DEGREES OF INCOMPUTABILITY, REALIZABILITY AND CONSTRUCTIVE REVERSE MATHEMATICS

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Abstract. There is a way of assigning a realizability notion to each degree of incomputability. In our setting, we make use of Weihrauch degrees (degrees of incomputability/discontinuity of partial multi-valued functions) to obtain Lifschitz-like relative realizability predicates. In this note, we present sample examples on how to lift some separation results on Weihrauch degrees to those over intuitionistic Zermelo-Fraenkel set theory **IZF**.

1. Introduction

1.1. **Summary.** This note is a contribution to *constructive reverse mathematics* initiated by Ishihara, cf. [13, 14]. For the overview of previous works on constructive reverse mathematics, we refer the reader to Diener [9]. However, contrary to [9], in our article, we do not include the axiom of countable choice $\mathsf{AC}_{\omega,\omega}$ in our base system of constructive reverse mathematics, because including $\mathsf{AC}_{\omega,\omega}$ makes it difficult to compare the results with Friedman/Simpson-style classical reverse mathematics [29]. For instance, weak König's lemma WKL is equivalent to the intermediate value theorem IVT over Biship-style constructive mathematics **BISH** (which entails countable choice), whereas WKL is strictly stronger than IVT in classical reverse mathematics. In order to avoid this difficulty, several base systems other than **BISH** have been proposed. For instance, Troelstra's elementary analysis $\mathbf{EL}_0$ is widely used as a base system of constructive reverse mathematics, e.g. in [3]. Here, "$\mathbf{EL}_0$ plus the law of excluded middle" is exactly the base system $\mathbf{RCA}_0$ of classical reverse mathematics; and moreover, "$\mathbf{EL}_0$ plus the axiom of countable choice" is considered as Biship-style constructive mathematics **BISH**. Thus, $\mathbf{EL}_0$ lies in the intersection of $\mathbf{RCA}_0$ and **BISH**.

Our aim is to separate various non-constructive principles which are equivalent under countable choice $\mathsf{AC}_{\omega,\omega}$, and our main tool is (a topological version of) Weihrauch reducibility (cf. [5]). The relationship between Weihrauch reducibility and intuitionistic systems are extensively studied, e.g. in [31, 35, 36]. Thus, some of separations may be automatically done using the previous works. Our main purpose in this note is to develop an intuitive and easily customizable framework constructing $\mathsf{AC}_{\omega,\omega}$-free models separating such non-constructive principles. In practice, establishing a convenient framework for separation arguments is sometimes more useful than giving a completeness result (for instance, look at classical reverse mathematics, where computability-theorists make use of Turing degree theory in order to construct ω-models separating various principles over $\mathbf{RCA}_0$, cf. [28]). Our work gives several sample separation results, not just over $\mathbf{EL}_0$, but even over intuitionistic Zermelo-Fraenkel set theory **IZF**.

1.1.1. *Variants of weak König's lemma.* In this article, we discuss a hierarchy between LLPO and WKL which collapse under the axiom $\mathsf{AC}_{\omega,\omega}$ of countable choice. Consider the following three principles:

- The binary expansion principle BE states that every regular Cauchy real has a binary expansion.
- The intermediate value theorem IVT states that for any continuous function $f\colon [0,1] \to [-1,1]$ if $f(0)$ and $f(1)$ have different signs then there is a regular Cauchy real $x \in [0,1]$ such that $f(x) = 0$.
- Weak König's lemma WKL states that every infinite binary tree has an infinite path.

Here, a regular Cauchy real is a real x which is represented by a sequence $(q_n)_{n\in\omega}$ of rational numbers such that $|q_n - q_m| < 2^{-n}$ for any $m \geq n$. As mentioned above, under countable choice $\mathsf{AC}_{\omega,2}$, we have

$$\mathsf{WKL} \leftrightarrow \mathsf{IVT} \leftrightarrow \mathsf{BE} \leftrightarrow \mathsf{LLPO}.$$

Even if countable choice, Markov's principle, etc. are absent, we have the forward implications, cf. Berger et al. [3]; however we will see that the implications are strict under the absence of countable choice.

Robust division principle RDIV. We examine the *division* principles for reals. It is known that *Markov's principle* (double negation elimination for Σ^0_1-formulas) is equivalent to the statement that, given regular Cauchy reals $x, y \in [0,1]$, we can divide x by y whenever y is nonzero; that is, if y is nonzero then $z = x/y$ for some regular Cauchy real z.

However, it is algorithmically undecidable if a given real y is zero or not. To overcome this difficulty, we consider any regular Cauchy real z such that if y is nonzero then $z = x/y$. If we require $x \leq y$ then such a z always satisfies $x = yz$ whatever y is (since $y = 0$ and $x \leq 0$ implies $x = 0$, so any z satisfies $0 = 0z$). To avoid the difficulty of deciding if y is nonzero or not, we always assume this additional requirement, and then call it the *robust division* principle RDIV. In other words, RDIV is the following statement:

$$(\forall x, y \in [0,1])\ [x \leq y \to\ (\exists z \in [0,1])\ x = yz]\,.$$

One may think that the condition $x \leq y$ is not decidable, but we can always replace x with $\min\{x, y\}$ without losing anything. Namely, the robust division principle RDIV is equivalent to the following:

$$(\forall x, y \in [0,1])(\exists z \in [0,1])\ \min\{x, y\} = yz.$$

The principle RDIV is known to be related to problems of finding Nash equilibria in bimatrix games [25] and of executing Gaussian elimination [16]. The following implications are known:

$$\begin{tikzcd}
 & & \mathsf{BE} \arrow[dr] & \\
\mathsf{WKL} \arrow[r] & \mathsf{IVT} \arrow[ur] \arrow[dr] & & \mathsf{LLPO} \\
 & & \mathsf{RDIV} \arrow[ur] &
\end{tikzcd}$$

1.2. **Main Theorem.** In this article, to construct $\mathsf{AC}_{\omega,\omega}$-free models of **IZF**, we incorporate Weihrauch reducibility (cf. [5]) into Chen-Rathjen's Lifschitz-like realizability [7]. In general, we introduce the notion of an idempotent jump operator on a relative partial combinatory algebra, and then we show that given such an operator always yields a realizability predicate which validates **IZF**. As sample results of our method, we describe realizability interpretations of the following:

Theorem 1.1. *The following are all realizable (with respect to suitable realizability predicates).*

(1) $\mathbf{IZF} + \mathsf{LLPO} + \neg\mathsf{RDIV} + \neg\mathsf{BE}$.
(2) $\mathbf{IZF} + \mathsf{RDIV} + \neg\mathsf{BE}$.
(3) $\mathbf{IZF} + \mathsf{BE} + \neg\mathsf{RDIV}$.
(4) $\mathbf{IZF} + \mathsf{RDIV} + \mathsf{BE} + \neg\mathsf{IVT}$.
(5) $\mathbf{IZF} + \mathsf{IVT} + \neg\mathsf{WKL}$.

2. Key technical notions

2.1. **Weihrauch reducibility.** We will make use of realizability to prove Theorem 1.1. A simple, but important, observation is that most principles which appear in Theorem 1.1 are of the $\forall\exists$-forms. First let us consider Kleene realizability (cf. [30, 33]). If the formula $S \equiv \forall x \in X\ [Q(x) \to \exists y P(x,y)]$ is realizable, then one can always obtain an (intentional) choice function for S; that is, from a realizer we obtain a function f such that, given (an expression $\underline{x}$ of) x where $Q(x)$ is realizable, the value $f(\underline{x})$ expresses some y with $P(x,y)$.

We view the $\forall\exists$-formula S as a *partial multi-valued function* (*multifunction*). Informally speaking, a (possibly false) statement $S \equiv \forall x \in X\ [Q(x) \to \exists y P(x,y)]$ is transformed into a partial multifunction $f_S :\subseteq X \rightrightarrows Y$ such that $\mathrm{dom}(f_S) = \{x \in X : Q(x)\}$ and $f_S(x) = \{y \in Y : P(x,y)\}$. Here, we consider formulas as partial multifunctions rather than relations in order to distinguish a *hardest* instance $f_S(x) = \emptyset$ (corresponding to a *false sentence*), and an *easiest* instance $x \in X \setminus \mathrm{dom}(f_S)$ (corresponding to a *vacuous truth*).

One may consider $f_S(x)$ as the set of solutions to (an instance of) a problem $x \in \mathrm{dom}(f)$. Then, roughly speaking, Kleene's number realizability only validates the principles which are computably solvable and refutes other non-computable principles, and Kleene's functional realizability validates the principles which are continuously solvable and refutes other discontinuous principles. Thus, the degree of difficulty of realizing a formula seems related to the degrees of difficulty of solving the problem represented by a partial multifunction. The main tool for measuring the degrees of difficulty of partial multifunctions in this article is Weihrauch reducibility.

2.1.1. *Weihrauch reducibility.* We say that f *is Weihrauch reducible to* g (written $f \leq_{\mathsf{W}} g$) if there are partial computable functions H and K such that, for any $x \in \mathrm{dom}(f)$, $y \in g(H(x))$ implies $K(x,y) \in f(x)$. For basics on Weihrauch reducibility, we refer the reader to [5]. The definition of Weihrauch reducibility $f \leq_{\mathsf{W}} g$ can be viewed as the following perfect information two-player game:

$$\begin{array}{lllll} \mathrm{I}: & x_0 \in \mathrm{dom}(f) & & x_1 \in g(y_0) & \\ \mathrm{II}: & & y_0 \in \mathrm{dom}(g) & & y_1 \in f(x_0) \end{array}$$

More precisely, each player chooses an element from ω^ω at each round, and Player II wins if there is a computable strategy τ for II which yields a play described above. Note that y_0 depends on x_0, and y_1 depends on x_0 and x_1, and moreover, a computable strategy τ for II yields partial computable maps $H\colon x_0 \mapsto y_0$ and $K\colon (x_0, x_1) \mapsto y_1$. Usually, H is called an *inner reduction* and K is called an *outer reduction.* If reductions H and K are allowed to be continuous, then we say that f *is continuously Weihrauch reducible to* g (written $f \leq^{c}_{\mathsf{W}} g$).

In (classical or intuitionistic) logic, when showing $P \to Q$, the weakening rule allows us to use the antecedent P more than once. However, the above reducibility notion essentially requires us to show Q by using P only once. To overcome this difficulty, there is an operation $h \star g$ on partial multifunctions g, h which allows us to use h after the use of g. For partial multi-valued functions g and h, define the composition $h \circ g$ as follows:

$$h \circ g(x) = \begin{cases} \bigcup\{h(y) : y \in g(x)\} & \text{if } g(x) \downarrow\subseteq \mathrm{dom}(h), \\ \uparrow & \text{otherwise.} \end{cases}$$

Note that the above definition of the composition of partial multifunctions is different from the usual definition of the composition of relations (so the partial multifunctions are completely different from the relations). Then it is shown that $\min_{\leq_{\mathsf{W}}}\{h_0 \circ g_0 : g_0 \leq_{\mathsf{W}} g \wedge h_0 \leq_{\mathsf{W}} h\}$ always exists, and a representative is denoted by $h \star g$; see Brattka-Pauly [6]. There is an explicit description of a representative $h \star g$, and it is indeed needed to prove results on $\star$. To explain the explicit definition, we consider the following play for "$f \leq_{\mathsf{W}} h \star g$":

$$\begin{array}{llllllll} \text{I:} & x_0 \in \mathrm{dom}(f) & & x_1 \in g(y_0) & & x_2 \in h(y_1) & \\ \text{II:} & & y_0 \in \mathrm{dom}(g) & & y_1 \in \mathrm{dom}(h) & & y_2 \in f(x_0) \end{array}$$

Again, a computable strategy for Player II yields functions $y_0 = H_0(x_0)$, $y_1 = H_1(x_0, x_1)$, and $y_2 = K(x_0, x_1, x_2)$. That is, II's strategy yields two inner reductions H_0, H_1 and an outer reduction K, where H_0 makes a query to g, and then, after seeing a result $x_1 \in g(y_0)$, the second reduction H_1 makes a query to h.

Now, the partial multifunction $h \star g$ takes, as an input, Player I's first move x_0 and a code for Player II's strategy τ (a code for $\langle H_0, H_1, K\rangle$) such that II's play according to the strategy τ obeys the rule unless I's play violates the rule at some previous round, where we say that Player I obeys the rule if $x_1 \in g(y_0)$ and $x_2 \in h(y_1)$, and Player II obeys the rule if $y_0 \in \mathrm{dom}(g)$ and $y_1 \in \mathrm{dom}(h)$. Then, $h \star g(x_0, \tau)$ returns $K(x_0, x_1, x_2)$ for I's some play (x_1, x_2) which obeys the rule. Note that there are many possible values for $K(x_0, x_1, x_2)$, which means that $h \star g$ is multi-valued. Without loss of generality, one can remove x_0 from the definition of $h \star g$ by considering a continuous strategy τ.

This operation $(g, h) \mapsto h \star g$ is called the *compositional product*; see also Brattka-Pauly [6]. For game-theoretic descriptions, see also Hirschfeldt-Jockusch [12] and Goh [10]. This idea extends to reduction games introduced below.

2.1.2. *$\star$-closure and reduction games.* In order to express "arbitrary use of an antecedent", Hirschfeldt-Jockusch [12] introduced *reduction games* and *generalized Weihrauch reducibility.* For partial multifunctions $f, g\colon\subseteq \omega^\omega \rightrightarrows \omega^\omega$, let us consider the following perfect information two-player game $G(f,g)$:

$$\begin{array}{llllllll} \text{I:} & x_0 & & x_1 & & x_2 & & \dots \\ \text{II:} & & y_0 & & y_1 & & y_2 & \dots \end{array}$$

Each player chooses an element from ω^ω at each round. Here, Players I and II need to obey the following rules.

- First, Player I chooses $x_0 \in \mathrm{dom}(f)$.
- At the nth round, Player II reacts with $y_n = \langle j, u_n \rangle$.
  - The choice $j = 0$ indicates that Player II makes a new query u_n to g. In this case, we require $u_n \in \mathrm{dom}(g)$.
  - The choice $j = 1$ indicates that Player II declares victory with u_n.
- At the $(n+1)$th round, Player I responds to the query made by Player II at the previous stage. This means that $x_{n+1} \in g(u_n)$.

Then, *Player II wins the game* $G(f,g)$ if either Player I violates the rule before Player II violates the rule or Player II obeys the rule and declares victory with $u_n \in f(x_0)$.

Hereafter, we require that Player II's moves are chosen in a continuous manner. In other words, Player II's strategy is a code τ of a partial *continuous* function $h_\tau \colon\subseteq (\omega^\omega)^{<\omega} \to \omega^\omega$. On the other hand, Player I's strategy is any partial function $\sigma\colon\subseteq (\omega^\omega)^{<\omega} \to \omega^\omega$ (which is not necessarily continuous). Given such strategies σ and τ yield a play $\sigma \otimes \tau$ in the following manner:

$$\begin{aligned} (\sigma\otimes\tau)(0) &= \sigma(\langle\rangle), \\ (\sigma\otimes\tau)(2n+1) &= h_\tau(\langle(\sigma\otimes\tau)(2m)\rangle_{m\le n}), \\ (\sigma\otimes\tau)(2n+2) &= \sigma(\langle(\sigma\otimes\tau)(2m+1)\rangle_{m\le n}). \end{aligned}$$

Player II's strategy τ is *winning* if Player II wins along $\sigma\otimes\tau$ whatever Player I's strategy σ is. We say that f *is generalized Weihrauch reducible to* g if Player II has a computable winning strategy for $G(f,g)$. In this case, we write $f \le_{\partial\mathsf{W}} g$. If Player II has a (continuous) winning strategy for $G(f,g)$, we write $f \le^c_{\partial\mathsf{W}} g$. Hirschfeldt-Jockusch [12] showed that the relation $\le_{\partial\mathsf{W}}$ is transitive.

Note that the rule of the above game does not mention f except for Player I's first move. Hence, if we skip Player I's first move, we can judge if a given play follows the rule without specifying f. For a partial multifunction $g\colon\subseteq \omega^\omega \rightrightarrows \omega^\omega$, we define $g^{\partial}\colon\subseteq \omega^\omega \rightrightarrows \omega^\omega$ as follows:

- $(x_0,\tau) \in \mathrm{dom}(g^{\partial})$ if and only if τ is Player II's strategy, and for Player I's any strategy σ with first move x_0, if Player II declares victory at some round along the play $\sigma\otimes\tau$.
- Then, $u \in g^{\partial}(x_0,\tau)$ if and only if Player II declares victory with u at some round along the play $\sigma\otimes\tau$ for some σ with first move x_0.

Here, the statement "Player II declares victory" does not necessarily mean "Player II wins". Indeed, the above definition is made before f is specified, so the statement "Player II wins" does not make any sense. Again, one can remove x_0 from an input for g^{∂} by considering a continuous strategy. The following is obvious by definition.

Observation 2.1. $f \le_{\mathsf{W}} g^{\partial} \iff f \le_{\partial\mathsf{W}} g$.

Note that Neumann-Pauly [24] used generalized register machines to define the $\star$-closure $g^\diamondsuit$ of g. This notion is also studied by Pauly-Tsuiki [26]. Neumann-Pauly [24] showed that $f \leq_{\mathsf{W}} g^\diamondsuit$ if and only if $f \leq_{\Game\mathsf{W}} g$. Therefore, we have $g^\Game \equiv_{\mathsf{W}} g^\diamondsuit$, but the game-theoretic definition is mathematically simpler and clearer than $\diamondsuit$ (and the symbol $\diamondsuit$ may be used in many contexts while $\Game$ is the standard symbol for the game quantifier), so we use $g^\Game$ throughout this paper.

By transitivity of $\leq_{\Game\mathsf{W}}$, we have $g^\Game \star g^\Game \equiv_{\mathsf{W}} g^\Game$. Moreover, Westrick [34] showed that $\diamondsuit$ (so $\Game$) gives the *least fixed point* of the operation $h \mapsto h \star h$ whenever $h \geq_{\mathsf{W}} \mathbf{1}$, where $\mathbf{1}$ is the Weihrauch degree of the identity function on ω^ω. More precisely, if $\mathbf{1} \leq_{\mathsf{W}} g$ then $g \leq_{\mathsf{W}} g \star g$ implies $g^\Game \leq_{\mathsf{W}} g$. For more details on the closure operator $\diamondsuit$, see Westrick [34].

2.1.3. *Recursion trick.* To prove various separation results on Weihrauch reducibility, *Kleene's recursion theorem* has been frequently used as a simple, but very strong, proof machinery, cf. [12, 16, 17]. We employ this machinery throughout this article, so we separate this argument, and call it *recursion trick.* In Kihara-Pauly [16], recursion trick is described in the context of two player games, where Player I is called Pro (the proponent) and II is called Opp (the opponent) because our purpose is to show Weihrauch separation $\not\leq_{\mathsf{W}}$. The formal description of recursion trick for short reduction games is described in Kihara-Pauly [17].

We first describe the idea of recursion trick in a slightly longer reduction game, i.e., for $f \leq_{\mathsf{W}} g \star h$. Note that, in this kind of reduction, any play of the reduction game ends at the third round of Player II:

$$\begin{array}{lcccccc} \text{I}: & x_0 & & x_1 & & x_2 & \\ \text{II}: & & y_0 & & y_1 & & y_2 \end{array}$$

If Player II's strategy is given, we may consider that the value of y_i depends on Player I's play; that is, $y_0 = y_0(x_0)$, $y_1 = y_1(x_0, x_1)$, and $y_2 = y_2(x_0, x_1, x_2)$. If we describe an algorithm constructing x_0' from (y_0, y_1, y_2) (with a parameter x_0), by Kleene's recursion theorem, the parameter x_0 can be interpreted as *self-reference*; that is, we may assume that $x_0' = x_0$. Hence, one can remove x_0 from the list of parameters in y_0, y_1 and y_2. In summary, to show that $f \not\leq_{\mathsf{W}} g \star h$, given y_0, y_1, y_2, it suffices to construct x_0 such that Player I wins along the following play

$$\begin{array}{lcccccc} \text{I}: & x_0 & & x_1 & & x_2 & \\ \text{II}: & & y_0 & & y_1(x_1) & & y_2(x_1, x_2) \end{array}$$

for some x_1 and x_2, that is, Player II violates the rule before I violates the rule, or Player I obeys the rule and $y_2 \not\in f(x_0)$. The essence of recursion trick is that *it allows us to construct Player I's first move x_0 later than II's moves* y_0, y_1, y_2. However, note that I's move x_0 has to obey the rule even if II violates the rule.

In general reduction games, II's strategy yields $y_n(x_0, x_1, \dots, x_n)$. Again we describe an algorithm $(y_n)_{n\in\omega} \mapsto x_0'$, and by Kleene's recursion theorem, consider x_0 as self-reference. As before, this allows us to construct Player I's first move x_0 later than II's moves $(y_n)_{n\in\omega}$, but I's move x_0 has to obey the rule even if II violates the rule.

Formal description of recursion trick: Now we rigorously describe what recursion trick is. Fix an oracle Z. Given Player II's Z-computable strategy τ, we get a uniform sequence of Z-computable functions $y^\tau_n(x_0, x_1, \dots, x_n)$ which determine the nth move of Player II. Now, assume that Player I's first move is the eth partial Z-computable function φ^Z_e on ω. Then, the reduction game proceeds as follows:

$$\begin{array}{lllllllll} \text{I:} & \varphi^Z_e & & x_1 & & x_2 & & \dots \\ \text{II:} & & y^\tau_0(\varphi^Z_e) & & y^\tau_1(\varphi^Z_e, x_1) & & y^\tau_2(\varphi^Z_e, x_1, x_2) & \dots \end{array}$$

Here, each move in the above play may be a *partial* function on ω, while each move in the original reduction game is a *total* function on ω.

Assume that we have constructed a Z-computable algorithm $e \mapsto x_0(e)$ constructing Player I's first move $x_0(e)$ using information of II's play $\langle \tau(\varphi^Z_e, x_1, \dots, x_n)\rangle_{n\in\omega}$. Clearly, there is a Z-computable function u such that $x_0(e) \simeq \varphi^Z_{u(e)}$. Moreover, assume that

($\star$) The move $x_0(e)$ is designed to defeat Player II's strategy $\langle \tau(\varphi^Z_e, x_1, \dots, x_n)\rangle_{n\in\omega}$; that is, for any e, there exist subsequent moves $(x_1, x_2, \dots)$ such that Player I wins along the following play:

$$\begin{array}{lllllllll} \text{I:} & x_0(e) \simeq \varphi^Z_{u(e)} & & x_1 & & x_2 & & \dots \\ \text{II:} & & y^\tau_0(\varphi^Z_e) & & y^\tau_1(\varphi^Z_e, x_1) & & y^\tau_2(\varphi^Z_e, x_1, x_2) & \dots \end{array}$$

Here, we do not need to find such $(x_1, x_2, \dots)$ effectively. By Kleene's recursion theorem, there exists r such that $x_0(r) \simeq \varphi^Z_{u(r)} \simeq \varphi^Z_r$. Note that, in the above play with $e = r$, Player II just follows the strategy τ. By the assumption ($\star$), I's play $(x_0(r), x_1, x_2, \dots)$ defeats Player II who follows τ, which means that τ is not a winning strategy.

In summary, if for any computable strategy τ for Player II (in the reduction game $G(f, g)$) one can construct a computable algorithm $e \mapsto x_0(e)$ satisfying ($\star$), then we obtain $f \not\leq_{\mathsf{W}} g^{\supset}$. Moreover, if this is possible relative to any oracle Z, then we also obtain the continuous Weihrauch separation $f \not\leq^c_{\mathsf{W}} g^{\supset}$. We often describe such a construction $e \mapsto x_0(e)$ as an algorithm describing $x_0(\tilde\tau)$ from given $\langle \tilde\tau(x_1, \dots, x_n)\rangle$, where $\tilde\tau(x_1, \dots, x_n) \simeq \tau(\varphi^Z_e, x_1, \dots, x_n)$ for some e.

2.2. **Realizability.** In this section, we construct a realizability predicate from a given Weihrauch degree, and moreover, if a given Weihrauch degree has a good closure property, then we show that the induced realizability predicate validates all axioms of intuitionistic Zermelo-Fraenkel set theory **IZF**.

2.2.1. *Partial combinatory algebra.* A partial magma is a pair $(M, *)$ of a set M and a partial binary operation $*$ on M. We often write xy instead of $x * y$, and as usual, we consider $*$ as a left-associative operation, that is, xyz stands for $(xy)z$. A partial magma is *combinatory complete* if, for any term $t(x_1, x_2, \dots, x_n)$, there is $a_t \in M$ such that $a_t x_1 x_2 \dots x_{n-1}\downarrow$ and $a_t x_1 x_2 \dots x_n \simeq t(x_1, x_2, \dots, x_n)$. For terms $t(x, y) = x$, and $u(x, y, z) = xzyz$, the corresponding elements $a_t, a_u \in M$ are usually written as k and s. A combinatory complete partial magma is called a *partial combinatory algebra* (abbreviated as *pca*). A *relative pca* is a triple $\mathbb{P} = (P, \underset{\sim}{\mathbf{P}}, *)$ such that $P \subseteq \underset{\sim}{\mathbf{P}}$, both $(\underset{\sim}{\mathbf{P}}, *)$ and $(P, * \upharpoonright P)$ are pcas, and share combinators s and k. For basics on a relative pca, we refer the reader to van Oosten [33, Sections 2.6.9 and 4.5]. In this article, the boldface algebra $\underset{\sim}{\mathbf{P}}$ is always the set ω^ω of all infinite sequences.

In Brouwerian intuitionistic mathematics, a relative pca may be considered as a pair of *lawlike* sequences and *lawless* sequences. In descriptive set theory, the idea of a relative pca is ubiquitous, which usually occurs as a pair of *lightface* and *boldface* pointclasses. Nobody seemed to explicitly mention before, but a large number of nontrivial, deep, examples of relative pcas have been (implicitly) studied in descriptive set theory [22]: By the *good parametrization lemma* in descriptive set theory (cf. Moschovakis [22, Lemma 3H.1]), any Σ^*-pointclass Γ — in particular, any Spector pointclass (cf. Moschovakis [22, Lemma 4C]), so many infinitary computation models including infinite time Turing machines (ITTMs) — yields a (relative) pca, since the good parametrization lemma is, roughly speaking, a generalized smn-theorem, so such a pointclass admits currying. Here, the *partial* Γ*-computable function application* form a lightface pca, and the *partial* $\boldsymbol{\Gamma}$*-measurable function application* form a boldface pca. In particular, we obtain a cartesian closed category whose morphisms are $\boldsymbol{\Gamma}$-measurably realizable functions (and even a topos, usually called a *realizability topos*, cf. van Oosten [33] or Pitts [27]).

Here we describe the details: If G is a good ω^ω-parametrization of a Σ^*-pointclass Γ (in particular, we have an indexing $(G_p)_{p\in\omega^\omega}$ of all $\boldsymbol{\Gamma}$ subsets of $\omega^\omega \times \omega \times \omega$), then one can define a partial application $*$ on $\mathbf{P}$ as follows:

$$p * x \downarrow= y \iff (\forall n \in \omega)\ [\langle x, n, y(n)\rangle \in G_p\ \wedge\ (\forall m \neq y(n))\ \langle x, n, m\rangle \notin G_p].$$

In other words, $p * x \downarrow= y$ holds if the pth $\boldsymbol{\Gamma}$ set G_p is the graph of a partial multifunction $\psi_p \colon\subseteq \omega^\omega \times \omega \rightrightarrows \omega$, and $\Psi_p \colon n \mapsto \psi_p(x, n)$ is exactly the function $y \colon \omega \to \omega$.

Then, goodness (i.e., the smn-property) $(\mathbf{P}, *)$ forms a pca, and moreover, the set $P \subseteq \mathbf{P}$ of all computable sequences with the partial application $* \restriction P$ forms a pca, too (by the good parametrization lemma). One can think of $p \in P$ as a code of a Γ-recursive function Φ_p (cf. [22, Sections 3D and 3G]), and then $p * x \downarrow= y$ means the function application $\Phi_p(x) \downarrow= y$. Similarly, consider $p \in \mathbf{P}$ as a code of a $\boldsymbol{\Gamma}$-measurable function Ψ_p, and then $p * x \downarrow= y$ means $\Psi_p(x) \downarrow= y$.

If $\Gamma = \Sigma^0_1$, the induced lightface pca is equivalent to *Kleene's first algebra* (associated with Kleene's number realizability), and the boldface pca is *Kleene's second algebra* (associated with Kleene's functional realizability). The relative pca induced from $\Gamma = \Sigma^0_1$ is known as Kleene-Vesley's algebra, cf. [33]. The pointclass $\Sigma^0_1(\emptyset^{(n)})$ is employed by Akama et al. [1] to show that the arithmetical hierarchy of the law of excluded middle does not collapse over Heyting arithmetic $\mathbf{HA}$. Here, note that Σ^0_n does not yield a pca on ω^ω since Σ^0_n-computable functions (on ω^ω) and $\boldsymbol{\Sigma}^0_n$-measurable functions are not closed under composition. In descriptive set theory, Borel pointclass whose associated measurable functions are closed under composition are studied under the name of Borel amenability, cf. [23]; for instance, the nth level Borel functions (a.k.a. $\Sigma_{n,n}$-functions) are closed under composition, and the induced reducibility notion is studied, e.g. by [18]. The pointclass Π^1_1 is the best-known example of a Spector pointclass, and the induced lightface pca obviously yields hyperarithmetical realizability. For the boldface pca, the associated total realizable functions are exactly the Borel measurable functions. If one considers Kleene realizability with such a pca (i.e., one obtained by a Spector pointclass), then it trivially validates Σ^0_n-$\mathsf{LEM}_\mathbb{R}$ for any $n \in \omega$. Bauer [2] also studied the pca (and the induced realizability topos) obtained from infinite time Turing machines (ITTMs). Here, recall that ITTMs form a Spector pointclass. One may also consider descriptive set theoretic versions of Scott's graph model using lightface/boldface pointclasses.

2.2.2. *Realizability predicates.* McCarty [21] introduced a realizability interpretation of the intuitionistic Zermero-Fraenkel set theory **IZF** which validates Church's thesis CT_0. Chen-Rathjen [7] combined McCarty-realizability with Lifschitz-realizability [20] to give a realizability interpretation of **IZF** which validates $\mathsf{CT}_0! + \neg\mathsf{CT}_0 + \mathsf{LLPO} + \neg\mathsf{LPO} + \neg\mathsf{AC}_{\omega,2}$.

The key idea in Lifschitz [20] (and van Oosten [32]) of validating $\mathsf{CT}_0! + \neg\mathsf{CT}_0$ is the use of *multifunction applications*, rather than single-valued applications. From the computability-theoretic perspective, their key idea is regarding Π^0_1 *classes* (or *effectively compact* Π^0_1 *sets*) as basic concepts rather than computable functions. More precisely, over the Kleene first algebra $(\omega, *)$, Lifschitz considered the partial multifunction $\mathbf{j}_{\mathrm{L}} :\subseteq \omega \rightrightarrows \omega$ defined by

$$\mathbf{j}_{\mathrm{L}}(\langle e, b\rangle) = \{n \in \omega : n < b \ \wedge \ e * n \uparrow\},$$

where $\mathbf{j}_{\mathrm{L}}(\langle b, e\rangle)\downarrow$ if and only if the set is nonempty. Obviously, Lifschitz's multifunction $\mathbf{j}_{\mathrm{L}}$ gives an effective enumeration of all bounded Π^0_1 subsets of ω. Van Oosten [32] generalized this notion to the Kleene second algebra $(\omega^\omega, *)$ by considering the following partial multifunction $\mathbf{j}_{\mathrm{vO}} :\subseteq \omega^\omega \rightrightarrows \omega^\omega$:

$$\mathbf{j}_{\mathrm{vO}}(\langle g, h\rangle) = \{x \in \omega^\omega : (\forall n \in \omega)\ x(n) < h(n) \ \wedge \ g * x \uparrow\},$$

It is again obvious that van Oosten's multifunction $\mathbf{j}_{\mathrm{vO}}$ gives a representation of all compact subsets of ω^ω, and $\mathbf{j}_{\mathrm{vO}}(\langle g, h\rangle)$ is an effectively compact Π^0_1 class relative to $\langle g, h\rangle$.

To define a set-theoretic realizability interpretation, we follow the argument in Chen-Rathjen [7]: First, we consider a set-theoretic universe with urlements $\mathbb{N}$. For a relative pca $\mathbb{P} = (P, \underset{\sim}{\mathbf{P}})$, as in the usual set-theoretic forcing argument, we consider a $\mathbb{P}$-*name*, which is any set x satisfying the following condition:

$$x \subseteq \{(p, u) : p \in \underset{\sim}{\mathbf{P}} \text{ and } (u \in \mathbb{N} \text{ or } u \text{ is a } \mathbb{P}\text{-name})\}.$$

The $\mathbb{P}$-names are used as our universe. This notion can also be defined as the cumulative hierarchy:

$$V^{\mathbb{P}}_0 = \emptyset, \quad V^{\mathbb{P}}_\alpha = \bigcup_{\beta<\alpha} \mathcal{P}(\underset{\sim}{\mathbf{P}} \times (V^{\mathbb{P}}_\beta \cup \mathbb{N})), \text{ and } V^{\mathbb{P}}_{\mathrm{set}} = \bigcup_{\alpha\in\mathrm{Ord}} V^{\mathbb{P}}_\alpha.$$

Note that the urelements $\mathbb{N}$ are disjoint from $V^{\mathbb{P}}_{\mathrm{set}}$. We define $V^{\mathbb{P}} = V^{\mathbb{P}}_{\mathrm{set}} \cup \mathbb{N}$. McCarty [21] used this notion to give a Kleene-like realizability interpretation for **IZF**. Then, as mentioned above, Chen-Rathjen [7] incorporated Lifschitz's multifunction $\mathbf{j}_{\mathrm{L}}$ into McCarty realizability. We now generalize Chen-Rathjen's realizability predicate to *any* partial multifunction $\mathbf{j}$. Fix a partial multifunction $\mathbf{j} \colon \subseteq \underset{\sim}{\mathbf{P}} \rightrightarrows \underset{\sim}{\mathbf{P}}$. For $e \in \underset{\sim}{\mathbf{P}}$ and a sentence φ of **IZF** from parameters from $V^{\mathbb{P}}$, we define a relation $e \Vdash_{\mathbb{P}} \varphi$. For relation symbols:

$$\begin{aligned} e \Vdash_{\mathbb{P}} R(\bar{a}) &\iff \mathbb{N} \models R(\bar{a}) \\ e \Vdash_{\mathbb{P}} \mathbf{N}(a) &\iff a \in \mathbb{N} \ \& \ e = \underline{a} \\ e \Vdash_{\mathbb{P}} \mathbf{Set}(a) &\iff a \in V^{\mathbb{P}}_{\mathrm{set}} \end{aligned}$$

where R is a primitive recursive relation. For set-theoretic symbols:

$$
\begin{aligned}
e \Vdash_{\mathbb{P}} a \in b &\iff \forall^+ d \in \mathbf{j}(e) \exists c\, [(\pi_0 d, c) \in b \ \wedge\ \pi_1 d \Vdash_{\mathbb{P}} a = c] \\
e \Vdash_{\mathbb{P}} a = b &\iff (a, b \in \mathbb{N} \ \wedge\ a = b) \vee (\mathbf{Set}(a) \ \wedge\ \mathbf{Set}(b) \wedge \\
&\qquad \forall^+ d \in \mathbf{j}(e) \forall p, c\, [(p, c) \in a \ \to\ \pi_0 dp \Vdash_{\mathbb{P}} c \in b] \wedge \\
&\qquad \forall^+ d \in \mathbf{j}(e) \forall p, c\, [(p, c) \in b \ \to\ \pi_1 dp \Vdash_{\mathbb{P}} c \in a].
\end{aligned}
$$

Here, we write $\forall^+ x \in X\ A(x)$ if both $X \neq \emptyset$ and $\forall x \in X\ A(x)$ hold. For logical connectives:

$$
\begin{aligned}
e \Vdash_{\mathbb{P}} A \wedge B &\iff \pi_0 e \Vdash_{\mathbb{P}} A \ \wedge\ \pi_1 e \Vdash_{\mathbb{P}} B \\
e \Vdash_{\mathbb{P}} A \vee B &\iff \forall^+ d \in \mathbf{j}(e)\ [(\pi_0 d = 0 \wedge \pi_1 d \Vdash_{\mathbb{P}} A) \vee (\pi_0 d = 1 \wedge \pi_1 d \Vdash_{\mathbb{P}} B)] \\
e \Vdash_{\mathbb{P}} \neg A &\iff (\forall a \in \underset{\sim}{\mathbf{P}})\ a \not\Vdash_{\mathbb{P}} A \\
e \Vdash_{\mathbb{P}} A \to B &\iff (\forall a \in \underset{\sim}{\mathbf{P}})\ [a \Vdash A \ \to\ ea \Vdash_{\mathbb{P}} B].
\end{aligned}
$$

For quantifiers:

$$
\begin{aligned}
e \Vdash_{\mathbb{P}} \forall x A &\iff (\forall^+ d \in \mathbf{j}(e))(\forall c \in V^{\mathbb{P}})\ e \Vdash_{\mathbb{P}} A[c/x] \\
e \Vdash_{\mathbb{P}} \exists x A &\iff (\forall^+ d \in \mathbf{j}(e))(\exists c \in V^{\mathbb{P}})\ e \Vdash_{\mathbb{P}} A[c/x].
\end{aligned}
$$

Then we say that a formula φ is $\mathbf{j}$*-realizable over* $\mathbb{P}$ if there is $e \in P$ such that $e \Vdash_{\mathbb{P}} \varphi$. Unfortunately, a $\mathbf{j}$-realizability predicate does not necessarily validate axioms of **IZF**. Next, we will discuss what condition for a multifunction $\mathbf{j}$ is needed in order to ensure all axioms of **IZF**.

2.2.3. *Jump operators.* Assume that $\mathbb{P} = (P, \underset{\sim}{\mathbf{P}})$ is a relative pca. We say that a partial multifunction $\mathbf{j} :\subseteq \underset{\sim}{\mathbf{P}} \rightrightarrows \underset{\sim}{\mathbf{P}}$ is an *idempotent jump operator on* $\mathbb{P}$ if

(1) There is $u \in P$ such that for any $a, x \in \underset{\sim}{\mathbf{P}}$, $a\mathbf{j}(x) = \mathbf{j}(uax)$.
(2) There is $\iota \in P$ such that for any $x \in \underset{\sim}{\mathbf{P}}$, $x = \mathbf{j}(\iota x)$.
(3) There is $j \in P$ such that for any $x \in \underset{\sim}{\mathbf{P}}$, $\mathbf{jj}(x) = \mathbf{j}(jx)$.

Here, $\mathbf{jj}$ is an abbreviation for the composition $\mathbf{j} \circ \mathbf{j}$, where recall the definition of the composition of multifunctions from Section 2.1.1. Also, if f is a multifunction on $\underset{\sim}{\mathbf{P}}$ and $a, x \in \underset{\sim}{\mathbf{P}}$, then define $af(x) = \{ay : y \in f(x)\}$.

A partial multifunction $f :\subseteq \underset{\sim}{\mathbf{P}} \rightrightarrows \underset{\sim}{\mathbf{P}}$ is $(P, \mathbf{j})$*-realizable* if there is $a \in P$ such that, for any $x \in \mathrm{dom}(f)$, we have $\mathbf{j}(ax) \subseteq f(x)$.

Remark. This notion (for operations satisfying (1) and (2)) is implicitly studied in the work on the jump of a represented space, e.g. by de Brecht [8]. From the viewpoint of the jump of a represented space, one may think of $\mathbf{j}$ as an endofunctor on the category **Rep** of represented spaces and realizable functions. Indeed, any idempotent jump operator $\mathbf{j}$ yields a *monad* on the category **Rep**: The condition (3) gives us a monad multiplication, and from (2) we get a unit. Thus, the $\mathbf{j}$-realizable functions on represented spaces are exactly the *Kleisli morphisms* for this monad.

One can obtain a lot of examples of idempotent jump operators from Weihrauch degrees. More precisely, for any partial multifunction f, note that f^{∂} is closed under the compositional product, i.e., $f^{\partial} \star f^{\partial} \equiv_{\mathsf{W}} f^{\partial}$, by transitivity of $\leq_{\partial\mathsf{W}}$ [12]. We show that such a Weihrauch degree always yields an idempotent jump operator. Recall that $\mathbf{1}$ is the Weihrauch degree of the identity function on ω^ω.

Lemma 2.2. *Let $\mathbf{d} \geq \mathbf{1}$ be a Weihrauch degree closed under the compositional product $\star$, i.e., $\mathbf{d} \star \mathbf{d} = \mathbf{d}$. Then, one can obtain an idempotent jump operator $\mathbf{j_d}$ on the Kleene-Vesley algebra such that the $(P, \mathbf{j_d})$-realizable partial multifunctions coincide with the partial multifunctions $\leq_W \mathbf{d}$.*

Proof. Let $\mathcal{F}_\mathbf{d}$ be the set of all multifunctions on ω^ω which is Weihrauch reducible to $\mathbf{d}$. Fix a partial multifunction $D \in \mathbf{d}$ on ω^ω. Let $(h_p, k_q)_{p,q\in\omega^\omega}$ be a standard list of pairs of partial continuous functions on ω^ω (e.g., $h_p(x) \simeq k_p(x) \simeq p * x$, where $*$ is the operation of the Kleene second algebra). Then, consider the following partial multifunction $\mathbf{j_d}$ on ω^ω:

$$\mathrm{dom}(\mathbf{j_d}) = \{(p,q,x) : h_p(x)\downarrow \ \wedge \forall y \in D(h_p(x))\ k_q(x,y)\downarrow\},$$
$$\mathbf{j_d}(p,q,x) = \{k_q(x,y) : y \in D(h_p(x))\}.$$

Clearly, $\mathbf{j_d}$ is Weihrauch reducible to D via computable reductions $(p,q,x) \mapsto h_p(x)$ and $(p,q,x,y) \mapsto k_q(x,y)$. Moreover, if $f \leq_\mathsf{W} D$ via h_p and k_q then $f \leq_\mathsf{W} \mathbf{j_d}$ via $x \mapsto (p,q,x)$ and id. As the outer reduction is identity, this indeed implies that f is $\mathbf{j_d}$-realizable. Hence, the $\mathbf{j_d}$-realizable multifunctions coincide with the multifunctions $\leq_W \mathbf{d}$.

It remains to check that $\mathbf{j_d}$ is an idempotent jump operator: (1) It is each to check that $\mathbf{j_d}(p, a{*}q, x) = \{a{*}z : z \in \mathbf{j}(p,q,x)\}$. Then, let $u \in P$ be such that $u\langle p,q,x\rangle = \langle p, uq, x\rangle$. (2) As id $\in \mathbf{1} \leq_\mathsf{W} \mathbf{d}$, there are i,j such that $\mathbf{j_d}(i,j,x) = \{x\}$. Then, let $\iota \in P$ be such that $\iota x = \langle i,j,x\rangle$. (3) Since $\mathbf{j_d} \leq_\mathsf{W} D$ and $\mathbf{d}$ is closed under compositional product, we have $\mathbf{j_d} \circ \mathbf{j_d} \leq_\mathsf{W} D \star D \leq_\mathsf{W} D$. If (d,e) are reductions witnessing $\mathbf{j_d} \circ \mathbf{j_d} \leq_\mathsf{W} D$, then we have $\mathbf{j_d} \circ \mathbf{j_d}(x) = \mathbf{j_d}(d,e,x)$. Thus, let $j \in P$ be such that $jx = \langle d,e,x\rangle$. ☐

If $\mathbf{j}$ is generated from (the finite parallelization of) LLPO, the $\mathbf{j}$-Lifshitz realizability is known as the *Lifschitz realizability* [20]. Van Oosten [32] also considered realizability obtained from WKL. Note that the property (1) in our definition of an idempotent jump operator corresponds to [20, Lemma 4], [32, Lemma 5.7], and [7, Lemma 4.4], the property (2) corresponds to [20, Lemma 2], [32, Lemma 5.4], and [7, Lemma 4.2], and (3) corresponds to [20, Lemma 3], [32, Lemma 5.6], and [7, Lemma 4.5].

For an idempotent jump $\mathbf{j}$ on a relative pca $\mathbb{P} = (P, \underset{\sim}{\mathbf{P}}, *)$, one can introduce a new partial application $*_\mathbf{j}$ on $\underset{\sim}{\mathbf{P}}$ defined by $a *_\mathbf{j} b \simeq a' * b$ if $\mathbf{j}(a) = \{a'\}$; otherwise $a *_\mathbf{j} b$ is undefined. Hereafter, we always write a' for the unique element of $\mathbf{j}(a)$ whenever $\mathbf{j}(a)$ is a singleton. Then, consider the following:

$$P_\mathbf{j} = \{a' : a \in P \text{ and } \mathbf{j}(a) \text{ is a singleton}\}.$$

Lemma 2.3. *$\mathbb{P}_\mathbf{j} = (P_\mathbf{j}, \underset{\sim}{\mathbf{P}}, *_\mathbf{j})$ is a relative pca. Moreover, (4) there is $\varepsilon \in P_\mathbf{j}$ such that for any $a \in \underset{\sim}{\mathbf{P}}$, if a' is defined, then $\varepsilon *_\mathbf{j} a = a'$.*

Proof. To see that $P_\mathbf{j}$ is closed under $*_\mathbf{j}$, for $a,b \in P$, we have $a *_\mathbf{j} b = a'b$. Then, $a'b = (\lambda x.xb)a' = (u(\lambda x.xb)a)'$, where u is from (1). As $u(\lambda x.xb)a \in P$, we have $a'b \in P_\mathbf{j}$.

Given a combinator k in P, define $\mathsf{k_j} = \iota(\lambda x.\iota(\mathsf{k}x))$, where ι is from (2). Then, $\mathsf{k_j} *_\mathbf{j} a *_\mathbf{j} b = (\mathsf{k}_\mathbf{j}'a)'b = (\iota(\mathsf{k}a))'b = \mathsf{k}ab = a$. Given a combinator s in P, define $\mathsf{s_j} = \iota(\lambda x.\iota(\lambda y.\iota(\mathsf{s}xy)))$. Then, $\mathsf{s_j} *_\mathbf{j} a *_\mathbf{j} b *_\mathbf{j} c = ((\mathsf{s}_\mathbf{j}'a)'b)'c = ((\iota(\lambda y.\iota(\mathsf{s}ay))'b)'c = (\iota(\mathsf{s}ab))'c = \mathsf{s}abc = ac(bc)$.

Thus, $\mathbb{P}_{\mathbf{j}}$ is a relative pca, and so we get a new $\lambda_{\mathbf{j}}$ by combinatory completeness. For the second assertion, define $\varepsilon = \lambda_{\mathbf{j}}x.u(\mathsf{k}x) *_{\mathbf{j}} 0$. Then, $\varepsilon *_{\mathbf{j}} a = u(\mathsf{k}a) *_{\mathbf{j}} 0 = (u(\mathsf{k}a))'0 = \mathsf{k}a'0 = a'$. $\square$

This is a procedure which makes a pca validate $\mathsf{AC}_{\omega,\omega}!$, *the axiom of unique choice on* ω. The condition (4) in Lemma 2.3 corresponds to [20, Lemma 1], [32, Lemma 5.3], and [7, Lemma 4.3]. We next show that the update of a given pca preserves the class of realizable functions.

Lemma 2.4. *The* $(P_{\mathbf{j}}, \mathbf{j})$*-realizable partial multifunctions are exactly the* $(P, \mathbf{j})$*-realizable partial multifunctions.*

Proof. By definition, every $(P, \mathbf{j})$-realizable partial multifunction is $(P_{\mathbf{j}}, \mathbf{j})$-realizable. Conversely, if f is $(P_{\mathbf{j}}, \mathbf{j})$-realizable, then there is $a \in P_{\mathbf{j}}$ such that $\mathbf{j}(ax) \subseteq f(x)$. By the definition of $P_{\mathbf{j}}$, there is $b \in P$ such that $a = b'$. Then $b'x = (\lambda y.yx)b' = (u(\lambda y.yx)b)'$, where u is from (1). Therefore, $\mathbf{j}(ax) = \mathbf{j}(b'x) = \mathbf{jj}(u(\lambda y.yx)b) = \mathbf{j}(j(u(\lambda y.yx)b))$ where j is from (3). Since $j(u(\lambda y.yx)b) \in P$, this shows that f is $(P, \mathbf{j})$-realizable. $\square$

Consequently, Lemma 2.2 still holds even if we relace the Kleene-Vesley algebra with the new pca $\mathbb{P}_{\mathbf{j}}$. Hereafter, we always assume the underlying pca $\mathbb{P}$ is $\mathbb{P}_{\mathbf{j}}$; so in particular, the properties (1), (2), (3) and (4) hold.

Lifschitz [20] showed that the above four properties (1), (2), (3) and (4) ensure that all axioms of the Heyting arithmetic **HA** are $\mathbf{j}_{\rm L}$-realizable. Moreover, van Oosten [32] showed that all axioms of Troelstra's elementary analysis **EL** are $\mathbf{j}_{\rm vO}$-realizable. Then Chen-Rathjen [7] showed that all axioms of **IZF** are $\mathbf{j}_{\rm L}$-realizable. Now, the properties (1), (2), (3), and (4) ensure the following general result.

Theorem 2.5. *If* $\mathbf{j}$ *is an idempotent jump operator on* $\mathbb{P}$*, then all axioms of* IZF *are* $\mathbf{j}$*-realizable over* $\mathbb{P}_{\mathbf{j}}$*.* $\square$

Alternatively, one can also use a topos-theoretic argument. Van Oosten (cf. [33, Section 2.6.8]) pointed out that the Lifschitz realizability predicate also yields a tripos (by considering "filters" on $\mathbb{P}$ as nonstandard truth-values), and so by the usual tripos-to-topos construction (see [33] or [27]) we obtain a *Lifschitz realizability topos*. One can adopt a similar construction to obtain a "$\mathbf{j}$-realizability topos".

2.2.4. *The internal Baire space and Weihraruch degrees.* Now we check what the internal Baire space $(\mathbb{N}^{\mathbb{N}})^{\mathbb{P}}$ is in our realizability model. First note that $e \Vdash_{\mathbb{P}} x \in \mathbb{N}^{\mathbb{N}}$ means that $e \Vdash_{\mathbb{P}} \forall n \in \mathbb{N} \exists! m \in \mathbb{N}\ \langle n, m\rangle \in x$. Then, it is easy to check that there is a unique $\tilde{x} \in \mathbb{N}^{\mathbb{N}}$ such that $\langle n, \tilde{x}(n)\rangle$ is realizable. By Chen-Rathjen [7, Lemma 4.7], given such an e, one can effectively produce a p_e such that $p_e *_{\mathbf{j}} n = \{\tilde{x}(n)\}$, i.e., $(p_e n)' = \tilde{x}(n)$. Thus, in a sense, $\tilde{x}$ is obtained from the *parallelization* of the $'$-jump. Here, for a partial multifunction f, the parallelization $\widehat{f}$ is defined by $\widehat{f}(\bigoplus_{i\in\mathbb{N}} x_i) = \bigoplus_{i\in\mathbb{N}} f(x_i)$; see [5] for more details.

Conversely, from such p_e, one can effectively recover the realizer e. Hence, the internal Baire space $(\mathbb{N}^{\mathbb{N}})^{\mathbb{P}}$ is essentially the parallelized $'$-jump of the standard representation of Baire space; that is, we consider p_x as a name of $x \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{P}}$ if $(p_x n)' = \tilde{x}(n)$.

Next, consider a sentence $S \equiv \forall x \in \mathbb{N}^{\mathbb{N}}\ [\neg P(x) \to \exists y \in \mathbb{N}^{\mathbb{N}}\ \neg Q(x, y)]$ for some formulas P and Q. If S is $\mathbf{j}$-realizable then there is an element a witnessing that $P(p_x')$

implies $Q(p'_x, y)$ for any $y \in \mathbf{j}(ap_x)$. In particular, $p_x \mapsto y$ is $\mathbf{j}$-realizable on the Kleene-Vesley algebra. This means that if $\mathbf{j} = \mathbf{j_d}$ is obtained from a Weihrauch degree $\mathbf{d}$, by Lemma 2.2, and therefore, if S yields a Weihrauch degree $\mathbf{d}_S$, then $\mathbf{j_d}$-realizability for S implies the degree $\mathbf{d}_S$ is below the parallelization of $\mathbf{d}$.

The above argument ensures that, whenever $\mathbf{d}$ is closed under both the parallelization for single-valued functions and the compositional product, such a sentence S is $\mathbf{j_d}$-realizable if and only if the Weihrauch degree of realizing S is $\leq \mathbf{d}$. If $\mathbf{d}$ is not single-valued parallelizable (i.e., there is a single-valued $f \leq_{\mathsf{W}} \mathbf{d}$ whose parallelization is not below $\mathbf{d}$), then this characterization may fail. For instance, the discrete limit operator $\lim_{\mathbb{N}}$ (see [5]) is closed under the compositional product, but not parallelizable (note that $\lim_{\mathbb{N}}$ itself is single-valued). Indeed, the parallelization of $\lim_{\mathbb{N}}$ is the limit operator, lim, on Baire space. If $\mathbf{d} = \lim_{\mathbb{N}}$ then the $\mathbf{j_d}$-realizability predicate corresponds to $(\lim, \lim)$-realizability, meaning that $p_x \mapsto p_y$ is realizable where $\lim(p_x) = x$ and $\lim(p_y) = y$. However, the $(\lim, \lim)$-realizable functions are different from both $\lim_{\mathbb{N}}$ and lim; see Brattka [4, Corollaries 3.5 and 3.6].

The possible realizability predicates obtained from continuous Weihrauch degrees of single-valued functions validating the axiom of unique choice $\mathsf{AC}_{\omega,\omega}!$ are completely classified by Kihara [15] (under meta-theory $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}$, where DC is the axiom of dependent choice, and AD is the axiom of determinacy): The structure of parallelized continuous Weihrauch degrees of single-valued functions is well-ordered (with the order type Θ), and the smallest nontrivial one already entails the Turing jump operator (a universal Baire class 1 function). In particular, under meta-theory ZFC, by Martin's Borel determinacy theorem, there are ω_1 many Borel single-valued functions (which are exactly universal Baire class α functions).

2.3. **More non-constructive principles.**

2.3.1. *Law of excluded middle.* Next, we discuss the limitations of separation arguments using generalized Weihraruch degrees. We here consider central nonconstructive principles, the *law of excluded middle* LEM: $P \vee \neg P$, the *double negation elimination* DNE: $\neg\neg P \to P$, and *de Morgan's law* DML: $\neg(P \wedge Q) \to \neg P \vee \neg Q$.

We examine the arithmetical hierarchy for these nonconstructive principles (cf. Akama et al. [1]) for formulas of second order arithmetic with function variables $\bar{v}$ (i.e, variables from $\mathbb{N}^{\mathbb{N}}$). For instance, we consider the class Σ^0_n of formulas of the form

$$\exists a_1 \forall a_2 \dots Q a_n R(a_1, a_2, \dots, a_n),$$

where $Q = \exists$ if n is odd, $Q = \forall$ if n is even, each a_i ranges over $\mathbb{N}$, and R is a decidable formula. Note that every quantifier-free formula is decidable. These principles are also described as nonconstructive principles on the reals $\mathbb{R}$. For instance:

- Σ^0_1-$\mathsf{DNE}_{\mathbb{R}}$ is known as the *Markov principle* MP, which is also equivalent to the following statement: For any regular Cauchy reals $x, y \in \mathbb{R}$, if $y \neq 0$ then you can divide x by y; that is, there is a regular Cauchy real $z \in \mathbb{R}$ such that $x = yz$.
- Σ^0_1-$\mathsf{DML}_{\mathbb{R}}$ is equivalent to the *lessor limited principle of omniscience* LLPO, which states that for any regular Cauchy reals $x, y \in \mathbb{R}$, either $x \leq y$ or $y \leq x$ holds.
- Π^0_1-$\mathsf{LEM}_{\mathbb{R}}$ is equivalent to the *limited principle of omniscience* LPO, which states that for any regular Cauchy reals $x, y \in \mathbb{R}$, either $x = y$ or $x \neq y$ holds.

- We always have Σ^0_n-$\mathsf{DNE}_\mathbb{R}$ + Π^0_n-$\mathsf{LEM}_\mathbb{R}$ = Σ^0_n-$\mathsf{LEM}_\mathbb{R}$. In particular, Π^0_1-$\mathsf{LEM}_\mathbb{R} \leftrightarrow$ Σ^0_1-$\mathsf{LEM}_\mathbb{R}$ is realizable with respect to any Weihrauch degree.
- Σ^0_2-$\mathsf{DML}_\mathbb{R}$ is equivalent (assuming Markov's principle, cf. Proposition 3.2) to the *pigeonhole principle* for infinite sets (or RT^1_2 in the context of reverse mathematics), which states that for any (characteristic functions of) $A, B \subseteq \mathbb{N}$, if it is not true that both of A and B are bounded, then either A or B is unbounded.

The arithmetical hierarchy of the law of excluded middle is first studied in the context of limit computable mathematics (one standpoint of constructive mathematics which allows *trial and errors*, which may be considered as a special case of a realizability interpretation of constructive mathematics w.r.t. a suitable partial combinatory algebra, see also Section 2.2.1). The equivalence between Σ^0_2-$\mathsf{DNE}_\mathbb{R}$ and Hilbert's basis theorem (Dickson's lemma), cf. [11, Section 5], can also be considered as an early result in constructive reverse mathematics.

If we have the axiom $\mathsf{AC}_{\omega,\omega}!$ of unique choice on natural numbers, then Π^0_1-$\mathsf{LEM}_\mathbb{R}$ implies the existence of the Turing jump of a given real. This implies the axiom scheme ACA of arithmetical comprehension, which clearly entails the law of excluded middle for each level of the arithmetical hierarchy. By this reason, *under unique choice* $\mathsf{AC}_{\omega,\omega}!$*, the arithmetical hierarchy of the law of excluded middle collapses.*

A realizability model always validates the unique choice $\mathsf{AC}_{\omega,\omega}!$. Thus, for instance, the following seems still open:

Question 1. *For $n > 0$, does EL_0 prove Σ^0_n-$\mathsf{LEM}_\mathbb{R} \to \Sigma^0_{n+1}$-$\mathsf{LEM}_\mathbb{R}$?*

We now go back to the discussion in Section 2.2.4: The discrete limit operation $\lim_\mathbb{N}$ is closed under the compositional product, but not parallelizable, and its parallelization $\lim$ is not closed under the compositional product. Obviously, its $\star$-closure $\lim^\partial$ dominates the whole arithmetical hierarchy. A similar argument also applies for LPO. Roughly speaking, $\lim_\mathbb{N}$ can be thought of as the law of double negation elimination for Σ^0_2-formulas, Σ^0_2-DNE (see Proposition 3.1). The parallelization of a single-valued function is essentially the axiom $\mathsf{AC}_{\omega,\omega}!$ of unique choice on natural numbers. The unique choice property for our realizability notions comes from the condition (4) in Lemma 2.3. This is another way of explaining the collapsing phenomenon of the arithmetical hierarchy of the law of excluded middle.

In summary, in $\mathsf{AC}_{\omega,\omega}!$-models, LPO implies ACA, so in particular WKL. However, it is easy to see that $\mathsf{WKL} \not\leq_{\partial\mathsf{W}} \mathsf{LPO}$. This means that a $\leq_{\partial\mathsf{W}}$-separation result does not always imply a separation result in constructive reverse mathematics. Then, we are also interested in how $\mathsf{AC}_{\omega,\omega}!$-models and generalized Weihrauch reducibility $\leq_{\partial\mathsf{W}}$ are different. This is what we will deal with in the later sections.

2.3.2. *Auxiliary principles.* To understand the structure of generalized Weihrauch reducibility $\leq_{\partial\mathsf{W}}$, we also consider the following principles $\mathsf{BE}_\mathbb{Q}$ and $\mathsf{IVT}_{\rm lin}$. These two principles may look weird, but we consider them since they naturally occur in our proofs. Indeed, all of these principles turn out to be equivalent to very weak variants of weak König's lemma (and indeed equivalent to weak König's lemma under countable choice); see also Section 3.1.2.

Binary expansion for regular Cauchy rationals $\mathsf{BE}_\mathbb{Q}$. We next consider the binary expansion principle $\mathsf{BE}_\mathbb{Q}$ for regular Cauchy rationals (more precisely, for non-irrationals), which states that, for any regular Cauchy real, if it happens to be a rational, then it has a binary expansion:

$$(\forall x \in [0,1]) \left[\neg(\forall a, b \in \mathbb{Z},\ ax \neq b) \longrightarrow \exists f\colon \mathbb{N} \to \{0,1\},\ x = \sum_{n=1}^{\infty} 2^{-f(n)}\right].$$

Intermediate value theorem for piecewise linear maps $\mathsf{IVT}_{\rm lin}$. Finally, we consider a (weird) variant of the intermediate value theorem. A rational piecewise linear map on $\mathbb{R}$ is determined by finitely many rational points $\bar{p} = (p_0, \dots, p_\ell)$ in the plane. Then, any continuous function on $\mathbb{R}$ can be represented as the limit of a sequence $(f_s)_{s\in\omega}$ of rational piecewise linear maps coded by $(\bar{p}^s)_{s\in\omega}$, where any p_i^{s+1} belongs to the 2^{-s-1}-neighborhood of the graph of f_s. That is, the sequence $(\bar{p}^s)_{s\in\omega}$ codes an approximation procedure of a continuous function f. Then, consider the case that a code $(\bar{p}^s)_{s\in\omega}$ of an approximation stabilizes, in the sense that $\bar{p}^s = \bar{p}^t$ for sufficiently large s, t. Then, of course, it converges to a rational piecewise linear map.

In other words, we represent a continuous function as usual, but we consider the case that such a continuous function *happens to be a rational piecewise linear map*. Formally, if f is coded by $(\bar{p}^s)_{s\in\omega}$ which does not change infinitely often (that is, $\neg\forall s \exists t > s\ \bar{p}^t \neq \bar{p}^s$) then we call f *a rational piecewise linear map as a discrete limit*. The principle $\mathsf{IVT}_{\rm lin}$ states that if $f\colon [0,1] \to \mathbb{R}$ is a rational piecewise linear map as a discrete limit such that $f(0) < 0 < f(1)$ then there is $x \in [0,1]$ such that $f(x) = 0$.

2.3.3. *Discussion.* For the relationship among non-constructive principles, see also Figure 1. Here, we explain some nontrivial implications: For Σ^0_2-$\mathsf{LEM}_{\mathbb{R}} \to \mathsf{IVT}$, the standard proof of the intermediate value theorem employs the nested interval argument with the conditional branching on whether $f(x) = 0$ for some $x \in \mathbb{Q}$. This shows that Σ^0_2-$\mathsf{LEM}_{\mathbb{R}}$ implies IVT, and hence BE. For Π^0_1-$\mathsf{LEM}_{\mathbb{R}} \to \mathsf{RDIV}$, one can decide if y is zero or not by using Π^0_1-$\mathsf{LEM}_{\mathbb{R}}$. If $y = 0$ then put $z = 0$; otherwise define $z = x/y$. Thus, Π^0_1-$\mathsf{LEM}_{\mathbb{R}}$ implies RDIV. For other nontrivial implications, see Section 3.1.3.

All principles mentioned above, except for Markov's principle, are *discontinuous* principles; that is, there are no continuous realizers for these principles. For instance, a realizer for the law of excluded middle Π^0_2-$\mathsf{LEM}_{\mathbb{R}}$ for Π^0_2 formulas is clearly related to *Dirichlet's nowhere continuous function* $\chi_{\mathbb{Q}}$. Similarly, any total binary expansion $\mathbb{R} \to 2^\omega$ is discontinuous, and indeed, there is no continuous function which, given a regular Cauchy representation of a real, retunes its binary presentation. Thus, all of these principles fail in a well-known model of **IZF** (indeed a topos) in which all functions on ω^ω are continuous.

However, there are differences among the *degrees of discontinuity* for these principles. For instance, Dirichlet's function is σ-continuous (i.e., decomposable into countably many continuous functions), and of Baire class 2 (i.e., a double pointwise limit of continuous functions), but not of Baire class 1, while weak König's lemma WKL has no σ-continuous realizer, but has a realizer of Baire class 1. Note also that WKL (even IVT) has no $\utilde{\mathbf{\Delta}}^0_2$-measurable realizer, where recall that there are functions of Baire class 1 (equivalently F_σ-measurable) which are not $\utilde{\mathbf{\Delta}}^0_2$-measurable. On the other hand, RDIV has a $\utilde{\mathbf{\Delta}}^0_2$-measurable realizer.

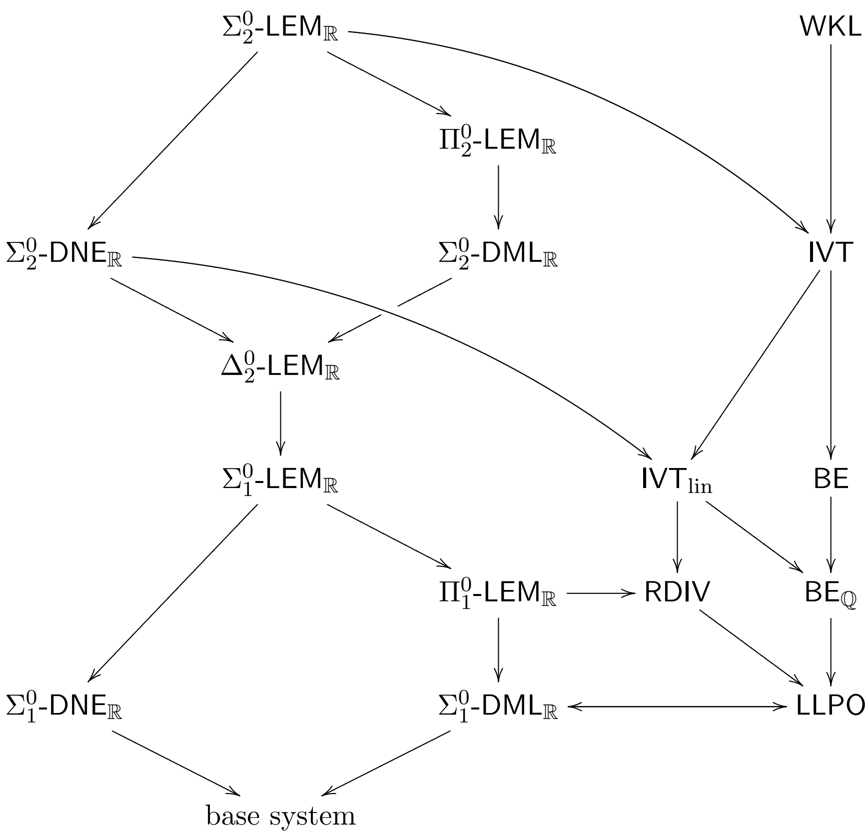

Figure 1. Nonconstructive principles over a base intuitionistic system

These observations indicate that these principles are all different. However, one of the questions is, given a level $\mathbf{d}$ of discontinuity (i.e., a continuous Weihrauch degree), how one can construct a model of constructive mathematics in which the functions on ω^ω are exactly those with discontinuity level at most $\mathbf{d}$.

In later sections, we will show $\leq^c_{\Game\mathsf{W}}$-separation results for most principles in Figure 1 (where recall that $\leq^c_{\Game\mathsf{W}}$ stands for continuous generalized Weihrauch reducibility):

Theorem 2.6.

(1) $\mathsf{RDIV} \not\leq^c_{\Game\mathsf{W}} \mathsf{BE}$.
(2) $\mathsf{BE}_{\mathbb{Q}} \not\leq^c_{\Game\mathsf{W}} \mathsf{RDIV}$.
(3) $\mathsf{IVT}_{\mathrm{lin}} \not\leq^c_{\Game\mathsf{W}} \mathsf{RDIV} \times \mathsf{BE}$.
(4) $\mathsf{BE} \not\leq^c_{\Game\mathsf{W}} \Sigma^0_2\text{-}\mathsf{DML}_{\mathbb{R}} \times \Sigma^0_2\text{-}\mathsf{DNE}_{\mathbb{R}}$.
(5) $\mathsf{IVT} \not\leq^c_{\Game\mathsf{W}} \Sigma^0_2\text{-}\mathsf{DML}_{\mathbb{R}} \times \Sigma^0_2\text{-}\mathsf{DNE}_{\mathbb{R}} \times \mathsf{BE}$.

However, our realizability model validates $\mathsf{AC}_{\omega,\omega}!$, so not every such $\leq^c_{\Game\mathsf{W}}$-separation result yields an **IZF**-separation result as discussed in Section 2.3.1.

3. Main proofs

3.1. **Weihrauch principles.** We first show some implications and equivalences among non-constructive principles in the context of Weihrauch degrees.

3.1.1. *Law of excluded middle.* Hereafter, we consider principles Σ^0_1-$\mathsf{DML}_{\mathbb{R}}$, Σ^0_2-$\mathsf{DNE}_{\mathbb{R}}$, and Σ^0_2-$\mathsf{DML}_{\mathbb{R}}$ as partial multifunctions. First, recall that de Morgan's law Σ^0_1-$\mathsf{DML}_{\mathbb{R}}$ is formulated as LLPO.

As a multifunction, Σ^0_2-$\mathsf{DNE}_{\mathbb{R}}$ is equivalent to the discrete limit operation $\mathsf{Lim}_{\mathbb{N}}$ (or equivalently, learnability with finite mind changes) in the Weihrauch context, where the discrete limit function $\mathsf{Lim}_{\mathbb{N}}$ is formalized as follows: Given a sequence $(a_i)_{i\in\mathbb{N}}$ of natural numbers, if $a_i = a_j$ holds for sufficiently large i, j, then $\mathsf{Lim}_{\mathbb{N}}$ returns the value of such an a_i. That is, $\mathsf{Lim}_{\mathbb{N}}$ is exactly the limit operation on the discrete space $\mathbb{N}$.

Proposition 3.1. Σ^0_2-$\mathsf{DNE}_{\mathbb{R}} \equiv_{\mathsf{W}} \mathsf{Lim}_{\mathbb{N}}$.

Proof. Assume that a sentence $\neg\neg\exists n\forall m R(n,m,\alpha)$ is given, where R is decidable. Search for the least n such that $\forall m R(n,m,\alpha)$ is true. Begin with $n = 0$, and if $\forall m R(n,m,\alpha)$ fails, then one can eventually find m such that $R(n,m,\alpha)$ by Markov's principle. If such an m is found, proceed this algorithm with $n+1$. If $\neg\neg\exists n\forall m R(n,m,\alpha)$ is true, then this algorithm eventually halts with the correct n such that $\forall m R(n,m,\alpha)$ is true. □

First note that de Morgan's law Σ^0_2-$\mathsf{DML}_{\mathbb{R}}$ for Σ^0_2 formulas is essentially equivalent to the pigeonhole principle on $\mathbb{N}$, or Ramsey's theorem RT^1_2 for singletons and two colors. Here, RT^1_2 (as a multifunction) states that, given a function $f\colon \mathbb{N}\to 2$ (which is called a 2-*coloring*), returns $h\colon \mathbb{N}\to 2$ such that f is constant on $H = \{n : h(n) = 1\}$, where H is called a *homogeneous set*.

Proposition 3.2. Σ^0_2-$\mathsf{DML}_{\mathbb{R}} \equiv_{\mathsf{W}} \mathsf{RT}^1_2$.

Proof. For $\mathsf{RT}^1_2 \leq_{\mathsf{W}} \Sigma^0_2$-$\mathsf{DML}_{\mathbb{R}}$, given a coloring $f\colon \mathbb{N}\to 2$, consider (the characteristic function of) the set $C_i = \{n : f(n) = i\}$. Either C_0 or C_1 is unbounded, so Σ^0_2-$\mathsf{DML}_{\mathbb{R}}$ choses an index j such that C_j is unbounded. Then, C_j must be an infinite homogeneous set.

For Σ^0_2-$\mathsf{DML}_{\mathbb{R}} \leq_{\mathsf{W}} \mathsf{RT}^1_2$, assume that Σ^0_2 formulas $\psi_i \equiv \exists a\forall b\varphi_i(a,b)$ are given. If $\psi_0\wedge\psi_1$ is false, then by Markov's principle, for any a, either $\exists b\neg\varphi_0(a,b)$ or $\exists b\neg\varphi_1(a,b)$. Hence, given a, search for the least b such that either $\neg\varphi_0(a,b)$ or $\neg\varphi_1(a,b)$ holds. For such b, if $\varphi_0(a,b)$ fails, then put $f(a) = 0$; otherwise put $f(a) = 1$. By RT^1_2, we get an infinite homogeneous set H for f. Then given any $a\in H$, compute the value $f(a) = j$. This j is a solution to Σ^0_2-$\mathsf{DML}_{\mathbb{R}}$ since for any $a\in H$ we have $\exists b\neg\varphi_j(a,b)$. □

3.1.2. *Weak König's lemma.* In this section, we consider weak variants of weak König's lemma. A *binary tree* is a subset of $2^{<\omega}$ closed under taking initial segments. A binary tree T is *infinite* if for any ℓ there is a node $\sigma\in T$ of length ℓ. We consider the following properties for trees.

- A 2-*tree* is a binary tree T such that, for any $\ell\in\mathbb{N}$, T has exactly two strings of length ℓ.
- A *rational* 2-*tree* is a 2-tree which does *not* have infinitely many branching nodes.
- An *all-or-unique tree* or simply an *aou-tree* is a binary tree T such that, for any $\ell\in\mathbb{N}$, T has either all binary strings or only one string of length ℓ.
- A *convex tree* is a binary tree T such that, for any $\ell\in\omega$, if two strings σ and τ of length ℓ are contained in T, then all length ℓ strings between σ and τ are also contained in T.

$$\begin{tikzcd}
 & \mathsf{WKL} \arrow[d] \\
 & \mathsf{IVT} \arrow[dl] \arrow[d] \\
\mathsf{IVT}_{\mathrm{lin}} \arrow[d] \arrow[dr] & \mathsf{BE} \arrow[d] \\
\mathsf{RDIV} \arrow[dr] & \mathsf{BE}_{\mathbb{Q}} \arrow[d] \\
 & \mathsf{LLPO}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathsf{WKL} \arrow[d] \\
 & \mathsf{WKL}_{\mathrm{conv}} \arrow[dl] \arrow[d] \\
\mathsf{WKL}_{\mathrm{clop}} \arrow[d] \arrow[dr] & \mathsf{WKL}_{\leq 2} \arrow[d] \\
\mathsf{WKL}_{\mathrm{aou}} \arrow[dr] & \mathsf{WKL}_{=2} \arrow[d] \\
 & \mathsf{LLPO}
\end{tikzcd}$$

FIGURE 2. Nonconstructive principles as variants of weak König's lemma

- A *clopen tree* is a binary tree T satisfying the double negation of the following: There are finitely many binary strings $\sigma_0, \dots, \sigma_\ell$ such that $\tau \in T$ if and only if τ extends σ_i for some $i \leq \ell$. If $\ell = 0$, then we say that T is a *basic clopen tree*.

The notion of a clopen tree is a bit weird, but we use this notion as it is useful for understanding the relationship among non-constructive principles. For a collection $\mathcal{T}$ of trees, we mean by *weak König's lemma for $\mathcal{T}$* the statement that every infinite tree in $\mathcal{T}$ has an infinite path. Then, we denote by $\mathsf{WKL}_{\leq 2}$, $\mathsf{WKL}_{=2}$, $\mathsf{WKL}_{\mathrm{aou}}$, $\mathsf{WKL}_{\mathrm{conv}}$, and $\mathsf{WKL}_{\mathrm{clop}}$ weak König's lemma for 2-trees, rational 2-trees, aou-trees, convex trees, and convex clopen trees, respectively.

It is known that, over elementary analysis EL_0, the binary expansion principle BE is equivalent to weak König's lemma for 2-trees, $\mathsf{WKL}_{\leq 2}$, and the intermediate value theorem IVT is equivalent to weak König's lemma for convex trees, $\mathsf{WKL}_{\mathrm{conv}}$, cf. [3]. For the relationship among these principles, see Figure 2 and Section 3.1.3 for the proofs.

In the context of Weihrauch degrees, these notions are formalized as multifunctions. First, *weak König's lemma for 2-trees*, $\mathsf{WKL}_{\leq 2} :\subseteq 2^{2^{<\omega}} \rightrightarrows 2^\omega$, is defined as follows:

$$\mathrm{dom}(\mathsf{WKL}_{\leq 2}) = \{T \subseteq 2^{<\omega} : T \text{ is a 2-tree}\},$$
$$\mathsf{WKL}_{\leq 2}(T) = [T] := \{p \in 2^\omega : p \text{ is an infinite path through } T\}.$$

Classically, a rational 2-tree has *exactly two* infinite paths. Then, define the multifunction $\mathsf{WKL}_{=2}$ as $\mathsf{WKL}_{\leq 2}$ restricted to the rational 2-trees.

In the study of Weihrauch degrees, this kind of notion is treated as a choice principle. Consider the *choice principle for finite closed sets in X with at most n elements* $\mathsf{C}_{X,\#\leq n}$ *(with exactly n elements $\mathsf{C}_{X,\#=n}$)*, which states that, given (a code of) a closed subset P of a space X if P is nonempty, but has at most n elements (exactly n elements), then P has an element; see [19]. The principles $\mathsf{WKL}_{\leq 2}$ and $\mathsf{C}_{X,\#\leq 2}$ are not constructively equivalent, where $X = 2^\omega$. However, if we consider $\mathsf{WKL}_{\leq 2}$ as a multifunction, then they are Weihrauch equivalent. Similarly, $\mathsf{WKL}_{=2}$ and $\mathsf{C}_{X,\#=2}$ are Weihrauch equivalent.

Next, the *weak König's lemma for convex trees* is a multifunction $\mathsf{WKL}_{\rm conv} :\subseteq 2^{2^{<\omega}} \rightrightarrows 2^\omega$ defined as follows:

$$\mathrm{dom}(\mathsf{WKL}_{\rm conv}) = \{T \subseteq 2^{<\omega} : T \text{ is a convex tree}\},$$
$$\mathsf{WKL}_{\rm conv}(T) = [T] := \{p \in 2^\omega : p \text{ is an infinite path through } T\}.$$

In the context of Weihrauch degrees, a clopen tree is exactly a binary tree T such that $[T]$ is clopen, and a basic clopen tree is a tree of the form $\{\tau \in 2^{<\omega} : \tau \succ \sigma\}$ for some binary string σ. It is clear that every basic clopen tree is a convex clopen tree. Then, define the multifunction $\mathsf{WKL}_{\rm clop}$ as $\mathsf{WKL}_{\rm conv}$ restricted to the convex clopen trees.

As mentioned above, the intermediate value theorem IVT is known to be constructively equivalent to weak König's lemma for convex trees, which is also Weihrauch equivalent to the convex choice XC_X for $X = 2^\omega$, where XC_X states that, given (a code of) a closed subset P of X, if P is convex, then P has an element, cf. [19, 17].

Finally, as a multifunction, *weak König's lemma for aou-trees*, $\mathsf{WKL}_{\rm aou} :\subseteq 2^{2^{<\omega}} \rightrightarrows 2^\omega$, is defined as follows:

$$\mathrm{dom}(\mathsf{WKL}_{\rm aou}) = \{T \subseteq 2^{<\omega} : T \text{ is an aou-tree}\},$$
$$\mathsf{WKL}_{\rm aou}(T) = [T] := \{p \in 2^\omega : p \text{ is an infinite path through } T\}.$$

As a multifunction, the robust division principle RDIV is Weihrauch equivalent to the *all-or-unique choice* AoUC (or equivalently, the *totalization of unique choice*), which takes, as an input (a code of), a closed subset P of Cantor space 2^ω such that either $P = X$ or P is a singleton, and then any element of P is a possible output, cf. [16]. This is, of course, Weihrauch equivalent to weak König's lemma for aou-trees.

3.1.3. *Implications.* As seen above, the principles introduced in Section 1.1.1 are written as some variants of weak König's lemma; see Figure 2. We show some nontrivial implications in Figure 2.

Proposition 3.3. $\mathsf{WKL}_{\rm clop} \to \mathsf{WKL}_{\rm aou}$.

Proof. Let T be an aou-tree. If T is found to be unique σ at height s, then let $H(T) = \{\tau : \tau \succ \sigma\}$, and for any infinite path p, define $K(p) = (p(0), p(1), \dots, p(s-2), q(s-1), q(s), q(s{+}1), \dots)$, where q is the unique infinite path through T. If it is not witnessed, i.e., $T = 2^{<\omega}$, then $H(T)$ becomes $2^{<\omega}$, and $K(p) = p$ for any infinite path p.

More formally, we define a basic clopen tree T^* and a function $k : \omega \times 2 \to 2$ as follows: First put the empty string into T^*. Given $s > 0$, if T contains all binary strings of length s then T^* contains all strings of length s, and define $k(s-1, i) = i$ for each $i < 2$. If T has only one binary string σ of length s but it is not true for all $t < s$, then σ is also the unique binary string of length s in T^*, and define $k(s-1, i) = \sigma(s-1)$ for each $i < 2$. Otherwise, for each $\tau \in T^*$ of length $s-1$, put $\tau 0$ and $\tau 1$ into T, and define $k(s-1, i) = \sigma(s-1)$ for each $i < 2$. Then, T^* is clearly a basic clopen tree. Therefore, by $\mathsf{WKL}_{\rm clop}$, the tree T^* has an infinite path p. Then, define $p^*(n) = k(n, p(n))$. Then, p^* is of the form $\langle p(0), p(1), \dots, p(t-1), \sigma_t(t), \sigma_{t+1}(t+1), \dots\rangle$, where σ_t is a string of length $t+1$ in T. It is easy to check that p^* is an infinite path through T. $\square$

Proposition 3.4. $\mathsf{WKL}_{\rm clop} \to \mathsf{WKL}_{=2}$.

Proof. Let T be a rational 2-tree. We define a basic clopen tree T^* as follows: For each $s > 0$, T has exactly two nodes ℓ_s, r_s of length s. For $s = 0$, let $\ell_s = r_s$ be the empty string. If both ℓ_s and r_s extend ℓ_{s-1} and r_{s-1}, respectively, then for each $\tau \in T$ of length $s-1$ put $\tau 0$ and $\tau 1$ into T. Otherwise, ℓ_s and r_s is of the form $\ell_s = b_s 0$ and $r_s = b_s 1$ for some b_s. Then, put both $\ell_s = b_s 0$ and $r_s = b_s 1$ into T^*. Clearly, T^* is the clopen tree of the form $T^* = \{\tau \in 2^\omega : \tau \succ b\}$, where b is the last branching node of T.

By $\mathsf{WKL}_{\mathsf{clop}}$, the tree T^* has an infinite path p. For each s, check if $p \upharpoonright s+1 \in T$ or not. If it is true, define $p^*(s) = p(s)$. Otherwise, if s is the least such number, then $p \upharpoonright s \in T$, and there is no branching node above $p \upharpoonright s$. Thus, for any $t \geq s$, there is the unique string $\sigma_t \in T$ of length t extending $p \upharpoonright s$. Hence, for such σ_t, define $p^*(t) = \sigma_t(t)$. Then, it is easy to see that p^* is an infinite path through T. □

Proposition 3.5. Σ^0_2-$\mathsf{DNE}_{\mathbb{R}} \to \mathsf{WKL}_{\mathrm{clop}}$.

Proof. Let T be a basic clopen tree. Then, we have the double negation of the existence of σ such that $\tau \in T$ for all $\tau \succeq \sigma$. By Σ^0_2-$\mathsf{DNE}_{\mathbb{R}}$, we get such a σ. Then, $\sigma^\frown\langle 0, 0, \dots\rangle$ is an infinite path through T. □

Proposition 3.6. $\mathsf{BE}_{\mathbb{Q}} \longleftrightarrow \mathsf{WKL}_{=2}$.

Proof. For the forward direction, from a given rational 2-tree T, one can easily construct a regular Cauchy real which happens to be a dyadic rational whose binary expansions are exactly infinite paths through T. For the backward direction, from a regular Cauchy real α, it is also easy to construct a 2-tree T whose infinite paths are binary expansions of α as usual. Although the binary expansion of a rational α is periodic, i.e., of the form $\sigma^\frown\tau^\frown\tau^\frown\tau^\frown\dots$, this is insufficient for ensuring T to be rational, so we need to construct another rational 2-tree T^* from T.

Use ℓ^*_s to denote the leftmost node of length s in T^*, and r^*_s to denote the rightmost node. We inductively ensure that for any s, every node of length $s+1$ in T is a successor of a node of length s in T^*. Fix s. Let b_{s+1} be the last branching node of length $\leq s+1$ in T. Consider all decompositions $b_{s+1} = \sigma^\frown\tau^\frown\tau^\frown\dots{}^\frown\tau^\frown(\tau \upharpoonright j)$, and then take the shortest $\sigma^\frown\tau$. Put $t = 2|b_{s+1}| + 1$. If either $b_t = b_{s+1}$ or $\sigma^\frown\tau$ is updated from the previous one, then define $\ell^*_s = \ell_s$ and $r^*_s = r_s$. Otherwise, $b_{s+1} \neq b_t$, so either $\ell_{s+1} \preceq b_t$ or $r_{s+1} \preceq b_t$ holds. If the former holds then, by induction hypothesis, ℓ_{s+1} is a successor of u^*_s, where $u \in \{\ell, r\}$. Put $\overline{u} = r$ if $u = \ell$; otherwise $\overline{u} = \ell$. Then, define $u^*_{s+1} = \ell_{s+1}$ and $\overline{u}^*_{s+1} = \overline{u}^*_s 0$. This ensures that $\ell^*_{s+1} \preceq b_t \preceq \ell_t, r_t$. If $r_{s+1} \preceq b_t$ holds, replace ℓ_{s+1} with r_{s+1} and vice versa. This procedure adds no new branching node. By periodicity of a binary expansion, $\sigma^\frown\tau$ cannot be updated infinitely often. Moreover, there cannot be infinitely many different b_{s+1} such that $b_t = b_{s+1}$ happens. This is because $b_t = b_{s+1}$ implies that $\ell_t = b_{s+1}01^{|b_{s+1}|}$ and $r_t = b_{s+1}10^{|b_{s+1}|}$, which witnesses that there is no seed $\sigma^\frown\tau$ of periodicity of α in b_{s+1}. Hence, if there are infinitely many such b_{s+1}, then α cannot be periodic. Thus, T^* is a rational 2-tree.

Note that if T is already rational then $T^* = T$. Otherwise, T has infinitely many branching nodes, and thus, we check arbitrary long decompositions, so the last branching node of T^* is the first branching node in T after finding the correct seed $\sigma^\frown\tau$ of periodicity of α. In particular, any infinite path through T^* extends $\sigma^\frown\tau$. Now, given an infinite path p through T^*, for each s, check if $p \upharpoonright s \in T$. If s is the least number such that $p \upharpoonright s+1 \notin T$ then T is not rational, and thus p extends the first branching node $b \in T$

after finding the correct seed $\sigma^\frown\tau$. In particular, we must have $\sigma^\frown\tau \preceq b \preceq p \upharpoonright s \in T$. Now, consider the all decompositions $p \upharpoonright s = \mu^\frown\nu^\frown\nu^\frown \dots {}^\frown\nu^\frown(\nu \upharpoonright j)$. There are only finitely many such decompositions, and if a decomposition is incorrect, it is witnessed after seeing T up to some finite height (which is effectively calculated from s). Thus, after checking such a height, we see that all surviving decompositions are equivalent. Therefore, a seed $\mu^\frown\nu$ of such a decomposition is actually equivalent to the correct seed $\sigma^\frown\tau$; hence $\mu^\frown\nu^\frown\nu^\frown\nu^\frown \dots$ is an infinite path through T. □

Proposition 3.7. $\mathsf{IVT}_{\rm lin} \longleftrightarrow \mathsf{WKL}_{\rm clop}$.

Proof. For the forward direction, one can assign an interval I_σ to each binary string σ in a standard manner: let $\tilde{\sigma} = \sigma[2/1]$ be the result of replacing every occurrence of 1 in σ with 2, and then the endpoints of I_σ are defined by $0.\tilde{\sigma}1$ and $0.\tilde{\sigma}2$ under the ternary expansions. Given a basic clopen tree $T = \{\tau : \tau \succeq \sigma\}$, it is easy to construct a piecewise linear map whose zeros are exactly I_σ.

For the backward direction, let (f_s) be a descrete approximation of a rational piecewise linear map f with $f(0) < 0 < f(1)$. Note that each f_s is determined by finitely many points in the plane. However, for each s, we only need to pay attention on at most four rational points $(x_i, f_s(x_i))_{i<4}$, where $x_0 \leq x_1 < x_2 \leq x_3$, and $-2^{-s} > f_s(x_0) \leq f_s(x_1) \leq 0 \leq f(x_2) \leq f(x_3) > 2^{-s}$. If $f(x_1) = f(x_2)$ then consider $I_s = [x_1, x_2]$. Otherwise, by linearity, one can compute a unique rational $x_1 < a_s < x_3$ such that $f(a_s) = 0$. Define I_s as a sufficiently small rational neighborhood of a_s, and one can ensure that (I_s) is a decreasing sequence.

Then, we define the tree T^* of binary expansions of elements of $\bigcap_n I_n$. If the rational interval I_s is updated because of $f(x_1) = f(x_2)$, then one can easily compute finitely many binary strings $\sigma_0, \dots, \sigma_\ell$ of length s such that any binary expansion of a real in I_s extends some σ_i. Then the nodes of length s in T^* are exactly $\sigma_0, \dots, \sigma_\ell$. If we currently guess $f(a_s) = 0$, then we proceed the argument in the proof of Proposition 3.6 to construct a basic clopen tree which determines a binary expansion of a_s.

Given an infinite path p through T^*, for each s, check if $p \upharpoonright s$ has an extension in I_s. If not, let s be the least number such that $p \upharpoonright s+1 \notin T$. Note that if I_t is updated because of $f(x_1) = f(x_2)$ for some $t > s$ then we take some nodes $\sigma_0, \dots, \sigma_\ell$ of length t and $p \upharpoonright s \preceq p \upharpoonright t = \sigma_i$ for some i, but σ_i extends to some element in $I_t \subseteq I_s$. Thus, this never happens. If we guess $f(a_s) = 0$, then as in the proof of Proposition 3.6 we get a binary expansion of a_s, so a regular Cauchy representation of a_s. Otherwise, the current guess I_s is already a correct solution, so just take any extension of $p \upharpoonright s$ which belongs to I_s. □

3.2. **Weihrauch separations.** In order to prove Theorem 1.1, we will lift separation results on Weihrauch degrees to the results on **IZF**. In this section, we formalize and prove statements on Weihrauch degrees which are needed to prove Theorem 1.1.

3.2.1. *The strength of choice for finite sets.* We first examine the strength of the game reinforcement of weak König's lemma for 2-trees, $(\mathsf{WKL}_{\leq 2})^{\circlearrowright}$. It is not hard to see that $(\mathsf{WKL}_{\leq 2})^{\circlearrowright}$ can be reduced to the finite parallelization $\mathsf{WKL}^*_{\leq 2}$ of $\mathsf{WKL}_{\leq 2}$; see Le Roux-Pauly [19] (where $\mathsf{WKL}_{\leq n} \star \mathsf{WKL}_{\leq m} \leq_{\mathsf{W}} \mathsf{WKL}_{\leq nm}$ is claimed) or Pauly-Tsuiki [26]: $\mathsf{C}^{\circlearrowright}_{2^\omega,\#\leq 2} \equiv_{\mathsf{W}} \bigsqcup_n \mathsf{C}_{2^\omega,\#\leq n}$.

Fact 3.8. $(\mathsf{WKL}_{\leq 2})^{\supset} \equiv_{\mathsf{W}} (\mathsf{WKL}_{\leq 2})^*$.

We first show that $(\mathsf{WKL}_{\leq 2})^{\supset}$ is not strong enough for solving weak König's lemma $\mathsf{WKL}_{\rm aou}$ for all-or-unique trees:

Proposition 3.9. $\mathsf{WKL}_{\rm aou} \not\leq^c_{\mathsf{W}} (\mathsf{WKL}_{\leq 2})^{\supset}$.

Proof. By Fact 3.8, it suffices to show that $\mathsf{WKL}_{\rm aou} \not\leq_{\mathsf{W}} (\mathsf{WKL}_{\leq 2})^*$. We now consider the following play:

$$\begin{array}{lllll} \text{I:} & T & & x = \langle x_0, \dots, x_n\rangle & \\ \text{II:} & & \langle S_0, \dots, S_n\rangle & & \alpha_x \end{array}$$

Player II tries to construct a sequence $S = \langle S_0, \dots, S_n\rangle$ of 2-trees, and a path α_x through I's aou-tree T, where the second move α_x depends on Player I's second move x. As usual, each partial continuous function f is coded as an element p of $(\omega \cup \{\perp\})^\omega$, so one can read a finite portion $p[s] = \langle p(0), p(1), \dots, p(s-1)\rangle$ of p by stage s. More precisely, $p(s) = \langle n, m\rangle$ indicates that we obtain the information $f(n)\downarrow= m$ at stage $s+1$, but if $p(s) = \perp$ we get no information.

By *recursion trick* (Section 2.1.3), it is sufficient to describe an algorithm constructing an aou-tree T from given S, α such that Player I wins along the above play for some x. Now, the algorithm is given as follows:

(1) At each stage s, either T contains all binary strings of length s or T only has a single node of length s. If we do not act at stage s (while our algorithm works), we always assume that T contains all binary strings of length $s+1$.
(2) Wait for n (in Player II's first move) being determined. Then wait for $\alpha_x \upharpoonright n+2$ being defined for any correct x; that is, any x with $x_i \in [S_i]$. By compactness, if it happens, it is witnessed by some finite stage.
(3) As we consider $n+1$ many 2-trees $S_0, \dots, S_n$, there are at most 2^{n+1} many correct x's, so there are at most 2^{n+1} many α_x's. Clearly, there are 2^{n+2} nodes of length $n+2$, so choose a binary string σ of length $n+2$ which is different from any $\alpha_x \upharpoonright n+2$. Then, we declare that $T = \{\sigma^\frown 0^\omega\}$, and our algorithm halts.

Note that if the procedure arrives at (3) then α_x cannot be a path through T. Otherwise, the algorithm waits at (2) forever, which means that α_x is not total for some correct x, and by (1) we have $T = 2^{<\omega}$, so Player I obeys the rule.

To apply recursion trick, even if Player II violates the rule, the first move T of Player I needs to obey the rule. To ensure this, one may assume that we proceed the step (3) in our algorithm only if Player II's partial trees look like 2-trees. Moreover, our algorithm performs some action on T only once; hence, in any case, T must be an aou-tree. Hence, Player I wins. □

3.2.2. *The strength of all-of-unique choice.* We next examine the strength of the game reinforcement of weak König's lemma for all-or-unique trees, $(\mathsf{WKL}_{\rm aou})^{\supset}$. Kihara-Pauly [16] showed that the hierarchy of compositional products of the finite parallelization AoUC^* of the all-or-unique choice AoUC collapses after the second level (where recall that AoUC is Weihrauch equivalent to $\mathsf{WKL}_{\rm aou}$). In particular, we have the following:

Fact 3.10. $(\mathsf{WKL}_{\rm aou})^{\supset} \equiv_{\mathsf{W}} (\mathsf{WKL}_{\rm aou})^* \star (\mathsf{WKL}_{\rm aou})^*$

We first verify that $(\mathsf{WKL}_{\mathrm{aou}})^{\partial}$ is not strong enough for solving weak König's lemma $\mathsf{WKL}_{=2}$ for rational 2-trees:

Proposition 3.11. $\mathsf{WKL}_{=2} \not\leq^{c}_{\mathsf{W}} (\mathsf{WKL}_{\mathrm{aou}})^{\partial}$.

Proof. By Kihara-Pauly [16], it suffices to show that $\mathsf{WKL}_{\leq 2} \not\leq_{\mathsf{W}} (\mathsf{WKL}_{\mathrm{aou}})^{*} \star (\mathsf{WKL}_{\mathrm{aou}})^{*}$. We now consider the following play:

$$\begin{array}{llllll} \text{I:} & T & & x=\langle x_0,\dots,x_n\rangle & & y=\langle y_0,\dots,y_{\ell(x)}\rangle & \\ \text{II:} & & \langle S_0,\dots,S_n\rangle & & \langle R^0_x,\dots,R^{\ell(x)}_x\rangle & & \alpha_{x,y} \end{array}$$

Player II tries to construct two sequences $S=\langle S_0,\dots,S_n\rangle$ and $R(x)=\langle R^0_x,\dots,R^{\ell(x)}_x\rangle$ of aou-trees, and a path $\alpha_{x,y}$ through I's rational 2-tree T, where the second move $R(x)$ depends on Player I's second move x, and the third move $\alpha_{x,y}$ depends on Player I's second and third moves (x,y). As in Proposition 3.9, each partial continuous function f is coded as an element p of $(\omega\cup\{\bot\})^{\omega}$, so one can read a finite portion $p[s]=\langle p(0),p(1),\dots,p(s-1)\rangle$ of p by stage s.

By *recursion trick* (Section 2.1.3), it is sufficient to describe an algorithm constructing a rational 2-tree T from given S,R,α such that Player I wins along the above play for some x and y. Now, the algorithm is given as follows:

(1) At each stage s, T has two nodes ℓ_s, r_s of length s, where ℓ_s is left to r_s. Let $v_s=\ell_s\wedge r_s$ be the current branching node of T. If we do not act at stage s, we always put $\ell_{s+1}=\ell_s1$ and $r_{s+1}=r_s0$.
(2) Wait for $\alpha_{x,y}(|v_s|)$ being defined for any correct x and y; that is, any (x,y) with $x_i\in[S_i]$ and $y_j\in[R^j_x]$. By compactness, if it happens, it is witnessed by some finite stage.
(3) Suppose that it happens at stage s. If $\alpha_{x,y}(|v_s|)=0$ for some correct x and y then $\alpha_{x,y}$ is incomparable with r_s, so we put $\ell_{s+1}=r_s0$ and $r_{s+1}=r_s1$. Otherwise, $\alpha_{x,y}$ for any correct x and y is incomparable with ℓ_s, so we put $\ell_{s+1}=\ell_s0$ and $r_{s+1}=r_s1$. In any case, this action ensures that $\alpha_{x,y}$ for some correct x and y is not a path through T.
(4) In order for Player II to win, II needs to make (x,y) incorrect; that is, II needs to remove either x_i from S_i for some i or y_j from R^j_x for some j.
(5) If Player II decided to remove x_i from S_i, then this action forces S_i to have at most one path as S_i is aou-tree. If it happens, go back to (2). Note that we can arrive at (5) at most $n+1$ times, since we only have $n+1$ trees $S_0,\dots,S_n$.
(6) If Player II decided to remove y_j from R^j_x, then this action also forces R^j_x to have at most one path as R^j_x is aou-tree. By continuity of R, there is a clopen neighborhood C of x such that R^j_z has a single path for any $z\in[\sigma]$. We may assume that ℓ takes a constant value c on C by making such a neighborhood C sufficiently small. Then we now go to the following $(2')$ with this C.
$(2')$ Wait for $\alpha_{x,y}(v_s)$ being defined for any correct $x\in C$ and y; that is, any (x,y) with $x_i\in[S_i]\cap C$ and $y_j\in[R^j_x]$. By compactness, if it happens, it is witnessed by some finite stage.
$(3')$ Suppose that it happens at stage s. If $\alpha_{x,y}(|\sigma|)=0$ for some correct $x\in U$ and y then $\alpha_{x,y}$ is incomparable with r_s, so we put $\ell_{s+1}=r_s0$ and $r_{s+1}=r_s1$. Otherwise, $\alpha_{x,y}$ for any correct x and y is incomparable with ℓ_s, so we put

$\ell_{s+1} = \ell_s 0$ and $r_{s+1} = r_s 1$. In any case, this action ensures that $\alpha_{x,y}$ for some correct x and y is not a path through T.

(7) As before, in order for Player II to win, II needs to make (x, y) incorrect; that is, II needs to remove either x_i from $S_i \cap C$ for some i or y_j from R^j_x for some j. In other words, Player II chooses either (5) or (6′) below at the next step. Here, after going to (5), the parameter C is initialized.

(6′) The action of this step is the same as (6), but a clopen neighborhood C' of x is chosen as a subset of C. The action of (6′) forces some R^j_z to have at most one path for any $z \in C'$. Then go back to (2′). Note that, as the value c of ℓ on C is determined at the step (6), we only have $c + 1$ many trees $T^0_z, \dots, T^c_z$ for any $z \in C$, so we can arrive at (6′) at most c many times unless going to (5).

Consequently, the procedure arrives (5) or (6′) at most finitely often. This means that Player I eventually defeats Player II; that is, either $\alpha_{x,y}$ is not a path through T for some x, y or else Player II violates the rule by making something which is not an aou-tree.

To apply recursion trick, even if Player II violates the rule, the first move T of Player I needs to obey the rule. To ensure this, one may assume that we proceed our algorithm at stage s only if Player II's partial trees look like aou-trees at all levels below s. Then if Player II violates the rule then our algorithm performs only finitely many actions, so T must be a rational 2-tree. □

3.2.3. *The strength of their product.* We now consider the product $\mathsf{WKL}_{\leq 2} \times \mathsf{WKL}_{\rm aou}$ of choice for finite sets and all-or-unique choice. To examine the strength of $(\mathsf{WKL}_{\leq 2} \times \mathsf{WKL}_{\rm aou})^{\supset}$, we consider the following compactness argument:

In general, to show either $f \leq_{\mathsf{W}} g^{\supset}$ or $f \not\leq_{\mathsf{W}} g^{\supset}$, we need to consider a reduction game of arbitrary length. However, if everything involved in this game is compact, then one can pre-determine a bound of the length of a given game. To explain more details, let $g(y) \subseteq 2^\omega$ be compact uniformly in y. For instance, one can take $\mathsf{WKL}_{\leq 2}$, $\mathsf{WKL}_{\rm aou}$, $\mathsf{WKL}_{\rm conv}$, and WKL. These examples are actually effectively compact. Then, consider the reduction game $G(f, g)$:

$$\begin{array}{lccccccc} \text{I:} & x_0 & & x_1 & & x_2 & & \dots \\ \text{II:} & & \langle j_0, y_0 \rangle & & \langle j_1, y_1 \rangle & & \langle j_2, y_2 \rangle & \dots \end{array}$$

Assume that Player II has a winning strategy τ, which yields continuous functions $\tilde{j}$ and $\tilde{y}$ such that $j_n = \tilde{j}(x_0, x_1, \dots, x_n)$ and $y_n = \tilde{y}(x_0, x_1, \dots, x_n)$. Now, consider the set $P_\tau(x_0)$ of Player I's all plays $x = (x_1, \dots)$ (where the first move x_0 is fixed) against II's strategy τ, where x obeys the rule (and $x_n \in 2^\omega$ is arbitrary if $j_m = 1$ for some $m < n$; that is, Player II has already declared victory). More precisely, consider

$$P_\tau(x_0) = \{x \in (2^\omega)^\omega : x_{n+1} \in g(\tilde{y}(x_0, \dots, x_n)) \text{ or } (\exists m < n)\ \tilde{j}(x_0, \dots, x_m) = 1\}.$$

By continuity of $\tilde{j}$ and $\tilde{y}$, and compactness of $g(y)$, the set $P_\tau(x_0)$ is compact. Hence, again by continuity of $\tilde{j}$ and compactness of $P_\tau(x_0)$, one can easily verify that there is a bound n of the length of any play; that is, for any $x = (x_i) \in P_\tau(x_0)$ there is m such that $\tilde{j}(x_0, \dots, x_m) = 1$. Moreover, $x_0 \mapsto n$ is continuous. If the strategy τ is computable, and $g(y)$ is effectively compact uniformly in y, then $P_\tau(x_0)$ is also effectively compact, and moreover, $x_0 \mapsto n$ is computable. This concludes that

$$f \leq_{\mathsf{W}} g^{\supseteq} \implies f \leq_{\mathsf{W}} \bigsqcup_n g^{(n)},$$

where $g^{(n)}$ is the nth iteration of the compositional product; that is, $g^{(1)} = g$ and $g^{(n+1)} = g \star g^{(n)}$. Furthermore, if $\mathrm{dom}(f)$ is effectively compact, one can also ensure that $f \leq_{\mathsf{W}} \bigsqcup_n g^{(n)}$ implies $f \leq_{\mathsf{W}} g^{(n)}$ for some $n \in \omega$. By the above argument, for instance, one can ensure the following:

Observation 3.12. $(\mathsf{WKL}_{\leq 2} \times \mathsf{WKL}_{\mathrm{aou}})^{\supseteq} \equiv_{\mathsf{W}} \bigsqcup_n (\mathsf{WKL}_{\leq 2} \times \mathsf{WKL}_{\mathrm{aou}})^{(n)}$.

Then we show that $(\mathsf{WKL}_{\leq 2} \times \mathsf{WKL}_{\mathrm{aou}})^{\supseteq}$ is not strong enough for solving $\mathsf{WKL}_{\mathrm{clop}}$, weak König's lemma for clopen convex trees.

Proposition 3.13. $\mathsf{WKL}_{\mathrm{clop}} \not\leq^{c}_{\mathsf{W}} (\mathsf{WKL}_{\mathrm{aou}} \times \mathsf{WKL}_{\leq 2})^{\supseteq}$.

Proof. By Observation 3.12, it suffices to show that $\mathsf{WKL}_{\mathrm{clop}} \not\leq_{\mathsf{W}} \bigsqcup_n (\mathsf{WKL}_{\mathrm{aou}} \times \mathsf{WKL}_{\leq 2})^{(n)}$. We now consider the following play:

$$\begin{array}{llllllll} \text{I:} & T & & x(1) = \langle x^1_A, x^1_B\rangle & & \dots & x(n) = \langle x^n_A, x^n_B\rangle & \\ \text{II:} & & n, \langle A, B\rangle & & \langle A_{x(1)}, B_{x(1)}\rangle & \dots & & \alpha_{x(1)\dots x(n)} \end{array}$$

Player II tries to construct a sequence of pairs of aou-trees $A_\bullet$ and 2-trees $B_\bullet$, and a path $\alpha_{x(1)\dots x(n)}$ through a basic clopen tree T, where the ith move $\langle A_{x(1)\dots x(i-1)}, B_{x(1)\dots x(i-1)}\rangle$ depends on Player I's moves $x(1), \dots, x(i-1)$. As in Proposition 3.9, each partial continuous function f is coded as an element p of $(\omega \cup \{\perp\})^\omega$, so one can read a finite portion $p[s] = \langle p(0), p(1), \dots, p(s-1)\rangle$ of p by stage s.

By *recursion trick* (Section 2.1.3), it is sufficient to describe an algorithm constructing a basic clopen tree T from given $n, \langle A_\bullet, B_\bullet\rangle, \alpha$ such that Player I wins along the above play for some $x(1), \dots, x(n)$. Now, the algorithm is given as follows:

(1) At each stage s, there is a binary string σ such that T contains all binary strings of length s extending σ. If we do not act at stage s, we always assume that T contains all binary strings of length $s+1$ extending σ.
(2) Wait for n (in Player II's first move) being determined. Then wait for $\alpha_{x(1)\dots x(n)} \upharpoonright n+2$ being defined for any correct $x(1), \dots, x(n)$, which means that $x^{i+1}_A \in A_{x(1)\dots x(i)}$ and $x^{i+1}_B \in B_{x(1)\dots x(i)}$. By compactness, if it happens, it is witnessed by some finite stage.
(3) For any i, there are two possibilities for x^i_B for a given $B_{x(i-1)}$ since $B_\bullet$ is a 2-tree. So, we consider a non-deterministic play of Player I, which takes both possibilities for x^i_B, but for x^i_A chooses just one element. In other words, we consider (at most) 2^n many $x = x(1)\dots x(n)$'s, so there are at most 2^{n+1} many α_x's. Clearly, there are 2^{n+2} nodes of length $n+2$, so choose a binary string σ of length $n+2$ which is different from any $\alpha_x \upharpoonright n+2$. Then, we declare that T only contains binary strings of length $s+1$ extending σ at stage $s+1$. This ensures that α_x is not a path through T.
(4) In order for Player II to win, II needs to remove x^i_A from $A_{x(1)\dots x(i-1)}$ for some i. This action forces $A_{x(1)\dots x(i-1)}$ to have at most one path as $A_\bullet$ is an aou-tree. Player I re-chooses x^i_A as a unique element in $A_{x(1)\dots x(i-1)}$, and go back to (2) with $k(n+2)$ instead of $n+2$, where k is the number of times which we have reached the step (4) so far.

When we arrive at (4), Player II need to declare that some of $A_\bullet$ has at most one path. Thus, it is easy to ensure that the algorithm can arrive at (4) at most finitely often. Hence, x^i_A is eventually stabilized for any i, and by our action at (3), this ensures that α_x cannot be a path through T. The string σ is also eventually stabilized, and thus T becomes a clopen tree. Thus, Player I obeys the rule.

To apply recursion trick, even if Player II violates the rule, the first move T of Player I needs to obey the rule. To ensure this, one may assume that we proceed our algorithm at stage s only if Player II's partial trees look like pairs of aou-trees and 2-trees at all levels below s. Then if Player II violates the rule then our algorithm performs only finitely many actions, so T must be a basic clopen tree. In any case, Player I obeys the rule, and so Player I defeats Player II. ☐

3.2.4. *The product of de Morgan's law and the double negation elimination.* We now examine the strength of $(\mathsf{RT}^1_2 \times \mathsf{Lim}_\mathbb{N})^\supsetdot$, the game closure of the product of Ramsey's theorem for singletons and two-colors (the infinite pigeonhole principle) and the discrete limit operation. We say that a partial finite-valued function $f\colon\subseteq \omega^\omega \rightrightarrows \omega^\omega$ has a *semicontinuous bound* if there is a partial continuous function $g\colon\subseteq \omega^\omega \times \omega \to \omega^\omega$ such that $f(x) \subseteq \{g(x,n) : n < b(x)\}$ for any $x \in \mathrm{dom}(f)$, where $b\colon\subseteq \omega^\omega \to \omega$ is lower semicontinuous.

Lemma 3.14. $(\mathsf{RT}^1_2 \times \mathsf{Lim}_\mathbb{N})^\supsetdot$ *has a semicontinuous bound.*

Proof. Let (x,τ) be an instance of $(\mathsf{RT}^1_2 \times \mathsf{Lim}_\mathbb{N})^\supsetdot$. Consider a play of a reduction game according to II's strategy τ:

$$\begin{array}{llllllll} \mathrm{I:} & x & & \sigma(0)=(j_0,k_0) & & \sigma(1)=(j_1,k_1) & & \dots \\ \mathrm{II:} & & (c,\alpha) & & (c_{\sigma(0)},\alpha_{\sigma(0)}) & & (c_{\sigma(0)\sigma(1)},\alpha_{\sigma(0)\sigma(1)}) & \dots \end{array}$$

Here, Player II's moves are automatically generated from the strategy τ. For each round, Player I needs to answer a query c_* to RT^1_2 and a query α_* to $\mathsf{Lim}_\mathbb{N}$ made by Player II. As an answer to the RT^1_2-query c_* is either 0 or 1, and to the $\mathsf{Lim}_\mathbb{N}$-query α_* is a natural number, one may assume that Player I's moves are restricted to $2\times\omega$. We define the *tree T of Player I's possible plays* as $T=(2\times\omega)^{<\omega}$.

We view each node $\sigma\in T$ as a sequence such that $j(n)\in 2$ and $k(n)\in\omega$ alternatively appear (i.e., $\sigma(2n)\in\{0,1\}$ and $\sigma(2n+1)\in\omega$) rather than a sequence of pairs from $2\times\omega$. Then, a node of T of even length is called a RT^1_2-*node*, and a node of odd length is called a $\mathsf{Lim}_\mathbb{N}$-*node*. The strategy τ automatically assigns a query c_σ to each RT^1_2-node $\sigma\in T_s$, where $c_\sigma\colon\mathbb{N}\to 2$ is a two-coloring. Then, each RT^1_2-node σ has at most two immediate successors $\sigma 0,\sigma 1$, and moving to the node σi indicates that Player I chose i as an answer to c_σ. Similarly, the strategy τ automatically assigns a query α_σ to each $\mathsf{Lim}_\mathbb{N}$-node $\sigma\in T_s$, where $\alpha_\sigma\in\omega^\omega$ is a (convergent) sequence of natural numbers.

As mentioned above, Player I's next move is restricted to ω (whenever I obeys the rule), but the uniqueness of the limit, there is an exactly one correct answer. Since $\mathsf{Lim}_\mathbb{N}$ is the discrete limit operation, if Player II obeys the rule, then α_σ must stabilize, that is, $\alpha_\sigma(s)=\alpha_\sigma(t)$ for any sufficiently large s,t, and this value is the correct answer to α_σ. Thus, if $\alpha_\sigma(s)=k$, we say that k is the *current true outcome* of the $\mathsf{Lim}_\mathbb{N}$-node σ at stage s, and only the edge $[\sigma,\sigma k]$ is *passable* at this stage.

Similarly, for a RT^1_2-node, we declare that the edge $[\sigma,\sigma i]$ is *passable* if i looks like a correct answer (which means that $c_\sigma(n)=i$ for infinitely many n). More precisely, the edge $[\sigma, \sigma i]$ is *passable at stage s* if $c_\sigma(t_\sigma(s)) = i$, where, each node σ has its own clock t_σ, and it moves only when σ is accessible, where we say that σ is *accessible at stage s* if $|\sigma| < s$ and all edges in the path $[\langle\rangle, \sigma]$ are passable at stage s (where $\langle\rangle$ is the root of T). Note that there are only finitely many accessible nodes at each stage.

If Player I obeys the rule, then, as (x, τ) is an instance of $(\mathsf{RT}^1_2 \times \mathsf{Lim}_{\mathbb{N}})^\supset$, Player II (according to the strategy τ) declares victory at some round. This means that if we restrict the tree T to the nodes σ at which Player I obeys the rule (i.e., Player I gives a correct answer to each query made by Player II), then the restriction R forms a well-founded binary tree; that is, a finite binary tree. Moreover, any $\sigma \notin R$ is never accessible after some stage. Indeed, if $\sigma \in R$ and $\sigma i \notin R$, then the edge $[\sigma, \sigma i]$ is never passable after some stage, and there are only finite such nodes as R is finite.

Let A_s be the finite tree of all accessible nodes at stage s. The above argument shows that for a sufficiently large t, we always have $A_t \subseteq R$; hence $A = \bigcup_s A_s$ is finite. We also note that if $\sigma \in R$ then there are infinitely many t such that $\sigma \in A_t$. At stage s, for each $\sigma \in A_s$, check if Player II (according to the strategy τ) declares victory with some value u_σ at the next round. If so, enumerate u_σ into an auxiliary set B. Then we get an (x, τ)-computable increasing sequence $(B_s)_{s\in\omega}$ of finite sets such that $B = \bigcup_s B_s$ is finite, and contains all possible outputs u_σ of $(\mathsf{RT}^1_2 \times \mathsf{Lim}_{\mathbb{N}})^\supset(x, \tau)$. ☐

We apply the above lemma to show that $(\mathsf{RT}^1_2 \times \mathsf{Lim}_{\mathbb{N}})^\supset$ is not strong enough for solving weak König's lemma for 2-trees:

Proposition 3.15. *If $f :\subseteq \omega^\omega \rightrightarrows \omega$ has a semi-continuous bound, then $\mathsf{WKL}_{\leq 2} \not\leq^c_{\mathsf{W}} f$. In particular, $\mathsf{WKL}_{\leq 2} \not\leq^c_{\mathsf{W}} (\mathsf{RT}^1_2 \times \mathsf{Lim}_{\mathbb{N}})^\supset$.*

Proof. If $\mathsf{WKL}_{\leq 2} \leq^c_{\mathsf{W}} f$, where f has a semi-continuous bound $(\tilde{g}, \tilde{b})$, then there are continuous functions H, K such that if $\tilde{T}$ is a 2-tree then, there is $n < b := \tilde{b}(H(\tilde{T}))$ such that $\tilde{g}(H(\tilde{T}), n)$ is defined, and $g(n) := K(\tilde{T}, \tilde{g}(H(\tilde{T}), n))$ is a path through $\tilde{T}$. If we describe an algorithm constructing T from (g, b) (with a parameter $\tilde{T}$), by Kleene's recursion theorem, $\tilde{T}$ can be interpreted as self-reference; that is, we may assume that $T = \tilde{T}$. This is the simplest version of *recursion trick* (Section 2.1.3).

Thus, it is sufficient to describe an algorithm constructing a 2-tree T from given (g, b):

(1) First we declare that each $n \in \mathbb{N}$ is active. At each stage s, T has two nodes ℓ_s, r_s of length s, where ℓ_s is left to r_s. Let $v_s = \ell_s \wedge r_s$ be the current branching node of T. If we do not act at stage s, we always put $\ell_{s+1} = \ell_s 1$ and $r_{s+1} = r_s 0$.
(2) Then, ask if $g(n)(|v_s|)$ is already defined by stage s for some active $n < b_s$, where b_s is the stage s approximation of b.
(3) If such n exists, then the least such n receives attention. For this n, if $g(n)(|v_s|) = 0$, then $g(n)$ is incomparable with the rightmost node r_s, so we put $\ell_{s+1} = r_s 0$ and $r_{s+1} = r_s 1$. Similarly, if $g(n)(|v_s|) = 1$, then $g(n)$ is incomparable with the leftmost node ℓ_s, so we put $\ell_{s+1} = r_s 0$ and $r_{s+1} = r_s 1$. This procedure ensures that $g(n)$ is incomparable with ℓ_{s+1} and r_{s+1}. Then, we declare that n is not active anymore.

Assume that Player II obeys the rule, i.e., (b_s) converges to a finite value b. Then, since there are only finitely many n such that $g(n)$ is total and $n < b$, such n receives attention at some stage s. By our strategy described above, for any such n, if p is a path through T then we must have $g(n) \neq p$. This contradicts our assumption that $g(n)$ is a path through T for some $n < b$.

To apply recursion trick, even if Player II violates the rule (i.e., (b_s) does not converge), the first move T of Player I needs to obey the rule. In this case, our algorithm may arrive at (3) infinitely often, but this just implies that T converges to a tree which has a unique path. In any case, T is a 2-tree. Hence, we have $\mathsf{WKL}_{\leq 2} \not\leq^c_{\mathsf{W}} f$. ☐

3.2.5. *More products.* We finally examine the strength of the game closure of $\mathsf{RT}^1_2 \times \mathsf{Lim}_{\mathbb{N}} \times \mathsf{WKL}_{\leq 2}$. We show that it is not strong enough for solving weak König's lemma $\mathsf{WKL}_{\mathrm{conv}}$ for convex trees.

Proposition 3.16. $\mathsf{WKL}_{\mathrm{conv}} \not\leq^c_{\mathsf{W}} (\mathsf{RT}^1_2 \times \mathsf{Lim}_{\mathbb{N}} \times \mathsf{WKL}_{\leq 2})^{\supset}$.

Proof. We now consider the following play:

$$\begin{array}{llllll} \text{I:} & T & & \sigma(0)=(j_0,k_0,x_0) & & \sigma(1)=(j_1,k_1,x_1) \ \dots \\ \text{II:} & & (c,\alpha,B) & & (c_{\sigma(0)},\alpha_{\sigma(0)},B_{\sigma(0)}) & \qquad\qquad \dots \end{array}$$

Again, by *recursion trick* (Section 2.1.3), it is sufficient to describe an algorithm constructing a basic clopen tree T from Player II's given strategy $c_\bullet, \alpha_\bullet, B_\bullet, \dots$ such that Player I wins along the above play for some $j_0, k_0, x_0, j_1, k_1, x_1, \dots$ The difficulty here is again lack of compactness.

Recall the tree of I's possible plays from Proposition 3.14. In the present game, each node of our tree essentially codes Player I's play $\langle j_0, k_0, x_0, j_1, k_1, x_1, \dots \rangle$; however, now our tree is slightly different from the previous one. The first entry is $j = j_0$, which is an answer to the query c, and the second entry is $k = k_0$, which codes an answer to the query α (assigned to the node $\langle j \rangle$).

However, although the 2-tree B (assigned to the node $\langle jk \rangle$) has at most two paths, we have a lot of candidates for the answer $x(n) \in 2^\omega$. We assign infinitely many possible outcomes $(\infty, 0, 1, \dots)$ to the node $\langle jk \rangle$. The outcome ∞ indicates that B has at most one path, and the outcome n indicates that B_j has exactly two paths which branch at height n. Then we consider a non-deterministic play of Player I, which means that I chooses both paths x, x' through B whenever $\langle jk \rangle$ has finite outcome. Thus, the next entry of Player I's move may be a pair $\langle j_1, j_1' \rangle$ of answers to two different queries c_{jkx} and $c_{jkx'}$. Then the tree is constructed by continuing this procedure.

We call a node of length $3n$ an RT^1_2-*node*, a node of length $3n+1$ an $\mathsf{Lim}_{\mathbb{N}}$-*node*, and a node of length $3n+2$ a $\mathsf{WKL}_{\leq 2}$-*node*. Note that an RT^1_2-node has at most finitely many immediate successors, while a $\mathsf{Lim}_{\mathbb{N}}$-node and a $\mathsf{WKL}_{\leq 2}$-node has infinitely many immediate successors. In general, many queries are assigned to each node. For each query on a RT^1_2-node, the outcomes are defined as in the proof of Proposition 3.15. For each query α on a $\mathsf{Lim}_{\mathbb{N}}$-node, we assign infinitely many possible finite outcomes $n \in \mathbb{N}$ to this query, where the outcome n indicates that α stabilizes after n; that is, n is the least number such that $\alpha(k) = \alpha(n)$ for any $k \geq n$ (which is slightly different from Proposition 3.15). The outcomes of each query B on a $\mathsf{WKL}_{\leq 2}$-node are the same as described as above: The outcome ∞ indicates that B_j has at most one path, and the outcome n indicates that B has exactly two paths which branch at height n. Then we consider a non-deterministic play of Player I, which means that I chooses both paths through B whenever the $\mathsf{WKL}_{\leq 2}$-query B has finite outcome. An outcome of a node σ is a sequence of outcomes of queries assigned to σ.

As in Proposition 3.14, each node σ has its own clock t_σ. Each $\mathsf{WKL}_{\leq 2}$-query B (which is assigned to a $\mathsf{WKL}_{\leq 2}$-node σ) has a unique *current true outcome* at stage $t = t_\sigma(s)$ in the following sense: The 2-tree B has exactly two strings ℓ_t and r_t of length t. If ℓ_t extends ℓ_{t-1} and r_t extends r_{t-1}, then the finite outcome n is currently true, where n is the height where ℓ_t and r_t branch; otherwise ∞ is currently true. Although the current true outcome is unique for each $\mathsf{WKL}_{\leq 2}$-query, the immediate successor may involve two RT^1_2-queries corresponding to II's responses to I's possible moves ℓ_t and r_t. In this sense, our tree consists of possible *non-deterministic* plays. Similarly, each $\mathsf{Lim}_{\mathbb{N}}$-query α (which is assigned to a $\mathsf{Lim}_{\mathbb{N}}$-node σ) has a unique current true outcome at stage $t = t_\sigma(s)$ in the following sense: The outcome n is currently true if α stabilizes between stages n and t; that is, n is the least number such that $\alpha(k) = \alpha(n)$ whenever $n \leq k \leq t$. Then, Player I's corresponding move to the outcome n is $\alpha(n)$. Thus, $\mathsf{Lim}_{\mathbb{N}}$-moves are deterministic.

A *current true multi-path* at stage s is a finite string of length s defined as follows: At an RT^1_2-node, choose a passable edge for each assigned RT^1_2-query (where an RT^1_2-query can have two passable edges). At a $\mathsf{WKL}_{\leq 2}$-node, choose a (unique) current true outcome for each assigned $\mathsf{WKL}_{\leq 2}$-query. Recall that we consider a non-deterministic play, so many queries may be assigned to each node, and we need to choose a passable edge and a current true outcome for each query. Therefore, there may be a lot of (but finitely many) current true multi-paths.

A *true multi-path* is defined as follows: At an RT^1_2-node, choose correct answers to the assigned RT^1_2-queries. At a $\mathsf{Lim}_{\mathbb{N}}$-node, choose the unique true outcome for each assigned query. Similarly, at a $\mathsf{WKL}_{\leq 2}$-node, choose true outcomes; i.e., n if an assigned $\mathsf{WKL}_{\leq 2}$-query B has two nodes which branch at height n; otherwise ∞. Note that a true multi-path α is non-deterministic, but two-branching, and every single path γ through α yields Player I's play which obeys the rule (Note also that there are many true multi-paths because each RT^1_2-query may have two true outcomes, and every true multi-path contains many paths because each true $\mathsf{WKL}_{\leq 2}$-outcome may yield two moves). As Player I obeys the rule along γ, if Player II wins, then Player II declares victory along γ. This means that a true multi-path α is well-founded; hence, finite as α is binary. The finiteness of a true multi-path α ensures that α becomes a current true multi-path at infinitely many stages.

Now, the algorithm constructing a basic clopen tree T is given as follows:

(1) At each stage s, there is a binary string σ such that T contains all binary strings of length s extending σ. If we do not act at stage s, we always assume that T contains all binary strings of length $s+1$ extending σ.
(2) Wait for some stage when Player II declares victory with $(u_i)_{i\in I}$ along all plays $(\gamma_i)_{i\in I}$ on a current true path, and $u_i(|\sigma|+|I|)$ is defined for any $i \in I$.
(3) If there is such a current true path α, then choose τ extending σ such that $\tau(|\sigma|+|I|) \neq u_i(|\sigma|+|I|)$ for any $i \in I$. Then we declare that, at stage $s+1$, all binary strings in T of length $s+1$ extend τ. This action ensures that u_i is not a path through T for any $i \in I$.
(4) If Player II wins, for each single path γ of α, some outcome in γ must be incorrect. Assume that some $\mathsf{WKL}_{\leq 2}$-outcome is incorrect. If a node $\sigma \prec \gamma$ made an incorrect outcome n, the corresponding $\mathsf{WKL}_{\leq 2}$-query B has ℓ_s and r_s

which branch at length n, but at some later stage $t > s$, either $\ell_t, r_t \succ \ell_s$ or $\ell_t, r_t \succ r_s$. However, the multi-path after σ is automatically generated from ℓ_s, r_s by definition, so ℓ_t, r_t yield a sub-multi-path of α. A similar argument holds for ∞. Thus, some RT^1_2- or $\mathsf{Lim}_\mathbb{N}$-outcome has to be incorrect.

In (4), as in the proof of Proposition 3.14, accessible nodes are eventually contained in the positions where Player I obeys the rule. This means that all RT^1_2- and $\mathsf{Lim}_\mathbb{N}$-outcomes along a current true multi-path are correct after some stage. Therefore, Player II cannot win.

To apply recursion trick, even if Player II violates the rule, the first move T of Player I needs to obey the rule. In this case, our algorithm may arrive at (3) infinitely often, but this just implies that T converges to a tree which has a unique path. In any case, T is a convex tree, which means that Player I obeys the rule, and so Player I defeats Player II. □

Now, Theorem 2.6 follows from Propositions 3.9, 3.11, 3.13, 3.15, and 3.16.

3.3. **Lifting Weihrauch separations to IZF-separations.** In order to prove Theorem 1.1, we will lift separation results on Weihrauch degrees to the results on **IZF**. In this section, we formalize and prove statements on Weihrauch degrees which are associated to items of Theorem 1.1.

We say that a sentence A is *f-realizable* if A is $\mathbf{j_d}$-realizable over $\mathbb{P}_\mathbf{j}$, where $\mathbf{d}$ is the Weihrauch degree of f. We also say that A is *boldface f-realizable* if A is $\mathbf{j_d}$-realizable over $\underset{\sim}{\mathbf{P}}$, i.e., there is $e \in \underset{\sim}{\mathbf{P}}$ such that $e \Vdash_\mathbb{P} A$. By definition, $\neg A$ is f-realizable if and only if A is not boldface f-realizable. Recall that $f^\circlearrowright$ is closed under compositional product, and therefore, by Lemma 2.2 and Theorem 2.5, **IZF** is $f^\circlearrowright$-realizable.

Proof of Theorem 1.1. We prove the following results which are stronger than Theorem 1.1:

(1) $\mathbf{IZF} + \Sigma^0_1\text{-}\mathsf{DML}_\mathbb{R} + \neg\mathsf{RDIV} + \neg\mathsf{BE}_\mathbb{Q}$ is $\mathsf{LLPO}^\circlearrowright$-realizable.
(2) $\mathbf{IZF} + \neg\Pi^0_1\text{-}\mathsf{LEM}_\mathbb{R} + \mathsf{BE}_\mathbb{Q} + \neg\mathsf{BE} + \neg\mathsf{RDIV}$ is $(\mathsf{WKL}_{=2})^\circlearrowright$-realizable.
(3) $\mathbf{IZF} + \neg\Pi^0_1\text{-}\mathsf{LEM}_\mathbb{R} + \mathsf{RDIV} + \neg\mathsf{BE}_\mathbb{Q}$ is $(\mathsf{WKL}_{\rm aou})^\circlearrowright$-realizable.
(4) $\mathbf{IZF} + \neg\Pi^0_1\text{-}\mathsf{LEM}_\mathbb{R} + \mathsf{IVT}_{\rm lin} + \neg\mathsf{BE}$ is $(\mathsf{WKL}_{\rm clop})^\circlearrowright$-realizable.
(5) $\mathbf{IZF} + \mathsf{BE} + \neg\mathsf{RDIV}$ is $(\mathsf{WKL}_{\leq 2})^\circlearrowright$-realizable.
(6) $\mathbf{IZF}+\neg\Pi^0_1\text{-}\mathsf{LEM}_\mathbb{R}+\mathsf{BE}_\mathbb{Q}+\mathsf{RDIV}+\neg\mathsf{IVT}_{\rm lin}+\neg\mathsf{BE}$ is $(\mathsf{WKL}_{=2}\times\mathsf{WKL}_{\rm aou})^\circlearrowright$-realizable.
(7) $\mathbf{IZF} + \neg\Pi^0_1\text{-}\mathsf{LEM}_\mathbb{R} + \mathsf{BE} + \mathsf{RDIV} + \neg\mathsf{IVT}_{\rm lin}$ is $(\mathsf{WKL}_{\leq 2} \times \mathsf{WKL}_{\rm aou})^\circlearrowright$-realizable.
(8) $\mathbf{IZF} + \neg\Pi^0_1\text{-}\mathsf{LEM}_\mathbb{R} + \mathsf{BE} + \mathsf{IVT}_{\rm lin} + \neg\mathsf{IVT}$ is $(\mathsf{WKL}_{\leq 2} \times \mathsf{WKL}_{\rm clop})^\circlearrowright$-realizable.
(9) $\mathbf{IZF} + \neg\Pi^0_1\text{-}\mathsf{LEM}_\mathbb{R} + \mathsf{IVT} + \neg\mathsf{WKL}$ is $(\mathsf{WKL}_{\sf conv})^\circlearrowright$-realizable.

(1) We use $\mathsf{LLPO}^\circlearrowright$-realizability (i.e., Lifshitz realizability; see below) to realize $\Sigma^0_1\text{-}\mathsf{DML}_\mathbb{R} + \neg\mathsf{RDIV} + \neg\mathsf{BE}_\mathbb{Q}$.

Recall that de Morgan's law $\Sigma^0_1\text{-}\mathsf{DML}_\mathbb{R}$ is formulated as LLPO, so it is clear that $\mathsf{LLPO}^\circlearrowright$-realizability validates $\Sigma^0_1\text{-}\mathsf{DML}_\mathbb{R}$. Moreover, $\mathsf{LLPO}^\circlearrowright$ is known to be Weihrauch equivalent to the *choice principle* $\mathsf{K}_\mathbb{N}$ *for compact sets in* $\mathbb{N}$, which is a multifunction that, given a bound $b \in \mathbb{N}$ and a sequence $a = (a_i)$ of natural numbers, if $V_{a,b} = \{n < b : (\forall i)\ n \neq a_i\}$ is nonempty, return any element from $V_{a,b}$.

Fact 3.17 ([26])**.** $\mathsf{LLPO}^\circlearrowright \equiv_\mathsf{W} \mathsf{K}_\mathbb{N}$.

To prove (1), we need to show that LLPO^{∂}-realizability refutes BE (equivalent to $\mathsf{WKL}_{\leq 2}$) and RDIV (equivalent to $\mathsf{WKL}_{\mathrm{aou}}$). This essentially follows from the following:

Lemma 3.18. $\mathsf{WKL}_{=2} \not\leq^{c}_{\mathsf{W}} \mathsf{K}_{\mathbb{N}}$ *and* $\mathsf{WKL}_{\mathrm{aou}} \not\leq^{c}_{\mathsf{W}} \mathsf{K}_{\mathbb{N}}$*.*

The former follows from Proposition 3.11 as $\mathsf{LLPO} \leq_{\mathsf{W}} \mathsf{WKL}_{\mathrm{aou}}$. The latter is a known fact, which also follows from Proposition 3.9 as $\mathsf{LLPO} \leq_{\mathsf{W}} \mathsf{WKL}_{\leq 2}$.

Now, we discuss on realizability. By Fact 3.17, LLPO^{∂}-realizability is equivalent to $\mathsf{K}_{\mathbb{N}}$-realizability, which may be considered as *Lifschitz realizability* (over the Kleene-Vesley algebra), and so the $\star$-closure property of $\mathsf{K}_{\mathbb{N}}$ has implicitly been known in the context of realizability (cf. Lifschitz [20, Lemma 3]). It is well-known that Lifschitz realizability validates LLPO.

If $\mathsf{BE}_{\mathbb{Q}}$ is boldface LLPO^{∂}-realizable, so is $\mathsf{WKL}_{\leq 2}$ by Proposition 3.6. Then, as in the argument in Section 2.2.4, one can see that there is a boldface LLPO^{∂}-realizable function which, given (T, a, b), where T is a tree, a witnesses that T is infinite, and b witnesses that T has at most two nodes at each level, returns an infinite path through T. However given such a T one can always recover a and b in an effective manner, so they have no extra information. This means that $\mathsf{WKL}_{=2} \leq^{c}_{\mathsf{W}} \mathsf{K}_{\mathbb{N}}$, which contradicts Lemma 3.18.

Similarly, if RDIV is boldface $\mathsf{K}_{\mathbb{N}}$-realizable, then weak König's lemma for aou-trees is also $\mathsf{K}_{\mathbb{N}}$-realizable, so there is a boldface $\mathsf{K}_{\mathbb{N}}$-realizable function which, given (T, a, b), where T is a tree, a witnesses that T is infinite, and b witnesses that T has all nodes or a single node at each level, returns an infinite path through T. However given such a T one can always recover a and b in an effective manner, so they have no extra information. This means that $\mathsf{WKL}_{\mathrm{aou}} \leq^{c}_{\mathsf{W}} \mathsf{K}_{\mathbb{N}}$, which contradicts Lemma 3.18. This verifies the item (1).

(2) We use $(\mathsf{WKL}_{=2})^{\partial}$-realizability to realize $\neg\Pi^0_1$-$\mathsf{LEM}_{\mathbb{R}} + \mathsf{BE}_{\mathbb{Q}} + \neg\mathsf{BE} + \neg\mathsf{RDIV}$. It is evident that $(\mathsf{WKL}_{=2})^{\partial}$-realizability validates $\mathsf{BE}_{\mathbb{Q}}$ by Proposition 3.6. Thus, to prove (2), it suffices to show the following:

Lemma 3.19. $\mathsf{LPO} \not\leq^{c}_{\mathsf{W}} (\mathsf{WKL}_{=2})^{\partial}$*,* $\mathsf{WKL}_{\leq 2} \not\leq^{c}_{\mathsf{W}} (\mathsf{WKL}_{=2})^{\partial}$ *and* $\mathsf{WKL}_{\mathrm{aou}} \not\leq^{c}_{\mathsf{W}} (\mathsf{WKL}_{=2})^{\partial}$*.*

The first assertion is trivial: It is known that $\mathsf{WKL}^{\partial} \equiv_{\mathsf{W}} \mathsf{WKL}$. This means that $P \leq_{\mathsf{W}} \mathsf{WKL}$ implies $P^{\partial} \leq_{\mathsf{W}} \mathsf{WKL}^{\partial} \equiv_{\mathsf{W}} \mathsf{WKL}$. It is also well-known that $\mathsf{LPO} \not\leq_{\mathsf{W}} \mathsf{WKL}$, where recall that LPO is equivalent to Π^0_1-$\mathsf{LEM}_{\mathbb{R}}$. The second assertion follows from Proposition 3.15 since $\mathsf{WKL}_{=2} \leq_{\mathsf{W}} \mathsf{Lim}_{\mathbb{N}}$ by Propositions 3.4 and 3.5. The last assertion clearly follows from Proposition 3.9. This proves Lemma 3.19; hence, as in (1), one can easily verify the item (2).

(3) We use $(\mathsf{WKL}_{\mathrm{aou}})^{\partial}$-realizability to realize $\neg\Pi^0_1$-$\mathsf{LEM}_{\mathbb{R}} + \mathsf{RDIV} + \neg\mathsf{BE}_{\mathbb{Q}}$. As $\mathsf{WKL}_{\mathrm{aou}}$ is equivalent to RDIV, it is evident that $(\mathsf{WKL}_{\mathrm{aou}})^{\partial}$-realizability validates RDIV. Thus, to prove (3), it suffices to show the following:

Lemma 3.20. $\mathsf{LPO} \not\leq_{\mathsf{W}} (\mathsf{WKL}_{\mathrm{aou}})^{\partial}$*, and* $\mathsf{WKL}_{=2} \not\leq_{\mathsf{W}} (\mathsf{WKL}_{\mathrm{aou}})^{\partial}$*.*

The first assertion is trivial as in Lemma 3.19. The second assertion clearly follows from Proposition 3.11. This proves Lemma 3.20; hence, as in (1), one can easily verify the item (3).

(4) We use $(\mathsf{WKL}_{\rm clop})^{\partial}$-realizability to realize $\neg\Pi^0_1$-$\mathsf{LEM}_{\mathbb{R}}+\mathsf{IVT}_{\rm lin}+\neg\mathsf{BE}$. It is evident that $(\mathsf{WKL}_{\rm clop})^{\partial}$-realizability validates $\mathsf{IVT}_{\rm lin}$ by Proposition 3.7. To prove (4), it suffices to show the following:

Lemma 3.21. $\mathsf{LPO}\not\leq^c_{\mathsf{W}}(\mathsf{WKL}_{\rm clop})^{\partial}$*, and* $\mathsf{WKL}_{\leq 2}\not\leq^c_{\mathsf{W}}(\mathsf{WKL}_{\rm clop})^{\partial}$*.*

The first assertion is trivial as in Lemma 3.19. The second assertion follows from Proposition 3.15 since $\mathsf{WKL}_{\rm clop}\leq_{\mathsf{W}}\mathsf{Lim}_{\mathbb{N}}$ by Proposition 3.5. This proves Lemma 3.21; hence, as in (1), one can easily verify the item (4).

(5) We use $(\mathsf{WKL}_{\leq 2})^{\partial}$-realizability to realize $\mathsf{BE}+\neg\mathsf{RDIV}$. As $\mathsf{WKL}_{\leq 2}$ is equivalent to BE, it is evident that $(\mathsf{WKL}_{\leq 2})^{\partial}$-realizability validates BE. Moreover, $(\mathsf{WKL}_{\leq 2})^{\partial}$-realizability refutes RDIV by Proposition 3.9. Hence, as in (1), one can easily verify the item (5).

(6) We use $(\mathsf{WKL}_{=2}\times\mathsf{WKL}_{\rm aou})^{\partial}$-realizability to realize $\neg\Pi^0_1$-$\mathsf{LEM}_{\mathbb{R}}+\mathsf{BE}_{\mathbb{Q}}+\mathsf{RDIV}+\neg\mathsf{IVT}_{\rm lin}+\neg\mathsf{BE}$. It is evident that $(\mathsf{WKL}_{=2}\times\mathsf{WKL}_{\rm aou})^{\partial}$-realizability validates $\mathsf{BE}_{\mathbb{Q}}+\mathsf{RDIV}$. To prove (6), it suffices to show the following:

Lemma 3.22. $\mathsf{LPO}\not\leq^c_{\mathsf{W}}(\mathsf{WKL}_{=2}\times\mathsf{WKL}_{\rm aou})^{\partial}$*,* $\mathsf{WKL}_{\rm clop}\not\leq^c_{\mathsf{W}}(\mathsf{WKL}_{=2}\times\mathsf{WKL}_{\rm aou})^{\partial}$*, and* $\mathsf{WKL}_{\leq 2}\not\leq^c_{\mathsf{W}}(\mathsf{WKL}_{=2}\times\mathsf{WKL}_{\rm aou})^{\partial}$*.*

The first assertion is trivial as in Lemma 3.19. The second assertion clearly follows from Proposition 3.13. The last assertion follows from Proposition 3.15 since $\mathsf{WKL}_{=2},\mathsf{WKL}_{\rm aou}\leq_{\mathsf{W}}\mathsf{Lim}_{\mathbb{N}}$ by Proposition 3.5. This proves Lemma 3.22; hence, as in (1), one can easily verify the item (6).

(7) We use $(\mathsf{WKL}_{\leq 2}\times\mathsf{WKL}_{\rm aou})^{\partial}$-realizability to realize $\neg\Pi^0_1$-$\mathsf{LEM}_{\mathbb{R}}+\mathsf{BE}+\mathsf{RDIV}+\neg\mathsf{IVT}_{\rm lin}$. It is evident that $(\mathsf{WKL}_{\leq 2}\times\mathsf{WKL}_{\rm aou})^{\partial}$-realizability validates $\mathsf{BE}+\mathsf{RDIV}$. To prove (7), it suffices to show the following:

Lemma 3.23. $\mathsf{LPO}\not\leq^c_{\mathsf{W}}(\mathsf{WKL}_{\leq 2}\times\mathsf{WKL}_{\rm aou})^{\partial}$*, and* $\mathsf{WKL}_{\rm clop}\not\leq^c_{\mathsf{W}}(\mathsf{WKL}_{\leq 2}\times\mathsf{WKL}_{\rm aou})^{\partial}$*.*

The first assertion is trivial as in Lemma 3.19. The second assertion follows from Proposition 3.16 since $\mathsf{WKL}_{\rm clop}\leq_{\mathsf{W}}\mathsf{Lim}_{\mathbb{N}}$ by Proposition 3.5. This proves Lemma 3.23; hence, as in (1), one can easily verify the item (7).

(8) We use $(\mathsf{WKL}_{\leq 2}\times\mathsf{WKL}_{\rm clop})^{\partial}$-realizability to realize $\neg\Pi^0_1$-$\mathsf{LEM}_{\mathbb{R}}+\mathsf{BE}+\mathsf{IVT}_{\rm lin}+\neg\mathsf{IVT}$. It is evident that $(\mathsf{WKL}_{\leq 2}\times\mathsf{WKL}_{\rm clop})^{\partial}$-realizability validates $\mathsf{BE}+\mathsf{IVT}_{\mathsf{lin}}$ by Proposition 3.5. To prove (8), it suffices to show the following:

Lemma 3.24. $\mathsf{LPO}\not\leq^c_{\mathsf{W}}(\mathsf{WKL}_{\leq 2}\times\mathsf{WKL}_{\rm clop})^{\partial}$*, and* $\mathsf{WKL}_{\rm conv}\not\leq^c_{\mathsf{W}}(\mathsf{WKL}_{\leq 2}\times\mathsf{WKL}_{\rm clop})^{\partial}$*.*

The first assertion is trivial as in Lemma 3.19. The second assertion clearly follows from Proposition 3.13. This proves Lemma 3.24; hence, as in (1), one can easily verify the item (8).

(9): Straightforward. □

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