

# A Hybridization of two Baire Hierarchies of Borel Functions

Takayuki Kihara

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## Abstract

In this paper, we introduce the notion of the hybrid Baire hierarchy to generalize the classical result by Lebesgue, Hausdorff, and Banach on the equivalence between the Baire hierarchy and the hierarchy of Borel measurable functions. Moreover, we show some variants of the Jayne-Rogers Theorem at transfinite levels of the hierarchy of Borel measurable functions. To prove these theorems, the Shore-Slaman Join Theorem on the Turing degrees is essentially used.

## 1 Introduction

### 1.1 Summary

Lusin's old problem was whether every Borel measurable function is countably continuous: If the preimage of each Borel set under a function is again Borel, then can the function be decomposed into countably many continuous functions? The Lusin problem has been negatively answered in the old days. But then, *which Borel measurable functions are countably continuous?* Hausdorff introduced the well-known hierarchy of Borel sets of length  $\omega_1$ . Where is the boundary of countable continuity in the hierarchy of Borel measurable functions? For countable ordinals  $\xi, \zeta < \omega_1$ , a function is called  $(\Sigma_\xi^0, \Sigma_\zeta^0)$ -measurable if the preimage of each  $\Sigma_\zeta^0$  set is  $\Sigma_\xi^0$ . If  $\xi < \zeta$  or  $\zeta \geq \omega$ , then it is easy to construct a  $(\Sigma_\xi^0, \Sigma_\zeta^0)$ -measurable function which is *not* countably continuous. A remarkable theorem proved by Jayne-Rogers [4] states that the  $(\Sigma_2^0, \Sigma_2^0)$ -measurability is equivalent to the  $\Pi_1^0$ -piecewise continuity, i.e., the decomposability into countably many continuous functions with closed domains. Moreover, Semmes [8] showed that, on Baire space, the  $(\Sigma_3^0, \Sigma_3^0)$ -measurability is equivalent to the  $\Pi_2^0$ -piecewise continuity. The countable continuity and the  $\omega$ -decomposability at all finite levels of Borel hierarchy was studied by Pawlikowska-Saboka [6] and Motto Ros [7]. The STRONG GENERALIZATION CONJECTURE ([7, Conjecture 1.6]) states that, for every function on separable metric spaces with analytic domain, the  $(\Sigma_{n+1}^0, \Sigma_{n+k+1}^0)$ -measurability is equivalent to the  $\Pi_{n+k}^0$ -piecewise  $\Sigma_{k+1}^0$ -measurability, i.e., the decomposability into countably many  $\Sigma_{k+1}^0$ -measurable functions with  $\Pi_{n+k}^0$  domains, at all finite levels  $n, k \in \mathbb{N}$ .

In this paper, we introduce the notion of the *continuous*  $(\Sigma_\xi^0, \Sigma_\zeta^0)$ -measurability and the *hybrid Baire hierarchy* to generalize the classical result by Lebesgue, Hausdorff, and Banach on the equivalence between the Baire hierarchy and the hierarchy of Borel measurable functions. We show that a function is continuously  $(\Sigma_{n+1}^0, \Sigma_{n+k+1}^0)$ -measurable if and only if it is  $\Pi_{n+k}^0$ -piecewise  $\Sigma_{k+1}^0$ -measurable if and only if it is of hybrid Baire class  $(n, k)$ , at finite levels  $k < n \neq 1$ . Moreover, we show some variants of the STRONG GENERALIZATION CONJECTURE on the Jayne-Rogers Theorem at transfinite levels of the hierarchy of Borel measurable functions.

### 1.2 Preliminary

**Definition 1.** Let  $\Gamma, \Xi$  be some levels of the Borel hierarchy. Call  $f : \mathcal{X} \rightarrow \mathcal{Y}$   $(\Gamma, \Xi)$ -measurable if  $f^{-1}(A) \in \Xi$  for every  $A \in \Gamma$ . Call  $f : \mathcal{X} \rightarrow \mathcal{Y}$  *continuously*  $(\Gamma, \Xi)$ -measurable, if it is  $(\Gamma, \Xi)$ -measurable, and the function  $f^{-1} : \Gamma|_{\mathcal{Y}} \rightarrow \Xi|_{\mathcal{X}}$  is continuous, i.e., there is a continuous function which maps a given Borel code of each  $A \in \Gamma$  to a Borel code of  $f^{-1}(A) \in \Xi$ . If a function is (continuously)  $(\Sigma_1^0, \Xi)$ -measurable, then it is also called (continuously)  $\Xi$ -measurable.

**Remark.** Brattka [1] pointed out that a function is  $\Sigma_n^0$ -measurable if and only if it is continuously  $\Sigma_n^0$ -measurable. Thus, if a function is  $\Sigma_n^0$ -measurable, then it is continuously  $(\Sigma_{m+1}^0, \Sigma_{n+m}^0)$ -measurable.

Pauly-de Brecht [5] also pointed out that a function is  $(\Sigma_2^0, \Sigma_2^0)$ -measurable if and only if it is continuously  $(\Sigma_2^0, \Sigma_2^0)$ -measurable.

**Definition 2.** For a point class  $\Gamma$ , and a property  $\Lambda$  of functions, a function  $f : \mathcal{X} \rightarrow \mathcal{Y}$  is  $\Gamma$ -piecewise  $\Lambda$  if there is a countable partition  $\{Q_i\}_{i \in \mathbb{N}}$  of  $\text{dom}(f)$  such that  $Q_i \in \Gamma$ , and  $f \upharpoonright Q_i$  satisfies  $\Lambda$  for each  $i \in \mathbb{N}$ . A function is *countably*  $\Lambda$  if it is  $\Gamma$ -piecewise  $\Lambda$  for some  $\Gamma$ .

**Definition 3** (Pointwise discrete limit). Fix a discrete space  $\mathbb{D}$ , and topological spaces  $\mathcal{X}$ ,  $\mathcal{X}_n$ ,  $\mathcal{X}_n^*$ , and  $\mathcal{Y}$ , where  $\mathcal{X}_n \subseteq \mathcal{X}_n^* \subseteq \mathcal{X}$ . A pair  $(f, \delta)$  of a sequence  $f = \{f_n\}_{n \in \mathbb{N}}$  of functions  $f_n : \mathcal{X}_n \rightarrow \mathcal{Y}$  and a sequence  $\delta = \{\delta_n\}_{n \in \mathbb{N}}$  of functions  $\delta_n : \mathcal{X}_n^* \rightarrow \mathbb{D}$  is *discrete* if  $\delta_n(x) = \delta_m(x)$  implies  $f_n(x) = f_m(x)$  for any  $x \in \mathcal{X}$  and  $m, n \in \mathbb{N}$ . The first projection  $g : \subseteq \mathcal{X} \rightarrow \mathcal{Y}$  of the pointwise limit of the discrete pair  $\{(f_n, \delta_n)\}_{n \in \mathbb{N}}$  is called the *pointwise discrete limit* of  $(f, \delta)$ , where  $(f_n, \delta_n) : \mathcal{X}_n \rightarrow \mathcal{Y} \times \mathbb{D}$  for each  $n \in \mathbb{N}$  is defined as  $(f_n, \delta_n)(x) = (f_n(x), \delta_n(x))$  for each  $x \in \mathcal{X}_n$ . Then, we also say that the discrete pair  $(f, \delta)$  *converges to*  $g$ .

The discrete Baire hierarchy of total functions was introduced by Császár-Laczkovich [3] (see also Bukovský [2]). We introduce the discrete Baire hierarchy for partial functions.

**Definition 4** (Discrete Baire hierarchy). Let  $\mathcal{X}, \mathcal{Y}$  be topological spaces, and  $f$  be a function from  $\mathcal{X}$  into  $\mathcal{Y}$ . The function  $f$  is *of discrete Baire class 0* if it is continuous. For any countable ordinal  $\xi > 0$ , the function  $f$  is *of discrete Baire class  $\xi$*  if it is the discrete pointwise limit of a discrete pair  $\{(f_n, \delta_n)\}_{n \in \mathbb{N}}$  of sequences of functions  $f_n$  and  $\delta_n$  of discrete Baire class less than  $\xi$  with  $\text{dom}(f_n) \subseteq \text{dom}(\delta_n) \supseteq \mathcal{X}$ .

Corollary 9 states for any  $n \in \mathbb{N}$  that a function is continuously  $(\Sigma_{n+1}^0, \Sigma_{n+1}^0)$ -measurable if and only if it is  $\Pi_n^0$ -piecewise continuous if and only if it is of discrete Baire class  $n$ .

**Definition 5** (Hybrid Baire hierarchy). Let  $\mathcal{X}, \mathcal{Y}$  be topological spaces, and  $f$  be a function from  $\mathcal{X}$  into  $\mathcal{Y}$ . Let  $\xi \leq \omega_1$  be a countable ordinal.

1. The function  $f$  is *of hybrid Baire class  $(\xi, 0)$*  if it is of Baire class  $\xi$ .
2. For a limit ordinal  $\xi$ , the function  $f$  is *of hybrid Baire class  $(< \xi, 0)$*  if it is of Baire class  $\zeta$  for some  $\zeta < \xi$ ; For a successor ordinal  $\xi$ , the function  $f$  is *of hybrid Baire class  $(< \xi, 0)$*  if it is of Baire class  $\xi$ .
3. For any countable ordinal  $\zeta > 0$ , the function  $f$  is *of hybrid Baire class  $(\xi, \zeta)$*  if it is the discrete pointwise limit of a discrete pair  $\{(f_n, \delta_n)\}_{n \in \mathbb{N}}$  of sequences of functions  $f_n$  and  $\delta_n$  of discrete Baire class  $(\xi, \eta)$  for some  $\eta < \zeta$  with  $\text{dom}(f_n) \subseteq \text{dom}(\delta_n) \supseteq \mathcal{X}$ .
4. The notion of hybrid Baire class  $(< \xi, \zeta)$  is also defined by the same way.

Our Main Theorem 8 states for certain countable ordinals  $\xi, \zeta < \omega_1$  that a function is continuously  $(\Sigma_{\xi+1}^0, \Sigma_{\xi+\zeta+1}^0)$ -measurable if and only if it is of  $\Pi_{\xi+\zeta}^0$ -piecewise Baire class  $< \delta(\xi, \zeta)$  if and only if it is of hybrid Baire class  $(< \delta(\xi, \zeta), \xi)$ . Here,  $\delta(\xi, \zeta)$  is the least ordinal  $\delta$  such that  $\delta + \xi > \xi + \zeta$ .

## 2 Main Theorem

**Definition 6.**  $\xi, \zeta < \omega_1$  be countable ordinals. Then the ordinal  $\delta(\xi, \zeta)$  is defined as follows:

$$\delta(\xi, \zeta) = \min\{\delta < \omega_1 : \delta + \xi > \xi + \zeta\}.$$

**Example 7.**  $\delta(m, k) = k + 1$  for any natural numbers  $m, k \in \mathbb{N}$ ; and  $\delta(\omega, k) = \omega$  for the least infinite ordinal  $\omega$  and any natural number  $k \in \mathbb{N}$ ; If  $\zeta < \xi$ , then  $\delta(\xi, \zeta) \leq \xi$ .

**Theorem 8.** For any countable ordinals  $\xi, \zeta < \omega_1$  with  $\zeta < \xi \neq 1$ , and for any partial function  $F : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$  the following four conditions are equivalent.

1.  $F$  is continuously  $(\Sigma_{\xi+1}^0, \Sigma_{\xi+\zeta+1}^0)$ -measurable.
2.  $F$  is continuously  $(\Pi_{\delta(\xi, \zeta)}^0, \Sigma_{\xi+\zeta+1}^0)$ -measurable, and countably  $\Sigma_{< \delta(\xi, \zeta)}^0(X)$ -measurable.

3.  $F$  is of hybrid Baire class  $(\delta(\xi, \zeta), \xi)$ .
4.  $F$  is of  $\Pi_{\xi+\zeta}^0$ -piecewise Baire class  $< \delta(\xi, \zeta)$

**Corollary 9.** For any  $n \geq 2$  and for any partial function  $F : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$  the following four conditions are pairwise equivalent.

1.  $F$  is continuously  $(\Sigma_{n+1}^0, \Sigma_{n+1}^0)$ -measurable.
2.  $F$  is continuously  $(\Pi_1^0, \Sigma_{n+1}^0)$ -measurable, and countably continuous.
3.  $F$  is of discrete Baire class  $n$ .
4.  $F$  is  $\Pi_n^0$ -piecewise continuous.

**Corollary 10.** A function is  $(\Sigma_3^0, \Sigma_3^0)$ -measurable if and only if it is continuously  $(\Sigma_3^0, \Sigma_3^0)$ -measurable.

*Proof.* Semmes [8] showed that a function is  $(\Sigma_3^0, \Sigma_3^0)$ -measurable if and only if it is  $\Pi_2^0$ -piecewise continuous.  $\square$

**Corollary 11.** A  $\Sigma_{n+1}^0$ -measurable function is countably continuous if and only if  $\Pi_{n+1}^0$ -piecewise continuous.

*Proof.* Clearly, if a function is  $\Pi_{n+1}^0$ -piecewise continuous, then it is countably continuous. Every  $\Sigma_{n+1}^0$ -measurable function is continuously  $(\Pi_1^0, \Sigma_{n+2}^0)$ -measurable. Hence, by Theorem 8, if such a function is countably continuous, then it is  $\Pi_{n+1}^0$ -piecewise continuous.  $\square$

**Proposition 12.** There is a function on  $[0, 1]$  which is  $\Sigma_{n+1}^0$ -measurable, and  $\Pi_{n+1}^0$ -piecewise continuous, but not  $(\Pi_1^0, \Sigma_{n+1}^0)$ -measurable (hence, not  $\Pi_n^0$ -piecewise continuous).

*Proof.* Fix a  $\Pi_{n+1}^0$ -complete set  $P \subseteq 2^{\mathbb{N}}$ . Then there is a partial computable function  $\Phi$  such that  $P = \{x \in 2^{\mathbb{N}} : \#\text{dom}(\Phi(x^{(n-1)})) = \infty\}$ . Set  $2^{-\infty} = 0$ . The function  $F : 2^{\mathbb{N}} \rightarrow [0, 1]$  defined by

$$F(x) = 2^{-\#\text{dom}(\Phi(x^{(n-1)}))}$$

is (effectively)  $\Sigma_{n+1}^0$ -measurable, and  $\Pi_{n+1}^0$ -piecewise continuous, but not  $(\Pi_1^0, \Sigma_{n+1}^0)$ -measurable. To see the  $\Sigma_{n+1}^0$ -measurability of  $F$ , note that, for each  $m \in \mathbb{N}$ , the conditions  $\#\text{dom}(\Phi(x^{(n-1)})) < m$  and  $\#\text{dom}(\Phi(x^{(n-1)})) > m$  are  $\Pi_1^0(x^{(n-1)})$  and  $\Sigma_1^0(x^{(n-1)})$  (hence  $\Pi_n^0(x)$  and  $\Sigma_n^0(x)$ ) respectively. Thus,  $F$  is  $\Sigma_{n+1}^0$ -measurable. Moreover,  $F$  is  $\Pi_{n+1}^0$ -piecewise continuous (indeed,  $\Pi_{n+1}^0$ -piecewise computable), where a countable partition of  $\text{dom}(F)$  is given by  $\{F^{-1}(\{2^{-n}\})\}_{n \in \mathbb{N} \cup \{\infty\}}$ . The preimage  $F^{-1}(\{0\})$  of the  $\Pi_1^0$  singleton  $\{0\}$  is exactly  $P$  which is  $\Pi_{n+1}^0$ -complete. Hence,  $F^{-1}(\{0\})$  is not  $\Sigma_{n+1}^0$ .  $\square$

### 3 Kumabe-Slaman Forcing

The key lemma used in our proof of Theorem 8 is the following result on Turing degrees.

**Theorem 13** (Shore-Slaman Join Theorem [9]). Let  $\xi$  be a computable ordinal. The Turing degree structure  $(\mathcal{D}_T, \leq, ', \oplus)$  satisfies the following formula, for each  $k \in \mathbb{N}$ .

$$(\forall \mathbf{a}, \mathbf{b})(\exists \mathbf{c} \geq \mathbf{a}) [(\forall \zeta < \xi) \mathbf{b} \not\leq \mathbf{a}^{(\zeta)} \rightarrow (\mathbf{c}^{(\xi)} \leq \mathbf{b} \oplus \mathbf{a}^{(\xi)} \leq \mathbf{b} \oplus \mathbf{c})].$$

For  $\xi = 1$ , it is just the Posner-Robinson Join Theorem. For finite ordinals  $\xi < \omega$ , it was the problem proposed by Jockusch and Shore. The main tool to show it was introduced by Kumabe and Slaman who showed the join theorem for  $\xi = \omega$ . Finally, Shore and Slaman proved the join theorem for all computable ordinals  $\xi$ . By combining with the Slaman-Woodin double jump definability theorem, they applied the join theorem for  $\xi = 2$  to show that the Turing jump is first-order definable in the partial ordering  $(\mathcal{D}_T, \leq)$  of Turing degrees.

We use the Shore-Slaman Join Theorem to show our main theorem. For Corollary 9, we only require the Shore-Slaman Join Theorem for  $\xi = 1$ , i.e., the Posner-Robinson Join Theorem. To show the theorem 8 on all levels of Borel hierarchy, we need the Shore-Slaman Join Theorem for all countable ordinals  $\xi < \omega_1$ . By analyzing the proof of Shore-Slaman [9], it is not difficult to see the theorem for countable ordinals  $\xi < \omega_1^R$ , for any  $R \in \mathbb{N}^{\mathbb{N}}$ . Here,  $\omega_1^R$  is the least countable ordinal which is not computable in  $R$ .

**Lemma 14.** Let  $R \in \mathbb{N}^{\mathbb{N}}$  be a real of Turing degree  $\mathbf{r}$ , and  $\xi < \omega_1^R$  be a countable ordinal. The Turing degree structure  $(\mathcal{D}_T, \leq, ', \oplus)$  satisfies the following formula, for each  $k \in \mathbb{N}$ .

$$(\forall \mathbf{a}, \mathbf{b})(\exists \mathbf{c} \geq \mathbf{a} \oplus \mathbf{r}) [((\forall \zeta < \xi) \mathbf{b} \not\leq (\mathbf{a} \oplus \mathbf{r})^{(\zeta)}) \rightarrow (\mathbf{c}^{(\xi)} \leq \mathbf{b} \oplus (\mathbf{a} \oplus \mathbf{r})^{(\xi)} \leq \mathbf{b} \oplus \mathbf{c})].$$

Here, the notion  $\mathbf{d}^{(\xi)}$  could be ill-defined. Therefore, if  $R$  is fixed, by  $\mathbf{d}^{(\xi:R)}$  only for  $\mathbf{d} \geq \mathbf{r}$ , we denote the  $\xi$ -th Turing jump relative to  $R$  which is defined along with Kleene's system  $\mathcal{O}^R$  of ordinal notations relative to  $R$ . Then, the notion  $\mathbf{d}^{(\xi)} = \mathbf{d}^{(\xi:R)}$  for  $\mathbf{d} \geq \mathbf{r}$  and  $\xi < \omega_1^R$  turns out to be well-defined, as the Spector uniqueness theorem (but,  $\mathbf{d}^{(\xi:R)}$  and  $\mathbf{d}^{(\xi:S)}$  could be different, for  $S \neq R$ , even if  $\mathbf{d} \geq \max\{\mathbf{r}, \mathbf{s}\}$  and  $\xi < \min\{\omega_1^R, \omega_1^S\}$ ).

## 4 Lightface Hierarchy and Relativization Lemmata

Our representation of the Borel sets is essentially same as Brattka [1].

**Definition 15** (Effective Borel sets). Let  $(\mathcal{X}, d)$  be a separable metric space with a countable base  $\{B_n\}_{n \in \mathbb{N}}$ . Fix an oracle  $R$ , and let  $\mathcal{O}^R$  denote Kleene's system of ordinal notations relative to  $R$ . Here we assume that  $1 \in \mathcal{O}^R$  represents the ordinal 1. We define representations  $\rho[\Sigma_{a:R}^0] : \mathbb{N}^{\mathbb{N}} \rightarrow \Sigma_{\xi}^0|_{\mathcal{X}}$ ,  $\rho[\Pi_{a:R}^0] : \mathbb{N}^{\mathbb{N}} \rightarrow \Pi_{\xi}^0|_{\mathcal{X}}$ , and  $\rho[\Delta_{a:R}^0] : \mathbb{N}^{\mathbb{N}} \rightarrow \Delta_{\xi}^0|_{\mathcal{X}}$  for each  $a \in \mathcal{O}^R$ . Here,  $\xi < \omega_1^R$  is the ordinal represented by  $a$ . Let  $p \in \mathbb{N}^{\mathbb{N}}$  be a real, and  $a \in \mathcal{O}^R$ .

1.  $\rho[\Sigma_{1:R}^0](p) = \bigcup_{n \in \mathbb{N}} B_{p(n)}$ .
2.  $\rho[\Pi_{a:R}^0](p) = \mathcal{X} \setminus \rho[\Sigma_{a:R}^0](p)$ .
3.  $\rho[\Delta_{a:R}^0](p, q) = \rho[\Sigma_{a:R}^0](p)$ , if and only if  $\rho[\Sigma_{a:R}^0](p) = \rho[\Pi_{a:R}^0](q)$ .
4. If  $a = 2^b$ , then  $\rho[\Sigma_{a:R}^0](p_0, p_1, \dots) = \bigcup_{n \in \mathbb{N}} \rho[\Pi_{a:R}^0](p_n)$ .
5. If  $a = 3 \cdot 5^b$ , then  $\rho[\Sigma_{a:R}^0](p_0, p_1, \dots) = \bigcup_{n \in \mathbb{N}} \rho[\Pi_{\Phi_e(R;n):R}^0](p_n)$ .

If  $R$  is fixed, for  $\xi = |a|_{\mathcal{O}^R}$ , we simply write  $\Sigma_{\xi}^0 = \Sigma_{a:R}^0$ ,  $\Pi_{\xi}^0 = \Pi_{a:R}^0$ , and  $\Delta_{\xi}^0 = \Delta_{a:R}^0$ . For  $\Gamma \in \{\Sigma, \Pi, \Delta\}$ , and  $X \geq_T R$ , a set  $A \in \Gamma_{\xi}^0$  is represented by an  $X$ -computable real  $p \in \mathbb{N}^{\mathbb{N}}$ , i.e.,  $\rho[\Gamma_{\xi}^0](p) = A$ , then  $A$  is called a  $\Gamma_{\xi}^0(X)$  set (for instance, if  $\Gamma = \Sigma$ , then it is a  $\Sigma_{\xi}^0(X)$  set). For such  $p$ , if  $p = \Phi_e(X)$ , then  $e$  is an  $\Gamma_{\xi}^0(X)$ -index of  $A$ , and  $A$  is also called the  $e$ -th  $\Gamma_{\xi}^0(X)$  set.

For  $\Gamma, \Lambda \in \{\Sigma, \Pi, \Delta\}$ ,  $\xi, \zeta < \omega_1^R$ , and  $X \geq_T R$ , a function  $F : \Gamma_{\xi}^0 \rightarrow \Lambda_{\zeta}^0$  is *continuous* (resp.  *$X$ -computable*) if there is a continuous (resp.  $X$ -computable) function  $f : \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$  such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{N}^{\mathbb{N}} & \xrightarrow{f} & \mathbb{N}^{\mathbb{N}} \\ \rho[\Gamma_{\xi}^0] \downarrow & & \downarrow \rho[\Lambda_{\zeta}^0] \\ \Gamma_{\xi}^0 & \xrightarrow{F} & \Lambda_{\zeta}^0 \end{array}$$

The effective hierarchy of Borel measurable functions has been studied by Brattka [1].

**Definition 16.** Let  $R, X, Y \in \mathbb{N}^{\mathbb{N}}$  be reals,  $\xi, \zeta < \omega_1^R$  be ordinals, and  $f$  be a function between computable metric spaces  $\mathcal{X}$  and  $\mathcal{Y}$ .

1.  $f$  is  $X$ -Markov  $(\Sigma_{\xi}^0, \Sigma_{\zeta}^0(Y))$ -measurable if the preimage  $f^{-1}(A) \in \Sigma_{\zeta}^0(X \oplus Y)$  for any  $A \in \Sigma_{\xi}^0(X)$ , and  $f^{-1} : \Sigma_{\xi}^0(X)|_{\mathcal{Y}} \rightarrow \Sigma_{\zeta}^0(X \oplus Y)|_{\mathcal{X}}$  is  $X$ -computable. In other words, there is an  $X$ -computable function  $d : \mathbb{N} \rightarrow \mathbb{N}$  such that  $d(e)$  is a  $\Sigma_{\zeta}^0(X \oplus Y)$ -index of the preimage  $f^{-1}(A)$  of the  $e$ -th  $\Sigma_{\xi}^0(X)$  set  $A \subseteq \mathcal{Y}$ , for any  $e \in \mathbb{N}$ .
2.  $f$  is  $X$ -effectively  $(\Sigma_{\xi}^0, \Sigma_{\zeta}^0)$ -measurable if it is  $(\Sigma_{\xi}^0, \Sigma_{\zeta}^0)$ -measurable, and the function  $f^{-1} : \Sigma_{\zeta}^0|_{\mathcal{Y}} \rightarrow \Sigma_{\xi}^0|_{\mathcal{X}}$  is  $X$ -computable. In other words, there is an  $X$ -computable function  $d : \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \rightarrow \mathbb{N}$  such that  $d(Y, e)$  is a  $\Sigma_{\zeta}^0(X \oplus Y)$ -index of the preimage  $f^{-1}(A)$  of the  $e$ -th  $\Sigma_{\xi}^0(Y)$  set  $A \subseteq \mathcal{Y}$ , for any  $(Y, e) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}$ .

**Proposition 17.** Let  $R \in \mathbb{N}^{\mathbb{N}}$  be a real, and  $\xi < \omega_1^R$  be an ordinal. Assume  $X \geq_T R$ . Then, a function is  $X$ -effectively  $\Sigma_\xi^0$ -measurable if and only if it is  $X$ -Markov  $\Sigma_\xi^0$ -measurable.  $\square$

**Lemma 18** (Relativization I). Let  $\xi, \zeta < \omega_1$  be countable ordinals. A partial function  $f$  is continuously  $(\Sigma_\xi^0, \Sigma_\zeta^0)$ -measurable if and only if it is  $R$ -effectively  $(\Sigma_{\xi;R}^0, \Sigma_{\zeta;R}^0)$ -measurable for some  $R \in \mathbb{N}^{\mathbb{N}}$  with  $\xi, \zeta < \omega_1^R$ .  $\square$

**Definition 19.** Let  $\Gamma$  be a lightface pointclass, and  $\Lambda$  be a property of functions. A function  $f : \mathcal{X} \rightarrow \mathcal{Y}$  is  $\Gamma$ -piecewise  $\Lambda$  if there is a covering  $\{Q_i\}_{i \in \mathbb{N}}$  of  $\mathcal{X}$  such that the sequence  $\{Q_i\}_{i \in \mathbb{N}}$  is uniformly  $\Gamma$ , and  $f \upharpoonright Q_i$  satisfies  $\Lambda$  uniformly in  $i \in \mathbb{N}$ .

**Lemma 20** (Relativization II). Let  $\xi < \omega_1$  be a countable ordinal. Let  $\Gamma$  be a lightface pointclass, and  $\mathbf{\Gamma}$  be its boldface version. A function is  $\mathbf{\Gamma}$ -piecewise  $\Sigma_\xi^0$ -measurable if and only if it is  $\Gamma(R)$ -piecewise  $R$ -effectively  $\Sigma_{\xi;R}^0$ -measurable for some  $R \in \mathbb{N}^{\mathbb{N}}$  with  $\xi < \omega_1^R$ .

*Proof.* If  $f$  is  $\mathbf{\Gamma}$ -piecewise  $\Sigma_\xi^0$ -measurable, then we have a countable partition  $\{X_i\}_{i \in \mathbb{N}}$  such that  $X_i \in \mathbf{\Gamma}$  and  $f_i = f \upharpoonright X_i$  is  $\Sigma_\xi^0$ -measurable for each  $i \in \mathbb{N}$ . Choose a real  $Q \in \mathbb{N}^{\mathbb{N}}$  such that  $\xi < \omega_1^Q$ . Then, for each  $i \in \mathbb{N}$ ,  $X_i$  is  $\Gamma(x_i)$  for some real  $x_i \in \mathbb{N}^{\mathbb{N}}$ , and  $f_i$  is  $(c_i \oplus Q)$ -effectively  $\Sigma_{\xi;Q}^0$ -measurable for some real  $c_i \in \mathbb{N}^{\mathbb{N}}$ . Consider functions  $p, q : \mathbb{N} \rightarrow \mathbb{N}$  such that  $X_i$  is the  $p(i)$ -th  $\Gamma(x_i)$  set, and  $f_i$  is the  $q(i)$ -th partial  $(c_i \oplus Q)$ -effectively  $\Sigma_\xi^0$ -measurable function, for each  $i \in \mathbb{N}$ . Put  $R = Q \oplus p \oplus q \oplus \bigoplus_{i \in \mathbb{N}} x_i \oplus \bigoplus_{i \in \mathbb{N}} c_i$ . Then,  $\{X_i\}_{i \in \mathbb{N}}$  is uniformly  $\Gamma(R)$ , and  $\{f_i\}_{i \in \mathbb{N}}$  is uniformly  $R$ -effectively  $\Sigma_{\xi;R}^0$ -measurable. Hence,  $f$  is  $\Gamma(R)$ -piecewise  $R$ -effectively  $\Sigma_{\xi;R}^0$ -measurable.  $\square$

**Lemma 21** (Relativization III). Let  $\xi < \omega_1$  be a countable ordinal. A function  $f$  is countably  $\Sigma_{\xi+1}^0$ -measurable if and only if there is  $R \in \mathbb{N}^{\mathbb{N}}$  with  $\xi < \omega_1^R$  satisfying the following condition:

$$f(x) \leq_T (x \oplus R)^{(\xi)}, \text{ for any } x \in \text{dom}(f).$$

*Proof.* By Lemma 20, if  $f$  is countably  $\Sigma_{\xi+1}^0$ -measurable, then there is a pointclass  $\Gamma$  such that  $f$  is  $\Gamma(R)$ -piecewise  $R$ -effectively  $\Sigma_{\xi+1}^0$ -measurable for some  $R$ . If  $g$  is  $R$ -effectively  $\Sigma_{\xi+1}^0$ -measurable, then  $g(x) \leq_T (x \oplus R)^{(\xi)}$  is satisfied for any  $x \in \text{dom}(g)$  by the universality of the Turing jump. Conversely, for some  $R \in \mathbb{N}^{\mathbb{N}}$ , if a function satisfies  $f(x) \leq_T (x \oplus R)^{(\xi)}$  for any  $x \in \text{dom}(f)$ , then define  $Q_e = \{x \in \text{dom}(f) : \Phi_e((x \oplus R)^{(\xi)}) = f(x)\}$ . Note that  $F_e(x) = \Phi_e((x \oplus R)^{(\xi)})$  is  $R$ -effectively  $\Sigma_{\xi+1}^0$ -measurable. Clearly,  $\bigcup_e Q_e = \text{dom}(f)$ , and  $f \upharpoonright Q_e = F_e \upharpoonright Q_e$ .  $\square$

**Definition 22** (Effective hybrid Baire hierarchy). Fix  $R \in \mathbb{N}^{\mathbb{N}}$ ,  $X \geq_T R$ , and let  $\xi \leq \omega_1^R$  be a countable ordinal. Let  $f$  be a function between computable metric spaces.

1.  $f$  is of  $X$ -effective hybrid Baire class  $(0, 0)$  if it is  $X$ -computable.
2.  $f$  is of  $X$ -effective hybrid Baire class  $(\xi, 0)$  if there is a uniform sequence  $\{G_k\}_{k \in \mathbb{N}}$  of functions of  $X$ -effective hybrid Baire class  $(\zeta, 0)$  for  $\zeta < \xi$  which converges to  $f$ .
3. For a limit ordinal  $\xi$ , the function  $f$  is of hybrid Baire class  $(< \xi, 0)$  if it is of Baire class  $\zeta$  for some  $\zeta < \xi$ ; For a successor ordinal  $\xi$ , the function  $f$  is of hybrid Baire class  $(< \xi, 0)$  if it is of Baire class  $\xi$ .
4. For any nonzero countable ordinal  $\zeta < \omega_1^R$ , the function  $f$  is of  $X$ -effective hybrid Baire class  $(\xi, \zeta)$  if it is the discrete pointwise limit of a discrete pair  $\{(f_n, \delta_n)\}_{n \in \mathbb{N}}$  of uniform sequences of functions  $f_n$  and  $\delta_n$  of discrete Baire class  $(\xi, \eta)$  for some  $\eta < \zeta$  with  $\text{dom}(f_n) \subseteq \text{dom}(\delta_n) \supseteq \text{dom}(f)$ .
5. The notion of  $X$ -effective hybrid Baire class  $(< \xi, \zeta)$  is also defined by the same way.

A partial function  $F$  is of  $\beta$ -effective Baire class  $n$  if it is of  $\beta$ -effective hybrid Baire class  $(n, 0)$ . A partial function  $F$  is of  $\beta$ -effective discrete Baire class  $n$  if it is of  $\beta$ -effective hybrid Baire class  $(0, n)$ .

**Lemma 23** (Relativization IV). Let  $\xi, \zeta < \omega_1$  be countable ordinals. A function is of hybrid Baire class  $(\xi, \zeta)$  if and only if it is of  $R$ -effective hybrid Baire class  $(\xi, \zeta)$  for some  $R \in \mathbb{N}^{\mathbb{N}}$  with  $\xi, \zeta < \omega_1^R$ .

*Proof.* If a function is of hybrid Baire class  $(\xi, \zeta)$ , then it is constructed from countably many continuous functions. Hence, we choose an oracle  $R$  in which these continuous functions are relatively uniformly computable.  $\square$

## 5 Proof of Main Theorem

**Lemma 24.** *Let  $R, X, Y \in \mathbb{N}^{\mathbb{N}}$  be reals, and  $\xi, \zeta < \omega_1^R$  be ordinals. The following are equivalent for any partial function  $F : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$ .*

1.  $F$  is of  $X$ -effective hybrid Baire class  $(< \delta(\xi, \zeta), \xi)$ .
2.  $F$  is  $\Pi_{\xi+\zeta}^0(X)$ -piecewise  $X$ -effectively  $\Sigma_{<\delta(\xi, \zeta)}^0$ -measurable.
3. For every  $x \in \text{dom}(F)$ , there is  $n \in \mathbb{N}$  such that  $F(x) \leq_T (x \oplus X)^{(\eta(n))}$ , where  $\{\eta(n)\}_{n \in \mathbb{N}}$  is an  $R$ -computable increasing sequence of ordinals with  $\lim_n(\eta(n) + 1) = \delta(\xi, \zeta)$ . Moreover, its index and  $n$  is  $(x \oplus X)^{(\xi+\zeta)}$ -computable uniformly in  $x \in \mathbb{N}^{\mathbb{N}}$ , i.e., there is a computable function  $\Psi, \Delta : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}$  such that, for any  $x \in \text{dom}(F)$ ,

$$F(x) = \Phi_{\Psi((x \oplus X)^{(\xi+\zeta)})}((x \oplus X)^{(\eta(\Delta((x \oplus X)^{(\xi+\zeta)})))}).$$

*Proof.* We only describe the case for finite  $\xi = n, \zeta = k \in \mathbb{N}$ .

(1)→(2): Assume that  $F$  is a function of  $X$ -effective hybrid Baire class  $(k, n)$ . For a discrete function  $(G, \delta) : \mathbb{N} \times \mathcal{X} \rightarrow \mathcal{Y} \times \mathbb{N}$ , we denote the pointwise discrete limit of  $(G, \delta) = \{(G_n, \delta_n)\}_{n \in \mathbb{N}}$  by  $\lim(G, \delta)$ , and write  $\lim_{n \rightarrow \infty}(G_n, \delta_n) = (\lim(G, \delta), \lim(\delta, \delta))$ , where note that  $(\delta, \delta)$  is a discrete function whenever  $(G, \delta)$  is discrete. Since  $F$  is of  $X$ -effective hybrid Baire class  $(k, n)$ , there is a  $X$ -effective  $\Sigma_k^0$ -measurable functions  $(G, \delta) = (G_{\vec{s}}, \delta_{\vec{s}})_{\vec{s} \in \mathbb{N}^n} : \mathbb{N}^n \times X \rightarrow Y \times \mathbb{N}^n$ , which are discrete and

$$F = \lim_{v_1 \rightarrow \infty} \dots \lim_{v_n \rightarrow \infty} (G_{v_1, \dots, v_n}, \delta_{v_1, \dots, v_n}).$$

Then,  $(G, \delta) = \{(G_v, \delta_v)\}_{v \in \mathbb{N}}$ , where  $(G_v, \delta_v) = \lim_{v_2 \rightarrow \infty} \dots \lim_{v_n \rightarrow \infty} (G_{v, v_2, \dots, v_n}, \delta_{v, v_2, \dots, v_n})$ , is of  $X$ -effective hybrid Baire class  $(k, n-1)$ . It is easy to see that, an  $(x \oplus X)^{(k)}$ -index of  $(G_v, \delta_v)(x)$  is  $(x \oplus X)^{(n+k-1)}$ -computable, uniformly in  $x$  and  $v$ . We also have a total  $(x \oplus X)^{(n+k-1)}$ -computable function  $\delta^x : \mathbb{N} \rightarrow \mathbb{N}$  such that, if  $\lim_v \delta^x(v)$  exists, then  $\lim_v \delta_v(x)$  exists and  $\lim_v \delta^x(v) = \lim_v \delta_v(x)$ , by defining  $\delta^x(v)$  to be the  $\delta_u(x)$  for the greatest  $u \leq v$  such that  $\delta_u(x)$  is defined by stage  $v$  in the  $(x \oplus X)^{(n+k-1)}$ -relative computation of  $\{\delta_v(x)\}_{v \in \mathbb{N}}$ . By uniformity and totality, the graph  $\{(x, v, e) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^2 : \delta^x(v) = e\}$  is  $\Delta_{n+k}^0(X)$ . Consider the following set  $P_{e,s}$  for each  $e, s \in \mathbb{N}$ .

$$P_{e,s} = \{x \in \mathbb{N}^{\mathbb{N}} : (\forall t \geq s) \delta^x(t) = e\}.$$

This set is  $\Pi_{n+k}^0(X)$ , and its index is uniformly calculated from  $e, s$ . Moreover,  $G_e \upharpoonright P_{e,s} = F \upharpoonright P_{e,s}$ , by our definition of  $\delta^x$ . Consequently,  $F$  is  $\Pi_{n+k}^0(X)$ -piecewise  $X$ -effectively  $\Sigma_k^0$ -measurable via  $\{P_{e,s}; G_e\}_{e,s \in \mathbb{N}}$ .

(2)→(3): Assume that  $F$  is  $\Pi_{n+k}^0(X)$ -piecewise  $X$ -effectively  $\Sigma_{k+1}^0$ -measurable, via a uniform sequence  $\{P_e\}_{e \in \mathbb{N}}$  of  $\Pi_{n+k}^0(X)$  sets and a computable sequence  $\{g_e\}_{e \in \mathbb{N}}$  of  $X$ -effectively  $\Sigma_{k+1}^0$ -measurable functions with  $F \upharpoonright P_e = g_e \upharpoonright P_e$ . Then, clearly,  $g_e(x)$  is  $(x \oplus X)^{(k)}$ -computable, and its index is computable uniformly in  $e \in \mathbb{N}$  and  $x \in \text{dom}(g_e)$ . Let  $d : \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \rightarrow \mathbb{N}$  be a computable function such that  $d(x, e)$  returns an  $(x \oplus X)^{(k)}$ -index of  $g_e(x)$ . As  $\{P_e\}_{e \in \mathbb{N}}$  is uniformly  $\Pi_{n+k}^0(X)$ , we can also compute the least  $e$  with  $x \in P_e$ , by an  $(x \oplus X)^{(n+k)}$ -computable way, uniformly in  $x$ . By  $c(x)$ , we denote its index. Then,  $E((x \oplus X)^{(n+k)})$  is defined to be  $d(x, c(x))$ . If  $e$  is the least number with  $x \in P_e$ , then it is easy to see the condition

$$\Phi_{E((x \oplus X)^{(n+k)})}((x \oplus X)^{(k)}) = \Phi_{d(x,e)}((x \oplus X)^{(k)}) = g_e(x) = F(x),$$

as desired.

(3)→(1): For each natural number  $z \in \mathbb{N}$ , fix a computable function  $J^{(z)} : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^z \rightarrow \mathbb{N}^{\mathbb{N}}$  satisfying the following condition:

$$y^{(z)} = \lim_{u_1 \rightarrow \infty} \dots \lim_{u_z \rightarrow \infty} J^{(z)}(y, u_1, \dots, u_z).$$

For each  $\vec{s} = \langle s_1, \dots, s_n \rangle$  and  $x \in \mathbb{N}^{\mathbb{N}}$ , define  $G_{\vec{s}}(x)$  and  $\delta_{\vec{s}}(x)$  as follows:

$$\begin{aligned} G_{\vec{s}}(x) &= \lim_{s_{n+1} \rightarrow \infty} \dots \lim_{s_{n+k} \rightarrow \infty} \Phi_{\Psi(J^{(n+k)}(x \oplus X, \vec{s}, s_{n+1}, \dots, s_{n+k}))} (J^{(k)}(x \oplus X, s_{n+1}, s_{n+2}, \dots, s_{n+k})). \\ \delta_{\vec{s}}(x) &= \lim_{s_{n+1} \rightarrow \infty} \dots \lim_{s_{n+k} \rightarrow \infty} \Psi(J^{(n+k)}(x \oplus X, \vec{s}, s_{n+1}, \dots, s_{n+k})) \end{aligned}$$

Clearly  $(G_{\vec{s}}, \delta_{\vec{s}})$  is  $X$ -effectively  $\Sigma_{k+1}^0$ -measurable uniformly in  $\vec{s}$ , and  $G_{\vec{s}}(x) = \Phi_{\delta_{\vec{s}}(x)}((x \oplus X)^{(k)})$  holds. Note that, for any  $x$ ,  $\vec{s}$  and  $\vec{t}$ , if  $\delta_{\vec{s}}(x) \neq \delta_{\vec{t}}(x)$ , then  $G_{\vec{s}}(x) \neq G_{\vec{t}}(x)$ . For  $\vec{s} \in \mathbb{N}^{n-1}$ , we first define  $(G_{\vec{s}}, \delta_{\vec{s}})(t, x) = (G_{\vec{s}, t}(x), \delta_{\vec{s}, t}(x))$ , and take its pointwise discrete limit. By iterating this procedure,  $F$  is easily constructed as pointwise discrete limits of  $\{(G_{\vec{s}}, \delta_{\vec{s}})\}_{\vec{s} \in \mathbb{N}^{<n}}$ . Hence,  $F$  is of  $X$ -effectively hybrid Baire class  $(k, n)$ .  $\square$

**Lemma 25.** *Let  $R, X, Y \in \mathbb{N}^{\mathbb{N}}$  be reals, and  $\xi, \zeta < \omega_1^R$  be ordinals. The following are equivalent for any partial function  $f : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$ .*

1.  $f$  is  $X$ -Markov  $(\Sigma_{\xi+1}^0, \Sigma_{\xi+\zeta+1}^0(Y))$ -measurable.
2.  $f$  is  $X$ -Markov  $(\Pi_{\xi}^0, \Sigma_{\xi+\zeta+1}^0(Y))$ -measurable.
3.  $f$  is  $X$ -Markov  $(\Sigma_{\xi}^0, \Delta_{\xi+\zeta+1}^0(Y))$ -measurable.

*Proof.* The equivalence of the conditions 1 and 2 is easy. Assume that  $f$  is  $X$ -Markov  $(\Sigma_{\xi+1}^0, \Sigma_{\xi+\zeta+1}^0(Y))$ -measurable. Then, by an  $X$ -computable way, we can calculate  $\Sigma_{\xi+\zeta+1}^0(X \oplus Y)$ -indices of  $f^{-1}(A)$  and  $f^{-1}(\mathbb{N}^{\mathbb{N}} \setminus A)$  from a  $\Sigma_{\xi}^0(X)$ -index of  $A \subseteq \mathbb{N}^{\mathbb{N}}$  and a  $\Pi_{\xi}^0(X)$ -index of  $\mathbb{N}^{\mathbb{N}} \setminus A$ . Clearly, the whole space  $\mathbb{N}^{\mathbb{N}}$  is the disjoint union of  $f^{-1}(A)$  and  $f^{-1}(\mathbb{N}^{\mathbb{N}} \setminus A)$ . Therefore,  $f^{-1}(A)$  is  $\Delta_{\xi+\zeta+1}^0(X \oplus Y)$ , effectively in  $\Sigma_{\xi}^0(X)$  sets  $A \subseteq \mathbb{N}^{\mathbb{N}}$ . Conversely, assume that  $f$  satisfies the condition 3. Then, for a given  $\Sigma_{\xi}^0(X)$  set  $S = \bigcup_{k \in \mathbb{N}} P_k$ , we can calculate a  $\Delta_{\xi+\zeta+1}^0(X \oplus Y)$  index of  $f^{-1}(P_k)$ , uniformly in  $k \in \mathbb{N}$ . Thus,  $f^{-1}(S) = \bigcup_{k \in \mathbb{N}} f^{-1}(P_k)$  is  $\Sigma_{\xi+\zeta+1}^0(X \oplus Y)$ , effectively from an index of  $S$ .  $\square$

**Lemma 26.** *Let  $R, X, Y \in \mathbb{N}^{\mathbb{N}}$  be reals, and  $\xi, \zeta < \omega_1^R$  be ordinals. Every  $X$ -Markov  $(\Sigma_{\xi+1}^0, \Sigma_{\xi+\zeta+1}^0(Y))$ -measurable function  $f : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$  satisfies the condition  $(f(x) \oplus X)^{(\xi)} \leq_T (x \oplus X \oplus Y)^{(\xi+\zeta)}$  for any  $x \in \text{dom}(f)$ .*

*Proof.* By  $S_e^{(\xi), X}$ ,  $P_e^{(\xi), X}$ , and  $D_e^{(\xi), X}$ , we denote the  $e$ -th  $\Sigma_{\xi}^0(X)$  set,  $\Pi_{\xi}^0(X)$  set, and  $\Delta_{\xi}^0(X)$  set, respectively. Fix a computable function  $\pi : \mathbb{N} \rightarrow \mathbb{N}$  such that  $S_{\pi(e)}^{(\xi), X} = \{x \in \mathbb{N}^{\mathbb{N}} : x \oplus X \in S_e^{(\xi), X}\}$ . By  $X$ -Markov  $(\Sigma_{\xi+1}^0, \Sigma_{\xi+\zeta+1}^0(Y))$ -measurability of  $f$  and Lemma 25, there is an  $X$ -computable function  $d : \mathbb{N} \rightarrow \mathbb{N}$ ,  $x \in D_{d(e)}^{(\xi+\zeta+1), X \oplus Y}$  if and only if  $f(x) \in S_{\pi(e)}^{(\xi), X}$ , for any  $e \in \mathbb{N}$  and  $x \in \text{dom}(f)$ . Then, we can easily calculate an index  $d_0^*(e)$ ,  $d_1^*(e)$  such that

$$x \oplus X \oplus Y \in S_{d_0^*(e)}^{(\xi+\zeta+1)} \iff x \oplus X \oplus Y \in P_{d_1^*(e)}^{(\xi+\zeta+1)} \iff f(x) \oplus X \in S_e^{(\xi)}.$$

Therefore, we can determine the value of  $(f(x) \oplus X)^{(\xi)}(e)$  by a uniformly  $(x \oplus X \oplus Y)^{(\xi+\zeta)}$ -computable way. In other words,  $(f(x) \oplus X)^{(\xi)} \leq_T (x \oplus X \oplus Y)^{(\xi+\zeta)}$ .  $\square$

Recall that  $\delta(\xi, \zeta)$  is the least ordinal  $\delta$  such that  $\delta + \xi > \xi + \zeta$ .

**Lemma 27.** *Let  $R, X, Y \in \mathbb{N}^{\mathbb{N}}$  be reals, and  $\xi, \zeta < \omega_1^R$  be ordinals. Assume that  $f : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$  is an  $X$ -effectively  $(\Sigma_{\xi+1}^0, \Sigma_{\xi+\zeta+1}^0)$ -measurable function. Then, for every  $x \in \text{dom}(f)$ , there is an ordinal  $\gamma < \delta(\xi, \zeta)$  such that  $f(x) \leq_T (x \oplus X)^{(\gamma)}$  holds.*

*Proof.* Fix an  $X$ -effectively  $(\Sigma_{\xi+1}^0, \Sigma_{\xi+\zeta+1}^0)$ -measurable function  $f : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$ . Then, clearly,  $f$  is  $(X \oplus z)$ -Markov  $(\Sigma_{\xi+1}^0, \Sigma_{\xi+\zeta+1}^0(X))$ -measurable, for any  $z \in \mathbb{N}^{\mathbb{N}}$ . By Lemma 26, the function  $f$  must satisfy the following inequality:

$$(f(x) \oplus X \oplus z)^{(\xi)} \leq_T (x \oplus X \oplus z)^{(\xi+\zeta)}, \quad (1)$$

for any  $z \in \mathbb{N}^{\mathbb{N}}$  and  $x \in \text{dom}(f)$ . Assume that, there is  $x \in \text{dom}(f)$  such that  $f(x) \not\leq_T (x \oplus X)^{(\gamma)}$  holds for any  $\gamma < \delta = \delta(\xi, \zeta)$ . Put  $y_X = f(x)$ , and  $x_X = x \oplus X$ . By the inequality (1) with  $z = \emptyset$ , and from the definition of  $\delta$ , we have  $y_X^{(\xi)} \leq_T x_X^{(\xi+\zeta)} <_T x_X^{(\delta+\xi)}$ . Therefore, there is an ordinal  $\eta < \xi$  such that  $y_X^{(\eta)} \leq_T x_X^{(\delta+\eta)}$ . Fix the least such ordinal  $\eta < \xi$ . We now have the following condition:

$$y_X^{(\theta)} \not\leq_T x_X^{(\delta+\theta)} \text{ for all } \theta < \eta, \text{ and } y_X^{(\eta)} \leq_T x_X^{(\delta+\eta)}.$$

Let  $\mathbf{a}$  be the Turing degree of  $x_X$ , and  $\mathbf{b}$  be the Turing degree of  $y_X^{(\eta)}$ . By Shore-Slaman Join Theorem 14 with such  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\delta + \eta$ , and by choosing  $z \in \mathbf{c}$ , we have  $z \geq_T x_X$  such that  $z^{(\delta+\eta)} \equiv_T x_X^{(\delta+\eta)} \equiv_T y_X^{(\eta)} \oplus z$ , since  $\eta$  is the least ordinal such that  $y_X^{(\eta)} \leq_T x_X^{(\delta+\eta)}$ . As  $\eta < \xi$ , there is an ordinal  $\theta \leq \xi$  such that  $\eta + \theta = \xi$ . Then, we obtain the following:

$$x_X^{(\xi+\zeta)} <_T x_X^{(\delta+\xi)} \equiv_T (x_X^{(\delta+\eta)})^{(\theta)} \equiv_T (y_X^{(\eta)} \oplus z)^{(\theta)} \leq_T (y_X \oplus z)^{(\eta+\theta)} \equiv_T (y_X \oplus z)^{(\xi)}.$$

Moreover, by the inequality (1), and by the property of  $x$ ,

$$(y_X \oplus z)^{(\xi)} \leq_T (x_X \oplus z)^{(\xi+\zeta)} \equiv_T z^{(\xi+\zeta)} \equiv_T x_X^{(\xi+\zeta)},$$

since  $\delta + \eta \leq \xi + \zeta$ . By combining the above two inequalities, we have  $x_X^{(\xi+\zeta)} < x_X^{(\xi+\zeta)}$  which is a contradiction. Therefore, for any  $x \in \text{dom}(f)$ , there must exist an ordinal  $\gamma < \delta$  such that  $f(x) \leq_T (x \oplus X)^{(\gamma)}$  holds.  $\square$

**Theorem 28.** *Let  $R, X \in \mathbb{N}^{\mathbb{N}}$  be reals. For any ordinals  $\xi, \zeta < \omega_1^R$  with  $\zeta < \xi \neq 1$ , and for any partial function  $F : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$  the following five conditions are pairwise equivalent.*

1.  $F$  is  $X$ -effectively  $(\Sigma_{\xi+1}^0, \Sigma_{\xi+\zeta+1}^0)$ -measurable.
2.  $F$  is  $X$ -effectively  $(\Pi_{\delta(\xi, \zeta)}^0, \Sigma_{\xi+\zeta+1}^0)$ -measurable, and, for any  $x \in \text{dom}(F)$ , there is  $\gamma < \delta(\xi, \zeta)$  such that  $F(x) \leq_T (x \oplus X)^{(\gamma)}$ .
3.  $F$  is of  $X$ -effective hybrid Baire class  $(< \delta(\xi, \zeta), \xi)$ .
4.  $F$  is  $\Pi_{\xi+\zeta}^0(X)$ -piecewise effectively  $\Sigma_{< \delta(\xi, \zeta)}^0(X)$ -measurable.

*Proof.* If  $\xi = 0$ , then it is the effectivization of Lebesgue-Hausdorff-Banach theorem. See Brattka [1]. So we assume  $\xi \geq 2$ .

(1)  $\rightarrow$  (2): By Lemma 27 and  $\xi > \zeta$ . (3)  $\leftrightarrow$  (4): By Lemma 24.

(2)  $\rightarrow$  (4): Assume that  $F$  is  $X$ -effectively  $(\Pi_{\xi+1}^0, \Sigma_{\xi+\zeta+1}^0)$ -measurable, and, for any  $x \in \text{dom}(F)$ , there is  $\gamma < \delta(\xi, \zeta)$  such that  $F(x) \leq_T (x \oplus X)^{(\gamma)}$ .

First we assume that  $\delta = \delta(\xi, \zeta)$  is a successor ordinal, i.e.,  $\delta = \eta + 1$  for an ordinal  $\eta < \delta$ . Then,  $F(x) \leq_T (x \oplus X)^{(\eta)}$  for all  $x \in \text{dom}(F)$ . For each  $x \in \mathbb{N}^{\mathbb{N}}$  and  $e \in \mathbb{N}$ , define  $I_e^x$  to be the singleton  $\{\Phi_e((x \oplus X)^{(\eta)})\}$  if  $\Phi_e((x \oplus X)^{(\eta)}; m)$  is defined for all  $m \in \mathbb{N}$ . Otherwise,  $I_e^x$  is defined to be the clopen set generated by the maximal initial segment of  $\Phi_e((x \oplus X)^{(\eta)})$ . Then we can calculate a  $\Pi_{\eta+1}^0(x \oplus X)$ -index of  $I_e^x$  uniformly in  $e$  and  $x$ .

Next, we assume that  $\delta = \delta(\xi, \zeta)$  is a limit ordinal. Pick  $d \in \mathcal{O}^R$  representing  $\delta$ , and write  $d(n) = \Phi_d(n)$  which represents an ordinal  $\eta(n) < \delta$  with  $\lim_n \eta(n) = \delta$ . Then, for every  $x \in \text{dom}(F)$ , there is  $n \in \mathbb{N}$  such that  $F(x) \leq_T (x \oplus X)^{(\eta(n))}$ . For each  $x \in \mathbb{N}^{\mathbb{N}}$  and  $e, n \in \mathbb{N}$ , define  $I_{e,n}^x$  to be the singleton  $\{\Phi_e((x \oplus X)^{(\eta(n))})\}$  if  $\Phi_e((x \oplus X)^{(\eta(n))}; m)$  is defined for all  $m \in \mathbb{N}$ . Otherwise,  $I_{e,n}^x$  is defined to be the clopen set generated by the maximal initial segment of  $\Phi_e((x \oplus X)^{(\eta(n))})$ . Then we can calculate a  $\Pi_{\eta(n)+1}^0(x \oplus X)$ -index of  $I_{e,n}^x$  uniformly in  $e, n$  and  $x$ .

We only describe the case that  $\delta = \delta(\xi, \zeta)$  is a limit ordinal. By  $X$ -effective  $(\Pi_{\delta(\xi, \zeta)}^0, \Sigma_{\xi+\zeta+1}^0)$ -measurability of  $F$ , we can also calculate a  $\Sigma_{\xi+\zeta+1}^0(x \oplus X)$ -index of  $F^{-1}(I_{e,n}^x)$  uniformly in  $e$  and  $x$ . Thus, we can  $X$ -effectively find a sequence of  $\Pi_{\xi+\zeta}^0(x \oplus X)$  sets  $\{P_i^x\}_{i \in \mathbb{N}}$  such that  $F^{-1}(I_{e,n}^x) = \bigcup_{i \in \mathbb{N}} P_i^x$ . Hence, we must have a  $\Pi_{\xi+\zeta}^0(X)$  formula  $\varphi^X$  satisfying the following condition.

$$x \in F^{-1}(I_{e,n}^x) \iff (\exists i \in \mathbb{N}) \varphi^X(i, e, n, x).$$

Then, consider the following set  $Q_{e,i}$  for each  $e, i \in \mathbb{N}$ .

$$Q_{e,n,i} = \{x \in \mathbb{N}^{\mathbb{N}} : \Phi_e((x \oplus X)^{(\eta(n))}; m) \text{ is defined for all } m \in \mathbb{N}, \text{ and } \varphi^X(i, e, n, x)\}.$$

Note that  $Q_{e,n,i}$  is the intersection of a  $\Pi_{\eta(n)+2}^0(X)$  set and a  $\Pi_{\xi+\zeta}^0(X)$  set. As  $\xi \geq 2$ , the set  $Q_{e,n,i}$  is  $\Pi_{\xi+\zeta}^0(X)$  uniformly in  $i, e, n \in \mathbb{N}$ . For every  $x \in \mathbb{N}^{\mathbb{N}}$ , if  $x \in Q_{e,n,i}$  then  $F(x) = \Phi_e((x \oplus X)^{(\eta(n))})$ , since, if  $\Phi_e((x \oplus X)^{(\eta(n))})$  is totally defined, then  $F(x) \in I_{e,n}^x = \{\Phi_e((x \oplus X)^{(\eta(n))})\}$ . Moreover, since we have

already seen the property that, for every  $x \in \text{dom}(F)$ ,  $F(x) \leq_T (x \oplus X)^{(\eta(n))}$  for some  $n \in \mathbb{N}$ , every  $x \in \text{dom}(F)$  is contained in  $Q_{e,n,i}$  for some  $e, n, i \in \mathbb{N}$ . Thus,  $F(x)$  agrees with the  $X$ -effectively  $\Sigma_{\eta(n)+1}^0(X)$ -measurable function  $\Phi_e((x \oplus X)^{(\eta(n))})$  on each piece  $Q_{e,n,i}$ . Therefore,  $F$  is  $\Pi_{\xi+\zeta}^0(X)$ -piecewise effectively  $\Sigma_{<\delta}^0$ -measurable, via  $\{Q_{e,n,i}\}_{e,n,i \in \mathbb{N}}$  and  $\{\Phi_e((x \oplus X)^{(\eta(n))})\}_{e,n \in \mathbb{N}}$ .

(5)→(1): Assume that  $F$  is  $\Pi_{\xi+\zeta}^0(X)$ -piecewise  $X$ -effectively  $\Sigma_{<\delta}^0$ -measurable via an  $X$ -computable sequence  $\{P_e\}_{e \in \mathbb{N}}$  of  $\Pi_{\xi+\zeta}^0(X)$  sets and an  $X$ -computable sequence  $\{H_e\}_{e \in \mathbb{N}}$  of  $X$ -effectively  $\Sigma_{<\delta}^0$ -measurable functions, where  $\delta = \delta(\xi, \zeta)$ . Then, for each  $Y \in \mathbb{N}^{\mathbb{N}}$  and for each  $\Sigma_{\xi+1}^0(Y)$  set  $S \subseteq \mathbb{N}^{\mathbb{N}}$ , the preimage  $F^{-1}(S)$  is the union of  $\{H_e^{-1}(S) \cap P_e\}$ . Here, as  $H_e$  is  $X$ -effectively  $\Sigma_{\eta}^0$ -measurable for some  $\eta < \delta$ , it is  $X$ -effectively  $(\Sigma_{\eta+1}^0, \Sigma_{\eta+\xi+1}^0)$ -measurable. By our definition of  $\delta$ , we have  $\eta + \xi \leq \xi + \zeta$  since  $\eta < \delta(\xi, \zeta)$ . Hence, the preimage  $H_e^{-1}(S)$  is  $\Sigma_{\xi+\zeta+1}^0(X \oplus Y)$ , and its index is computed from any index of  $S$  and  $e$ . Thus,  $F^{-1}(S) = \bigcup_e (H_e^{-1}(S) \cap P_e)$  is  $\Sigma_{\xi+\zeta+1}^0(X \oplus Y)$ , and we can effectively calculate its index. Hence,  $F$  is  $X$ -effectively  $(\Sigma_{\xi+1}^0, \Sigma_{\xi+\zeta+1}^0)$ -measurable.  $\square$

**Corollary 29.** *For any  $n \in \mathbb{N}$  and for any partial function  $F : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$  the following five conditions are pairwise equivalent.*

1.  $F$  is effectively  $(\Sigma_{n+1}^0, \Sigma_{n+1}^0)$ -measurable.
2.  $F$  is effectively  $(\Pi_1^0, \Sigma_{n+1}^0)$ -measurable, and nonuniformly computable.
3.  $F$  is of effective discrete Baire class  $n$ .
4.  $F$  is  $\Pi_n^0$ -piecewise computable.

*Proof.* For  $n = 1$ , it was proved by Pauly-de Brecht [5] By Lemma 28 with  $X = \emptyset$ .  $\square$

*Proof of Theorem 8.* By Lemma 28.  $\square$

## 5.1 Open Question

**Question 30.** *Does Theorem 8 hold for all ordinals  $\xi, \zeta \in \mathbb{N}$ ?*

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