61:697–720, 1965.

[Sel83] V. L. Selivanov. Hierarchies of hyperarithmetic sets and functions. *Algebra i Logika*, 22(6):666–692, 720, 1983.

[Sel95] Victor L. Selivanov. Fine hierarchies and Boolean terms. *J. Symb. Log.*, 60(1):289–317, 1995.

[Sel07] Victor L. Selivanov. Hierarchies of Δ^0_2-measurable k-partitions. *MLQ Math. Log. Q.*, 53(4-5):446–461, 2007.

[Sel11] Victor Selivanov. A fine hierarchy of ω-regular k-partitions. In *Models of computation in context*, volume 6735 of *Lecture Notes in Comput. Sci.*, pages 260–269. Springer, Heidelberg, 2011.

[Sel17a] Victor Selivanov. Towards a descriptive theory of cb_0-spaces. *Mathematical Structures in Computer Science*, 27(8):1553–1580, 2017.

[Sel17b] Victor L. Selivanov. Extending Wadge theory to k-partitions. In *Unveiling Dynamics and Complexity - 13th Conference on Computability in Europe*, volume 10307 of *Lecture Notes in Computer Science*, pages 387–399. Springer, 2017.

[Sim85] Stephen G. Simpson. Bqo-theory and Fraïssé's conjecture. Chapter 9 of [MW85], 1985.

[vEMS87] Fons van Engelen, Arnold W. Miller, and John Steel. Rigid Borel sets and better quasi-order theory. In *Logic and combinatorics (Arcata, Calif., 1985)*, volume 65 of *Contemp. Math.*, pages 199–222. Amer. Math. Soc., Providence, RI, 1987.

[VW78] Robert Van Wesep. Wadge degrees and descriptive set theory. In *Cabal Seminar 76–77 (Proc. Caltech-UCLA Logic Sem., 1976–77)*, volume 689 of *Lecture Notes in Math.*, pages 151–170. Springer, Berlin, 1978.

[Wad83] William Wilfred Wadge. *Reducibility and Determinateness on the Baire Space*. ProQuest LLC, Ann Arbor, MI, 1983. Thesis (Ph.D.)–University of California, Berkeley.

(Takayuki Kihara) Graduate School of Informatics, Nagoya University, Japan
E-mail address: kihara@i.nagoya-u.ac.jp
URL: math.mi.i.nagoya-u.ac.jp/∼kihara

(Antonio Montalbán) Department of Mathematics, University of California, Berkeley, United States
E-mail address: antonio@math.berkeley.edu
URL: www.math.berkeley.edu/∼antonio