A SYNTACTIC APPROACH TO BOREL FUNCTIONS: SOME EXTENSIONS OF LOUVEAU'S THEOREM

TAKAYUKI KIHARA AND KENTA SASAKI

Abstract. Louveau showed that if a Borel set in a Polish space happens to be in a Borel Wadge class Γ, then its Γ-code can be obtained from its Borel code in a hyperarithmetical manner. We extend Louveau's theorem to Borel functions: If a Borel function on a Polish space happens to be a $\utilde{\mathbf{\Sigma}}_t$-function, then one can find its $\utilde{\mathbf{\Sigma}}_t$-code hyperarithmetically relative to its Borel code. More generally, we prove extension-type, domination-type, and decomposition-type variants of Louveau's theorem for Borel functions.

1. Introduction

1.1. **Background.** In [21], Louveau showed that if a Borel set in a Polish space happens to be a $\utilde{\mathbf{\Sigma}}^0_\xi$ set, then its $\utilde{\mathbf{\Sigma}}^0_\xi$-code can be obtained from its Borel code in a hyperarithmetical manner. This is derived from the so-called Louveau separation theorem [21], which states that if a disjoint pair of Σ^1_1 sets is separated by a $\utilde{\mathbf{\Sigma}}^0_\xi$ set, then it is separated by a $\utilde{\mathbf{\Sigma}}^0_\xi$ set which has a hyperarithmetical $\utilde{\mathbf{\Sigma}}^0_\xi$-code. Louveau applied his result to solve the section problem on Borel sets. This result is useful for extracting information about uniformity from a non-uniform condition. For instance, using Louveau's theorem, Solecki [35] obtained an inequality for cardinal invariants related to decomposability of Borel functions, and Fujita-Mátrai [10] solved Laczkovich's problem on differences of Borel functions. Louveau's theorem is also known to be a powerful tool which enables us to use effective methods in topological arguments. For instance, Gregoriades-Kihara-Ng [11] used it as a tool to reduce a problem on descriptive set theory to a problem on computability theory, and gave a partial solution to the decomposability problem on Borel functions.

In [22], Louveau revisited his theorem in the context of the Wadge hierarchy. The notion of Wadge degrees [37] provides us an ultimate refinement of all known hierarchies in descriptive set theory, such as the Borel hierarchy and the Hausdorff-Kuratowski difference hierarchy. However, the original definition of the Wadge hierarchy does not tell us the way to obtain each Wadge class. To address this issue, Louveau [22] introduced a set of basic ω-ary Boolean operations, and observed that each blueprint u for how to combine these operations determines a Wadge class $\utilde{\mathbf{\Gamma}}_u$. Indeed, Louveau [22] showed that any Borel Wadge class in the Baire space can be obtained as a combination of these ω-ary Boolean operations, and moreover, he showed that if a Borel set in a Polish space happens to be a $\utilde{\mathbf{\Gamma}}_u$ set, then its $\utilde{\mathbf{\Gamma}}_u$-code can be obtained from its Borel code in a hyperarithmetical manner. Louveau's explicit description of each Borel Wadge class

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has been applied to prove Borel Wadge determinacy within second order arithmetic [23]. Similar notions have also been investigated, e.g. in [31, 8]. For instance, Duparc [8, 7] introduced a slightly different set of operations, on the basis of which he gave the normal form of each Borel Wadge degree, even in non-separable spaces.

The notion of Wadge reducibility has been extended to functions, and extensively studied, e.g. in [36, 32, 1, 19, 13]. For Borel functions with better-quasi-ordered ranges, Kihara-Montalbán [19] obtained a full characterization of the Wadge degrees. In order to obtain the result, Kihara-Montalbán [19] introduced a language consisting of a few basic algebraic operations, inspired by Duparc's operations [8, 7], for describing any Wadge class of Borel functions. As before, each blueprint t for how to combine these operations determines a Wadge class $\undertilde{\mathbf{\Sigma}}_t$. The main difference with Louveau's work is not merely that it is for functions, but that the blueprints (i.e., the terms in the language) naturally form a quasi-ordered algebraic structure, which can be viewed as the structured collection of (suitably extended) nested labeled trees and forests [19], and forgetting the algebraic information from the term model yields the ordering of Wadge degrees; see also [33].

In this article, we show that if a Borel function between Polish spaces happens to be a $\undertilde{\mathbf{\Sigma}}_t$ function, then its $\undertilde{\mathbf{\Sigma}}_t$-code can be obtained from its Borel code in a hyperarithmetical manner. To show this, we rewrite the definition of $\undertilde{\mathbf{\Sigma}}_t$ as the mechanism controlled by certain flowcharts only involving conditional branching. We observe that the flowchart representation of Borel functions is a powerful tool for clarifying various arguments on the class $\undertilde{\mathbf{\Sigma}}_t$. Indeed, this flowchart definition makes our proof simpler than Louveau's original one in [22], even though our theorem is a far extension of Louveau's one.

1.2. **Summary.** In Section 2 we introduce Kihara-Montalbán's signature $\mathcal{L}_{\mathrm{Veb}}(Q)$ and Wadge classes $\undertilde{\mathbf{\Sigma}}_t$ for $\mathcal{L}_{\mathrm{Veb}}(Q)$-terms t (see [19]), and reorganize Kihara-Montalbán's theory of $\undertilde{\mathbf{\Sigma}}_t$, paying special attention to its syntactic aspects. In Sections 2 and 3, we provide four definitions of $\undertilde{\mathbf{\Sigma}}_t$, one of which corresponds to Kihara-Montalbán's definition [19] and one to Selivanov's definition [33]. In Section 3, we pay particular attention to the flowchart-based definition of $\undertilde{\mathbf{\Sigma}}_t$ and give a rigorous proof of the equivalence of the four definitions of $\undertilde{\mathbf{\Sigma}}_t$. Indeed, the real purpose of this article is to popularize these definitions of $\undertilde{\mathbf{\Sigma}}_t$, thus making it more accessible to, for example, Selivanov's series of studies (e.g. [31, 33]) and broadening the base of research.

Since these notions have their origin in the theory of Wadge degrees, Section 4 looks at an application of the flowchart representation of $\undertilde{\mathbf{\Sigma}}_t$ in the theory of Wadge degrees. Specifically, we see that the topological and symbolic complexity of Borel functions on Polish spaces coincide: the symbolic Wadge degrees à la Pequignot [26] on bqo-valued Borel functions on a (possibly higher-dimensional) Polish space is exactly the hierarchy obtained from the topological Wadge classes $\undertilde{\mathbf{\Sigma}}_t$ (Theorem 4.5). Finally, in Section 5, we present several extensions of Louveau's theorem: Let X and Q be computable Polish spaces, and t be a hyperarithmetical $\mathcal{L}_{\mathrm{Veb}}(Q)$-term.

(1) If $f\colon X \to Q$ is both $\undertilde{\mathbf{\Sigma}}_t$ and hyperarithmetical, then f has a hyperarithmetical $\undertilde{\mathbf{\Sigma}}_t$-code (Theorem 5.1).
(2) Let $f\colon\subseteq \omega^\omega \to Q$ be a partial Π^1_1-measurable function with a Σ^1_1 domain, and suppose that f can be extended to a total $\undertilde{\mathbf{\Sigma}}_t$ function $g\colon \omega^\omega \to Q$. Then, f can

be extended to a total $\underset{\sim}{\mathbf{\Sigma}}_t$ function $g^\star\colon \omega^\omega \to Q$ which has a hyperarithmetical $\underset{\sim}{\mathbf{\Sigma}}_t$-code (Theorem 5.2).

(3) Let $\leq_Q$ be a Π^1_1 quasi-order on Q, and $f\colon\subseteq \omega^\omega \to Q$ be a partial Π^1_1-measurable function with a Σ^1_1 domain. Suppose that f is $\leq_Q$-dominated by some total $\underset{\sim}{\mathbf{\Sigma}}_t$ function $g\colon \omega^\omega \to Q$. Then, f is $\leq_Q$-dominated by some total $\underset{\sim}{\mathbf{\Sigma}}_t$ function $g^\star\colon \omega^\omega \to Q$ which has a hyperarithmetical $\underset{\sim}{\mathbf{\Sigma}}_t$-code (Theorem 5.3).

We also prove a decomposition-type variant of Louveau's theorem (Theorem 5.4).

1.3. **Preliminaries.** In this article, we assume that the reader is familiar with elementary facts about descriptive set theory. For the basics of (effective) descriptive set theory, we refer the reader to Moschovakis [25].

For a function $f\colon X \to Y$ and $A \subseteq X$, we use the symbol $f \upharpoonright A$ denote the restriction of f to A. We denote a partial function from X to Y as $f\colon\subseteq X \to Y$. We also use the following notations on strings: For finite strings $\sigma, \tau \in \omega^{<\omega}$, we write $\sigma \preceq \tau$ if σ is an initial segment of τ, and write $\sigma \prec \tau$ if σ is a proper initial segment of τ. We also use the same notation even if τ is an infinite string, i.e., $\tau \in \omega^\omega$. For $\sigma \in \omega^{<\omega}$ and $\ell \in \omega$, define $\sigma \upharpoonright \ell$ as the initial segment of σ of length ℓ. For finite strings $\sigma, \tau \in \omega^{<\omega}$, let $\sigma^\frown\tau$ be the concatenation of σ and τ. If τ is a string of length 1, i.e., τ is of the form $\langle n\rangle$ for some $n \in \omega$, then $\sigma^\frown\langle n\rangle$ is abbreviated to $\sigma^\frown n$. We always assume that ω^ω is equipped with the standard Baire topology, that is, the ω-product of the discrete topology on ω. For $\sigma \in \omega^{<\omega}$, let $[\sigma]$ be the clopen set generated by σ, i.e., $[\sigma] = \{x \in \omega^\omega : \sigma \prec x\}$.

For Polish spaces X and Y, for a pointclass Γ, we say that $f\colon X \to Y$ is *Γ-measurable* if $f^{-1}[U] \in \Gamma$ for any open set $U \subseteq Y$. A *computable Polish space* or a *recursively presented Polish space* is a triple (X, d, α), where (X, d) is a Polish space, $(\alpha_n)_{n\in\omega}$ is a dense sequence in X, and the map $(n, m) \mapsto d(\alpha_n, \alpha_m)$ has a nice computability-theoretic property; see [25, Section 3I] and [2]. If Γ is a lightface pointclass on computable Polish spaces, then we say that $f\colon X \to Y$ is *effectively Γ-measurable*, or simply, *Γ-measurable* if the relation $R(x, e)$ defined by $f(x) \in B_e$ is in Γ, where B_e is the e-th rational open ball in Y. Let WO be the set of all well-orders on ω. For $\alpha \in \mathsf{WO}$, we define $|\alpha|$ as the order type of α.

Let X and Y be topological spaces, and Q be a quasi-ordered set. A function $f\colon X \to Q$ is *Wadge reducible to* $g\colon Y \to Q$ (written $f \leq_W g$) if there exists a continuous function $\theta\colon X \to Y$ such that $f(x) \leq_Q g(\theta(x))$ for any $x \in X$.

Some of the notion addressed in this article involve "codes". For such notions, as usual, by the expression "given X one can effectively find Y such that Z holds" we mean "there exists a computable function φ such that if c is a code of X then $\varphi(c)$ is a code of some Y such that Z holds".

2. Describing Borel Wadge classes

2.1. **Syntax.** As is well-known, the processes of approximating functions by finite mind-changes can be represented by labeled well-founded trees and forests; see e.g. [12, 32, 13]. As an algebraic aspect of such precesses, it is known that one can describe well-founded forests as terms in the signature $\{\sqcup, \rightsquigarrow\}$ over an (infinitary) equational theory (Figure 1; see also [33]). However, as shown in [19], a control mechanism for Borel functions is more complicated, and it cannot be described by just a tree or a forest. Instead, it is described by a matryoshka of trees: Within each node of the tree there is a tree, and

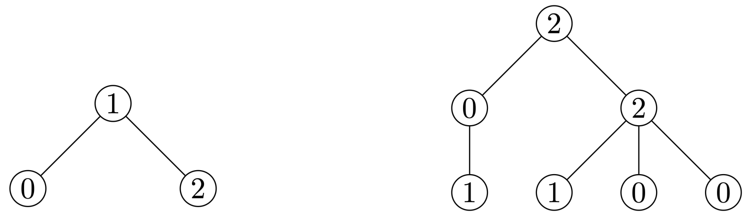

FIGURE 1. (left) The tree $1 \rightsquigarrow (0 \sqcup 2)$; (right) The tree $2 \rightsquigarrow ((0 \rightsquigarrow 1) \sqcup (2 \rightsquigarrow (1 \sqcup 0 \sqcup 0)))$

within each node of that tree there is a tree, and within each node of that tree there is another tree, and so on. Kihara-Montalbán [19] introduced the language for describing matryoshkas of trees as follows:

Definition 2.1 (Kihara-Montalbán [19])**.** For a set Q, the signature $\mathcal{L}_{\mathrm{Veb}}(Q)$ consists of a constant symbol q for each $q \in Q$, a binary function symbol $\rightsquigarrow$, an ω-ary function symbol $\sqcup$, and a unary function symbol $\langle\cdot\rangle^{\omega^\alpha}$ for each $\alpha < \omega_1$. If $Q = \emptyset$, we write $\mathcal{L}_{\mathrm{Veb}}$ for $\mathcal{L}_{\mathrm{Veb}}(\emptyset)$.

Here, the original language introduced by Kihara-Montalbán [19] uses the symbol $^{\rightarrow}$ instead of $\rightsquigarrow$ (the reason for using the symbol $\rightsquigarrow$ here is to avoid confusion, since the usual arrow symbol is used for many other purposes). As in [19], we abbreviate the symbol $\langle\cdot\rangle^{\omega^0}$ as $\langle\cdot\rangle$. The description $\langle t\rangle$ represents a node labeled by t. Not only that, but the label types are ranked, e.g. $\langle t\rangle^{\omega^\alpha}$ is a node labelled with t, and the rank of its label is ω^α. Hereafter, we use the symbol ϕ_α to denote $\langle\cdot\rangle^{\omega^\alpha}$. As the function symbol $\phi_\alpha = \langle\cdot\rangle^{\omega^\alpha}$ is specified by the fixed point axiom $\phi_\beta(\phi_\alpha(t)) = \phi_\alpha(t)$ for any $\beta < \alpha$, we say that ϕ_α is the *α-th Veblen function symbol*, see also [20], where the symbol s_α is used in [20, 33] instead of $\langle\cdot\rangle^{\omega^\alpha}$ or ϕ_α.

The notion of terms in a given signature is inductively defined as usual in logic, universal algebra, and other areas:

Definition 2.2 (Term)**.** Let $\mathcal{L}$ be a (possibly infinitary) signature.

- A variable symbol x and a constant symbol $c \in \mathcal{L}$ are $\mathcal{L}$-terms.
- If f is an I-ary function symbol in $\mathcal{L}$, and t_i is an $\mathcal{L}$-term for any $i \in I$, then $f(\langle t_i\rangle_{i\in I})$ is an $\mathcal{L}$-term.

When considering the signature $\mathcal{L}_{\mathrm{Veb}}(Q)$, the terms $\rightsquigarrow(\langle s, t\rangle)$ and $\sqcup(\langle t_i\rangle_{i\in\omega})$ are abbreviated as $s \rightsquigarrow t$ and $\sqcup_{i\in\omega} t_i$ as usual. Recall that a term is *closed* if it does not contain variable symbols. As in Kihara-Montalbán [19], we consider only closed $\mathcal{L}_{\mathrm{Veb}}(Q)$-terms in most cases, so we refer to a closed $\mathcal{L}_{\mathrm{Veb}}(Q)$-term simply as an $\mathcal{L}_{\mathrm{Veb}}(Q)$-term. When dealing with an $\mathcal{L}_{\mathrm{Veb}}(Q)$-term which may contain variable symbols, we refer to it emphatically as an *open* $\mathcal{L}_{\mathrm{Veb}}(Q)$-term. In this article, the analysis of the syntax of terms plays an important role in various aspects.

Definition 2.3 (Syntax tree)**.** Let $\mathcal{L}$ be a (possibly infinitary) signature. Any $\mathcal{L}$-term t defines a labeled tree $\mathtt{Syn}_t = (\mathtt{S}_t, \mathtt{l}_t)$ called the *syntax tree* of the term t as follows:

- If t is a constant or variable symbol x, then the syntax tree $\mathtt{Syn}_t$ is the singleton $\mathtt{S}_t = \{\varepsilon\}$ labeled by $\mathtt{l}_t(\varepsilon) = x$, i.e., $\mathtt{Syn}_t$ can be written as $\langle x\rangle$ in terms of the language of labeled trees.

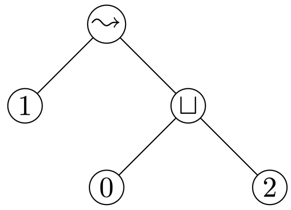

FIGURE 2. The syntax tree of $1 \rightsquigarrow (0 \sqcup 2)$

- If t is of the form $f(\langle s_i \rangle_{i \in I})$ for some function symbol $f \in \mathcal{L}$ and $\mathcal{L}$-terms $\langle s_i \rangle_{i \in I}$, then $\mathtt{Syn}_t$ is the result by adding a root ε and edges from ε to $\mathtt{Syn}_{s_i}$ for $i \in I$, where the root ε is labeled by f, i.e., $\mathtt{Syn}_t$ can be written as $\langle f \rangle \rightsquigarrow \sqcup_{i \in I} \mathtt{Syn}_{s_i}$ in terms of the language of labeled trees.

Note that the syntax tree of an $\mathcal{L}$-term is always well-founded. When considering $\mathcal{L}_{\mathrm{Veb}}(Q)$-terms, be careful not to confuse the syntax tree of a term t (e.g., Figure 2) with the tree represented by t (e.g., Figure 1). If t is an $\mathcal{L}_{\mathrm{Veb}}(Q)$-term, then one can always think of $\mathtt{Syn}_t$ as a labeled subtree of $\omega^{<\omega}$, since the signature $\mathcal{L}_{\mathrm{Veb}}(Q)$ consists of at most ω-ary function symbols. More explicitly, for any $\mathcal{L}_{\mathrm{Veb}}(Q)$-term t, one may assume that $\mathtt{S}_t$ is a subtree of $\omega^{<\omega}$ as follows:

- If t is either a constant or variable symbol then $\mathtt{S}_t = \{\varepsilon\}$, where ε is the empty string.
- If t is of the form $u \rightsquigarrow s$ then $\mathtt{S}_t = \{0^\frown\sigma : \sigma \in \mathtt{S}_u\} \cup \{1^\frown\sigma : \sigma \in \mathtt{S}_s\}$.
- If t is of the form $\sqcup_{n \in \omega} s_n$ then $\mathtt{S}_t = \{n^\frown\sigma : n \in \omega \text{ and } \sigma \in \mathtt{S}_{s_n}\}$.
- If t is of the form $\phi_\alpha(s)$ then $\mathtt{S}_t = \{0^\frown\sigma : \sigma \in \mathtt{S}_s\}$.

Definition 2.4. An $\mathcal{L}_{\mathrm{Veb}}(Q)$-term t is *well-formed* if t contains no subterm of the form $\phi_\alpha(\sqcup_{n \in \omega} s_n)$; and t is *normal* if the symbol $\rightsquigarrow$ always occurs as of the form $\phi_\alpha(u) \rightsquigarrow \sqcup_{n \in \omega} s_n$ or $q \rightsquigarrow \sqcup_{n \in \omega} s_n$.

In other words, using a syntax tree, an $\mathcal{L}_{\mathrm{Veb}}(Q)$-term t is well-formed if and only if, for any $\sigma \in \mathtt{Syn}_t$, the following holds:

$$\sigma \text{ is labeled by } \phi_\alpha \implies \sigma^\frown 0 \text{ is not labeled by } \sqcup.$$

Similarly, an $\mathcal{L}_{\mathrm{Veb}}(Q)$-term t is normal if and only if, for any $\sigma \in \mathtt{Syn}_t$, the following holds:

$$\sigma \text{ is labeled by } \rightsquigarrow \implies \begin{cases} \sigma^\frown 0 \text{ is either a leaf or labeled by } \phi_\alpha, \\ \sigma^\frown 1 \text{ is labeled by } \sqcup. \end{cases}$$

Remark. Let us add a supplementary note on the difference in terminology between this article and Kihara-Montalbán [19]. First, for signature, the symbol $\mathcal{L}(Q)$ is used in [19] instead of $\mathcal{L}_{\mathrm{Veb}}(Q)$. Also note that an $\mathcal{L}(Q)$-term in the sense of [19] is a normal well-formed $\mathcal{L}_{\mathrm{Veb}}(Q)$-term in the sense of this article. The items (3) and (5) in [19, Definition 3.19] guarantee wellformedness and normality, respectively. Therefore, the notation ${}^{\sqcup}\mathrm{Tree}^{\omega_1}(\mathcal{Q})$ used in [19] refers to the set of all normal well-formed $\mathcal{L}_{\mathrm{Veb}}(Q)$-terms.

Let $\trianglelefteq$ be the nested homomorphic quasi-order on the normal well-formed $\mathcal{L}_{\mathrm{Veb}}(Q)$-terms introduced by Kihara-Montalbán [19, Definition 3.20] to characterize the order type of the Wadge degrees of Q-valued Borel functions. For a quasi-order, its quotient by the induced equivalence relation is called the poset reflection.

Fact 2.5 (Kihara-Montalbán [19, Theorem 1.5]). *Let Q be a better-quasi-ordered set. Then, the Wadge degrees of Borel functions $\omega^\omega \to Q$ is isomorphic to the poset reflection of the quasi-order $\trianglelefteq$ on the normal well-formed $\mathcal{L}_{\mathrm{Veb}}(Q)$-terms.*

Coding a syntax tree: As usual, a subset of $\omega^{<\omega}$ can be viewed as a subset of ω via an effective bijection $\omega^{<\omega} \simeq \omega$. We assume that the constant symbols Q and the variable symbols are indexed by ω^ω, i.e., $Q = \{q_z\}_{z\in\omega^\omega}$ and $\mathrm{Var} = \{x_z\}_{z\in\omega^\omega}$, and then we consider the following coding:

$$\mathtt{c}(0, z) = q_z, \quad \mathtt{c}(1, z) = x_z, \quad \mathtt{c}(2, z) = \rightsquigarrow, \quad \mathtt{c}(3, z) = \sqcup, \quad \mathtt{c}(4, z) = \phi_{|z|},$$

where $\mathtt{c}(4, z)$ is defined only when $z \in \mathsf{WO}$. The function $\mathtt{c}$ gives a representation of the $\mathcal{L}_{\mathrm{Veb}}(Q)$-symbols. A realizer of a labeling function $\mathtt{l}_t$ is a function $\lambda_t \colon \mathtt{S}_t \to \omega \times \omega^\omega$ such that $\mathtt{l}_t = \mathtt{c} \circ \lambda_t$. As we consider $\mathtt{S}_t$ as a subset of ω, a realizer λ_t of a labeling function can be viewed as an element of $(\omega \times \omega^\omega)^\omega \simeq \omega^\omega$. Via this identification, we call such a pair $(\mathtt{S}_t, \lambda_t)$ (as an element of ω^ω) a code of the syntax tree $\mathtt{Syn}_t = (\mathtt{S}_t, \mathtt{l}_t)$.

As for effectivity, we say that an $\mathcal{L}_{\mathrm{Veb}}(Q)$-term t is hyperarithmetic or Δ^1_1 if its syntax tree $\mathtt{Syn}_t$ has a Δ^1_1-code.

2.2. **Semantics.** The essential idea of Kihara-Montalbán's Theorem (Fact 2.5) is to map the $\mathcal{L}_{\mathrm{Veb}}(Q)$-terms to the control mechanisms of Borel functions. More precisely, Kihara-Montalbán assigned to each $\mathcal{L}_{\mathrm{Veb}}(Q)$-term t a class $\utilde{\Sigma}_t$ of Borel functions, which may be thought of as an ultimate refinement of the Borel/Baire hierarchy.

Definition 2.6 (First definition of $\utilde{\Sigma}_t$). Let Z be a zero-dimensional Polish space. For each $\mathcal{L}_{\mathrm{Veb}}(Q)$-term t, we inductively define the classes $\utilde{\Sigma}_t(Z)$ and $\utilde{\Sigma}_t(Y; Z)$ of Q-valued functions on Polish spaces Z and $Y \subseteq Z$ as follows:

(1) $\utilde{\Sigma}_q(Z)$ consists only of the constant function $x \mapsto q \colon Z \to Q$.
(2) If $t = \sqcup_{i\in\omega} s_i$, then $f \in \utilde{\Sigma}_t(Z)$ if and only if there is an open cover $(U_i)_{i\in\omega}$ of Z such that $f \restriction U_i \in \utilde{\Sigma}_{s_i}(U_i; Z)$ for each $i \in \omega$.
(3) If $t = s \rightsquigarrow u$, then $f \in \utilde{\Sigma}_t(Z)$ if and only if there is an open set $V \subseteq Z$ such that $f \restriction V \in \utilde{\Sigma}_u(V; Z)$ and $f \restriction (Z \setminus V) = g \restriction (Z \setminus V)$ for some $g \in \utilde{\Sigma}_s(Z)$.
(4) If $t = \phi_\alpha(s)$, then $f \in \utilde{\Sigma}_t(Z)$ if and only if there is a $\utilde{\Sigma}^0_{1+\omega^\alpha}$-measurable function $\beta \colon Z \to Z$ and a $\utilde{\Sigma}_s(Z)$ function $g \colon Z \to Q$ such that $f = g \circ \beta$.

Here, a function $f \colon Y \to Q$ is in $\utilde{\Sigma}_t(Y; X)$ if there exists a continuous function $\gamma \colon Y \to X$ such that $f = g \circ \gamma$ for some $g \in \utilde{\Sigma}_t(X)$.

In this article, we introduce various equivalent definitions of $\utilde{\Sigma}_t$. When we emphasize that it is $\utilde{\Sigma}_t$ in the sense of Definition 2.6, we write it as $\utilde{\Sigma}^W_t$; the superscript W suggests that this is the one associated with a Wadge class. Indeed, Kihara-Montalbán's theorem in [19] shows that any Borel Wadge class can be described as $\utilde{\Sigma}^W_t$ for some $\mathcal{L}_{\mathrm{Veb}}(Q)$-term t. That is to say, the essential ingredient of Fact 2.5 is not simply that the two structures are isomorphic, but more precisely that the complexity of the control mechanism of the Borel functions represented by a term corresponds exactly to the Wadge degree. This can be stated as follows:

Fact 2.7 (Kihara-Montalbán [19, Propositions 1.7 and 1.8])**.** *Let Q be a better-quasi-ordered set, and t be an $\mathcal{L}_{\rm Veb}(Q)$-term. If $g\colon \omega^\omega \to Q$ is a $\utilde{\mathbf{\Sigma}}^W_t$-function, then for any $f\colon \omega^\omega \to Q$,*

$$f \leq_W g \iff f \in \utilde{\mathbf{\Sigma}}^W_s(\omega^\omega) \text{ for some } s \trianglelefteq t.$$

Note that Definition 2.6 is slightly different from Kihara-Montalbán's original definition [19] of the class $\utilde{\mathbf{\Sigma}}_t$. To explain the original definition of the class $\utilde{\mathbf{\Sigma}}_t$, we assume $Z = \omega^\omega$, and let us consider the following: First, given a nonempty open set $V \subseteq Z$ one can effectively construct a map $e_V\colon \omega \to \omega^{<\omega}$ such that $V = \bigcup_{n\in\omega}[e_V(n)]$, and $e_V(n)$ is incomparable with $e_V(m)$ whenever $n \neq m$. Then, define $\mathtt{in}_V(n^\frown x) = e_V(n)^\frown x$. It is easy to see that $\mathtt{in}_V\colon \omega^\omega \simeq V$ is a homeomorphism. Next, by zero-dimensionality of Z, one can also effectively find a continuous retraction $\mathtt{out}_V\colon \omega^\omega \to (\omega^\omega \setminus V)$. See also Kihara-Montalbán [19, Observations 3.5 and 3.6] for the details.

Definition 2.8 (Second definition of $\utilde{\mathbf{\Sigma}}_t$ [19, Definition 3.7])**.** For each $\mathcal{L}_{\rm Veb}(Q)$-term t, we inductively define $\utilde{\mathbf{\Sigma}}_t(Z)$ as the class fulfilling (1), (2′), (3′) and (4):

(2′) If $t = \sqcup_i s_i$, then $f \in \utilde{\mathbf{\Sigma}}_t(Z)$ if and only if there is a clopen partition $(C_i)_{i\in\omega}$ of Z such that $f \circ \mathtt{in}_{C_i} \in \utilde{\mathbf{\Sigma}}_{s_i}(Z)$ for each $i \in \omega$.

(3′) If $t = s \rightsquigarrow u$, then $f \in \utilde{\mathbf{\Sigma}}_t(Z)$ if and only if there is an open set $V \subseteq Z$ such that $f \circ \mathtt{in}_V \in \utilde{\mathbf{\Sigma}}_u(Z)$ and $f \circ \mathtt{out}_V \in \utilde{\mathbf{\Sigma}}_s(Z)$.

To avoid confusion, we write $\utilde{\mathbf{\Sigma}}^{W\circ}_t$ for $\utilde{\mathbf{\Sigma}}_t$ in the sense of Definition 2.8.

Observation 2.9. *If $Z = \omega^\omega$, the first and second definitions of $\utilde{\mathbf{\Sigma}}_t(Z)$ coincide, i.e., $\utilde{\mathbf{\Sigma}}^W_t(Z) = \utilde{\mathbf{\Sigma}}^{W\circ}_t(Z)$.*

Proof. We first show the inclusion $\utilde{\mathbf{\Sigma}}^W_t(Z) \subseteq \utilde{\mathbf{\Sigma}}^{W\circ}_t(Z)$ by induction. If $t = \sqcup_{i\in\omega} s_i$, and $f \in \utilde{\mathbf{\Sigma}}^W_t(Z)$ via an open cover (U_i) of Z, then $f \upharpoonright U_i = g_i \circ \gamma_i$ for some $g_i \in \utilde{\mathbf{\Sigma}}^W_{s_i}(Z)$ and γ_i. By the induction hypothesis, we have $g_i \in \utilde{\mathbf{\Sigma}}^{W\circ}_{s_i}(Z)$. Since Z is zero-dimensional, one can effectively find a clopen partition (C_i) such that $C_i \subseteq U_i$ and $\bigcup_i C_i = \bigcup_i U_i$. Then, we get $f \circ \mathtt{in}_{C_i} = g_i \circ \gamma_i \circ \mathtt{in}_{C_i} \in \utilde{\mathbf{\Sigma}}^{W\circ}_{s_i}(Z)$ since $\gamma_i \circ \mathtt{in}_{C_i}\colon Z \to Z$ is continuous. Hence, $f \in \utilde{\mathbf{\Sigma}}^{W\circ}_t(Z)$. If $t = s \rightsquigarrow u$, and $f \in \utilde{\mathbf{\Sigma}}^W_t(Z)$ via an open set $V \subseteq Z$, then $f \upharpoonright (Z \setminus V) = g_0 \circ \gamma_0$ for some $g_0 \in \utilde{\mathbf{\Sigma}}^W_s(Z)$ and γ_0, and $f \upharpoonright V = g_1 \circ \gamma_1$ for some $g_1 \in \utilde{\mathbf{\Sigma}}^W_u(Z)$ and γ_1. By the induction hypothesis, we have $g_0 \in \utilde{\mathbf{\Sigma}}^{W\circ}_s(Z)$ and $g_1 \in \utilde{\mathbf{\Sigma}}^{W\circ}_u(Z)$. Then, we get $f \circ \mathtt{out}_V = g_0 \circ \gamma_0 \circ \mathtt{out}_V \in \utilde{\mathbf{\Sigma}}^W_s(Z)$ since $\gamma_0 \circ \mathtt{out}_V\colon Z \to Z$ is continuous, and similarly, we also get $f \circ \mathtt{in}_V = g_1 \circ \gamma_1 \circ \mathtt{in}_V \in \utilde{\mathbf{\Sigma}}^W_u(Z)$ since $\gamma_1 \circ \mathtt{in}_V\colon Z \to Z$ is continuous.

Next, we show the inclusion $\utilde{\mathbf{\Sigma}}^{W\circ}_t(Z) \subseteq \utilde{\mathbf{\Sigma}}^W_t(Z)$ by induction. If $t = \sqcup_{i\in\omega} s_i$, and if $f \in \utilde{\mathbf{\Sigma}}^{W\circ}_t(Z)$ via a clopen partition $(C_i)_{i\in\omega}$ of Z, then by the induction hypothesis, one can see that $f \upharpoonright C_i \in \utilde{\mathbf{\Sigma}}^W_{s_i}(C_i; Z)$ via $g_i = f \circ \mathtt{in}_{C_i}$ and $\gamma_i = \mathtt{in}_{C_i}^{-1}$. Hence, $f \in \utilde{\mathbf{\Sigma}}^W_t(Z)$. If $t = s \rightsquigarrow u$, and $f \in \utilde{\mathbf{\Sigma}}^{W\circ}_t(Z)$ via an open set $V \subseteq Z$, then by the induction hypothesis, one can see that $f \upharpoonright V \in \utilde{\mathbf{\Sigma}}^W_u(V; Z)$ via $g_1 = f \circ \mathtt{in}_V$ and $\gamma_1 = \mathtt{in}_V^{-1}$, and $f \upharpoonright (Z \setminus V) \in \utilde{\mathbf{\Sigma}}^W_s(Z \setminus V; Z)$ via $g_0 = f \circ \mathtt{out}_V$ and $\gamma_0 = \mathrm{id}$. Hence, $f \in \utilde{\mathbf{\Sigma}}^W_t(Z)$. □

2.3. **Command.** Next, we decompose the inductive definition of $\utilde{\mathbf{\Sigma}}_t$ along the syntax tree of t. This decomposition reveals the algorithmic aspect of the definition of $\utilde{\mathbf{\Sigma}}_t$. Let t be an $\mathcal{L}_{\rm Veb}(Q)$-term, and consider its syntax tree $\mathtt{Syn}_t$.

Definition 2.10. A *command* on the $\mathcal{L}_{\mathrm{Veb}}(Q)$-term t over a zero-dimensional Polish space Z is a family $\mathbf{U} = (U_\sigma, u_\sigma)_{\sigma \in \mathtt{Syn}_t}$ indexed by the syntax tree $\mathtt{Syn}_t$ satisfying the following condition:

(1) For a leaf $\sigma \in \mathtt{Syn}_t$, then $U_\sigma = Z$.
(2) If $\sigma \in \mathtt{Syn}_t$ is labeled by $\rightsquigarrow$, then U_σ is an open subset of Z, $u_{\sigma^\frown 0}$ is the identity map, and $u_{\sigma^\frown 1} \colon U_\sigma \to Z$ is a continuous function.
(3) If $\sigma \in \mathtt{Syn}_t$ is labeled by $\sqcup$, then U_σ is a sequence $(U_{\sigma,n})_{n\in\omega}$ of open subsets of Z, and $u_{\sigma^\frown n} \colon U_{\sigma,n} \to Z$ is a continuous function for each $n \in \omega$.
(4) If $\sigma \in \mathtt{Syn}_t$ is labeled by ϕ_α, then $U_\sigma = Z$, and $u_{\sigma^\frown 0} \colon Z \to Z$ is a $\underset{\sim}{\mathbf{\Sigma}}^0_{1+\omega^\alpha}$-measurable function.

Note that Definition 2.6 (2) requires an additional condition: We say that a command $\mathbf{U}$ is *strongly total* if $(U_{\sigma,n})_{n\in\omega}$ is an open cover of Z whenever σ is labeled by $\sqcup$.

A command may be thought of as a control flow that defines a function: Feed a value into an input variable $\mathtt{x}$. Then follow the command from the root to a leaf, where each U_σ represents a condition branch depending on $\mathtt{x} \in U_\sigma$ in the case (2) and on $\mathtt{x} \in U_{\sigma,n}$ in the case (3), and $u_{\sigma^\frown n}$ represents a reassignment $\mathtt{x} \leftarrow u_{\sigma^\frown n}(\mathtt{x})$. When we reach a leaf, output the label of the leaf, where note that any leaf must be labeled by a constant symbol $q \in Q$.

To give a formal definition of the above argument, for a node $\sigma \in \mathtt{Syn}_t$ of length $\ell > 0$, put

$$\mathtt{val}_\sigma = u_\sigma \circ u_{\sigma\restriction \ell-1} \circ \cdots \circ u_{\sigma\restriction 2} \circ u_{\sigma\restriction 1} \colon\subseteq Z \to Z,$$

and $\mathtt{val}_\varepsilon = \mathrm{id}$, where ε is the empty string. In other words, if x is the first value fed into variable $\mathtt{x}$, then $\mathtt{val}_\sigma(x)$ is the value stored in variable $\mathtt{x}$ when we reach σ. Next, we introduce the notion of a *true position* for $x \in Z$ (with respect to the command $\mathbf{U}$) in the following inductive manner, where we consider $\mathtt{Syn}_t$ as a labeled subtree of $\omega^{<\omega}$:

- The root of the syntax tree $\mathtt{Syn}_t$ is a true position for x.
- Assume that $\sigma \in \mathtt{Syn}_t$ is a true position for x, and is labeled by $\rightsquigarrow$. If $\mathtt{val}_\sigma(x) \in U_\sigma$ then $\sigma^\frown 1$ is a true position for x, and if $\mathtt{val}_\sigma(x) \notin U_\sigma$ then $\sigma^\frown 0$ is a true position for x.
- Assume that $\sigma \in \mathtt{Syn}_t$ is a true position for x, and is labeled by $\sqcup$. If $\mathtt{val}_\sigma(x) \in U_{\sigma,n}$ then $\sigma^\frown n$ is a true position for x.
- Assume that $\sigma \in \mathtt{Syn}_t$ is a true position for x, and is labeled by ϕ_α. Then, the unique immediate successor $\sigma^\frown 0$ of σ is a true position for x.

If a leaf $\rho \in \mathtt{Syn}_t$ is a true position for x, then we call it a *true path for x* (with respect to $\mathbf{U}$). A label of a true path for x is considered as an output of the function determined by the command $\mathbf{U}$.

Notation. We write $[\![\mathbf{U}]\!](x) = q$ if q is a label of a true path for x with respect to $\mathbf{U}$.

However, it may happen that there are many true paths for x or there is no true path for x. If, for any $x \in Z$, there is a true path for x, that is, $[\![\mathbf{U}]\!](x)$ is defined, then we call $\mathbf{U}$ *total*. It is easy to see that a strongly total command is total. If, for any $x \in Z$, there are at most one value q such that $[\![\mathbf{U}]\!](x) = q$, then $\mathbf{U}$ is called a *deterministic command*. Then, the function $[\![\mathbf{U}]\!] \colon Z \to Q$ is called the *evaluation* of the command $\mathbf{U}$.

Observation 2.11. *Let Z be a zero-dimensional Polish space. For any $\mathcal{L}_{\rm Veb}(Q)$-term t, the class $\underset{\sim}{\mathbf{\Sigma}}_t^W(Z)$ is the set of functions obtained as the evaluations $[\![\mathbf{U}]\!]$ of deterministic strongly total commands $\mathbf{U}$ on the term t over the space Z.*

One may consider $[\![\mathbf{U}]\!]$ even if $\mathbf{U}$ is non-deterministic, and in such a case, $[\![\mathbf{U}]\!]$ is multi-valued. Next, we say that a command $\mathbf{U}$ is *simple* if $u_{\sigma^\frown i}$ is the identity map for each i whenever σ is labeled by either $\sqcup$ or $\rightsquigarrow$. Later we will see that the deterministic total simple commands also yield the same class. This notion induces the third inductive definition of $\underset{\sim}{\mathbf{\Sigma}}_t$:

Definition 2.12 (Third definition of $\underset{\sim}{\mathbf{\Sigma}}_t$)**.** For each $\mathcal{L}_{\rm Veb}(Q)$-term t, we inductively define $\underset{\sim}{\mathbf{\Sigma}}_t(Z)$ as the class fulfilling (1), (2″), (3″) and (4):

(2″) If $t = \sqcup_i s_i$, then $f \in \underset{\sim}{\mathbf{\Sigma}}_t(Z)$ if and only if there is an open cover $(U_i)_{i\in\omega}$ of Z such that $f \upharpoonright U_i \in \underset{\sim}{\mathbf{\Sigma}}_{s_i}(U_i)$ for each $i \in \omega$.

(3″) If $t = s \rightsquigarrow u$, then $f \in \underset{\sim}{\mathbf{\Sigma}}_t(Z)$ if and only if there is an open set $V \subseteq X$ such that $f \upharpoonright V \in \underset{\sim}{\mathbf{\Sigma}}_u(V)$ and $f \upharpoonright (Z \setminus V) \in \underset{\sim}{\mathbf{\Sigma}}_s(Z \setminus V)$.

To avoid confusion, let us write $\underset{\sim}{\mathbf{\Sigma}}'_t$ for $\underset{\sim}{\mathbf{\Sigma}}_t$ in the sense of Definition 2.12.

Observation 2.13. *Let Z be a zero-dimensional Polish space. For any $\mathcal{L}_{\rm Veb}(Q)$-term t, the class $\underset{\sim}{\mathbf{\Sigma}}'_t(Z)$ is the set of functions obtained as the evaluations $[\![\mathbf{U}]\!]$ of deterministic total simple commands $\mathbf{U}$ on the term t over the space Z.*

This third definition seems to be the most natural definition of $\underset{\sim}{\mathbf{\Sigma}}_t$ among the ones given so far. It should be noted, however, that this third definition has a very different feature from the first two: In the process of defining a function, a partial function with a complicated domain may appear. Because of this feature, proving the equivalence of first and second definitions and the third definition is not an easy task (see Theorem 3.6), and for this reason, the relevance of the third definition to the Wadge theory is not immediately obvious.

Borel rank: We define the *Borel rank* of a node $\sigma \in \mathtt{Syn}_t$ as follows. First, enumerate all proper initial segments of σ which are labeled by Veblen function symbols ϕ_α:

$$\tau_0 \prec \tau_1 \prec \tau_2 \prec \cdots \prec \tau_\ell \prec \sigma,$$

where τ_i is labeled by ϕ_{α_i}. We call $(\tau_i)_{i\leq\ell}$ the *Veblen initial segments of* σ. Then, the Borel rank of $\sigma \in \mathtt{Syn}_t$ is defined as the following ordinal:

$$\mathrm{rank}(\sigma) = 1 + \omega^{\alpha_0} + \omega^{\alpha_1} + \cdots + \omega^{\alpha_\ell}.$$

Observation 2.14. *The function $\mathtt{val}_\sigma \colon\subseteq Z \to Z$ is $\underset{\sim}{\mathbf{\Sigma}}^0_{\mathrm{rank}(\sigma)}$-measurable. Moreover, the domain of $\mathtt{val}_\sigma \colon\subseteq Z \to Z$ is $\underset{\sim}{\mathbf{\Sigma}}^0_{\mathrm{rank}(\sigma)}$.*

Proof. The first assertion is clear. It is easy to show the second assertion by induction since the domain of u_σ is open for each $\sigma \in \mathtt{Syn}_t$. □

Coding a command: Using the standard codings of the open sets and the $\underset{\sim}{\mathbf{\Sigma}}^0_{1+\omega^\alpha}$-measurable functions, one can define the notion of a code of a command $\mathbf{U}$ on an $\mathcal{L}_{\rm Veb}(Q)$-term t in the straightforward manner. More precisely, a *code* of a command $\mathbf{U} = (U_\sigma, u_\sigma)_{\sigma\in\mathtt{Syn}_t}$ is a pair of a code of the syntax tree $\mathtt{Syn}_t = (\mathtt{S}_t, \mathtt{l}_t)$ and an $\mathtt{S}_t$-indexed collection $(c_\sigma)_{\sigma\in\mathtt{S}_t}$ of elements of ω^ω satisfying the following conditions:

(1) If σ is a leaf, then c_σ is arbitrary.
(2) If σ is labeled by $\rightsquigarrow$, then c_σ is a triple of $\underset{\sim}{\mathbf{\Sigma}}^0_1$-codes of $(U_\sigma, u_{\sigma,0}, u_{\sigma,1})$.
(3) If σ is labeled by $\sqcup$, then c_σ is a sequence of $\underset{\sim}{\mathbf{\Sigma}}^0_1$-codes of elements of $(U_{\sigma,n}, u_{\sigma^\frown n})_{n\in\omega}$.
(4) If σ is labeled by ϕ_α, then c_σ is a $\underset{\sim}{\mathbf{\Sigma}}^0_{1+\omega^\alpha}$-code of $u_{\sigma^\frown 0}$.

As before, $(c_\sigma)_{\sigma\in\mathtt{S}_t}$ can be considered as an element of ω^ω.

3. Reassignment-elimination

3.1. **Flowchart.** The working mechanism of a command is a bit unclear, since it involves reassignment instructions $\mathtt{x} \leftarrow u(\mathtt{x})$ by $\underset{\sim}{\mathbf{\Sigma}}^0_\alpha$-measurable functions u. Fortunately, however, it is possible to eliminate the reassignment instructions: Let us introduce the notion of a flowchart, which is a mechanism for defining functions using only conditional branching. This notion is particularly useful when dealing with higher-dimensional spaces. A similar concept has been studied, e.g. in Kihara [18] and Selivanov [34].

Definition 3.1. A *flowchart* on an $\mathcal{L}_{\mathrm{Veb}}(Q)$-term t over a topological space X is a family $\mathbf{S} = (S_\sigma)_{\sigma\in\mathtt{Syn}_t}$ indexed by the syntax tree $\mathtt{Syn}_t$ satisfying the following conditions:

(1) For a leaf $\sigma \in \mathtt{Syn}_t$, then $S_\sigma = X$.
(2) If $\sigma \in \mathtt{Syn}_t$ is labeled by $\rightsquigarrow$, then S_σ is a $\underset{\sim}{\mathbf{\Sigma}}^0_{\mathrm{rank}(\sigma)}$ subset of X.
(3) If $\sigma \in \mathtt{Syn}_t$ is labeled by $\sqcup$, then S_σ is a sequence $(S_{\sigma,n})_{n\in\omega}$ of $\underset{\sim}{\mathbf{\Sigma}}^0_{\mathrm{rank}(\sigma)}$ subsets of X.
(4) If $\sigma \in \mathtt{Syn}_t$ is labeled by ϕ_α, then $S_\sigma = X$.

We introduce the notion of a *true position* for $x \in X$ (with respect to a flowchart $\mathbf{S}$) in the following inductive manner:

- The root of the syntax tree $\mathtt{Syn}_t$ is a true position for x.
- Assume that $\sigma \in \mathtt{Syn}_t$ is a true position for x, and is labeled by $\rightsquigarrow$. Then, if $x \in S_\sigma$ then $\sigma^\frown 1$ is a true position for x, and if $x \notin S_\sigma$ then $\sigma^\frown 0$ is a true position for x.
- Assume that $\sigma \in \mathtt{Syn}_t$ is a true position for x, and is labeled by $\sqcup$. If $x \in S_{\sigma,n}$ then $\sigma^\frown n$ is a true position for x.
- Assume that $\sigma \in \mathtt{Syn}_t$ is a true position for x, and is labeled by ϕ_α. Then, the unique immediate successor $\sigma^\frown 0$ of σ is a true position for x.

If a leaf $\rho \in \mathtt{Syn}_t$ is a true position for x, then we call it a *true path for x* (with respect to $\mathbf{S}$). A label of a true path for x is considered as an output of the function determined by the flowchart $\mathbf{S}$.

Notation. We write $[\![\mathbf{S}]\!](x) = q$ if q is a label of a true path for x with respect to $\mathbf{S}$.

As before, it may happen that there are many true paths for x or there is no true path for x. If, for any $x \in X$, there is a true path for x, that is, $[\![\mathbf{S}]\!](x)$ is defined, then we call $\mathbf{S}$ *total.* If, for any $x \in X$, there are at most one value q such that $[\![\mathbf{S}]\!](x) = q$, then $\mathbf{S}$ is called a *deterministic flowchart.* Then, the function $[\![\mathbf{S}]\!]\colon X \to Q$ is called the *evaluation* of the flowchart $\mathbf{S}$.

Definition 3.2 (Fourth definition of $\underset{\sim}{\mathbf{\Sigma}}_t$)**.** Let X be a topological space. For any $\mathcal{L}_{\mathrm{Veb}}(Q)$-term t, define $\underset{\sim}{\mathbf{\Sigma}}_t(X)$ as the set of functions obtained as the evaluations $[\![\mathbf{S}]\!]$ of deterministic total flowcharts $\mathbf{S}$ on the term t over the space X.

As mentioned above, this notion is easy to handle even in higher-dimensional spaces, and therefore, in this article, we declare this to be the correct definition of $\undertilde{\mathbf{\Sigma}}_t$. As before, one may consider $[\![\mathbf{S}]\!]$ even if $\mathbf{S}$ is non-deterministic, and in such a case, $[\![\mathbf{S}]\!]$ is multi-valued. In this case, we also say that a multi-valued function g is $\undertilde{\mathbf{\Sigma}}_t$ if g coincides with $[\![\mathbf{S}]\!]$ for a flowchart $\mathbf{S}$ on t.

Coding a flowchart: If X is second-countable, using the standard codings of the $\undertilde{\mathbf{\Sigma}}^0_\xi$ sets, one can define the notion of a code of a flowchart $\mathbf{S}$ on an $\mathcal{L}_{\rm Veb}(Q)$-term t in the straightforward manner. More precisely, a *code* of a flowchart $\mathbf{S} = (S_\sigma)_{\sigma \in \mathtt{Syn}_t}$ is a pair of a code of the syntax tree $\mathtt{Syn}_t = (\mathtt{S}_t, \mathtt{l}_t)$ and an $\mathtt{S}_t$-indexed collection $(c_\sigma)_{\sigma \in \mathtt{S}_t}$ of elements of ω^ω satisfying the following conditions:

(1) If σ is a leaf, then c_σ is arbitrary.
(2) If σ is labeled by $\rightsquigarrow$, then c_σ is a $\undertilde{\mathbf{\Sigma}}^0_{\mathrm{rank}(\sigma)}$-code of S_σ.
(3) If σ is labeled by $\sqcup$, then c_σ is a sequence of $\undertilde{\mathbf{\Sigma}}^0_{\mathrm{rank}(\sigma)}$-codes of elements of $(S_{\sigma,n})_{n\in\omega}$.
(4) If σ is labeled by ϕ_α, then c_σ is arbitrary.

As before, $(c_\sigma)_{\sigma \in \mathtt{S}_t}$ can be considered as an element of ω^ω. Since there is a total representation of $\undertilde{\mathbf{\Sigma}}^0_\xi$ sets for each $\xi < \omega_1$, any sequence $(c_\sigma)_{\sigma \in \mathtt{S}_t}$ can be considered as a code of a flowchart. If t is a hyperarithmetical $\mathcal{L}_{\rm Veb}(Q)$-term, we say that a flowchart is Δ^1_1 if it has a Δ^1_1 code. Define $\Sigma_t(\Delta^1_1; X)$ as the set of all functions $g\colon X \to Q$ determined by a Δ^1_1 flowchart. If the underlying space is clear from the context, we simply write $\Sigma_t(\Delta^1_1)$ instead of $\Sigma_t(\Delta^1_1; X)$. We will discuss the complexity of the codes of total and deterministic flowcharts in Lemma 3.3.

3.2. **Technical lemmata.** We introduce here some useful concepts on flowcharts that we will use later, but not immediately. First, one can define the notion of a true position by assigning a set $D_\sigma \subseteq X$ to each $\sigma \in \mathtt{Syn}_t$.

(1) For the root $\langle\rangle$ of $\mathtt{Syn}_t$, define $D_{\langle\rangle} = X$.
(2) If σ is labeled by $\rightsquigarrow$, then define $D_{\sigma^\frown 0} = D_\sigma \setminus S_\sigma$ and $D_{\sigma^\frown 1} = D_\sigma \cap S_\sigma$.
(3) If σ is labeled by $\sqcup$, then define $D_{\sigma^\frown n} = D_\sigma \cap S_{\sigma,n}$.
(4) If σ is labeled by ϕ_α, then define $D_{\sigma^\frown 0} = D_\sigma$.

We call $(D_\sigma)_{\sigma \in \mathtt{Syn}_t}$ the *domain assignment* to $\mathbf{S} = (S_\sigma)_{\sigma \in \mathtt{Syn}_t}$. It is easy to see that $\sigma \in \mathtt{Syn}_t$ is a true position for x with respect to $\mathbf{S}$ if and only if $x \in D_\sigma$. Hence, $\mathbf{S}$ is total if and only if $(S_{\sigma,n})_{n\in\omega}$ covers D_σ whenever σ is labeled by $\sqcup$. Using this notion, let us show a few complexity results on flowcharts.

Lemma 3.3. *Let X be a computable Polish space, and t be a hyperarithmetical $\mathcal{L}_{\rm Veb}(Q)$-term.*

(1) *Every partial $\Sigma_t(\Delta^1_1)$-function $g\colon\subseteq X \to Q$ can be extended to a $\Sigma_t(\Delta^1_1)$-function $\widehat{g}\colon\subseteq X \to Q$ whose domain is Δ^1_1.*
(2) *If $g\colon\subseteq X \to Q$ is $\Sigma_t(\Delta^1_1)$, then there exists a Δ^1_1-measurable function $G\colon\subseteq X \to \omega^\omega$ such that $g(x) = q_{G(x)}$ for any $x \in \mathrm{dom}(g)$, where Q is indexed as $\{q_z\}_{z\in\omega^\omega}$.*
(3) *The set of all codes of deterministic flowcharts on t over X is Π^1_1.*
(4) *If A is a Σ^1_1 subset of X, then the set of all codes of flowcharts $\mathbf{S}$ on t over X such that the domain of $[\![\mathbf{S}]\!]$ includes A is Π^1_1. In particular, the set of all codes of total flowcharts on t over X is Π^1_1.*

Proof. (1) Given a code c of a flowchart $\mathbf{S}$, one can easily see that the domain assignment $(D_\sigma)_{\sigma\in\mathtt{Syn}_t}$ for $\mathbf{S}$ is uniformly Δ^1_1 relative to the code c. Moreover, a node $\sigma\in\mathtt{Syn}_t$ is a true position for x if and only if $x\in\bigcap_{\tau\preceq\sigma}D_\tau$. This is a Δ^1_1 property relative to c. Now, let $\mathbf{S}$ be a Δ^1_1-coded flowchart over $\mathrm{dom}(g)\subseteq X$ determining g. Note that $\mathbf{S}$ can also be thought of as a flowchart over X. Then the set D of all $x\in X$ such that there exists a unique true path for x is Δ^1_1 in X since $\mathbf{S}$ has a Δ^1_1 code. Now it is easy to observe that $\mathbf{S}$ determines a $\Sigma_t(\Delta^1_1)$-function $\widehat{g}\colon\subseteq D\to Q$ extending g.

(2) Recall that a code of a syntax tree contains the information about what each node of the tree is labeled. In particular, one can effectively recover information about the label of σ. Let $G(x)$ return a code of the label of a true path for x, and then G is clearly Δ^1_1-measurable relative to c.

(3) By definition, c is a code of a deterministic flowchart if and only if for any $x\in X$ and any $\sigma,\tau\in\mathtt{Syn}_t$, if both σ and τ are true paths for x w.r.t. the flowchart coded by c, then both σ and τ are labeled by the same symbol. This is a Π^1_1 property.

(4) One can see that the domain of $[\![\mathbf{S}]\!]$ includes A if and only if, whenever σ is labeled by $\sqcup$, the sequence $(S_{\sigma,n})_{n\in\omega}$ covers $A\cap D_\sigma$, i.e., for any $x\in X$, if $x\in A\cap D_\sigma$ then $x\in S_{\sigma,n}$ for some $n\in\omega$. Since A is Σ^1_1, and D_σ and $(S_{\sigma,n})_{n\in\omega}$ are Δ^1_1 relative to a given code, this is a Π^1_1 property. □

We say that a flowchart $\mathbf{S}$ is *monotone* if any set assigned to $\sigma\in\mathtt{Syn}_t$ by $\mathbf{S}$ is a subset of D_σ. In other words, if σ is labeled by $\rightsquigarrow$, then $D_{\sigma^\frown 1}=S_\sigma$; and if σ is labeled by $\sqcup$, then $D_{\sigma^\frown n}=S_{\sigma,n}$.

Lemma 3.4 (see also Selivanov [34])**.** *Given a flowchart $\mathbf{S}$ on a normal $\mathcal{L}_{\rm Veb}(Q)$-term t one can effectively find a monotone flowchart $\mathbf{S}'$ on t determining the same function as $\mathbf{S}$.*

Proof. Let $(D_\sigma)_{\sigma\in\mathtt{Syn}_t}$ be the domain assignment for $\mathbf{S}$. We inductively show that, if $\sigma\in\mathtt{Syn}_t$ is not a leaf, then D_σ is a $\undertilde{\mathbf{\Sigma}}_{\mathrm{rank}(\sigma)}$ subset of X. If σ is labeled by $\sqcup$, then $D_{\sigma^\frown n}=D_\sigma\cap S_{\sigma,n}$ is a $\undertilde{\mathbf{\Sigma}}_{\mathrm{rank}(\sigma)}$ subset of X since both D_σ and $S_{\sigma,n}$ are $\undertilde{\mathbf{\Sigma}}_{\mathrm{rank}(\sigma)}$ by induction hypothesis. Since $\mathrm{rank}(\sigma)\leq\mathrm{rank}(\sigma^\frown n)$, we get $D_{\sigma^\frown n}\in\undertilde{\mathbf{\Sigma}}_{\mathrm{rank}(\sigma^\frown n)}$. If σ is labeled by $\rightsquigarrow$, then $D_{\sigma^\frown 1}=D_\sigma\cap S_\sigma$ is $\undertilde{\mathbf{\Sigma}}_{\mathrm{rank}(\sigma^\frown 1)}$ as above. One can also see that $D_{\sigma^\frown 0}=D_\sigma\setminus S_\sigma$ is a $\undertilde{\mathbf{\Sigma}}_{\mathrm{rank}(\sigma)+1}$ subset of X. By normality of t, $\sigma^\frown 0$ is either a leaf or labeled by ϕ_α. If $\sigma^\frown 0$ is a leaf, there is nothing to do. If $\sigma^\frown 0$ is labeled by ϕ_α, we have $\mathrm{rank}(\sigma)+1\leq\mathrm{rank}(\sigma^\frown 0)$, and therefore, we get $D_{\sigma^\frown 0}\in\undertilde{\mathbf{\Sigma}}_{\mathrm{rank}(\sigma^\frown 0)}$.

If σ is labeled by $\rightsquigarrow$, define $S'_\sigma=D_\sigma\cap S_\sigma$, which is $\undertilde{\mathbf{\Sigma}}^0_{\mathrm{rank}(\sigma)}$; and if σ is labeled by $\sqcup$, define $S'_{\sigma,n}=D_\sigma\cap S_{\sigma,n}$, which is $\undertilde{\mathbf{\Sigma}}^0_{\mathrm{rank}(\sigma)}$. Hence, $(\mathbf{S}'_\sigma)_{\sigma\in\mathtt{Syn}_t}$ gives a monotone flowchart on t, which clearly satisfies $[\![\mathbf{S}]\!]=[\![\mathbf{S}']\!]$. □

We next see that, if the space X is zero-dimensional, then for any node $\sigma\in\mathtt{Syn}_t$ labeled by $\sqcup$, the assigned sequence $(U_{\sigma,n})_{n\in\omega}$ can be pairwise disjoint. Note that this property is required in the original definition of $\undertilde{\mathbf{\Sigma}}_t$ (Definition 2.8). Such a flowchart may be called a reduced flowchart, following the terminology in [33]. This property is particularly useful when combining more than one deterministic flowchart to make a new deterministic flowchart; see also the proof of Theorem 5.4.

Proposition 3.5. *Let X be a subspace of a zero-dimensional Polish space, and let $\mathbf{S}$ be a flowchart on an $\mathcal{L}_{\rm Veb}(Q)$-term t over X. Then, in a hyperarithmetical manner, one* *can find a flowchart* $\mathbf{S}'$ *on* t *such that for any node* $\sigma \in \mathtt{Syn}_t$ *labeled by* $\sqcup$*, the assigned sequence* $(U_{\sigma,n})_{n\in\omega}$ *is pairwise disjoint, and* $\mathbf{S}'$ *determines the same function as* $\mathbf{S}$*.*

Proof. It suffices to show that, for any $\xi < \omega_1^{\rm CK}$, given a uniform Δ^1_1-sequence of $\underset{\sim}{\mathbf{\Sigma}}^0_\xi$ sets $(R_n)_{n\in\omega}$, one can effectively find a Δ^1_1-sequence $(R^*_n)_{n\in\omega}$ of pairwise disjoint $\underset{\sim}{\mathbf{\Sigma}}^0_\xi$ sets such that $R^*_n \subseteq R_n$ for each $n \in \omega$ and $\bigcup_{n\in\omega} R^*_n = \bigcup_{n\in\omega} R_n$. To see this, let R be the $\underset{\sim}{\mathbf{\Sigma}}^0_\xi(\Delta^1_1)$ set such that $(x, n) \in R$ if and only if $x \in R_n$. If P is a universal Σ^0_ξ set, there exists a Δ^1_1 element $\varepsilon \in \omega^\omega$ such that $(x, n) \in R$ if and only if $(\varepsilon, x, n) \in P$. By the uniformization property on $\Sigma^0_\xi(\varepsilon)$ (see [25, 3E.10]), we have some $R^* \in \Sigma^0_\xi(\varepsilon)$ which uniformizes R. Now, let R^*_n be the $\Sigma^0_\xi(\varepsilon)$ set such that $x \in R^*_n$ if and only if $(x, n) \in R^*$. Then, $(R^*_n)_{n\in\omega}$ has the required property. □

Remark. The flowchart definition of $\underset{\sim}{\mathbf{\Sigma}}_t$ is essentially the same as Selivanov's fine hierarchy [31, 34] over $\mathcal{L}_{\rm Veb}(Q)$-terms, although not at all obvious at first glance. More precisely, our $\underset{\sim}{\mathbf{\Sigma}}_t(X)$ (restricted to monotone flowcharts) corresponds to the t-family $\mathcal{L}(X, t)$ in the Borel bases in Selivanov [34, Definition 3.21]. One of the major differences between these definitions is that our flowchart definition is based on the syntax tree of a term, whereas Selivanov's definition is based rather on semantics of a term, i.e., induction on a nested tree. The latter semantical definition requires effort to understand its meaning because it is given by induction on a nested tree, with various notions intertwined. In comparison, our flowchart definition has the advantage of making the idea easy to understand even at first glance.

3.3. **Translation.** Now, for an $\mathcal{L}_{\rm Veb}(Q)$-term t, we will show that the four definitions of $\underset{\sim}{\mathbf{\Sigma}}_t$ introduced so far all define the same class of functions.

$$\underset{\sim}{\mathbf{\Sigma}}_t(\omega^\omega) = \underset{\sim}{\mathbf{\Sigma}}^W_t(\omega^\omega) = \underset{\sim}{\mathbf{\Sigma}}^{W\circ}_t(\omega^\omega) = \underset{\sim}{\mathbf{\Sigma}}'_t(\omega^\omega). \tag{3.1}$$

Note that the equivalence $\underset{\sim}{\mathbf{\Sigma}}_t(\omega^\omega) = \underset{\sim}{\mathbf{\Sigma}}'_t(\omega^\omega)$ has also been shown in [33, Theorem 4.10], although the terminology is slightly different. To prove the equivalence (3.1), it suffices to show the following:

Theorem 3.6. *Let* t *be an* $\mathcal{L}_{\rm Veb}(Q)$*-term, and assume that* $Z = \omega^\omega$*.*

(1) *Given a command* $\mathbf{U}$ *on* t *over* Z*, one can effectively find a flowchart* $\mathbf{S}$ *on* t *over* Z *such that* $[\![\mathbf{U}]\!] = [\![\mathbf{S}]\!]$*.*
(2) *Conversely, given a flowchart* $\mathbf{S}$ *on* t *over* Z*, there exists a simple command* $\mathbf{U}$ *on* t *over* Z *such that* $[\![\mathbf{S}]\!] = [\![\mathbf{U}]\!]$*. If* $\mathbf{S}$ *is total, then* $\mathbf{U}$ *is also total, and moreover there exists a strongly total command* $\mathbf{U}'$ *on* t *over* Z *such that* $[\![\mathbf{S}]\!] = [\![\mathbf{U}']\!]$*. Furthermore, if* $\mathbf{S}$ *has a* Δ^1_1*-code, so do* $\mathbf{U}$ *and* $\mathbf{U}'$*.*

Proof. (1) Given a command $\mathbf{U} = (U_\sigma, u_\sigma)_{\sigma\in\mathtt{Syn}_t}$, we construct a flowchart $\mathbf{S} = (S_\sigma)_{\sigma\in\mathtt{Syn}_t}$ as follows:

- For a leaf $\sigma \in \mathtt{Syn}_t$, then $S_\sigma = Z$.
- If $\sigma \in \mathtt{Syn}_t$ is labeled by $\rightsquigarrow$, then define $S_\sigma = \mathtt{val}_\sigma^{-1}[U_\sigma]$.
- If $\sigma \in \mathtt{Syn}_t$ is labeled by $\sqcup$, then define $S_{\sigma,n} = (\mathtt{val}_\sigma^{-1}[U_{\sigma,n}])_{n\in\omega}$.
- If $\sigma \in \mathtt{Syn}_t$ is labeled by ϕ_α, then $S_\sigma = Z$.

Then, S_σ and $S_{\sigma,n}$ are $\underset{\sim}{\mathbf{\Sigma}}^0_{\mathrm{rank}(\sigma)}$ subsets of Z since $\mathtt{val}_\sigma$ is a $\underset{\sim}{\mathbf{\Sigma}}^0_{\mathrm{rank}(\sigma)}$-measurable function with a $\underset{\sim}{\mathbf{\Sigma}}^0_{\mathrm{rank}(\sigma)}$ domain by Observation 2.14. Hence, $\mathbf{S}$ is a flowchart on the $\mathcal{L}_{\rm Veb}(Q)$-term t. Then $\mathbf{U}$ and $\mathbf{S}$ determine the same function, i.e., $[\![\mathbf{U}]\!] = [\![\mathbf{S}]\!]$. This is because, by definition, if σ is labeled by $\rightsquigarrow$, then $x \in S_\sigma$ if and only if $\mathtt{val}_\sigma(x) \in U_\sigma$; and if σ is labeled by $\sqcup$, then $x \in S_{\sigma,n}$ if and only if $\mathtt{val}_\sigma(x) \in U_{\sigma,n}$. By induction, this trivially ensures that σ is a true position for x with respect to $\mathbf{S}$ if and only if σ is a true position for x with respect to $\mathbf{U}$. In particular, $\sigma \in \mathtt{Syn}_t$ is a true path for x with respect to $\mathbf{S}$ if and only if σ is a true path for x with respect to $\mathbf{U}$. This means that $[\![\mathbf{S}]\!](x) = q$ if and only if $[\![\mathbf{U}]\!](x) = q$.

(2) First, we fix a sufficiently strong oracle δ, relative to which all the Borel sets attached to the given flowchart $\mathbf{S}$ are lightface. For any $\mathcal{L}_{\mathrm{Veb}}(Q)$-term t, its syntax tree is countable, and therefore, only countably many Borel sets are attached; hence such an oracle δ exists. Then, for any δ-computable ordinal α, let us consider the ω^α-th Turing jump operator $j_\alpha := \mathcal{J}^{\omega^\alpha,\delta} \colon Z \to Z$ with true stages relative to δ; see Kihara-Montalbán [19, Sections 4.1 and 6.1]. For the sake of brevity, hereafter, δ will be omitted from the notation. One of the key properties of the jump operator j_α with true stages is that its image $j_\alpha[Z]$ is closed. Moreover, a single index ensures the inequality $x \leq_T j_\alpha(x)$ for any x, and therefore, j_α has a computable left-inverse $j_\alpha^{-1} \colon j_\alpha[Z] \to Z$. By combining these properties, observe that, if $A \subseteq Z$ is closed, so is $j_\alpha[A]$.

By expressing an ordinal in Cantor normal form, any transfinite Turing jump can be represented as the composition $j_{\alpha_\ell} \circ j_{\alpha_{\ell-1}} \circ \cdots \circ j_{\alpha_1} \circ j_{\alpha_0}$; see [19, Section 6.1]. In this paper, we denote the α-th Turing jump by j^α. The $\utilde{\mathbf{\Sigma}}^0_{1+\alpha}$-measurability of j^α is trivial by definition, but we failed to find the exact reference, so here is the proof:

Observation 3.7 (Folklore)**.** *j^α is $\utilde{\mathbf{\Sigma}}^0_{1+\alpha}$-measurable.*

Proof. Recall that a transfinite Turing jump j^α is defined along an ordinal α equipped with a fundamental sequence $\{\alpha[n]\}_{n\in\omega}$ (see [19, Section 6.1]). We prove the assertion by induction. The $\utilde{\mathbf{\Sigma}}^0_2$-measurability of the Turing jump is trivial. If $\alpha = \beta + 1$, then $j^{\beta+1} = j^1 \circ j^\beta$ is the composition of a $\utilde{\mathbf{\Sigma}}^0_2$-measurable function and a $\utilde{\mathbf{\Sigma}}^0_{1+\beta}$-measurable function by the induction hypothesis, so $j^{\beta+1}$ is $\utilde{\mathbf{\Sigma}}^0_{1+\beta+1}$-measurable. If α is limit, $j^\alpha(x)$ is computable in $\langle j^{\alpha[n]}(x)\rangle_{n\in\omega}$, so consider $j^\alpha_*(x)(i,j) = j^{\alpha[i]}(x)(j)$. Clearly, j^α_* is $\utilde{\mathbf{\Sigma}}^0_{1+\alpha}$-measurable since $j^{\alpha[i]}$ is $\utilde{\mathbf{\Sigma}}^0_{1+\alpha[i]}$-measurable by the induction hypothesis. Thus, j^α is $\utilde{\mathbf{\Sigma}}^0_{1+\alpha}$-measurable as it is given by the composition of a computable function and j^α_*. □

By replacing δ with a more powerful oracle if necessary, by a straightforward trick, one can assume the following assertion:

Claim 1 (Universality from the right)**.** *Let S be any Borel set that appears in the flowchart $\mathbf{S}$. If $S \subseteq \omega^\omega$ is Σ^0_α then one can effectively find an open set $U \subseteq \omega^\omega$ such that $S = (j^\alpha)^{-1}[U]$.*

Proof. To prove this, we need to go a little further into the definition of transfinite Turing jumps. As in the proof of Observation 3.7, assume that an ordinal α is equipped with a fundamental sequence, and each $\alpha[n]$ is also an ordinal equipped with a fundamental sequence. Such a system can be thought of as a well-founded tree T_α whose rank is α. Here, for the rank of the tree, see for example [16, 25]. A limit ordinal β corresponds to an infinite branching node, and the n-th immediate successor corresponds to $\beta[n]$. When $\sigma \in T_\alpha$ is an infinite branching node, the rank of $\sigma^\frown m$ is greater than the rank of $\sigma^\frown n$ if $n < m$. If $\sigma \in T_\alpha$ corresponds to a successor ordinal, then it has only one immediate successor $\sigma^\frown 0$. Note that, in classical computability theory, such a tree is represented as a subset of the natural numbers, called Kleene's $\mathcal{O}$; see [27].

A Borel code c is a blueprint of how a Borel set is constructed from below, which also forms a well-founded tree T_c. The problem is that even if a Borel set S is Σ^0_α, the shape of the tree T_c of its Borel code and the tree T_α of the ordinal α may not match well. However, this can be solved by a *padding* process that inserts a sufficiently large number of empty sets inside the countable union operation. To be precise, by induction on the tree, a construction of a Borel set S is modified as follows: If S is Σ^0_α, then its code yields a sequence $(P_n)_{n\in\omega}$ such that $S = \bigcup_n P_n$ where P_n is $\Pi^0_{\beta_n}$ and $\beta_n < \alpha$. For each n, choose $m_n > m_{n-1}$ such that $\beta_n < \alpha[m_n]$ and define $P'_{m_n} = P_n$. If $k \neq m_n$ for any n, put $P'_k = \emptyset$. Obviously, $S = \bigcup_n P'_n$, and $P_n \in \Pi^0_{\alpha[n]}$. Since the syntax tree is countable (so only countably many Borel sets can be attached), this padding process can be performed with a strong enough oracle δ. In fact, such a δ can be Δ^1_1 relative to a given Borel code because both the calculation of the Borel rank of a node $\sigma \in T_c$ and the comparison of ordinals are hyperarithmetic (see e.g. [25, 27]). Of course, in order to show the claim, it is necessary to address all the Borel sets that appear in the flowchart $\mathbf{S}$. However, codes of the Borel sets appearing in the flowchart $\mathbf{S}$ can be uniformly recovered from a code c of $\mathbf{S}$, so δ can be chosen as Δ^1_1 relative to such c.

Hereafter, we assume that the above padding modification has already been performed. For any Q appearing in the construction of S, we show by induction that one can effectively find a Σ^0_1 set V such that $Q = (j^\beta)^{-1}[V]$ when Q is calculated to be Σ^0_β from the shape of the Borel code tree. Here, our oracle δ allows us to effectively compute such β. If $Q = \bigcup_n P_n$, by the induction hypothesis and the padding assumption, one can effectively find a Σ^0_1 set U_n such that $P_n = \omega^\omega \setminus (j^{\beta[n]})^{-1}[U_n]$. Determining whether $j^{\beta[n]}(x) \in U_n$ or not is computable relative to $j^{\beta[n]+1}(x)$, and therefore $j^\beta(x)$ can be used to determine it uniformly in n. This decision process is written as φ; that is, one can effectively find a computable function φ such that $\varphi(n, j^\beta(x)) = 0$ if and only if $j^{\beta[n]}(x) \notin U_n$, i.e., $x \in P_n$. Considering the Σ^0_1 set $V = \{y : \exists n\ \varphi(n, y) = 0\}$, it is easy to check that $Q = (j^\beta)^{-1}[V]$. Since our construction from $(U_n)_{n\in\omega}$ to V is effective, the usual effective transfinite recursion (see e.g. [27]) verifies the claim. □

Step 1. We first transform a flowchart $\mathbf{S} = (S_\sigma)_{\sigma\in\mathtt{Syn}_t}$ into a command $\mathbf{U}$. First we define $(u_\sigma)_{\sigma\in\mathtt{Syn}_t}$ as follows: If $\sigma \in \mathtt{Syn}_t$ is labeled by $\rightsquigarrow$ or $\sqcup$ then $u_{\sigma^\frown n} = \mathrm{id}$ for each n; and if σ is labeled by ϕ_α then $u_{\sigma^\frown 0} = j_\alpha$, which is $\underset{\sim}{\mathbf{\Sigma}}^0_{1+\alpha}$-measurable by Observation 3.7. Then we define a partial function $\mathtt{val}_\sigma$ as before. Let $(\tau_i)_{i\leq\ell}$ be the Veblen initial segments of a node $\sigma \in \mathtt{Syn}_t$, where τ_i is labeled by ϕ_{α_i}. Then, note that

$$\mathtt{val}_\sigma(x) = j_{\alpha_\ell} \circ j_{\alpha_{\ell-1}} \circ \cdots \circ j_{\alpha_1} \circ j_{\alpha_0}(x).$$

Next, we define $(U_\sigma)_{\sigma\in\mathtt{Syn}_t}$ as follows: If σ is labeled by $\rightsquigarrow$, since S_σ is a $\underset{\sim}{\mathbf{\Sigma}}^0_{\mathrm{rank}(\sigma)}$ set, by Claim 1, one can find an open set U_σ such that

$$S_\sigma = (j_{\alpha_\ell} \circ j_{\alpha_{\ell-1}} \circ \cdots \circ j_{\alpha_1} \circ j_{\alpha_0})^{-1}[U_\sigma].$$

Similarly, if σ is labeled by $\sqcup$, for any n, one can find an open set $U_{\sigma,n}$ such that $S_{\sigma,n} = (j_{\alpha_\ell} \circ j_{\alpha_{\ell-1}} \circ \cdots \circ j_{\alpha_1} \circ j_{\alpha_0})^{-1}[U_{\sigma,n}]$. If σ is labeled by ϕ_α, then put $U_\sigma = Z$. Then, $\mathbf{U} = (U_\sigma, u_\sigma)_{\sigma\in\mathtt{Syn}_t}$ gives a simple command on the $\mathcal{L}_{\mathrm{Veb}}(Q)$-term t.

Claim 2. $\mathbf{S}$ *and* $\mathbf{U}$ *determine the same function, i.e.,* $[\![\mathbf{S}]\!] = [\![\mathbf{U}]\!]$.

Proof. By definition, if σ is labeled by $\rightsquigarrow$, then $x \in S_\sigma$ if and only if $\mathtt{val}_\sigma(x) \in U_\sigma$; and if σ is labeled by $\sqcup$, then $x \in S_{\sigma,n}$ if and only if $\mathtt{val}_\sigma(x) \in U_{\sigma,n}$. Therefore, as in the proof of the item (1), one can easily see that $[\![\mathbf{S}]\!](x) = q$ if and only if $[\![\mathbf{U}]\!](x) = q$. □

Claim 3. *If* $\mathbf{S}$ *is total, so is* $\mathbf{U}$.

Proof. Assume that σ is a true position for x with respect to $\mathbf{S}$, and σ is labeled by $\sqcup$. By the proof of Claim 2, this implies that σ is also a true position for x with respect to $\mathbf{U}$. Since $\mathbf{S}$ is total, there exists $n \in \omega$ such that $\sigma^\frown n$ is a true position for x with respect to $\mathbf{S}$. This means that $x \in S_{\sigma,n}$ for some $n \in \omega$. By definition, $x \in S_{\sigma,n}$ if and only if $\mathtt{val}_\sigma(x) \in U_{\sigma,n}$. Thus, $\sigma^\frown n$ is a true position for x with respect to $\mathbf{U}$. By induction, this shows that $\mathbf{U}$ is total. □

Step 2. Next we transform the above total command $\mathbf{U}$ into a strongly total command $\mathbf{U}'$. We define the domain assignment of the command $\mathbf{U} = (U_\sigma)_{\sigma \in \mathtt{Syn}_t}$ as follows:

(1) For the root $\langle\rangle$ of $\mathtt{Syn}_t$, define $D_{\langle\rangle} = Z$.
(2) If σ is labeled by $\rightsquigarrow$, then define $D_{\sigma^\frown 0} = D_\sigma \setminus U_\sigma$ and $D_{\sigma^\frown 1} = D_\sigma \cap U_\sigma$.
(3) If σ is labeled by $\sqcup$, then define $D_{\sigma^\frown n} = D_\sigma \cap U_{\sigma,n}$.
(4) If σ is labeled by ϕ_α, then define $D_{\sigma^\frown 0} = j_\alpha[D_\sigma]$.

Note that we always have $\mathtt{val}_\sigma(x) \in D_\sigma$ whenever σ is a true position for x with respect to $\mathbf{U}$. We will inductively define a computable homeomorphism $\iota_\sigma \colon D_\sigma \simeq Z_\sigma$, where Z_σ is a closed subset of Z.

To describe our construction, first we note that the Turing jump operator has the so-called uniformly order preserving (UOP) property (see e.g. [24]): there exists a computable function $p \colon \omega \to \omega$ such that

$$x \leq_T y \text{ via } e \implies j_\alpha(x) \leq_T j_\alpha(y) \text{ via } p(e).$$

In other words, given a partial computable function $\varphi_e \colon \subseteq \omega^\omega \to \omega^\omega$, one can effectively find a partial computable function $\varphi_{p(e)} \colon \subseteq \omega^\omega \to \omega^\omega$ such that $\varphi_e(y) = x$ implies $\varphi_{p(e)}(j_\alpha(y)) = j_\alpha(x)$; that is, $\varphi_{p(e)} \circ j_\alpha(y) = j_\alpha \circ \varphi_e(y)$ if $\varphi_e(y)$ is defined. Therefore, if $\iota \colon A \simeq B$ is a computable homeomorphism (i.e., ι and ι^{-1} are computable bijections), then the UOP property of the jump gives a computable homeomorphism $\iota^\star \colon j_\alpha[A] \simeq j_\alpha[B]$ such that $j_\alpha \circ \iota(x) = \iota^\star \circ j_\alpha(x)$ for any $x \in A$.

We also recall that, for a nonempty open set $U \subseteq Z$, $\mathtt{in}_U \colon Z \simeq U$ denotes a homeomorphism between Z and U. For a function $f \colon A \to B$ and $C \subseteq B$, we use the symbol $f \downharpoonleft C$ to denote $f \upharpoonright f^{-1}[C]$, i.e., the restriction of f whose codomain is C. For a subspace Y of Z, if U is open in Y then there is an open set $\hat{U}$ in Z such that $U = \hat{U} \cap Y$. Then, the homeomorphism $\mathtt{in}_{\hat{U}} \colon Z \simeq \hat{U}$ induces another homeomorphism $\mathtt{in}_{\hat{U}} \downharpoonleft Y \colon \mathtt{in}_{\hat{U}}^{-1}[Y] \simeq U$. Note that if Y is closed in Z, so is the domain $\mathtt{in}_{\hat{U}}^{-1}[Y]$. In the following argument, we may think of $\mathtt{in}_{\hat{U}}^{-1}$ as a magnifying glass which enlarges $\hat{U}$ to the size of the whole space Z, and $\mathtt{in}_{\hat{U}}^{-1}[Y]$ as the view of $U = \hat{U} \cap Y$ under this scale.

Now, let us start the construction of $\iota_\sigma \colon D_\sigma \simeq Z_\sigma$. First put $\iota_\varepsilon = \mathrm{id}$ and $Z_\varepsilon = Z$, and assume that ι_σ and Z_σ have already been defined.

Case 1. If σ is labeled by $\rightsquigarrow$, then put $V'_\sigma = \iota_\sigma[D_\sigma \cap U_\sigma]$, and note that, by the induction hypothesis, $i_\sigma \colon D_\sigma \simeq Z_\sigma$ is a homeomorphism, so V'_σ is open in Z_σ since U_σ

is open. Then, there exists an open set $\hat{V}'_\sigma$ in Z such that $V'_\sigma = \hat{V}'_\sigma \cap Z_\sigma$. Then put

$$Z_{\sigma^\frown 1} = \mathtt{in}^{-1}_{\hat{V}'_\sigma}[Z_\sigma], \qquad\qquad u'_{\sigma^\frown 1} = \left(\mathtt{in}_{\hat{V}'_\sigma} \downharpoonleft Z_\sigma\right)^{-1} : V'_\sigma \simeq Z_{\sigma^\frown 1}.$$

As mentioned above, $Z_{\sigma^\frown 1}$ is closed in Z. Then define $\iota_{\sigma^\frown 1}$ as the restriction of $u'_{\sigma^\frown 1} \circ \iota_\sigma$ to U_σ. Since $D_{\sigma^\frown 1} = D_\sigma \cap U_\sigma$, the map $\iota_{\sigma^\frown 1} \colon D_{\sigma^\frown 1} \simeq Z_{\sigma^\frown 1}$ is a homeomorphism. Diagrammatically, this argument may be described as follows:

$$\begin{tikzcd}
D_\sigma \arrow[r, "\sim", "\iota_\sigma"'] & Z_\sigma & \\
D_\sigma \cap U_\sigma \arrow[u, hook] \arrow[r, "\sim", "\iota_\sigma \restriction U_\sigma"'] & \hat{V}'_\sigma \cap Z_\sigma \arrow[u, hook] \arrow[d, hook] & Z_{\sigma^\frown 1} \arrow[l, "\sim"', "\mathtt{in}_{\hat{V}'_\sigma} \downharpoonleft Z_\sigma"] \arrow[d, hook] \\
& \hat{V}'_\sigma & Z \arrow[l, "\sim"', "\mathtt{in}_{\hat{V}'_\sigma}"]
\end{tikzcd}$$

We also put $u'_{\sigma^\frown 0} = \mathrm{id}$, $Z_{\sigma^\frown 0} = Z_\sigma \setminus \hat{V}'_\sigma$, and $\iota_{\sigma^\frown 0} = \iota_\sigma \restriction D_{\sigma^\frown 0}$. Note that $\iota_\sigma[D_{\sigma^\frown 0}] = \iota_\sigma[D_\sigma \setminus U_\sigma] = Z_\sigma \setminus V'_\sigma = Z_{\sigma^\frown 0}$. Hence, $\iota_{\sigma^\frown 0} \colon D_{\sigma^\frown 0} \simeq Z_{\sigma^\frown 0}$ is a homeomorphism.

Case 2. If σ is labeled by $\sqcup$, then put $V'_{\sigma,n} = \iota_\sigma[D_\sigma \cap U_{\sigma,n}]$, and let $\hat{V}'_{\sigma,n}$ be an open set in Z such that $V'_{\sigma,n} = \hat{V}'_{\sigma,n} \cap Z_\sigma$ for each $n \in \omega$, as above. Then put

$$Z_{\sigma^\frown n} = \mathtt{in}^{-1}_{\hat{V}'_{\sigma,n}}[Z_\sigma], \qquad\qquad u'_{\sigma^\frown n} = \left(\mathtt{in}_{\hat{V}'_{\sigma,n}} \downharpoonleft Z_\sigma\right)^{-1} : V'_{\sigma,n} \simeq Z_{\sigma^\frown n}.$$

As before $Z_{\sigma^\frown n}$ is closed in Z. Then define $\iota_{\sigma^\frown n}$ as the restriction of $u'_{\sigma^\frown n} \circ \iota_\sigma$ to $U_{\sigma,n}$. Since $D_{\sigma^\frown n} = D_\sigma \cap U_{\sigma,n}$, the map $\iota_{\sigma^\frown n} \colon D_{\sigma^\frown n} \simeq Z_{\sigma^\frown n}$ is a homeomorphism.

Case 3. If σ is labeled by ϕ_α, then put $\hat{V}'_\sigma = Z$, $Z_{\sigma^\frown 0} = j_\alpha[Z_\sigma]$, $u'_{\sigma^\frown 0} = j_\alpha$. By the property of our specific Turing jump operator mentioned above, $Z_{\sigma^\frown 0}$ is closed in Z. Since $\iota_\sigma \colon D_\sigma \simeq Z_\sigma$ by the induction hypothesis, one can effectively find a homeomorphism $\iota^\star_\sigma \colon j_\alpha[D_\sigma] \simeq j_\alpha[Z_\sigma]$ by the UOP property. Then define $\iota_{\sigma^\frown 0} = \iota^\star_\sigma$. Diagrammatically,

$$\begin{tikzcd}
D_\sigma \arrow[r, "\sim", "\iota_\sigma"'] \arrow[d, "j_\alpha"'] & Z_\sigma \arrow[d, "j_\alpha"] \\
j_\alpha[D_\sigma] \arrow[r, "\sim", "\iota^\star_\sigma"'] & j_\alpha[Z_\sigma]
\end{tikzcd}$$

Finally, if $\sigma \in \mathtt{Syn}_t$ is labeled by $\rightsquigarrow$ or ϕ_α, we define $U'_\sigma = \hat{V}'_\sigma$; and if σ is labeled by $\sqcup$, we define $U'_{\sigma,0} = \hat{V}'_{\sigma,0} \cup (Z \setminus Z_\sigma)$ and $U'_{\sigma,n} = \hat{V}'_{\sigma,n}$ for each $n > 0$. Note that $U'_{\sigma,0}$ is open since Z_σ is closed.

Now, we claim that $\mathbf{U}' = (U'_\sigma, u'_\sigma)_{\sigma \in \mathtt{Syn}_t}$ can be thought of as a command. First note that, for example, if we consider the case where σ is labeled with $\sqcup$, the function $u'_{\sigma^\frown n}$ is only defined on $V'_{\sigma,n} \subseteq \hat{V}'_{\sigma,n} = U'_{\sigma,n}$, so it may not satisfy the condition of being a command. However, since Z_σ is closed and $V'_{\sigma,n} = \hat{V}'_{\sigma,n} \cap Z_\sigma$, any function on $V'_{\sigma,n}$ can be extended to a function on $\hat{V}'_\sigma$ by composing the retraction $\mathtt{out}_{\omega^\omega \setminus Z_\sigma} \colon \omega^\omega \to Z_\sigma$. Hence, we may assume that u'_σ is defined on U'_σ by replacing u'_σ with $u'_\sigma \circ \mathtt{out}_{\omega^\omega \setminus Z_\sigma}$ (restricted to $\hat{V}'_{\sigma,n}$) if necessarily.

Claim 4. *The command* $\mathbf{U}'$ *is strongly total.*

Proof. By the property of the domain assignment, if σ is labeled by $\sqcup$, then D_σ is covered by $(U_{\sigma,n})_{n\in\omega}$ since the command $\mathbf{U}$ is total. Therefore,

$$Z_\sigma = \iota_\sigma[D_\sigma] \subseteq \iota_\sigma \left[D_\sigma \cap \bigcup_{n\in\omega} U_{\sigma,n} \right] = \bigcup_{n\in\omega} \iota_\sigma[D_\sigma \cap U_{\sigma,n}] = \bigcup_{n\in\omega} V'_{\sigma,n} \subseteq \bigcup_{n\in\omega} U'_{\sigma,n}.$$

Moreover, we have $Z \setminus Z_\sigma \subseteq U'_{\sigma,0}$. Hence, $(U'_{\sigma,n})_{n\in\omega}$ is a cover of Z. This means that $\mathbf{U}'$ is strongly total. ◻

As before, for a node $\sigma \in \mathtt{Syn}_t$ of length ℓ, we define a partial function $\mathtt{val}'_\sigma$ as follows:

$$\mathtt{val}'_\sigma = u'_\sigma \circ u'_{\sigma\restriction(\ell-1)} \circ \cdots \circ u'_{\sigma\restriction 2} \circ u'_{\sigma\restriction 1}.$$

By induction, one can show that, if σ is a true position for x with respect to $\mathbf{U}'$, then $\mathtt{val}'_\sigma(x) \in Z_\sigma$ as follows: If σ is labeled by $\sqcup$ or ϕ_α then $u'_{\sigma^\frown n}(x) \in Z_\sigma$. If σ is labeled by $\rightsquigarrow$ then $u'_{\sigma^\frown 1}(x) \in Z_\sigma$. If $\sigma^\frown 0$ is a true position for x with respect to $\mathbf{U}'$ then $\mathtt{val}'_\sigma(x) \notin U'_\sigma = \hat{V}'_\sigma$ by the definition of true position. By the induction hypothesis, we have $\mathtt{val}'_\sigma(x) \in Z_\sigma$, so $\mathtt{val}'_\sigma(x) \in Z_{\sigma^\frown 0} = Z_\sigma \setminus \hat{V}'_\sigma$. Since $u'_{\sigma^\frown 0} = \mathrm{id}$, we conclude that $\mathtt{val}'_\sigma(x) = \mathtt{val}_\sigma(x) \in Z_{\sigma^\frown 0}$.

Claim 5. *The commands* $\mathbf{U}$ *and* $\mathbf{U}'$ *determine the same function, i.e.,* $[\![\mathbf{U}]\!] = [\![\mathbf{U}']\!]$.

Proof. We inductively show $\mathtt{val}'_\sigma(x) = \iota_\sigma \circ \mathtt{val}_\sigma(x)$ whenever σ is a true position for $x \in Z$ with respect to $\mathbf{U}$. If $\sigma = \varepsilon$, the assertion is clear. Let σ be labeled by $\sqcup$, and assume that $\sigma^\frown n$ is a true position for $x \in Z$ with respect to $\mathbf{U}$. In this case, σ is also a true position, so by the induction hypothesis, we have $\mathtt{val}'_{\sigma^\frown n}(x) = u'_{\sigma^\frown n} \circ \mathtt{val}'_\sigma(x) = u'_{\sigma^\frown n} \circ \iota_\sigma \circ \mathtt{val}_\sigma(x)$. By definition, we have $\iota_{\sigma^\frown n}(z) = u'_{\sigma^\frown n} \circ \iota_\sigma(z)$ for any $z \in D_\sigma \cap U_{\sigma,n}$. Moreover, since $\sigma^\frown n$ is a true position for $x \in Z$ with respect to $\mathbf{U}$ we have $\mathtt{val}_{\sigma^\frown n}(x) \in D_{\sigma^\frown n} = D_\sigma \cap U_{\sigma,n}$. Therefore,

$$\mathtt{val}'_{\sigma^\frown n}(x) = u'_{\sigma^\frown n} \circ \iota_\sigma \circ \mathtt{val}_\sigma(x) = \iota_{\sigma^\frown n} \circ \mathtt{val}_\sigma(x) = \iota_{\sigma^\frown n} \circ \mathtt{val}_{\sigma^\frown n}(x).$$

Here, the last equality holds because σ is labeled by $\sqcup$, so σ and $\sigma^\frown n$ have the same Veblen initial segments; hence simplicity of $\mathbf{U}$ implies $\mathtt{val}_{\sigma^\frown n} = \mathtt{val}_\sigma$. The same argument applies when σ is labeled by $\rightsquigarrow$. If σ is labeled by ϕ_α, then by the induction hypothesis and the UOP property,

$$\mathtt{val}'_{\sigma^\frown 0} = u'_{\sigma^\frown 0} \circ \mathtt{val}'_\sigma = j_\alpha \circ \iota_\sigma \circ \mathtt{val}_\sigma = \iota^\star_\sigma \circ j_\alpha \circ \mathtt{val}_\sigma = \iota_{\sigma^\frown 0} \circ \mathtt{val}_{\sigma^\frown 0}.$$

By induction we next prove that a node $\sigma \in \mathtt{Syn}_t$ is a true position for x with respect to $\mathbf{U}$ if and only if σ is a true position for x with respect to $\mathbf{U}'$. Assume that σ is a true position for x with respect to $\mathbf{U}$ if and only if σ is a true position for x with respect to $\mathbf{U}'$. The former condition is equivalent to that $\mathtt{val}_\sigma(x)$ is defined and contained in D_σ. If σ is labeled by $\sqcup$, then $\mathtt{val}_\sigma(x) \in D_\sigma \cap U_{\sigma,n}$ if and only if $\mathtt{val}'_\sigma(x) = \iota_\sigma \circ \mathtt{val}_\sigma(x) \in \iota_\sigma[D_\sigma \cap U_{\sigma,n}] = V'_{\sigma,n}$. Since $U'_{\sigma,n} \cap Z_\sigma = V'_{\sigma,n}$, one can see that $\sigma^\frown n$ is a true position for x with respect to $\mathbf{U}$ if and only if $\sigma^\frown n$ is a true position for x with respect to $\mathbf{U}'$. Here, the backward direction holds because the latter condition implies $\mathtt{val}'_\sigma(x) \in Z_\sigma$ as mentioned above. The same argument applies when σ is labeled by $\rightsquigarrow$ or ϕ_α.

In particular, $\sigma \in \mathtt{Syn}_t$ is a true path for x with respect to $\mathbf{U}$ if and only if σ is a true path for x with respect to $\mathbf{U}'$. This means that $[\![\mathbf{U}]\!](x) = q$ if and only if $[\![\mathbf{U}']\!](x) = q$. $\square$

It remains to show that the above construction preserves Δ^1_1-ness. If $\mathbf{S}$ has a Δ^1_1-code, then an oracle δ can be Δ^1_1 as mentioned in the proof of Claim 1. Moreover, $(u_\sigma)_{\sigma\in\mathtt{Syn}_t}$ is obviously Δ^1_1 and Claim 1 ensures that $(U_\sigma)_{\sigma\in\mathtt{Syn}_t}$ is also Δ^1_1. Hence $\mathbf{U}$ has a Δ^1_1-code.

As for the construction of $\mathbf{U}'$, we inductively assume that codes of Z_σ and ι_σ are known. Let us consider Case 1. First, the information on $\hat{V}'_\sigma$ can be recovered by a standard method. More precisely, since ι_σ is a computable homomorphism, there is a computation φ on $\omega^{<\omega}$ that simulates ι_σ^{-1}, so $\{\tau \in \omega^{<\omega} : [\varphi(\tau)] \subseteq U_\sigma\}$ generates the open set $\hat{V}'_\sigma$. Therefore, we know $\mathtt{in}_{\hat{V}'_\sigma}$ and its inverse, so we can effectively obtain codes of $Z_{\sigma^\frown 1}$, $u'_{\sigma^\frown 1}$ and $\iota_{\sigma^\frown 1}$. For $\sigma^\frown 0$, it is easy. By the same argument, the construction in Case 2 is also shown to be effective.

For Case 3, we know codes of $\widehat{V}'_\sigma = Z$ and $u'_{\sigma^\frown 0} = j_\alpha$, and the information $\iota_{\sigma^\frown 0} = \iota^\star_\sigma$ can be obtained effectively from ι_σ. The only thing that is not clear is $Z_\sigma{}^\frown 0 = j_\alpha[Z_\sigma]$, and for this, it is necessary to examine the property of the α-th Turing jump j_α. As mentioned above, j_α has a computable left-inverse, so let φ be an algorithm to compute it. For a given $z \in Z$, note that $z \in j_\alpha[Z_\sigma]$ if and only if $\varphi(z) \in Z_\sigma$ and $z = j_\alpha(\varphi(z))$. As j_α is Δ^1_1, it is clear that this decision can be performed hyperarithmetically relative to a code of Z_σ. We have mentioned that j_α has the property "jump operator with true stages" so that $j_\alpha[Z_\sigma]$ is closed, which actually means that when the above condition is refuted, it is verified in a finite step. Thus, this actually gives a hyperarithmetic code for the closed set $j_\alpha[Z_\sigma]$. This completes the proof of Theorem 3.6. $\square$

By this theorem, the equivalence (3.1) of the four classes can be shown as follows: First, we have already seen $\utilde{\mathbf{\Sigma}}^W_t(\omega^\omega) = \utilde{\mathbf{\Sigma}}^{W\circ}_t(\omega^\omega)$ by Observation 2.9. Second, Observations 2.11 and 2.13 give characterizations of classes $\utilde{\mathbf{\Sigma}}^W_t$ and $\utilde{\mathbf{\Sigma}}'_t$ in terms of command. Hence, Theorem 3.6 (1) implies $\utilde{\mathbf{\Sigma}}^W_t(\omega^\omega) \cup \utilde{\mathbf{\Sigma}}'_t(\omega^\omega) \subseteq \utilde{\mathbf{\Sigma}}_t(\omega^\omega)$. By Theorem 3.6 (2) we finally obtain $\utilde{\mathbf{\Sigma}}_t(\omega^\omega) \subseteq \utilde{\mathbf{\Sigma}}'_t(\omega^\omega) \subseteq \utilde{\mathbf{\Sigma}}^W_t(\omega^\omega)$. This completes the proof.

4. Symbolic Wadge reducibility

4.1. **Symbolic representation.** As an advantage of defining $\utilde{\mathbf{\Sigma}}_t$ by means of a flowchart, in this section, we argue that one can export many results on the term classes $\utilde{\mathbf{\Sigma}}_t$ on zero-dimensional Polish spaces to arbitrary Polish spaces. First, one of the basic ideas in modern computability theory is that various mathematical objects (such as real and complex numbers) can be represented by symbolic sequences. A symbolic sequence is an element of I^ω, where I is an alphabet, i.e., a set of symbols. In this article, we assume that $I = \omega$, and consider the set $Z = \omega^\omega$ of all symbolic sequences.

A *symbolic representation*, or simply a *representation*, of a set X is a partial surjection $\delta_X :\subseteq \omega^\omega \to X$, where ω^ω is as above. If $\delta_X(p) = x$, then p is called a *δ_X-name of x* (or simply, a *name of x* if δ_X is clear from the context). A pair of a set and its representation is called a *represented space*. A representation $\delta\colon \omega^\omega \to X$ is *admissible* if for any partial continuous function $f\colon\subseteq \omega^\omega \to X$ there exists a continuous function $g\colon\subseteq \omega^\omega \to \omega^\omega$ such that $f = \delta \circ g$. It is known that every second-countable T_0 space (indeed, every

T_0 space having a countable cs-network) has an admissible representation; see Schröder [30].

Example 4.1. Let $\mathcal{X} = (X, d, \alpha)$ be a separable metric space, where $\{\alpha_i\}_{i\in\omega}$ is a dense subset of X. Then, a *Cauchy name* of a point $x \in X$ is a sequence $p \in \omega^\omega$ such that $d(x, \alpha_{p(k)}) < 2^{-k}$ for any $k \in \omega$. This notion induces a partial surjection $\delta :\subseteq \omega^\omega \to X$ defined by

$$\delta(p) = x \iff p \text{ is a Cauchy name of } x.$$

This surjection δ is called the *Cauchy representation of* X (induced from (d, α)). One can show that δ is an admissible representation of $\mathcal{X}$.

Fact 4.2 (de Brecht [5, Theorem 49])**.** *Every Polish space has a total admissible representation. Indeed, every quasi-Polish space has a total admissible representation.*

For represented spaces $\mathcal{X} = (X, \delta_X)$ and $\mathcal{Y} = (Y, \delta_Y)$, we say that a function $f\colon \mathcal{X} \to \mathcal{Y}$ is *symbolically continuous* if there exists a continuous function $F\colon\subseteq \omega^\omega \to \omega^\omega$ such that, given a δ_X-name $\mathtt{x}$ of $x \in X$, $F(\mathtt{x})$ returns a δ_Y-name of $f(x)$. In other words, the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ \omega^\omega \arrow[u, "\delta_X"] \arrow[r, "F"'] & \omega^\omega \arrow[u, "\delta_Y"'] \end{tikzcd}$$

Such a function F is called a *realizer* of f. As long as we are dealing with admissible representations of Polish spaces (more generally, second countable T_0-spaces), there is no need to distinguish between symbolic continuity and topological continuity at all.

Fact 4.3 (Schröder [30, Theorem 4])**.** *If $\mathcal{X}$ and $\mathcal{Y}$ are admissibly represented spaces. Then, f is sequentially continuous if and only if f is symbolically continuous.*

Let $\mathcal{X} = (X, \delta_X)$ be a represented space, and Γ be a pointclass. We say that $A \subseteq X$ is *symbolically* Γ if the set $\delta_X^{-1}[A]$ of all names of elements in A is Γ in $\mathrm{dom}(\delta_X)$. As above, the symbolic Borel hierarchy and the topological Borel hierarchy coincide, and moreover, the symbolic difference hierarchy and the topological difference hierarchy coincide. More precisely, let $D_\beta(\undertilde{\mathbf{\Sigma}}^0_\alpha)$ be the β-th level of the difference hierarchy starting from $\undertilde{\mathbf{\Sigma}}^0_\alpha$ sets, and then we have the following:

Fact 4.4 (de Brecht [5, Theorem 68])**.** *Let $\mathcal{X}$ be an admissibly represented second-countable T_0-space. Then, a set $A \subseteq \mathcal{X}$ is $D_\beta(\undertilde{\mathbf{\Sigma}}^0_\alpha)$ if and only if A is symbolically $D_\beta(\undertilde{\mathbf{\Sigma}}^0_\alpha)$.*

As an important observation, these results show, in particular, that the notion of symbolic complexity does not depend on the choice of the admissible representation. Callard-Hoyrup [3, Theorem 4.1] has shown the effective version of Fact 4.4 (when α and β are finite).

We are interested in whether these results hold for the term classes $\undertilde{\mathbf{\Sigma}}_t$. Let (X, δ_X) be an admissibly represented space. The following result is due to Selivanov [34] (see also the discussion in the last paragraph of Section 4.1).

Theorem 4.5 (Selivanov [34, Theorem 4.6]). *Let δ be an open admissible representation of a second-countable T_0 space X with the domain $\|X\| \subseteq \omega^\omega$, and let t be a well-formed normal $\mathcal{L}_{\mathrm{Veb}}(Q)$-term. For a function $f\colon X \to Q$, f is a $\underset{\sim}{\mathbf{\Sigma}}_t(X)$-function if and only if $f \circ \delta\colon\subseteq \omega^\omega \to Q$ is a $\underset{\sim}{\mathbf{\Sigma}}_t(\|X\|)$-function.*

Let us say that a function $f\colon X \to Q$ is *symbolically* $\underset{\sim}{\mathbf{\Sigma}}_t$ if $f \circ \delta_X\colon \omega^\omega \to Q$ is $\underset{\sim}{\mathbf{\Sigma}}_t$, where δ_X is an admissible representation of X. Then, Theorem 4.5 can be rephrased as follows:

$$f \text{ is } \underset{\sim}{\mathbf{\Sigma}}_t \iff f \text{ is symbolically } \underset{\sim}{\mathbf{\Sigma}}_t.$$

Hence, the symbolic complexity (with respect to flowcharts) of a Borel function is always the same as its topological complexity. Moreover, this equivalence holds effectively.

The correspondence between Theorem 4.5 and Selivanov [34, Theorem 4.6] may not be obvious at first glance, but recall that our $\underset{\sim}{\mathbf{\Sigma}}_t$ and Selivanov's $\mathcal{L}(X,t)$ roughly correspond, as noted in the last Remark in Section 3.2. However, our definition of $\underset{\sim}{\mathbf{\Sigma}}_t$ is still slightly different from the definition of $\mathcal{L}(X,t)$ adopted in Selivanov [34], and we would also like to add one more remark about the conclusion that follows from this theorem (see Section 4.2). Therefore, for the sake of completeness, in Section 4.3, we will give a complete proof of Theorem 4.5, which also shows that a flowchart is a useful notion for making the proof crystal clear. This is also useful for clarifying that the proof of our main theorem in Section 5 has an analogous structure to that of Selivanov's Theorem 4.5, namely the transformation of the sets assigned to a flowchart.

4.2. **Symbolic Wadge reducibility.** One may apply this result to the Wadge theory on admissibly represented spaces introduced by Pequignot [26]. In this section, we show that, even in higher-dimensional Polish spaces, our definition of $\underset{\sim}{\mathbf{\Sigma}}_t$ by means of a flowchart can be thought of as a Wadge class.

In contrast to the case of zero-dimensional spaces, it is known that the Wadge degrees in higher-dimensional spaces behaves badly [15, 29]. For this reason, it is difficult to say that the Wadge degrees in higher-dimensional spaces is a useful measure of topological complexity. To overcome this difficulty, Pequignot [26] introduced a modified version of Wadge reducibility. Let (X,δ_X) and (Y,δ_Y) be admissibly represented spaces.

Definition 4.6 (Pequignot [26]). We say that $A \subseteq X$ is *symbolic Wadge reducible* to $B \subseteq Y$ (written $A/X \sqsubseteq_W B/Y$) if there exists a continuous function $\theta\colon\subseteq \omega^\omega \to \omega^\omega$ such that for any δ_X-name p,

$$p \text{ is a } \delta_X\text{-name of an element of } A \iff \theta(p) \text{ is a } \delta_Y\text{-name of an element of } B.$$

In other words, $A/X \sqsubseteq_W B/Y$ states that $\delta_X^{-1}[A]$ is Wadge reducible to $\delta_Y^{-1}[B]$, but it is sufficient if such a reduction is defined only on the domain of δ_X.

This notion has also been studied in Camerlo [4]. In general, we consider the following notion:

Definition 4.7. Let Q be a quasi-ordered set. We say that $f\colon X \to Q$ is *symbolic Wadge reducible* to $g\colon Y \to Q$ (written $f \sqsubseteq_W g$) if there exists a continuous function $\theta\colon\subseteq \omega^\omega \to \omega^\omega$ such that, whenever p is a δ_X-name of an element $x \in X$, $\theta(p)$ is a δ_Y-name of an element $y \in Y$, and

$$f(x) \leq_Q g(y).$$

In other words, $f \sqsubseteq_W g$ states that $f \circ \delta_X$ is Wadge reducible to $g \circ \delta_Y$.

If $Q = \{0,1\}$ is equipped with the discrete order, then Definitions 4.6 and 4.7 for Q coincide. Diagrammatically, the definition of $f \sqsubseteq_W g$ may be described as follows:

$$\begin{tikzcd}
\omega^\omega \arrow[r, "\delta_X"] \arrow[d, "\theta"'] & X \arrow[r, "f"] & Q \arrow[d, "\leq_Q"] \\
\omega^\omega \arrow[r, "\delta_Y"'] & Y \arrow[r, "g"'] & Q
\end{tikzcd}$$

It is easy to see that the definition of $\sqsubseteq_W$ is independent of the choice of the admissible representations (since two admissible representations are reducible to each other by definition). Note also that a quasi-Polish space always has an open admissible representation which has Polish fibers (i.e., the set of names of a point is always G_δ). This fact is explicitly stated in de Brecht [5, the proof of Theorem 68], but the reason in a nutshell is that the standard total admissible representation $\delta(p) = \{n \in \omega : (\exists m)\ p(m) = n+1\}$ of the universal quasi-Polish space $\mathcal{P}(\omega)$ is clearly open and has Polish fibers, and this property is inherited to any subspace. Combining these two facts shows that when dealing with symbolic Wadge reducibility on quasi-Polish spaces, without loss of generality, each space is represented by an open function having Polish fibers.

Recall from Facts 2.5 and 2.7, the Wadge classes of Borel functions $\omega^\omega \to Q$ coincide with the classes $\utilde{\boldsymbol{\Sigma}}^W_t(\omega^\omega)$ for $\mathcal{L}(Q)$-terms t. Recall also that $\trianglelefteq$ is the nested homomorphic quasi-order on the normal well-formed $\mathcal{L}_{\rm Veb}(Q)$-terms introduced by Kihara-Montalbán [19]. By Theorems 3.6 and 4.5, we obtain the following:

Corollary 4.8. *Let X and Y be Polish spaces, Q be a better-quasi-ordered set, and t be a well-formed normal $\mathcal{L}_{\rm Veb}(Q)$-term. Then, for any function $f\colon X \to Q$, if $g\colon Y \to Q$ is $\utilde{\boldsymbol{\Sigma}}_t(Y)$,*

$$f \sqsubseteq_W g \iff f \in \utilde{\boldsymbol{\Sigma}}_s(X) \text{ for some } s \trianglelefteq t.$$

Proof. Let δ_X and δ_Y be total admissible representations of X and Y, respectively. By definition, $f \sqsubseteq_W g$ if and only if $f \circ \delta_X \leq_W g \circ \delta_Y$. By Selivanov's Theorem 4.5, $g \in \utilde{\boldsymbol{\Sigma}}_t(Y)$ if and only if $g \circ \delta \in \utilde{\boldsymbol{\Sigma}}_t(\omega^\omega)$. By Theorem 3.6, the latter condition is equivalent to $g \circ \delta \in \utilde{\boldsymbol{\Sigma}}^W_t(\omega^\omega)$. Hence, by Kihara-Montalbán's Theorem (Fact 2.7),

$$f \circ \delta_X \leq_W g \circ \delta_Y \iff f \circ \delta_X \in \utilde{\boldsymbol{\Sigma}}^W_s(\omega^\omega) \text{ for some } s \trianglelefteq t.$$

Again by Theorem 3.6, $f \circ \delta_X \in \utilde{\boldsymbol{\Sigma}}^W_s(\omega^\omega)$ if and only if $f \circ \delta_X \in \utilde{\boldsymbol{\Sigma}}_s(\omega^\omega)$. Hence, by Selivanov's Theorem 4.5, the latter condition is equivalent to $f \in \utilde{\boldsymbol{\Sigma}}_s(X)$. This concludes the proof. $\square$

Let us describe a conclusion derived from Corollary 4.8. First, without knowing Corollary 4.8, it is easy to see that the subsets of Polish spaces are semi-well-ordered under symbolic Wadge reducibility; see also Pequignot [26]. Therefore, one can assign an ordinal rank to each subset of a Polish space X. However, it is not immediately obvious what a pointclass in X corresponding to a given ordinal rank α is. Now, Corollary 4.8 makes this clear: The pointclass in X corresponding to the ordinal rank α is exactly what we expect it to be.

Remark. The normality assumption of t in Corollary 4.8 can be removed. This is because, as mentioned in Remark in Section 2.1, Kihara-Montalbán [19] only deals with well-formed normal $\mathcal{L}_{\rm Veb}(Q)$-terms, but nevertheless shows that these exhaust all Wadge classes of Q-valued Borel functions (see the paragraph above Fact 2.7); that is, if a class Γ of Q-valued Borel functions is closed under continuous substitution and has a complete element, then $\Gamma = \utilde{\mathbf{\Sigma}}_t$ for some well-formed normal $\mathcal{L}_{\rm Veb}(Q)$-term. For any well-formed $\mathcal{L}_{\rm Veb}(Q)$-term, every $\utilde{\mathbf{\Sigma}}_t$-function is trivially Borel. Moreover, $\utilde{\mathbf{\Sigma}}_t$ is closed under continuous substitution, since for any flowchart $\mathbf{S} = (S_\sigma)_{\sigma\in\mathtt{Syn}_t}$ determining $f \in \utilde{\mathbf{\Sigma}}_t$, the flowchart $(\theta^{-1}[S_\sigma])_{\sigma\in\mathtt{Syn}_t}$ determines $f \circ \theta$. Note that in Kihara-Montalbán [19, Lemma 3.17], only well-formedness is used for the existence of a $\utilde{\mathbf{\Sigma}}^W_t$-complete function. Hence, by the above argument, we have $\utilde{\mathbf{\Sigma}}^W_t = \utilde{\mathbf{\Sigma}}_{t'}$ for some well-formed normal $\mathcal{L}_{\rm Veb}(Q)$-term t' (in fact, the semantics of the term t forms a nested labeled tree, whose shape can be described as a normal form, which is the desired term t').

4.3. **Vaught transform of a flowchart.** Let δ be an admissible representation for a second-countable T_0-space X. Then, for $x \in X$ we use $\|x\|$ to denote the set $\delta^{-1}\{x\}$ of all δ-names of x. As in de Brecht [5, Theorem 68] (recall also the discussion in the previous subsection), one can assume that δ is an open function and has Polish fibers. For a set $A \subseteq \omega^\omega$, define the *$\delta$-Vaught transform* $\delta^*[A]$ of A (cf. Saint-Raymond [28, Lemma 17], de Brecht [5, Theorem 68] and Callard-Hoyrup [3, Theorem 4.1]) as follows:

$$\delta^*[A] = \{x \in X : \|x\| \cap A \text{ is not meager in } \|x\|\}.$$

Fact 4.9 (Hyperarithmetical complexity of the forcing relation; cf. [28, 5, 3])**.** *Let X be a second countable T_0-space with an admissible representation δ. Then, for any set $A \subseteq X$,*

$$A \text{ is } \utilde{\mathbf{\Sigma}}^0_\alpha \implies \delta^*[A] \text{ is } \utilde{\mathbf{\Sigma}}^0_\alpha.$$

Indeed, given a $\utilde{\mathbf{\Sigma}}^0_\alpha$-code of A, one can effectively find a $\utilde{\mathbf{\Sigma}}^0_\alpha$-code of $\delta^*[A]$.

Let $\mathbf{S}$ be a normal flowchart on an $\mathcal{L}_{\rm Veb}(Q)$-term t over ω^ω. Then, we define the *δ-Vaught transform of* $\mathbf{S}$ as a flowchart $\mathbf{S}^\delta = (S^\delta_\sigma)_{\sigma\in\mathtt{Syn}_t}$ on t over X as follows:

(1) For a leaf $\sigma \in \mathtt{Syn}_t$, $S^\delta_\sigma = X$.
(2) If $\sigma \in \mathtt{Syn}_t$ is labeled by $\rightsquigarrow$, then define $S^\delta_\sigma = \delta^*[S_\sigma]$.
(3) If $\sigma \in \mathtt{Syn}_t$ is labeled by $\sqcup$, then define $S^\delta_{\sigma,n} = \delta^*[S_{\sigma,n}]$ for each $n \in \omega$.
(4) If $\sigma \in \mathtt{Syn}_t$ is labeled by ϕ_α, then $S^\delta_\sigma = X$.

By Fact 4.9, $\mathbf{S}^\delta$ is also a flowchart on t. By Lemma 3.4, one can assume that $\mathbf{S}$ is monotone.

Proposition 4.10 (Selivanov [34, Lemma 4.5])**.** *Let X be a second-countable T_0 space with an admissible representation δ, and $f\colon X \to Q$ be a function. If* $\mathbf{S}$ *is a monotone flowchart determining $f \circ \delta$, then* $\mathbf{S}^\delta$ *is a flowchart determining f.*

Proof. As above, one can assume that δ has Polish fibers, that is, $\|x\|$ is Polish for any $x \in X$. Let $(D_\sigma)_{\sigma\in\mathtt{Syn}_t}$ and $(E_\sigma)_{\sigma\in\mathtt{Syn}_t}$ be the domain assignments to $\mathbf{S}$ and $\mathbf{S}^\delta$, respectively. We first show the following claim:

Claim 6. $E_\sigma \subseteq \delta^*[D_\sigma]$.

Proof. We first observe that $\delta^*[A] \setminus \delta^*[B] \subseteq \delta^*[A \setminus B]$. To see this, assume that $x \in \delta^*[A] \setminus \delta^*[B]$. Then $x \in \delta^*[A]$ means that $\|x\| \cap A$ is not meager in $\|x\|$, and $x \notin \delta^*[B]$ means that $\|x\| \cap B$ is meager, so $\|x\| \setminus B$ is comeager, in $\|x\|$. Hence, by the Baire category theorem on the Polish fiber $\|x\|$, we see that $\|x\| \cap (A \setminus B)$ is not meager in $\|x\|$. Therefore, we have $x \in \delta^*[A \setminus B]$.

By induction on the syntax tree $\mathtt{Syn}_t$. If σ is labeled by $\rightsquigarrow$, then by induction hypothesis and the above observation on the difference of sets,

$$E_{\sigma^\frown 0} = E_\sigma \setminus \delta^*[S_\sigma] \subseteq \delta^*[D_\sigma] \setminus \delta^*[S_\sigma] \subseteq \delta^*[D_\sigma \setminus S_\sigma] = \delta^*[D_{\sigma^\frown 0}].$$

By monotonicity of $\mathbf{S}$, we have $D_{\sigma^\frown 1} = S_\sigma$. Hence, by induction hypothesis,

$$E_{\sigma^\frown 1} = E_\sigma \cap \delta^*[S_\sigma] \subseteq \delta^*[D_\sigma] \cap \delta^*[S_\sigma] = \delta^*[D_\sigma] \cap \delta^*[D_{\sigma^\frown 1}] = \delta^*[D_{\sigma^\frown 1}]$$

since $D_{\sigma^\frown 1} \subseteq D_\sigma$. The same argument applies when σ is labeled by $\sqcup$. $\square$

Next, note that $\mathbf{S}$ must be a total flowchart over X since f is total. Again, let $(D_\sigma)_{\sigma\in\mathtt{Syn}_t}$ and $(E_\sigma)_{\sigma\in\mathtt{Syn}_t}$ be the domain assignments to $\mathbf{S}$ and $\mathbf{S}^\delta$, respectively. To show that $\mathbf{S}^\delta$ is also total, it suffices to show that $E_\sigma \subseteq \bigcup_{n\in\omega} \delta^*[S_{\sigma,n}]$ whenever σ is labeled by $\sqcup$. By Claim 6, $x \in E_\sigma$ implies $x \in \delta^*[D_\sigma]$. Hence, $\|x\| \cap D_\sigma$ is not meager in $\|x\|$. By totality of $\mathbf{S}$, we have $D_\sigma \subseteq \bigcup_{n\in\omega} S_{\sigma,n}$. Therefore, by the Baire category theorem on $\|x\|$, there exists $n \in \omega$ such that $\|x\| \cap S_{\sigma,n}$ is not meager in $\|x\|$. This means that $x \in \delta^*[S_{\sigma,n}]$. This concludes that $\mathbf{S}^\delta$ is total.

Let ρ be a true path for x with respect to $\mathbf{S}^\delta$; that is, $x \in E_\rho$. If the leaf ρ is labeled by q_ρ, then we have $[\![\mathbf{S}^\delta]\!](x) = q_\rho$. By the above claim, we also have $x \in \delta^*[D_\rho]$. In particular, there is $p \in \delta^{-1}\{x\}$ such that $p \in D_\rho$. Thus, $[\![\mathbf{S}]\!](p) = q_\rho$. By our assumption, $\mathbf{S}$ is a flowchart determining $f \circ \delta$, and therefore, $f \circ \delta(p) = [\![\mathbf{S}]\!](p)$. Hence,

$$[\![\mathbf{S}^\delta]\!](x) = q_\rho = [\![\mathbf{S}]\!](p) = f \circ \delta(p) = f(x).$$

This shows that $\mathbf{S}^\delta$ is a flowchart determining f. $\square$

Proof of Theorem 4.5. As seen in the last Remark in Section 4.2, the class $\underset{\sim}{\mathbf{\Sigma}}_t$ is closed under continuous substitution; that is, if $f \in \underset{\sim}{\mathbf{\Sigma}}_t(X)$, then $f \circ \delta \in \underset{\sim}{\mathbf{\Sigma}}_t(\|X\|)$. For the converse direction, assume $f \circ \delta \in \underset{\sim}{\mathbf{\Sigma}}_t(\|X\|)$. Then we have a flowchart $\mathbf{S}$ on t determining $f \circ \delta$. By Proposition 4.10, its δ-Vaught transform $\mathbf{S}^\delta$ is a flowchart on t determining f. Consequently, f is a $\underset{\sim}{\mathbf{\Sigma}}_t$-function. $\square$

Remark. The assumptions of normality and well-formedness are not used in the proof, so they can be removed from the assumptions of Theorem 4.5.

5. Louveau-type effectivization

5.1. **Main results.** In this section, we show that, if a Borel function between Polish spaces happens to be a $\underset{\sim}{\mathbf{\Sigma}}_t$ function, then its $\underset{\sim}{\mathbf{\Sigma}}_t$-code can be obtained from its Borel code in a hyperarithmetical manner.

Theorem 5.1. *Let X be a computable Polish space, Q be a computable Polish space, and t be a hyperarithmetical $\mathcal{L}_{\mathrm{Veb}}(Q)$-term. Then, $\underset{\sim}{\mathbf{\Sigma}}_t(X) \cap \Delta^1_1 = \Sigma_t(\Delta^1_1; X)$.*

This can be viewed as an extension of Louveau's theorem [21, Theorem A], which states that, if $\underset{\sim}{\mathbf{\Gamma}}$ is a Wadge class of ω^ω which has some Δ^1_1-description, then $\Gamma(\Delta^1_1) = \underset{\sim}{\mathbf{\Gamma}} \cap \Delta^1_1$. Essentially this is a special case of Theorem 5.1 with $Q = \{0, 1\}$, although our term-description is different from Louveau's one. One can also show that Theorem 5.1 holds for computable quasi-Polish spaces. Here, roughly speaking, a computable quasi-Polish space is a represented second countable T_0 space which is the image of a computable open surjection from ω^ω (which almost corresponds to the effective version of Fact 4.2; see also [6, 14]).

To further generalize Louveau's theorem, let us note that separating pairwise disjoint sets is a special case of extending a partial function. To see this, let us identify a subset S of X with its characteristic function $\chi_S\colon X \to \{0,1\}$. Then, for example, the Lusin's separation theorem for $\utilde{\mathbf{\Sigma}}^1_1$ sets is interpreted as the statement saying that there exists a total 2-valued $\utilde{\mathbf{\Delta}}^1_1$-function which extends a given partial 2-valued $\utilde{\mathbf{\Sigma}}^1_1$-function.

Note that if the codomain Q is countable, $\utilde{\mathbf{\Sigma}}^1_1$-functions and $\utilde{\mathbf{\Pi}}^1_1$-functions are in fact the same class since $f(x) = q$ if and only if, for any $p \in Q$, $p \neq q$ implies $f(x) \neq q$. Hence, instead of $\utilde{\mathbf{\Sigma}}^1_1$-functions, one may consider $\utilde{\mathbf{\Pi}}^1_1$-functions with $\utilde{\mathbf{\Sigma}}^1_1$-domains. We show the following generalization of Louveau's separation theorem (see Fact 5.11):

Theorem 5.2 (Effective extension)**.** *Let Q be a computable Polish space, t be a hyperarithmetical $\mathcal{L}_{\rm Veb}(Q)$-term, and $f\colon\subseteq \omega^\omega \to Q$ be a partial Π^1_1-measurable function with a Σ^1_1 domain. Suppose that f can be extended to a total $\utilde{\mathbf{\Sigma}}_t$ function $g\colon \omega^\omega \to Q$. Then, f can be extended to a total $\Sigma_t(\Delta^1_1)$ function $g^\star\colon \omega^\omega \to Q$.*

Indeed, Theorem 5.2 can be viewed as a functional version of Louveau's theorem [21, Theorem 2.7] for Borel Wadge classes. However, our language $\mathcal{L}_{\rm Veb}(Q)$ is designed as a tool to discuss the case where Q is a quasi-ordered set. In such a case, it is natural to deal with the domination theorem rather than the extension theorem. Let $\leq_Q$ be a quasi-order on Q. For sets $A \subseteq B$ and functions $f\colon A \to Q$ and $g\colon B \to Q$, we say that *f is $\leq_Q$-dominated by g* if $f(x) \leq_Q g(x)$ for any $x \in A$.

Theorem 5.3 (Effective domination)**.** *Let $\leq_Q$ be a Π^1_1 quasi-order on a computable Polish space Q, $t(\bar{x})$ be a hyperarithmetical $\mathcal{L}_{\rm Veb}(Q)$-term, and $f\colon\subseteq \omega^\omega \to Q$ be a partial Π^1_1-measurable function with a Σ^1_1 domain. Suppose that f is $\leq_Q$-dominated by some total $\utilde{\mathbf{\Sigma}}_t$ function $g\colon \omega^\omega \to Q$. Then, f is $\leq_Q$-dominated by some total $\Sigma_t(\Delta^1_1)$ function $g^\star\colon \omega^\omega \to Q$.*

Note that g extends f if and only if f is $=$-dominated by g. Thus, Theorem 5.3 is more general than Theorem 5.2. Effective Domination Theorem 5.3 can be applied, for example, to ordinal-valued functions; see also Example 5.7. For studies on ordinal-valued functions in the context of Wadge degrees, see also [9, 1].

As another different type of application of our main techniques, we next consider decomposability of Borel functions. The study of decomposability of Borel functions originates from Lusin's old question asking whether any Borel function can be written as a union of countably many partial continuous functions. In the study of decomposability of Borel function, it is known that computability-theoretic analysis plays an important role, as shown in Kihara [17] and Gregoriades-Kihara-Ng [11]. The decomposition of a Borel function into partial continuous functions along a flowchart has been studied in [18]. One can express such a decomposition as an $\mathcal{L}_{\rm Veb}(\mathcal{C})$-term where each partial continuous function is considered as a constant symbol, i.e., $\mathcal{C}$ is the set of partial continuous functions.

More precisely, for an $\mathcal{L}_{\mathrm{Veb}}(\mathcal{C})$-term, a function $f\colon X \to Y$ is in $\utilde{\mathbf{\Sigma}}_t$ if there exists a flowchart $\mathbf{S}$ on t such that, for any $x \in X$ and true path ρ for x w.r.t. $\mathbf{S}$, if ρ is labeled by h, then $f(x) = h(x)$. Note that $\mathbf{S}$ is not necessarily deterministic in the sense of Section 3.1, but still deterministic in a certain sense. A $\utilde{\mathbf{\Sigma}}_t$-function is clearly Borel-piecewise continuous. However, this definition is too restrictive as the constant symbols in the term t already tell us what partial continuous functions it can be decomposed into. To weaken this restriction, we give another definition of $\utilde{\mathbf{\Sigma}}_t$-piecewise continuity using an open term, which make the part of partial continuous functions undetermined.

Recall that variable symbols are indexed as $(x_j)_{j\in\omega^\omega}$. For an open $\mathcal{L}_{\mathrm{Veb}}$-term $t(\bar{x})$ and a set Q, a *Q-valuation of $t(\bar{x})$* is a sequence $\bar{a} = (a_j)_{j\in\omega^\omega}$ where $a_j \in Q$ for any $j \in \omega^\omega$. Then, by $t(\bar{a})$, we denote the result of substituting a_j for x_j in $t(\bar{x})$ for each $j \in \omega^\omega$. We say that a function $f\colon\subseteq X \to Y$ is *$\utilde{\mathbf{\Sigma}}_{t(\bar{x})}$-piecewise continuous* if there exists a $\mathcal{C}$-valuation $\bar{a}$ such that f is in $\utilde{\mathbf{\Sigma}}_{t(\bar{a})}$. Then, f is *$\utilde{\mathbf{\Sigma}}_{t(\bar{x})}$-piecewise Δ^1_1-continuous* if such an a_j is a Δ^1_1-continuous function with a Δ^1_1-domain for any $j \in \omega^\omega$. We say that f is *$\Sigma_{t(\bar{x})}(\Delta^1_1)$-piecewise continuous* if such a $\mathcal{C}$-valuation $\bar{a}\colon \omega^\omega \to \mathcal{C}$ is Δ^1_1-measurable and f is in $\Sigma_{t(\bar{a})}(\Delta^1_1)$, where $\mathcal{C}$ is considered as a represented space as usual; that is, each partial continuous function is coded by an element of ω^ω, see e.g. [2, Sections 2.5.2, 9.2.12 and Definition 11.2.5 (5)]).

Theorem 5.4 (Effective decomposition)**.** *Let X and Y be computable Polish spaces and $t(\bar{x})$ be a hyperarithmetical open $\mathcal{L}_{\mathrm{Veb}}$-term. If $f\colon X \to Y$ is Δ^1_1-measurable and $\utilde{\mathbf{\Sigma}}_{t(\bar{x})}$-piecewise Δ^1_1-continuous, then it is $\Sigma_{t(\bar{x})}(\Delta^1_1)$-piecewise continuous.*

5.2. **Auxiliary tools.** As a common setting for dealing with our main results, we introduce a few auxiliary notions: For sets X, Y and Z, and a binary relation $\lhd \subseteq Y\times Z$, we say that a partial function $f\colon\subseteq X \to Y$ is *$\lhd$-dominated by* $g\colon X \to Z$ if $f(x) \lhd g(x)$ whenever $x \in \mathrm{dom}(f)$. Let X be a computable Polish space, and Q be a totally represented set, i.e., $Q = \{q_z\}_{z\in\omega^\omega}$. For a pointclass Γ, we say that f is *Γ-measurable w.r.t.* $\lhd$ if there exists a Γ set $R \subseteq X \times \omega^\omega$ such that for any $x \in X$,

$$x \in \mathrm{dom}(f) \implies (f(x) \lhd q_z \iff (x, z) \in R).$$

Example 5.5. For Polish spaces X and Y, if $f\colon X \to Y$ is a partial $\utilde{\mathbf{\Sigma}}^1_1$-measurable function, then f is $\utilde{\mathbf{\Pi}}^1_1$-measurable w.r.t. the equality. To see this, note that $f(x) = y$ if and only if $x \in f^{-1}\{y\}$, which is $\utilde{\mathbf{\Pi}}^1_1$ since any singleton is closed.

If $f\colon X \to Y$ is a partial $\utilde{\mathbf{\Pi}}^1_1$-measurable function, then f is $\utilde{\mathbf{\Delta}}^1_1$-measurable w.r.t. the equality. To see this, let $(B_e)_{e\in\omega}$ be an enumeration of rational open balls in X. Then, whenever $x \in \mathrm{dom}(f)$, we have the following:

$$\begin{aligned} f(x) = y &\iff (\forall e \in \omega)\ [y \in B_e \implies f(x) \in B_e], \\ f(x) \neq y &\iff (\exists d, e \in \omega)\ [B_d \cap B_e = \emptyset\ \wedge\ y \in B_d\ \wedge\ f(x) \in B_e]. \end{aligned}$$

It is easy to see that the both formulas are $\utilde{\mathbf{\Pi}}^1_1$ since f is $\utilde{\mathbf{\Pi}}^1_1$-measurable and B_e is open.

Example 5.6. Let X and Q be Polish spaces. Moreover, let $f\colon\subseteq X \to Q$ be a partial $\utilde{\mathbf{\Pi}}^1_1$-measurable function, and $\lhd$ be a $\utilde{\mathbf{\Pi}}^1_1$ binary relation on Q. Then, f is $\utilde{\mathbf{\Pi}}^1_1$-measurable w.r.t. $\lhd$, since whenever $x \in \mathrm{dom}(f)$ we have

$$f(x) \lhd z \iff (\forall q \in Q)\ [f(x) = q \implies q \lhd z].$$

Example 5.7. Let WO be the set of all well-orders on ω, and define a binary relation $\lhd$ on WO as follows:

$$\alpha \lhd \beta \iff \text{the order type of } \alpha \text{ is less than or equal to that of } \beta.$$

We think of WO as a subspace of the Polish space $2^{\omega\times\omega}$. By the usual ordinal comparison argument (see [25, Theorem 4A.2]), if $f :\subseteq X \to \mathsf{WO}$ is a partial $\boldsymbol{\Pi}^1_1$-measurable function, then f is $\boldsymbol{\Pi}^1_1$-measurable w.r.t. $\lhd$.

As a tool for proving our main result, we use the following topology, which is known to be useful in transforming boldface arguments into lightface ones.

Definition 5.8. For a computable Polish space X, the *Gandy-Harrington topology* T_∞ on X is the topology generated by Σ^1_1 subsets of X. We use T_ξ to denote the subtopology generated by sets in $\bigcup_{\eta<\xi} \boldsymbol{\Pi}^0_\eta \cap \Sigma^1_1$.

For the basics of the Gandy-Harrington topology T_∞ and its subtopology T_ξ, we refer the reader to Louveau [21].

Fact 5.9 (see [21, Proposition 6])**.** *The Gandy-Harrington topology T_∞ is Baire in the topological sense: Every nonempty T_∞-open subset of X is T_∞-nonmeager.*

In the following, we write $\forall^\infty x P(x)$ to denote that $\{x : \neg P(x)\}$ is T_∞-meager. In this case, we also say that P holds ∞-a.e. The following fact plays a key role in our proof.

Fact 5.10 (Louveau [21, Lemma 8])**.** *Let A be a $\boldsymbol{\Sigma}^0_\xi$ set in a computable Polish space X. Then, there exists a T_ξ-open set A' such that $A = A'$ ∞-a.e.*

Louveau used these facts to prove the so-called Louveau separation theorem:

Fact 5.11 (Louveau [21, Theorem B])**.** *If a disjoint pair of Σ^1_1 sets is separated by a $\boldsymbol{\Sigma}^0_\xi$ set, then it is separated by a $\Sigma^0_\xi(\Delta^1_1)$ set.*

We also use a few basic facts in classical computability theory. First, the following is known as Kreisel's Δ^1_1-selection theorem; see e.g. [25, 4B.5] or [27, Theorem II.2.3].

Fact 5.12 (Δ^1_1-selection)**.** *Let X be a computable Polish space, and $E \subseteq X \times \omega$ be a total Π^1_1 relation, i.e., for any $x \in X$ there exists $n \in \omega$ such that $E(x, n)$ holds. Then there exists a Δ^1_1 function $f : X \to \omega$ such that $(x, f(x)) \in E$ holds for any $x \in X$.*

A formula $\varphi(X)$ with a set variable is Π^1_1 on Π^1_1 if for each uniformly Π^1_1 sequence $(A_n)_{n\in\omega}$, the set $\{n \in \omega : \varphi(A_n)\}$ is Π^1_1. The following is known as the first Π^1_1-reflection theorem (the lightface version of [16, Theorem 35.10]):

Fact 5.13 (Π^1_1-reflection)**.** *Let φ be Π^1_1 on Π^1_1, and $A \in \Pi^1_1$. If $\varphi(A)$, then there is some $B \in \Delta^1_1$ such that $B \subseteq A$ and $\varphi(B)$.*

5.3. **Proof of Main Theorems.** In this section we prove Theorems 5.1, 5.2, 5.3 and 5.4 by induction on the complexity of $\mathcal{L}_{\mathrm{Veb}}(Q)$-terms. However, in order for the induction to work, we have to prove the following stronger claim:

Lemma 5.14. *Let H be a Σ^1_1 subset of ω^ω, Q be a set, t be a hyperarithmetical $\mathcal{L}_{\mathrm{Veb}}(Q)$-term, $\lhd \subseteq Y \times Q$ be a binary relation, and $f :\subseteq H \to Y$ be a partial function which* *is Π^1_1-measurable w.r.t. $\lhd$ on its Σ^1_1 domain. Suppose that f is ∞-a.e. $\lhd$-dominated by some $\underset{\sim}{\mathbf{\Sigma}}_t$ function $g\colon H \to Q$, i.e.,*

$$\{x \in X : x \in \mathrm{dom}(f) \ \wedge\ \neg(f(x) \lhd g(x))\} \text{ is } T_\infty\text{-meager.}$$

Then, f is $\lhd$-dominated by some $\Sigma_t(\Delta^1_1)$ function $g^\star\colon H \to Q$.

By effectivizing the arguments in Examples 5.5 and 5.6 to calculate the lightface complexity, we observe that Lemma 5.14 implies Theorems 5.2 and 5.3, which also deduces Theorem 5.1 for $X = \omega^\omega$. To prove Theorem 5.1 for computable Polish X, first note that if X is computable Polish (indeed, if X is computable quasi-Polish; see [6, 14]), then there exists a total computable open surjection $\delta\colon \omega^\omega \to X$; see also Fact 4.2. Then for a function $f\colon X \to Q$, $f \in \underset{\sim}{\mathbf{\Sigma}}_t(X) \cap \Delta^1_1$ implies $f \circ \delta \in \underset{\sim}{\mathbf{\Sigma}}_t(\omega^\omega) \cap \Delta^1_1$; hence by Lemma 5.14, we have $f \circ \delta \in \Sigma_t(\Delta^1_1; \omega^\omega)$, and thus $f \in \Sigma_t(\Delta^1_1; X)$ by Selivanov's Theorem 4.5.

To prove Theorem 5.4, we show the multi-valued version of Lemma 5.14 for any computable Polish space X. Recall from Section 3.1 that it is straightforward to define $\underset{\sim}{\mathbf{\Sigma}}_t$ and $\Sigma_t(\Delta^1_1)$ even for multi-valued functions by considering possibly non-deterministic flowcharts. If g is multi-valued, then we say that f is $\lhd$-dominated by g if, for any $x \in \mathrm{dom}(f)$ and any value z of $g(x)$, we have $f(x) \lhd z$.

Lemma 5.15. *Let H be a Σ^1_1 subset of a computable Polish space X, Q be a set, t be a hyperarithmetical $\mathcal{L}_{\mathrm{Veb}}(Q)$-term, $\lhd \subseteq Y \times Q$ be a binary relation, and $f\colon\subseteq H \to Y$ be a partial function which is Π^1_1-measurable w.r.t. $\lhd$ on its Σ^1_1 domain. Suppose that f is ∞-a.e. $\lhd$-dominated by some multi-valued $\underset{\sim}{\mathbf{\Sigma}}_t$ function $g\colon H \rightrightarrows Q$, i.e.,*

$$\{x \in X : x \in \mathrm{dom}(f) \ \wedge\ \exists z \in g(x)\ \neg(f(x) \lhd z)\} \text{ is } T_\infty\text{-meager.}$$

Then, f is $\lhd$-dominated by some multi-valued $\Sigma_t(\Delta^1_1)$ function $g^\star\colon H \rightrightarrows Q$.

Assuming Lemma 5.15, let us give the detailed proof of Theorem 5.4.

Proof of Theorem 5.4. Assume that f is $\underset{\sim}{\mathbf{\Sigma}}_{t(\bar{x})}$-piecewise Δ^1_1-continuous. Then there exists a $\mathcal{C}(\Delta^1_1)$-valuation $\bar{h} = (h_j)_{j\in\omega^\omega}$ such that f is a $\underset{\sim}{\mathbf{\Sigma}}_{t(\bar{h})}$-function. For each $e \in \omega$, let φ_e be the partial Π^1_1-continuous function from X to Y coded by e, and let $I \subseteq \omega$ be the Π^1_1 set of all codes e such that φ_e is indeed Δ^1_1. Then, there exists a function $d\colon \omega^\omega \to I$ such that $h_j = \varphi_{d(j)}$ for any $j \in \omega^\omega$. Put $\bar{d} = (d_j)_{j\in\omega^\omega}$ and then indeed, the definition of f involves a (possibly non-deterministic) flowchart $\mathbf{S}$ on the $\mathcal{L}_{\mathrm{Veb}}(\omega)$-term $t(\bar{d})$. Clearly, $\mathbf{S}$ yields a multi-valued $\underset{\sim}{\mathbf{\Sigma}}_{t(\bar{d})}$-function g such that, for any $x \in \mathrm{dom}(f)$ and any value e of $g(x)$, we have $e \in I$ and $f(x) = \varphi_e(x)$.

Define $\lhd \subseteq (X\times Y)\times\omega$ by $(x, y) \lhd e$ if and only if $e \in I$, $x \in \mathrm{dom}(\varphi_e)$ and $\varphi_e(x) = y$. Let us consider the function $(\mathrm{id}, f)\colon X \to X\times Y$ defined by $(\mathrm{id}, f)(x) = (x, f(x))$. Since f is Δ^1_1-measurable, and each φ_e is a Δ^1_1-continuous function with a Δ^1_1-domain, the conditions $e \in I$, $x \in \mathrm{dom}(\varphi_e)$ and $f(x) = \varphi_e(x)$ are Π^1_1; hence, (id, f) is Π^1_1-measurable w.r.t. $\lhd$. For a single-valued function h, (id, f) is $\lhd$-dominated by h if and only if $(x, f(x)) \lhd h(x)$ for any $x \in \mathrm{dom}(f)$ if and only if $h(x) \in I$ and $f(x) = \varphi_{h(x)}(x)$ for any $x \in \mathrm{dom}(f)$. Therefore, for a multi-valued function h, (id, f) is $\lhd$-dominated by h if and only if, for any $x \in \mathrm{dom}(f)$ and $e \in h(x)$, we have $e \in I$ and $f(x) = \varphi_e(x)$. Therefore, (id, f) is $\lhd$-dominated by g. Now, by Lemma 5.15, (id, f) is $\lhd$-dominated by a multi-valued $\Sigma_{t(\bar{d})}(\Delta^1_1)$-function $g^\star$. Then, we have $f(x) = \varphi_e(x)$ for any $e \in g^\star(x)$ as above. Let $\mathbf{S}'$ be a Δ^1_1 flowchart on $t(\bar{d})$ determining $g^\star$, and then define $\mathbf{S}''$ by replacing the label $e \in \omega$ of each leaf $\rho \in \mathtt{Syn}_{t(\bar{d})}$ with φ_e. Clearly, $\mathbf{S}''$ is a Δ^1_1 flowchart, and we also have that $f(x) = \varphi_e(x)$ if and only if $e \in g^\star(x)$ if and only if $[\![\mathbf{S}']\!](x) = e$ if and only if $[\![\mathbf{S}'']\!](x) = \varphi_e$. By the definition of $\underset{\sim}{\Sigma}_t$ for an $\mathcal{L}_{\mathrm{Veb}}(\mathcal{Q})$-term (see Section 5.1), this means that this flowchart $\mathbf{S}''$ witnesses that f is $\Sigma_{t(\bar{x})}(\Delta^1_1)$-piecewise continuous. $\square$

Before starting the proof of Lemma 5.14, let us look at the syntax tree of an $\mathcal{L}_{\mathrm{Veb}}(Q)$-term t. Then we can immediately see that the syntax tree $\mathtt{Syn}_t$ has the uppermost node σ' (i.e., the node closest to the root) which is not labeled by a Veblen function symbol. Let us call σ' the *neck node of* $\mathtt{Syn}_t$. In other words, the term t can always be written as

$$t = t' \quad \text{or} \quad t = \phi_{\alpha_0}(\phi_{\alpha_1}(\dots(\phi_{\alpha_\ell}(t')))),$$

where the root of the syntax tree $\mathtt{Syn}_{t'}$ of the subterm t' is not labeled by a Veblen function symbol. In this case, we write $\phi_{\alpha_0}(\phi_{\alpha_1}(\dots(\phi_{\alpha_\ell}(\cdot))))$ as $\phi_t(\cdot)$, and let us say that ϕ_t is the head of t, and t' is the body of t. Then, t is of the form $\phi_t(t')$. See also [34, Lemma 3.5].

Example 5.16. If $s = a \sqcup \phi_1(b)$ then the neck node of $\mathtt{Syn}_s$ is the root of $\mathtt{Syn}_s$. If $t = \phi_0(\phi_2(\phi_5(a) \rightsquigarrow (b \sqcup \phi_3(c))))$ then its neck node is the third node of $\mathtt{Syn}_t$ counting from the root: the first node is labeled by ϕ_0, the second node is labeled by ϕ_2, and the third node is labeled by $\rightsquigarrow$, so this is the neck node of $\mathtt{Syn}_t$. Moreover, $\phi_t = \phi_0 \circ \phi_2$.

It is straightforward to see the following:

Observation 5.17. *Let $f \colon X \to Q$ be a function, and ξ be the Borel rank of the neck node of an $\mathcal{L}_{\mathrm{Veb}}(Q)$-term t.*

1. *In the case where t is of the form $\phi_t(q)$ for some $q \in Q$, $f \in \underset{\sim}{\Sigma}_t$ if and only if $f \in \underset{\sim}{\Sigma}_q$.*
2. *In the case where t is of the form $\phi_t(s \rightsquigarrow u)$, $f \in \underset{\sim}{\Sigma}_t$ if and only if there exists a $\underset{\sim}{\Sigma}^0_\xi$ set $U \subseteq X$ such that $f \upharpoonright (X \setminus U) \in \underset{\sim}{\Sigma}_{\phi_t(s)}$ and $f \upharpoonright U \in \underset{\sim}{\Sigma}_{\phi_t(u)}$.*
3. *In the case where t is of the form $\phi_t(\sqcup_{i\in\omega} s_i)$, $f \in \underset{\sim}{\Sigma}_t$ if and only if there exists a $\underset{\sim}{\Sigma}^0_\xi$ cover $(U_n)_{n\in\omega}$ of X such that $f \upharpoonright U_n \in \underset{\sim}{\Sigma}_{\phi_t(s_n)}$ for any $n \in \omega$.*

Now, let us start the proof of Main Lemma 5.14.

Proof of Lemma 5.14. We prove the assertion by induction on the complexity of the body of $t = \phi_t(t')$, where t' is the body of t. Inductively assume that we have already shown the assertion for the term $\phi_t(t'')$ for any proper subterm t'' of t'. Put $X = \omega^\omega$.

Case 1. The neck node of $\mathtt{Syn}_t$ is labeled by a constant symbol q, i.e., t is of the form $\phi_t(q)$. Then, $\underset{\sim}{\Sigma}_t = \underset{\sim}{\Sigma}_q$. By our assumption, f is ∞-a.e. $\lhd$-dominated by a $\underset{\sim}{\Sigma}_t$ function. However, such a function must be the constant function $x \mapsto q$. Therefore, $\{x \in \mathrm{dom}(f) : \neg(f(x) \lhd q)\}$ is T_∞-meager. Since $\mathrm{dom}(f)$ is Σ^1_1 and f is Π^1_1-measurable w.r.t. $\lhd$, this set is Σ^1_1, and in particular, T_∞-open. Therefore, it must be empty by the Baire category theorem for T_∞ (Fact 5.9). Hence, we have $f(x) \lhd q$ for any $x \in \mathrm{dom}(f)$. This means that f is $\lhd$-dominated by the constant function $g^\star \colon x \mapsto q$ on any $H \subseteq X$. Clearly, $g^\star \in \Sigma_t(\Delta^1_1)$ in H.

Case 2. The neck node of $\mathtt{Syn}_t$ is labeled by $\rightsquigarrow$, i.e., t is of the form $\phi_t(s \rightsquigarrow u)$. By our assumption, f is ∞-a.e. $\lhd$-dominated by a total $\underset{\sim}{\Sigma}_t$ function $g \colon H \to Q$. Let

$\mathbf{S} = (S_\sigma)_{\sigma \in \mathtt{Syn}_t}$ be a flowchart on the term t which determines g, i.e., $[\![\mathbf{S}]\!] = g$. Let ξ be the Borel rank of the neck node of $\mathtt{Syn}_t$. Note that $\xi < \omega_1^{\mathrm{CK}}$ since t is Δ^1_1. Then, a $\utilde{\mathbf{\Sigma}}^0_\xi$ set $U \subseteq X$ is assigned to the neck node σ', i.e., $S_{\sigma'} = U$. By Observation 5.17, note that we have $[\![\mathbf{S}]\!] \upharpoonright (H \setminus U) \in \utilde{\mathbf{\Sigma}}_{\phi_t(s)}$ and $[\![\mathbf{S}]\!] \upharpoonright (H \cap U) \in \utilde{\mathbf{\Sigma}}_{\phi_t(u)}$. Then, define U^+ as the following T_ξ-open set:

$$x \in U^+ \iff (\exists O \in \Sigma^1_1 \cap \utilde{\mathbf{\Sigma}}^0_\xi)(\exists g \in \utilde{\mathbf{\Sigma}}_{\phi_t(u)}(H \cap O)) \\ [x \in O \ \wedge \ (\forall^\infty y)\ (y \in \mathrm{dom}(f) \cap O \ \longrightarrow \ f(y) \lhd g(y))].$$

The reason why it is T_ξ-open is that it can be written as the union of $\Sigma^1_1 \cap \utilde{\mathbf{\Sigma}}^0_\xi$ sets O. It is not straightforward, but one can also show that $U^+ \subseteq X$ is the largest T_ξ-open set such that some $\utilde{\mathbf{\Sigma}}_{\phi_t(u)}$-function on $U^+ \cap H$ ∞-a.e. $\lhd$-dominates f in the same way as in Claim 10 below. First we show the following equivalences:

$$\begin{aligned} x \in U^+ &\iff (\exists O \in \Sigma^1_1 \cap \utilde{\mathbf{\Sigma}}^0_\xi)(\exists g^\star \in \Sigma_{\phi_t(u)}(\Delta^1_1; H \cap O)) \\ &\qquad [x \in O \ \wedge \ (\forall y)\ (y \in \mathrm{dom}(f) \cap O \ \longrightarrow \ f(y) \lhd g^\star(y))] \\ &\iff (\exists O^\star \in \Sigma^0_\xi(\Delta^1_1))(\exists g^\star \in \Sigma_{\phi_t(u)}(\Delta^1_1; H \cap O^\star)) \\ &\qquad [x \in O^\star \ \wedge \ (\forall y)\ (y \in \mathrm{dom}(f) \cap O^\star \ \longrightarrow \ f(y) \lhd g^\star(y))] \end{aligned}$$

The first equivalence follows from the induction hypothesis restricted to the Σ^1_1-domain $H \cap O$. For the second equivalence, observe that a set O in the first equivalent formula is always disjoint from $L = \{y \in \mathrm{dom}(f) : \neg(f(y) \lhd g^\star(y))\} \cup (H \setminus \mathrm{dom}(g^\star))$. Since $g^\star \in \Sigma_{\phi_t(u)}(\Delta^1_1)$, by Lemma 3.3, one can assume that the domain of $g^\star$ is Δ^1_1 in X, and moreover, $g^\star$ has a Δ^1_1-measurable realizer $G :\subseteq H \to \omega^\omega$, i.e., $g^\star(y) = q_{G(y)}$. As for the complexity of L, note that

$$y \in L \iff \big(y \in \mathrm{dom}(f) \ \wedge \ (\exists z \in \omega^\omega)\ [G(y) = z \ \wedge \ \neg(f(y) \lhd q_z)]\big) \ \vee \ y \in H \setminus \mathrm{dom}(g^\star).$$

Therefore, the set L is Σ^1_1, since $\mathrm{dom}(f)$ and H are Σ^1_1, $\mathrm{dom}(g^\star)$ is Δ^1_1, and f is Π^1_1-measurable w.r.t. $\lhd$. Hence, given a set O in the first equivalent formula, since O is Σ^1_1 and $\utilde{\mathbf{\Sigma}}^0_\xi$, by the Louveau separation theorem (Fact 5.11), there exists a $\Sigma^0_\xi(\Delta^1_1)$ set $O^\star$ separating O from L. Then $x \in O \subseteq O^\star$, $H \cap O^\star \subseteq \mathrm{dom}(g^\star)$ and $f(y) \lhd g^\star(y)$ for any $y \in \mathrm{dom}(f) \cap O^\star$. This verifies the second equivalence. Next, we check that the second equivalent formula is Π^1_1. Indeed:

Claim 7. *U^+ can be obtained from a Π^1_1-sequence of Δ^1_1-indices of $\utilde{\mathbf{\Sigma}}^0_\xi$ sets.*

Proof. To verify the claim, let C be the set of codes of deterministic flowcharts $\mathbf{S}$ such that the domain of $[\![\mathbf{S}]\!]$ includes $H \cap O^\star$, and for $c \in C$, let $G_c :\subseteq X \to \omega^\omega$ be the realizer for the flowchart $\mathbf{S}_c$ coded by c, i.e., $[\![\mathbf{S}_c]\!](x) = q_{G_c(x)}$ for any x in the domain of $[\![\mathbf{S}_c]\!]$. Then, the condition for $O^\star$ in the second equivalent formula for $x \in U^+$ can be rewritten as follows:

$$(\exists c \in \Delta^1_1)\ [c \in C \ \wedge \ (\forall y, z)\, ((y \in \mathrm{dom}(f) \cap O^\star \ \wedge \ G_c(y) = z) \ \longrightarrow \ f(y) \lhd q_z)]$$

To see that the formula inside the square brackets is Π^1_1, recall from Lemma 3.3 that the set C is Π^1_1 since $O^\star \cap H$ is Σ^1_1, and G_c is Δ^1_1-measurable relative to c. Thus, since $\mathrm{dom}(f)$ is Σ^1_1 and f is Π^1_1-measurable w.r.t. $\lhd$, the inner formula is Π^1_1. Hence, by the usual hyperarithmetical quantification argument (see [27, Lemma III.3.1]), the whole formula is also Π^1_1. This verifies the claim. $\square$

Hereafter, we use H' to denote the domain of f.

Claim 8. *The function $f \upharpoonright H' \setminus U^+$ is $\lhd$-dominated by a $\Sigma_{\phi_t(s)}(\Delta^1_1)$-function on $H \setminus U^+$.*

Proof. By Fact 5.10, there exists a T_ξ-open set V which is equal to U ∞-a.e. Since V is T_ξ-open, if $x \in V$, then there exists $O \in \Sigma^1_1 \cap \undertilde{\mathbf{\Sigma}}^0_\xi$ such that $x \in O \subseteq V$. Note that O is ∞-a.e. included in U, and therefore, $f \upharpoonright H' \cap O$ is ∞-a.e. $\lhd$-dominated by $[\![\mathbf{S}]\!] \upharpoonright H \cap U \in \undertilde{\mathbf{\Sigma}}_{\phi_t(u)}$. Hence, by the definition of U^+, we have $O \subseteq U^+$, and therefore $x \in U^+$ for any $x \in V$. Since x is an arbitrary element of V, we have $U \subseteq U^+$ ∞-a.e. In particular, $H \setminus U^+ \subseteq H \setminus U$ ∞-a.e., and thus $f \upharpoonright H' \setminus U^+$ is ∞-a.e. $\lhd$-dominated by $[\![\mathbf{S}]\!] \upharpoonright (H \setminus U) \in \mathbf{\Sigma}_{\phi_t(s)}$. By Claim 7, U^+ is Π^1_1, and therefore, by the induction hypothesis restricted to the Σ^1_1-domain $H \setminus U^+$, the function $f \upharpoonright H' \setminus U^+$ is now $\lhd$-dominated by some function in $\Sigma_{\phi_t(s)}(\Delta^1_1)$ on $H \setminus U^+$. $\square$

Let $\psi \colon\subseteq \omega^2 \to \omega$ be a partial Π^1_1 function parametrizing all Δ^1_1 functions, i.e., for any Δ^1_1-function $d\colon \omega \to \omega$, there exists $e \in \omega$ such that $\psi(e,n) = d(n)$ for any $n \in \omega$. Let I be the set of all indices $e \in \omega$ such that the function ψ_e defined by $\psi_e(n) = \psi(e,n)$ is total. Clearly, I is Π^1_1. Given $P \subseteq I$, we use the notation $[P]_\xi$ to denote the union of $\undertilde{\mathbf{\Sigma}}^0_\xi$ sets whose codes have Δ^1_1-indices in P, i.e., $[P]_\xi = \bigcup_{e \in P} \delta_\xi(\psi_e)$, where $\delta_\xi(p)$ is the $\undertilde{\mathbf{\Sigma}}^0_\xi$ set coded by p. By Claim 7, there exists a Π^1_1 set $P^+ \subseteq I$ such that $U^+ = [P^+]_\xi$. Let $\Phi(Y)$ be the formula saying that $f \upharpoonright H' \setminus [Y \cap I]_\xi$ is $\lhd$-dominated by a $\Sigma_{\phi_t(s)}(\Delta^1_1)$ function on $H \setminus [Y \cap I]_\xi$.

Claim 9. *Φ is Π^1_1 on Π^1_1.*

Proof. Let $C(Y)$ be the set of codes of deterministic flowcharts $\mathbf{S}$ such that the domain of $[\![\mathbf{S}]\!]$ includes $H \setminus [Y \cap I]_\xi$. If $(Y_n)_{n\in\omega}$ is uniformly Π^1_1, then clearly $(H \setminus [Y_n \cap I]_\xi)_{n\in\omega}$ is uniformly Σ^1_1. Hence, by Lemma 3.3, $(C(Y_n))_{n\in\omega}$ is uniformly Π^1_1. Then, $\Phi(Y_n)$ holds if and only if

$$(\exists c \in \Delta^1_1)\ \big[c \in C(Y_n)\ \wedge\ (\forall y,z)\\ ((y \in \mathrm{dom}(f) \cap [Y_n \cap I]_\xi\ \wedge\ G_c(y) = z)\ \longrightarrow\ f(y) \lhd q_z)\big]$$

By the same argument as in the previous claim, one can see that this is a Π^1_1 property uniformly in $n \in \omega$. Hence, Φ is Π^1_1 on Π^1_1. $\square$

Recall from Claim 8 that $\Phi(P^+)$ holds since $U^+ = [P^+]_\xi$. Hence, by Π^1_1-reflection (Fact 5.13), there exists a Δ^1_1 set $R \subseteq P^+$ such that $\Phi(R)$ holds. Then, $[R]_\xi$ is $\Sigma^0_\xi(\Delta^1_1)$.

Claim 10. *The function $f \upharpoonright H' \cap [R]_\xi$ is $\lhd$-dominated by a $\Sigma_{\phi_t(u)}(\Delta^1_1)$-function on $H \cap [R]_\xi$.*

Proof. Fix a Δ^1_1-enumeration of $R = (r_n)_{n\in\omega}$, and put $O^\star_n = [\{r_n\}]_\xi$. Note that $(O^\star_n)_{n\in\omega}$ is a Δ^1_1-sequence of $\undertilde{\mathbf{\Sigma}}^0_\xi$ sets which covers $[R]_\xi$. Since $X = \omega^\omega$ is zero-dimensional, as seen in the proof of Proposition 3.5, one can effectively find a Δ^1_1-sequence of pairwise disjoint $\undertilde{\mathbf{\Sigma}}^0_\xi$ sets $(O'_n)_{n\in\omega}$ such that $O'_n \subseteq O_n$ for each $n \in \omega$ and $(O'_n)_{n\in\omega}$ covers $[R]_\xi$. Since $R \subseteq P^+$ and $O'_n \subseteq O^\star_n$, for any $n \in \omega$, the function $f \upharpoonright H' \cap O'_n$ is $\lhd$-dominated by a $\Sigma_{\phi_t(u)}(\Delta^1_1)$-function $g'_n\colon H \cap O'_n \to Q$.

Now, consider the formula $E(n,c)$ (where $n \in \omega$ and $c \in \omega^\omega$) stating that $c \in \Delta^1_1$ and $f \upharpoonright H' \cap O'_n$ is $\lhd$-dominated by the $\Sigma_{\phi_t(u)}(\Delta^1_1)$-function on $H \cap O'_n$ determined by the flowchart coded by c. Since the formula saying that c is Δ^1_1 is Π^1_1, and the set $H \cap O'_n$ is Σ^1_1 uniformly in $n \in \omega$, by Lemma 3.3, we observe that the formula $E(n,c)$ is Π^1_1. By the property of O'_n, E is a total relation, that is, for any $n \in \omega$ there exists $c \in \omega^\omega$ such that $E(n,c)$. Hence, by Δ^1_1-selection (Fact 5.12), there exists a Δ^1_1 function $h\colon \omega \to \omega^\omega$ such that $E(n, h(n))$ for any $n \in \omega$.

By the definition of E, this function h yields a uniform Δ^1_1-sequence of flowcharts $(\mathbf{S}_n)_{n\in\omega}$ on $\phi_t(u)$ such that each $\mathbf{S}_n$ determines a $\Sigma_{\phi_t(u)}(\Delta^1_1)$-function $g'_n\colon H \cap O'_n \to Q$ which $\lhd$-dominates $f \upharpoonright H' \cap O'_n$. We write J for the syntax tree of $\phi_t(u)$, and then each flowchart $\mathbf{S}_n$ is of the form $(S^n_\sigma)_{\sigma\in J}$. Then, for each $\sigma \in J$, if σ is labeled by $\rightsquigarrow$, we define $S'_\sigma = \bigcup_{n\in\omega} O'_n \cap S^n_\sigma$; and if σ is labeled by $\sqcup$, we define $S'_{\sigma,i} = \bigcup_{n\in\omega} O'_n \cap S^n_{\sigma,i}$ for each $i \in \omega$. Since such a node extends the neck node, its Borel rank is greater than or equal to ξ. Hence, S'_σ and $S'_{\sigma,i}$ are $\Sigma_{\mathrm{rank}(\sigma)}(\Delta^1_1)$ sets. Thus, $\mathbf{S}' = (S'_\sigma)_{\sigma\in J}$ is a Δ^1_1-flowchart on $\phi_t(u)$. Since $(O'_n)_{n\in\omega}$ is pairwise disjoint, for any $x \in [R]_\xi$ there exists a unique $n \in \omega$ such that $x \in O'_n$. Then, σ is a true path for x w.r.t. $\mathbf{S}'$ if and only if σ is a true path for x w.r.t. $\mathbf{S}_n$. Since $\mathbf{S}_n$ determines a function, it is deterministic; hence this shows that $\mathbf{S}'$ is also deterministic. Let g' be the function determined by the flowchart $\mathbf{S}'$. Then, g' is a $\Sigma_{\phi_t(u)}(\Delta^1_1)$-function whose domain includes $H \cap [R]_\xi$, and we have $g'(x) = [\![\mathbf{S}']\!](x) = [\![\mathbf{S}_n]\!](x) = g'_n(x)$ whenever $x \in H \cap O'_n$. By our choice of g'_n, this implies that $f \upharpoonright H' \cap O'_n$ is $\lhd$-dominated by $g' \upharpoonright H \cap O'_n$ for any $n \in \omega$. Consequently, $f \upharpoonright H' \cap [R]_\xi$ is $\lhd$-dominated by g'. $\square$

Since $\Phi(R)$ holds, there exists a $\Sigma_{\phi_t(s)}(\Delta^1_1)$-function $g''\colon H \setminus [R]_\xi \to Q$ which $\lhd$-dominates $f \upharpoonright H' \setminus [R]_\xi$. Let $\mathbf{S}''$ be a Δ^1_1-flowchart on $\phi_t(s)$ determining g'', and let $\mathbf{S}'$ be the Δ^1_1-flowchart on $\phi_t(u)$ obtained by Claim 10 which determines a $\Sigma_{\phi_t(u)}(\Delta^1_1)$-function $g\colon H \cap [R]_\xi \to Q$ which $\lhd$-dominates $f \upharpoonright H' \cap [R]_\xi$. Since $[R]_\xi$ is in $\Sigma^0_\xi(\Delta^1_1)$, the straightforward combination of the Δ^1_1-flowcharts $\mathbf{S}''$ and $\mathbf{S}'$ yields a Δ^1_1-flowchart $\mathbf{S}^\star$ on $t = \phi_t(s \rightsquigarrow u)$. More precisely, recall that σ' is the neck node of $\mathtt{Syn}_t$, and its Borel rank is ξ. Then, define $S^\star_{\sigma'} = [R]_\xi$. Moreover, for each node $\sigma \in \mathtt{Syn}_t$, if σ is of the form $\sigma'^\frown 0^\frown\tau$, define $S^\star_\sigma = S''_{\sigma'^\frown\tau}$; and if σ is of the form $\sigma'^\frown 1^\frown\tau$, define $S^\star_\sigma = S'_{\sigma'^\frown\tau}$. It is easy to see that $\mathbf{S}^\star = (S^\star_\sigma)_{\sigma\in\mathtt{Syn}_t}$ is a Δ^1_1-flowchart determining a $\Sigma_t(\Delta^1_1)$-function $g^\star\colon H \to Q$ which $\lhd$-dominates f. This completes the proof for Case 2.

Case 3. The neck node of $\mathtt{Syn}_t$ is labeled by $\sqcup$, i.e., t is of the form $\phi_t(\sqcup_{n\in\omega} s_n)$. By our assumption, f is ∞-a.e. $\lhd$-dominated by a total $\underset{\sim}{\mathbf{\Sigma}}_t$ function $g\colon H \to Q$. Let $\mathbf{S} = (S_\sigma)_{\sigma\in\mathtt{Syn}_t}$ be a flowchart on the term t which determines g, i.e., $[\![\mathbf{S}]\!] = g$. Let ξ be the Borel rank of the neck node of $\mathtt{Syn}_t$. Note that $\xi < \omega_1^{\mathrm{CK}}$ since t is Δ^1_1. Then, a $\underset{\sim}{\mathbf{\Sigma}}^0_\xi$ cover S_n of H is assigned to the neck node σ', i.e., $S_{\sigma'} = (S_n)_{n\in\omega}$. By Observation 5.17, note that we have $[\![\mathbf{S}]\!] \upharpoonright (H \cap S_n) \in \underset{\sim}{\mathbf{\Sigma}}_{\phi_t(s_n)}$. Then, define U^+_n as the following T_ξ-open set:

$$\begin{aligned} x \in U^+_n \iff & (\exists O \in \Sigma^1_1 \cap \underset{\sim}{\mathbf{\Sigma}}^0_\xi)(\exists g \in \underset{\sim}{\mathbf{\Sigma}}_{\phi_t(s_n)}(H \cap O)) \\ & [x \in O \ \wedge\ (\forall^\infty y)\ (y \in \mathrm{dom}(f) \cap O \ \longrightarrow\ f(y) \lhd g(y))]. \end{aligned}$$

As in the Case 2, intuitively, $U_n^+ \subseteq X$ is the largest T_ξ-open set such that some $\underset{\sim}{\mathbf{\Sigma}}_{\phi_t(s_n)}$-function on $H \cap U_n^+$ ∞-a.e. $\lhd$-dominates f. Then, we have the following equivalences:

$$\begin{aligned} x \in U_n^+ &\iff (\exists O \in \Sigma^1_1 \cap \underset{\sim}{\mathbf{\Sigma}}^0_\xi)(\exists g^\star \in \Sigma_{\phi_t(s_n)}(\Delta^1_1; H \cap O)) \\ &\qquad\qquad [x \in O \ \wedge\ (\forall y)\ (y \in \mathrm{dom}(f) \cap O \ \longrightarrow\ f(y) \lhd g^\star(y))] \\ &\iff (\exists O^\star \in \Sigma^0_\xi(\Delta^1_1))(\exists g^\star \in \Sigma_{\phi_t(s_n)}(\Delta^1_1; H \cap O^\star)) \\ &\qquad\qquad [x \in O^\star \ \wedge\ (\forall y)\ (y \in \mathrm{dom}(f) \cap O^\star \ \longrightarrow\ f(y) \lhd g^\star(y))] \end{aligned}$$

The first equivalence follows from the induction hypothesis restricted to the Σ^1_1-domain $H \cap O$. The second equivalence follows from Lemma 3.3 and Louveau's separation theorem (Fact 5.11) as in the Case 2. As in Claim 7, one can see that U_n^+ is obtained from a Π^1_1-sequence of Δ^1_1-indices of $\underset{\sim}{\mathbf{\Sigma}}^0_\xi$ sets, uniformly in $n \in \omega$. In other words, there exists a Π^1_1 set $P^+ \subseteq \omega \times I$ such that $U_n^+ = [P_n^+]_\xi$, where $P_n^+ = \{e \in I : (n, e) \in P^+\}$, where I is the set defined as in Case 2.

Claim 11. *$(U_n^+)_{n\in\omega}$ covers H.*

Proof. By Fact 5.10 there is some T_ξ open set which is equal to S_n ∞-a.e. for each n. Then, as in the proof of Claim 8, one can see that $S_n \subseteq U_n^+$ ∞-a.e. Since $(S_n)_{n\in\omega}$ covers H, we have $H \setminus \bigcup_{n\in\omega} U_n^+ \subseteq \bigcup_{n\in\omega} S_n \setminus \bigcup_{n\in\omega} U_n^+ \subseteq \bigcup_{n\in\omega}(S_n \setminus U_n^+)$. This set is T_∞-meager since a countable union of T_∞-meager sets is T_∞-meager. However, $H \setminus \bigcup_n U_n^+$ is Σ^1_1 and thus T_∞-open. By the Baire category theorem for T_∞ (Fact 5.9), this is in fact an empty set. Consequently, $(U_n^+)_{n\in\omega}$ covers H. □

For $Y \subseteq \omega^2$, put $(Y)_n = \{e \in \omega : (n, e) \in Y\}$ for each $n \in \omega$. Let $\Phi(Y)$ be the formula stating that $H \subseteq \bigcup_{n\in\omega}[(Y \cap (\omega \times I))_n]_\xi$. For any uniform sequence of Π^1_1 sets $(P_k)_{k\in\omega}$, it is easy to see that $\Phi(P_k)$ is Π^1_1 uniformly in $k \in \omega$ since H is Σ^1_1. Hence, Φ is Π^1_1 on Π^1_1. By Claim 11, $\Phi(P^+)$ holds. Therefore, by Π^1_1-reflection (Fact 5.13), there exists a Δ^1_1 set $R \subseteq P^+$ such that $\Phi(R)$ holds.

Fix a Δ^1_1-enumeration of $(R)_n = (r_{n,k})_{k\in\omega}$, and put $O^\star_{n,k} = [\{r_{n,k}\}]_\xi$. Note that $(O^\star_{n,k})_{n,k\in\omega}$ is a Δ^1_1-sequence of $\underset{\sim}{\mathbf{\Sigma}}^0_\xi$ sets which covers H. Since $X = \omega^\omega$ is zero-dimensional, as seen in the proof of Proposition 3.5, one can effectively find a Δ^1_1-sequence of pairwise disjoint $\underset{\sim}{\mathbf{\Sigma}}^0_\xi$ sets $(O'_{n,k})_{n,k\in\omega}$ such that $O'_{n,k} \subseteq O^\star_{n,k}$ for any $n, k \in \omega$ and $(O'_{n,k})_{n,k\in\omega}$ covers H. Since $(R)_n \subseteq P_n^+$ and $O'_{n,k} \subseteq O^\star_{n,k}$, for any $n, k \in \omega$, the function $f \restriction H' \cap O'_{n,k}$ is $\lhd$-dominated by a $\Sigma_{\phi_t(s_n)}(\Delta^1_1)$-function $g'_{n,k} \colon H \cap O'_{n,k} \to Q$. Here, H' is the domain of f as in Case 2.

Put $S^\star_n = \bigcup_{k\in\omega} O'_{n,k}$ for each $n \in \omega$. Then, $S^\star_n$ is $\Sigma^0_\xi(\Delta^1_1)$, and as in Claim 10, one can show that $f \restriction H' \cap S^\star_n$ is $\lhd$-dominated by a $\Sigma_{\phi_t(s_n)}(\Delta^1_1)$-function on $H \cap S^\star_n$ for each $n \in \omega$. Again, as in the proof of Claim 10, consider the formula $E(n, c)$ (where $n \in \omega$ and $c \in \omega^\omega$) stating that $c \in \Delta^1_1$ and $f \restriction H' \cap S^\star_n$ is $\lhd$-dominated by the $\Sigma_{\phi_t(s_n)}(\Delta^1_1)$-function on $H \cap S^\star_n$ determined by the flowchart coded by c. As seen in the proof of Claim 10, the formula $E(n, c)$ is Π^1_1, and by the property of $S^\star_n$, E is a total relation. Hence, by Δ^1_1-selection (Fact 5.12), there exists a Δ^1_1 function $h \colon \omega \to \omega^\omega$ such that $E(n, h(n))$ for any $n \in \omega$.

By the definition of E, this function h yields a uniform Δ^1_1-sequence of flowcharts $(\mathbf{S}_n)_{n\in\omega}$ such that each $\mathbf{S}_n = (S^n_\sigma)_\sigma$ determines a $\Sigma_{\phi_t(s_n)}(\Delta^1_1)$-function $g'_n \colon H \cap O'_n \to Q$ which $\lhd$-dominates $f \restriction H' \cap O'_n$. Since $S^\star_n$ is in $\Sigma^0_\xi(\Delta^1_1)$ uniformly in $n \in \omega$, the straightforward combination of the Δ^1_1-flowcharts $(\mathbf{S}_n)_{n\in\omega}$ yields a Δ^1_1-flowchart $\mathbf{S}^\star$ on $t = \phi_t(\sqcup_{n\in\omega} s_n)$. More precisely, recall that σ' is the neck node of $\mathtt{Syn}_t$, and its Borel rank is ξ. Then, define $S^\star_{\sigma',n} = S^\star_n$. Moreover, for each node $\sigma \in \mathtt{Syn}_t$, if σ is of the form $\sigma'^\frown n^\frown\tau$, define $S^\star_\sigma = S^n_{\sigma'^\frown\tau}$. Since $(S^\star_n)_{n\in\omega}$ is a Δ^1_1-sequence of pairwise disjoint $\utilde{\mathbf{\Sigma}}^0_\xi$ sets, as in Claim 10, one can show that $\mathbf{S}^\star = (S^\star_\sigma)_{\sigma\in\mathtt{Syn}_t}$ is a Δ^1_1-flowchart determining a $\Sigma_t(\Delta^1_1)$-function $g^\star\colon H \to Q$ which $\lhd$-dominates f. This completes the proof. $\square$

Proof of Lemma 5.15. In the proof of Lemma 5.14, the assumption $X = \omega^\omega$ (i.e., zero-dimensionality of the space) is only used to ensure that several flowcharts are deterministic. For this reason, if X is not necessarily zero-dimensional, then we cannot ensure that the resulting function $g^\star$ is single-valued. To verify the assertion, we need to modify the proof of Lemma 5.14 to conform to multi-valued functions; for instance, we need to replace the condition $G_c(y) = z$ in Claim 7 with $z \in G_c(y)$. However, as in the proof of Lemma 3.3, one can see that $q_z \in h(x)$ is a Δ^1_1 condition whenever h is in $\Sigma_s(\Delta^1_1)$ where s is an $\mathcal{L}_{\rm Veb}(Q)$-term. Thus, in any complexity calculation (e.g. Claims 7, 9 and 10), involving multi-valued functions does not increase the complexity. Therefore, we can make exactly the same argument as in the proof of Lemma 5.14. $\square$

Question 1. *Does Theorems 5.2 and 5.3 hold for an arbitrary computable Polish space instead of ω^ω?*

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(T. Kihara) Department of Mathematical Informatics, Graduate School of Informatics, Nagoya University, Nagoya 464-8601, Japan

E-mail address: `kihara@i.nagoya-u.ac.jp`

(K. Sasaki) Department of Mathematical Informatics, Graduate School of Informatics, Nagoya University, Nagoya 464-8601, Japan

E-mail address: `sasaki.kenta@c.mbox.nagoya-u.ac.jp`