Incomputability of Simply Connected Planar Continua

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Abstract. Le Roux and Ziegler asked whether every simply connected compact nonempty planar Π^0_1 set always contains a computable point. In this paper, we solve the problem of le Roux and Ziegler by showing that there exists a planar Π^0_1 dendroid without computable points. We also provide several pathological examples of tree-like Π^0_1 continua fulfilling certain global incomputability properties: there is a computable dendrite which does not $*$-include a Π^0_1 tree; there is a Π^0_1 dendrite which does not $*$-include a computable dendrite; there is a computable dendroid which does not $*$-include a Π^0_1 dendrite. Here, a continuum A *$*$-includes* a member of a class $\mathcal{P}$ of continua if, for every positive real ε, A includes a continuum $B \in \mathcal{P}$ such that the Hausdorff distance between A and B is smaller than ε.

Key words: Computable Analysis, Type-two-theory of effectivity, Effectively closed sets

1 Background

Every nonempty open set in a computable metric space (such as Euclidean n-space $\mathbb{R}^n$) contains a computable point. In contrast, the Non-Basis Theorem asserts that a nonempty *co-c.e. closed* set (also called a Π^0_1 set) in Cantor space (hence, even in Euclidean 1-space) can avoid any computable points. Non-Basis Theorems can shed new light on connections between *local* and *global* properties by incorporating the notions of *measure* and *category*. For instance, Kreisel-Lacombe [6] and Tanaka [17] showed that there is a Π^0_1 set with positive measure that contains no computable point. Recent exciting progress in *Computable Analysis* [18] naturally raises the question whether Non-Basis Theorems exist for *connected* Π^0_1 sets. However, we observe that, if a nonempty Π^0_1 subset of $\mathbb{R}^1$ contains no computable points, then it must be totally disconnected. Then, in higher dimensional Euclidean space, can there exist a connected Π^0_1 set containing no computable points? It is easy to construct a nonempty connected Π^0_1 subset of $[0,1]^2$ without computable points, and a nonempty simply connected Π^0_1 subset of $[0,1]^3$ without computable points. An open problem, formulated by Le Roux and Ziegler [13] was whether every nonempty simply connected

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compact planar Π^0_1 set contains a computable point. As mentioned in Penrose's book "*Emperor's New Mind*" [12], *the Mandelbrot set* is an example of a simply connected compact planar Π^0_1 set which contains a computable point, and he conjectured that the Mandelbrot set is not computable *as a closed set*. Hertling [5] observed that the Penrose conjecture has an implication for a famous open problem on local connectivity of the Mandelbrot set. Our interest is which topological assumption (especially, connectivity assumption) on a Π^0_1 set can force it to possess a given computability property. Miller [10] showed that every Π^0_1 sphere in $\mathbb{R}^n$ is computable, and so it contains a dense c.e. subset of computable points. He also showed that every Π^0_1 ball in $\mathbb{R}^n$ contains a dense subset of computable points. Iljazović [7] showed that chainable continua (e.g., arcs) in certain metric spaces are almost computable, and hence there always is a dense subset of computable points. In this paper, we show that *not* every Π^0_1 dendrite is almost computable, by using a tree-immune Π^0_1 class in Cantor space. This notion of immunity was introduced by Cenzer, Weber, Wu, and the author [4]. We also provide pathological examples of tree-like Π^0_1 continua fulfilling certain global incomputability properties: there is a computable dendrite which does not $*$-include a Π^0_1 tree; there is a computable dendroid which does not $*$-include a Π^0_1 dendrite. Finally, we solve the problem of Le Roux and Ziegler [13] by showing that there exists a planar Π^0_1 dendroid without computable points. Indeed, our planar dendroid is contractible. Hence, our dendroid is also the first example of a contractible Euclidean Π^0_1 set without computable points.

2 Preliminaries

Basic Notation: $2^{<\mathbb{N}}$ denotes the set of all finite binary strings. Let X be a topological space. For a subset $Y \subseteq X$, $cl(Y)$ ($int(Y)$, resp.) denotes the closure (the interior, resp.) of Y. Let $(X;d)$ be a metric space. For any $x \in X$ and $r \in \mathbb{R}$, $B(x;r)$ denotes the open ball $B(x;r) = \{y \in X : d(x,y) < r\}$. Then x is called *the center of* $B(x;r)$, and r is called *the radius of* $B(x;r)$. For a given open ball $B = B(x;r)$, $\hat{B}$ denotes the corresponding closed ball $\hat{B} = \{y \in X : d(x,y) \leq r\}$. For $a,b \in \mathbb{R}$, $[a,b]$ denotes the closed interval $[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$, (a,b) denotes the open interval $(a,b) = \{x \in \mathbb{R} : a < x < b\}$, and $\langle a,b\rangle$ denotes a point of Euclidean plane $\mathbb{R}^2$. For $X \subseteq \mathbb{R}^n$, $\mathrm{diam}(X)$ denotes $\max\{d(x,y) : x,y \in X\}$.

Continuum Theory: *A continuum* is a compact connected metric space. For basic terminology concerning *Continuum Theory*, see Nadler [11] and Illanes-Nadler [8].

Let X be a topological space. The set X is *a Peano continuum* if it is a locally connected continuum. The set X is *a dendrite* if it is a Peano continuum which contains no Jordan curve. The set X is *unicoherent* if $A \cap B$ is connected for every connected closed subsets $A, B \subseteq X$ with $A \cup B = X$. The set X is *hereditarily unicoherent* if every subcontinuum of X is unicoherent. The set X is *a dendroid* if it is an arcwise connected hereditarily unicoherent continuum. For a point x of a dendroid X, $r_X(x)$ denotes the cardinality of the set of arc-components of $X \setminus \{x\}$. If $r_X(x) \geq 3$ then x is said to be *a ramification point of* X.

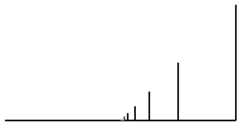

Fig. 1. The basic dendrite

Fig. 2. The harmonic comb

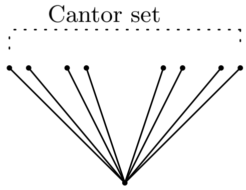

Fig. 3. The Cantor fan

The set X is *a tree* if it is dendrite with finitely many ramification points. Note that a topological space X is a dendrite if and only if it is a locally connected dendroid. Hahn-Mazurkiewicz's Theorem states that a Hausdorff space X is a Peano continuum if and only if X is an image of a continuous curve.

Example 1 (Planar Dendroids).

1. Put $\mathcal{B}_t = \{2^{-t}\} \times [0, 2^{-t}]$. Then the following set $\mathcal{B} \subseteq \mathbb{R}^2$ is dendrite.
$$\mathcal{B} = \bigcup_{t \in \mathbb{N}} \mathcal{B}_t \cup ([-1, 1] \times \{0\}).$$
We call $\mathcal{B}$ *the basic dendrite*. The set $\mathcal{B}_t$ is called *the t-th rising of $\mathcal{B}$*. See Fig. 1.
2. The set $\mathcal{H} = cl((\{1/n : n \in \mathbb{N}\} \times [0, 1]) \cup ([0, 1] \times \{0\}))$ is called *a harmonic comb*. Then $\mathcal{H}$ is a dendroid, but not a dendrite. The set $\{1/n\} \times [0, 1]$ is called *the n-th rising of the comb $\mathcal{H}$*, and the set $[0, 1] \times \{0\}$ is called *the grip of $\mathcal{H}$*. See Fig. 2.
3. Let $C \subseteq \mathbb{R}^1$ be the middle third Cantor set. Then the one-point compactification of $C \times (0, 1]$ is called *the Cantor fan*. (Equivalently, it is the quotient space $\mathrm{Cone}(C) = (C \times [0, 1])/(C \times \{0\})$.) The Cantor fan is a dendroid, but not a dendrite. See Fig. 3.

Let X be a topological space. X is *n-connected* if it is path-connected and $\pi_i(X) \equiv 0$ for any $1 \leq i \leq n$, where $\pi_i(X)$ is the i-th homotopy group of X. X is *simply connected* if X is 1-connected. X is *contractible* if the identity map on X is null-homotopic. Note that, if X is contractible, then X is n-connected for each $n \geq 1$. It is easy to see that the dendroids in Example 1 are contractible.

Computability Theory: We assume that the reader is familiar with Computability Theory on the natural numbers $\mathbb{N}$, Cantor space $2^{\mathbb{N}}$, and Baire space $\mathbb{N}^{\mathbb{N}}$ (see also Soare [16]). For basic terminology concerning *Computable Analysis*, see Weihrauch [18], Brattka-Weihrauch [3], and Brattka-Presser [2].

Hereafter, we fix a countable base for the Euclidean n-space $\mathbb{R}^n$ by $\rho = \{B(x; r) : x \in \mathbb{Q}^n \ \& \ r \in \mathbb{Q}^+\}$, where $\mathbb{Q}^+$ denotes the set of all positive rationals. Let $\{\rho_n\}_{n \in \mathbb{N}}$ be an effective enumeration of ρ. We say that a point $x \in \mathbb{R}^n$ is *computable* if the code of its principal filter $\mathcal{F}(x) = \{i \in \mathbb{N} : x \in \rho_i\}$ is computably enumerable (hereafter c.e.) A closed subset $F \subseteq \mathbb{R}^n$ is Π^0_1 if there is a c.e. set $W \subseteq \mathbb{N}$ such that $F = X \setminus \bigcup_{e \in W} \rho_e$. A closed subset $F \subseteq \mathbb{R}^n$ is *computably enumerable* (hereafter *c.e.*) if $\{e \in \mathbb{N} : F \cap \rho_e \neq \emptyset\}$ is c.e. A closed subset $F \subseteq \mathbb{R}^n$ is *computable* if it is Π^0_1 and c.e. on $\mathbb{R}^n$.

Almost Computability: Let A_0, A_1 be nonempty closed subsets of a metric space (X, d). Then *the Hausdorff distance* between A_0 and A_1 is defined by

$$d_H(A_0, A_1) = \max_{i<2} \sup_{x \in A_i} \inf_{y \in A_{1-i}} d(x, y).$$

Let $\mathcal{P}$ be a class of continua. We say that a continuum A *$*$-includes a member of $\mathcal{P}$* if $\inf\{d_H(A, B) : A \supseteq B \in \mathcal{P}\} = 0$.

Proposition 1. *Every Euclidean dendroid $*$-includes a tree.*

Proof. Fix a Euclidean dendroid $D \subseteq \mathbb{R}^n$, and a positive rational $\varepsilon \in \mathbb{Q}$. Then D is covered by finitely many open rational balls $\{B_i\}_{i<n}$ of radius $\varepsilon/2$. Choose $d_i \in D \cap B_i$ for each $i < n$ if B_i intersects with D. Note that $\{B(d_i; \varepsilon)\}_{i<n}$ covers D. Since D is a dendroid, there is a unique arc $\gamma_{i,j} \subseteq D$ connecting d_i and d_j for each $i, j < n$. Then, $E = \bigcup_{\{i,j\} \subseteq n} \gamma_{i,j}$ is connected and locally connected, since E is a union of finitely many arcs (i.e., it is a graph, in the sense of Continuum Theory; see also Nadler [11]). It is easy to see that E has no Jordan curve, since E is a subset of the dendroid D. Consequently, E is a tree. Moreover, clearly $d_H(E, D) < \varepsilon$, since $d_i \in E$ for each $i < n$. □

The class $\mathcal{P}$ has *the almost computability property* if every $A \in \mathcal{P}$ $*$-includes a computable member of $\mathcal{P}$ as a closed set. In this case, we simply say that *every $A \in \mathcal{P}$ is almost computable.* Iljazović [7] showed that every Π^0_1 chainable continuum is almost computable, hence every Π^0_1 arc is almost computable.

3 Incomputability of Dendrites

3.1 A Computable Dendrite Approximable by No Π^0_1 Trees

By Proposition 1, topologically, every planar dendrite $*$-includes a tree. However, if we try to effectivize this fact, we will find a counterexample.

Theorem 1. *Not every computable planar dendrite $*$-includes a Π^0_1 tree.*

Proof. Let $A \subseteq \mathbb{N}$ be an incomputable c.e. set. Thus, there is a total computable function $f_A : \mathbb{N} \to \mathbb{N}$ such that $\mathrm{range}(f_A) = A$. We may assume $f_A(s) \leq s$ for every $s \in \mathbb{N}$. Let A_s denote the finite set $\{f_A(u) : u \leq s\}$. Then $\mathrm{st}^A : \mathbb{N} \to \mathbb{N}$ is defined as $\mathrm{st}^A(n) = \min\{s \in \mathbb{N} : n \in A_s\}$. Note that $\mathrm{st}^A(n) \geq n$ by our assumption $f_A(s) \leq s$.

Construction. Recall the definition of the basic dendrite from Example 1. We construct a computable dendrite by modifying the basic dendrite $\mathcal{B}$. For every $t \in \mathbb{N}$, we introduce *the width of the t-rising $w(t)$* as follows:

$$w(t) = \begin{cases} 2^{-(2+\mathrm{st}^A(t))}, & \text{if } t \in A, \\ 0, & \text{otherwise.} \end{cases}$$

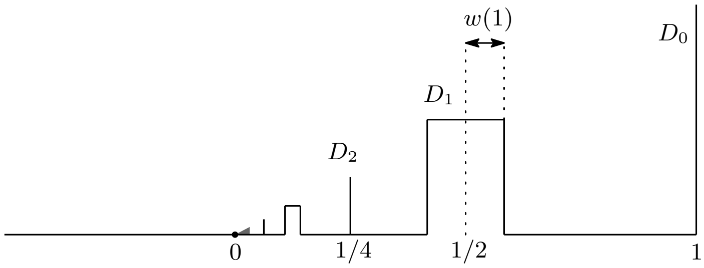

Fig. 4. The dendrite D for $0, 2, 4 \notin A$ and $1, 3 \in A$.

Let I_t be the closed interval $[2^{-t} - w(t), 2^{-t} + w(t)]$. Since $\mathrm{st}^A(n) \geq n$, we have $I_t \cap I_s = \emptyset$ whenever $t \neq s$. We observe that $\{w(t)\}_{t\in\mathbb{N}}$ is a uniformly computable sequence of real numbers. Now we define a computable dendrite $D \subseteq \mathbb{R}^2$ by:

$$
\begin{aligned}
D_t^0 &= (\{2^{-t} - w(t)\} \cup \{2^{-t} + w(t)\}) \times [0, 2^{-t}] \\
D_t^1 &= [2^{-t} - w(t), 2^{-t} + w(t)] \times \{2^{-t}\} \\
D_t^2 &= (2^{-t} - w(t), 2^{-t} + w(t)) \times (-1, 2^{-t}) \\
D &= \Big(\bigcup_{t\in\mathbb{N}} (D_t^0 \cup D_t^1)\Big) \cup \Big(([-1, 1] \times \{0\}) \setminus \bigcup_{t\in\mathbb{N}} D_{t,m}^2\Big).
\end{aligned}
$$

We call $D_t = D_t^0 \cup D_t^1$ *the t-th rising of D*. See Fig. 4.

Claim. The set D is a dendrite.

To prove D is a Peano continuum, by the Hahn-Mazurkiewicz Theorem, it suffices to show that $D = \mathrm{Im}(h)$ for some continuous curve $h : [-1, 1] \to \mathbb{R}^2$. We divide the unit interval $[0, 1]$ into infinitely many parts $I_t = [2^{-(t+1)}, 2^{-t}]$. Furthermore, we also divide each interval I_{2t} into three parts I_{2t}^0, I_{2t}^1, and I_{2t}^2, where $I_{2t}^i = [(5 - i) \cdot 3^{-1} \cdot 2^{-(2t+1)}, (6 - i) \cdot 3^{-1} \cdot 2^{-(2t+1)}]$ for each $i < 3$. Then we define a desired curve h as follows.

$$
h(x) \text{ moves in } \begin{cases}
\{2^{-t} + w(t)\} \times [0, 2^{-t}] & \text{if } x \in I_{2t}^0, \\
[2^{-t} - w(t), 2^{-t} + w(t)] \times \{2^{-t}\} & \text{if } x \in I_{2t}^1, \\
\{2^{-t} - w(t)\} \times [0, 2^{-t}] & \text{if } x \in I_{2t}^2, \\
[2^{-(t+1)} + w(t+1), 2^{-t} - w(t)] \times \{0\} & \text{if } x \in I_{2t+1}, \\
[-1, 0] \times \{0\} & \text{if } x \in [-1, 0].
\end{cases}
$$

Clearly, h can be continuous, and indeed computable, since the map $w : \mathbb{N} \to \mathbb{R}$ is computable. It is easy to see that $D = \mathrm{Im}(h)$. Moreover, $\mathrm{Im}(h)$ contains no Jordan curve since $\pi_0(h(x)) \leq \pi_0(h(y))$ whenever $x \leq y$, where $\pi_0(p)$ denotes the first coordinate of $p \in \mathbb{R}^2$. Consequently, D is a dendrite.

Moreover, by construction, it is easy to see that D is computable.

Claim. The computable dendrite D does not $*$-include a Π^0_1 tree.

Suppose that D contains a Π^0_1 subtree $T \subseteq D$. We consider a rational open ball B_t with center $\langle 2^{-t}, 2^{-t}\rangle$ and radius $2^{-(t+2)}$, for each $t \in \mathbb{N}$. Note that $B_t \cap D \subseteq D_t$ for every $t \in \mathbb{N}$. Since T is Π^0_1 in $\mathbb{R}^2$, $B = \{t \in \mathbb{N} : \hat{B}_t \cap T = \emptyset\}$ is c.e. If $w(t) > 0$ (i.e., $t \in A$) then $D \setminus (D_t \cap B_t)$ is disconnected. Therefore, either $T \subseteq [-1, 2^{-t}] \times \mathbb{R}$ or $T \subseteq [2^{-t}, 1] \times \mathbb{R}$ holds whenever $\hat{B}_t \cap T = \emptyset$ (i.e., $t \in B$), since T is connected. Thus, if the condition $\#(A \cap B) = \aleph_0$ is satisfied, then either $T \subseteq [-1, 0] \times \mathbb{R}$ or $T \subseteq [0, 1] \times \mathbb{R}$ holds. Consequently, we must have $d_H(T, D) \geq 1$.

Therefore, we may assume $\# A \cap B < \aleph_0$. Since A is coinfinite, D has infinitely many ramification points $\langle 2^{-t}, 0\rangle$ for $t \notin A$. However, by the definition of tree, T has only finitely many ramification points. Thus we must have $(D^0_t \cap T) \setminus \{\langle 2^{-t}, 0\rangle\} = \emptyset$ for almost all $t \notin A$. Since $\hat{B}_t \cap T \subseteq (D^0_t \cap T) \setminus \{\langle 2^{-t}, 0\rangle\}$, we have $t \in B$ for almost all $t \in \mathbb{N} \setminus A$. Consequently, we have $\#((\mathbb{N} \setminus A) \triangle B) < \aleph_0$. This implies that $\mathbb{N} \setminus A$ is also c.e., since B is c.e. This contradicts that A is incomputable. □

Note that a Hausdorff space (hence every metric space) is (locally) arcwise connected if and only if it is (locally) pathwise connected. However, Miller [10] pointed out that the effective versions of arcwise connectivity and pathwise connectivity do *not* coincide. Theorem 1 could give a result on effective connectivity properties. Note that *effectively pathwise connectivity* is defined by Brattka [1]. Clearly, the dendrite D is effectively pathwise connected. We now introduce a new effective version of arcwise connectivity property by thinking arcs as closed sets. Let $\mathcal{A}_-(X)$ denote the hyperspace of closed subsets of X with negative information (see also Brattka [1]).

Definition 1. *A computable metric space* (X, d, α) *is* semi-effectively arcwise connected *if there exists a total computable multi-valued function* $P : X^2 \rightrightarrows \mathcal{A}_-(X)$ *such that* $P(x, y)$ *is the set of all arcs* A *whose two end points are* x *and* y*, for any* $x, y \in X$*.*

Obviously D is not semi-effectively arcwise connected. Indeed, for every $\varepsilon > 0$ there exists $x_0, x_1 \in [0, 1]$ with $d(x_0, x_1) < \varepsilon$ such that $\langle x_0, 0\rangle, \langle x_1, 0\rangle \in D$ cannot be connected by any Π^0_1 arc. Thus, we have the following corollary.

Corollary 1. *There exists an effectively pathwise connected Euclidean continuum* D *such that* D *is not semi-effectively arcwise connected.*

3.2 Plotting Binary Trees on Euclid Plane

For a string $\sigma \in 2^{<\mathbb{N}}$, let $lh(\sigma)$ denote the length of σ. Then

$$\psi(\sigma) = \left\langle 2^{-1} \cdot 3^{-i} + 2 \sum_{i < lh(\sigma)\ \&\ \sigma(i) = 1} 3^{-(i+1)}, 2^{-lh(\sigma)} \right\rangle \in \mathbb{R}^2.$$

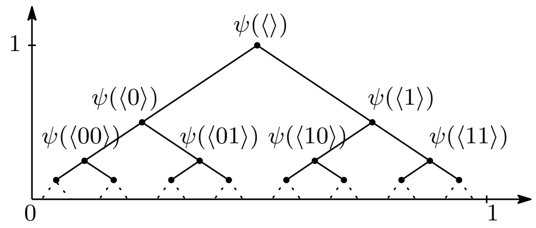

Fig. 5. The plotted tree $\Psi(2^{<\mathbb{N}})$.

For two points $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$, the closed line segment $L(\boldsymbol{x}, \boldsymbol{y})$ from $\boldsymbol{x}$ to $\boldsymbol{y}$ is defined by $L(\boldsymbol{x}, \boldsymbol{y}) = \{(1-t)\boldsymbol{x} + t\boldsymbol{y} : t \in [0,1]\}$. For a (possibly infinite) tree $T \subseteq 2^{<\mathbb{N}}$, we plot an embedded tree $\Psi(T) \subseteq \mathbb{R}^2$ by

$$\Psi(T) = cl\left(\bigcup\{L(\psi(\sigma), \psi(\tau)) : \sigma, \tau \in T \;\&\; lh(\sigma) = lh(\tau) + 1\}\right).$$

Then $\Psi(T)$ is a dendrite (but not necessarily a tree, in the sense of Continuum Theory), for any (possibly infinite) tree $T \subseteq 2^{\mathbb{N}}$. See Fig. 5.

We can easily prove the following lemmata.

Lemma 1. *Let T be a subtree of $2^{<\mathbb{N}}$, and D be a planar subset such that $\psi(\langle\rangle) \in D \subseteq \Psi(T)$ for the root $\langle\rangle \in 2^{<\mathbb{N}}$. Then D is a dendrite if and only if D is homeomorphic to $\Psi(S)$ for a subtree $S \subseteq T$.*

Proof. The "if" part is obvious. Let D be a dendrite. For a binary string σ which is not a root, let σ^- be an immediate predecessor of σ. We consider the set $S = \{\langle\rangle\} \cup \{\sigma \in 2^{<\mathbb{N}} : \sigma \neq \langle\rangle \;\&\; D \cap (L(\psi(\sigma^-), \psi(\sigma)) \setminus \{\psi(\sigma^-)\}) \neq \emptyset\}$. Since D is connected, S is a subtree of T. It is easy to see that D is homeomorphic to $\Psi(S)$. □

Lemma 2. *Let T be a subtree of $2^{<\mathbb{N}}$. Then T is Π^0_1 (c.e., computable, resp.) if and only if $\Psi(T)$ is a Π^0_1 (c.e., computable, resp.) dendrite in $\mathbb{R}^2$.*

Proof. With our definition of Ψ, the dendrite $\Psi(2^{<\mathbb{N}})$ is clearly a computable closed subset of $\mathbb{R}^2$. So, if T is Π^0_1, then it is easy to prove that $\Psi(T)$ is also Π^0_1. Assume that T is a c.e. tree. At stage s, we compute whether $L(\psi(\sigma^-), \psi(\sigma))$ intersects with the e-th open rational ball ρ_e, for any $e < s$ and any string σ which is already enumerated into T by stage s. If so, we enumerate e into W_T at stage s. Then $\{e \in \mathbb{N} : \Psi(T) \cap \rho_e \neq \emptyset\} = W_T$ is c.e.

Assume that $\Psi(T)$ is Π^0_1. Then, we consider an open rational ball $B_-(\sigma) = B(\psi(\sigma); 2^{-(lh(\sigma)+2)})$ for each $\sigma \in 2^{<\mathbb{N}}$. Note that $\hat{B}_-(\sigma) \cap \hat{B}_-(\tau) = \emptyset$ for $\sigma \neq \tau$. Since $\Psi(T)$ is Π^0_1, $T^* = \{\sigma \in 2^{<\mathbb{N}} : \Psi(T) \cap \hat{B}_-(\sigma) = \emptyset\}$ is c.e., and it is easy to see that $T = 2^{<\mathbb{N}} \setminus T^*$. Thus, T is a Π^0_1 tree of $2^{<\mathbb{N}}$. We next assume that $\Psi(T)$ is c.e. We can assume that $\Psi(T)$ contains the root $\psi(\langle\rangle)$, otherwise $T = \emptyset$, and clearly it is c.e. For a binary string σ which is not a root, let σ^- be an immediate predecessor of σ. Pick an open rational ball $B_+(\sigma)$ such that $\Psi(2^{<\mathbb{N}}) \cap B_+(\sigma) \subseteq L(\psi(\sigma^-), \psi(\sigma))$ for each σ. Since $\Psi(T)$ is c.e., $T^* = \{\sigma \in 2^{<\mathbb{N}} : \Psi(T) \cap B_+(\sigma) \neq \emptyset\}$ is c.e. If σ is not a root and $\sigma \in T$ then $L(\psi(\sigma^-), \psi(\sigma)) \subseteq \Psi(T)$, so $\Psi(T) \cap B_+(\sigma) \neq \emptyset$. We observe that if $\sigma \notin T$ then $L(\psi(\sigma^-), \psi(\sigma)) \cap \Psi(T) = \emptyset$, so $\Psi(T) \cap B_+(\sigma) = \emptyset$. Thus, we have $T = T^*$. In the case that $\Psi(T)$ is computable, $\Psi(T)$ is c.e. and Π^0_1, hence T is c.e. and Π^0_1, i.e., T is computable. ◻

Lemma 3. *Let D be a computable subdendrite of $\Psi(2^{<\mathbb{N}})$ with $\psi(\langle\rangle) \in D$. Then there exist computable subtrees $T^-, T^+ \subseteq 2^{<\mathbb{N}}$ such that $\Psi(T^-) \subseteq D \subseteq \Psi(T^+)$ and $([0,1] \times \{0\}) \cap D = ([0,1] \times \{0\}) \cap \Psi(T^-) = ([0,1] \times \{0\}) \cap \Psi(T^+)$.*

Proof. Again we consider an open ball $B_-(\sigma) = B(\psi(\sigma); 2^{-(lh(\sigma)+2)})$, and an open rational ball $B_+(\sigma)$ such that $\Psi(2^{<\mathbb{N}}) \cap B_+(\sigma) \subseteq L(\psi(\sigma^-), \psi(\sigma))$ for each $\sigma \in 2^{<\mathbb{N}}$. Since D is Π^0_1, $U^* = \{\sigma \in 2^{<\mathbb{N}} : D \cap \hat{B}_-(\sigma) = \emptyset\}$ is c.e. Since D is c.e., $T^* = \{\sigma \in 2^{<\mathbb{N}} : D \cap B_+(\sigma) \neq \emptyset\}$ is c.e., and it is a tree by Lemma 1. For every $\sigma \in 2^{<\mathbb{N}}$, either $D \cap \hat{B}_-(\sigma) = \emptyset$ or $D \cap B_+(\sigma) \neq \emptyset$ holds. Therefore, we have $T^* \cup U^* = 2^{<\mathbb{N}}$. Moreover, for the set of *leaves of* T^*, $L^*_T = \{\rho \in T^* : (\forall i < 2)\ \rho^\frown\langle i\rangle \notin T^*\}$, we observe that $T^* \cap U^* \subseteq L^*_T$. Recall that the pointclass Σ^0_1 has the reduction property, that is, for two c.e. sets T^* and U^*, there exist c.e. subsets $T \subseteq T^*$ and $U \subseteq U^*$ such that $T \cup U = T^* \cup U^*$ and $T \cap U = \emptyset$. This is because, for $\sigma \in T^* \cap U^*$, σ is enumerated into T when σ is enumerated into T^* before it is enumerated into U^*; σ is enumerated into U otherwise. Since $T^* \cap U^* \subseteq L^*_T$, T must be a tree. Furthermore, T is c.e., and $U = 2^{<\mathbb{N}} \setminus T$ is also c.e. Thus, T is a computable tree. Therefore, $T^- = \{\sigma^- \in 2^{<\mathbb{N}} : \sigma \in T \ \&\ (\forall i < 2)\ \sigma^\frown\langle i\rangle \notin T\}$ and $T^+ = \{\sigma^\frown\langle i\rangle : \sigma \in T \ \&\ i < 2\}$ are also computable trees. Then, $\Psi(T^-) \subseteq D \subseteq \Psi(T^+)$, and we have $([0,1] \times \{0\}) \cap D = ([0,1] \times \{0\}) \cap \Psi(T^-) = ([0,1] \times \{0\}) \cap \Psi(T^+)$ since the sets of all infinite paths of T, T^- and T^+ coincide. ◻

Lemma 4. *There is an infinite Π^0_1 tree $P \subseteq 2^{<\mathbb{N}}$ which has no infinite computable subtree $T \subseteq P$.*

Proof. Let $\langle A, B\rangle$ be a computably inseparable pair of pairwise disjoint c.e. subsets of $\mathbb{N}$, i.e., there is no set S satisfying $n \in A \to n \in S \to n \notin B$ for any $n \in \mathbb{N}$. Define P as follows:

$$P = \{\sigma \in 2^{<\mathbb{N}} : (\forall n < lh(\sigma))\ n \in A \to \sigma(n) = 1 \to n \notin B\}.$$

Assume that P has an infinite computable subtree T. Then, for each $n \in \mathbb{N}$, choose the leftmost string $S(n) \in T$ of length n. Clearly S is computable, and separates A and B. This contradicts our choice of $\langle A, B\rangle$. For more details on such trees, see also Cenzer-Kihara-Weber-Wu [4]. ◻

Lemma 5. *Let $P \subseteq 2^{<\mathbb{N}}$ be an infinite Π^0_1 tree without infinite computable subtrees, and let $D \subseteq \Psi(P)$ be any computable subdendrite. Then $([0,1] \times \{0\}) \cap D = \emptyset$ holds.*

Proof. We can assume $\psi(\langle\rangle) \in D$, otherwise we connect $\psi(\langle\rangle)$ and the root of D by a subarc of $\Psi(2^{<\mathbb{N}})$. By Lemma 3, there exist computable trees $T^-, T^+ \subseteq 2^{<\mathbb{N}}$ such that $\Psi(T^-) \subseteq D \subseteq \Psi(T^+)$ and both $\Psi(T^-)$ and $\Psi(T^+)$ agree with D on $[0,1] \times \{0\}$. Since $\Psi(T^-) \subseteq D$, we have $T^- \subseteq P$. By our assumption of P, the tree T^- must be finite. Hence, T^+ is also finite. By using weak König's lemma (or, equivalently, compactness of Cantor space), $T^+ \subseteq 2^l$ holds for some $l \in \mathbb{N}$. Thus, $D \subseteq \Psi(T^+) \subseteq [0,1] \times [2^{-l}, 1]$ as desired. □

Note that if T is an infinite binary subtree of $2^{<\mathbb{N}}$, then for every $\delta > 0$ it holds that $((0,1) \times (0,\delta)) \cap \Psi(T) \neq \emptyset$.

3.3 A Π^0_1 Dendrite Approximable by No Computable Dendrite

Theorem 2. *Not every Π^0_1 planar dendrite is almost computable.*[1]

Proof. Again, we adapt the construction in the proof of Theorem 1. We fix an infinite Π^0_1 tree $P \subseteq 2^{<\mathbb{N}}$ with no infinite computable subtree, as in Lemma 4. For $\sigma \in 2^{<\mathbb{N}}$, put $E(\sigma) = \{\tau \in 2^{<\mathbb{N}} : \tau \supseteq \sigma\}$. For each Π^0_1 tree $P \subseteq 2^{<\mathbb{N}}$, there exists a computable function $f_P : \mathbb{N} \to 2^{<\mathbb{N}}$ such that $P = 2^{<\mathbb{N}} \setminus \bigcup_n E(f_P(n))$ and such that for each $\sigma \in 2^{<\mathbb{N}}$ and $s \in \mathbb{N}$ we have $\sigma \in \bigcup_{t<s} E(f_P(t))$ whenever $\sigma^\frown 0, \sigma^\frown 1 \in \bigcup_{t<s} E(f_P(t))$. For such a computable function $f_P : \mathbb{N} \to 2^{<\mathbb{N}}$, we let P_s denote $2^{<\mathbb{N}} \setminus \bigcup_{t<s} E(f_P(t))$. Then P_s is a tree, and $\{P_s : s \in \mathbb{N}\}$ is computable uniformly in s.

Construction. Let $\boldsymbol{e}_1$ denote $\langle 1, 0\rangle \in \mathbb{R}^2$. For a tree $T \subseteq 2^{<\mathbb{N}}$ and $w \in \mathbb{Q}$, we define $\Psi(T; w)$, *the edge of the fat approximation of the tree T of width w*, by

$$\Psi(T;w) = cl\Big(\bigcup\Big\{L\left(\psi(\sigma) \pm (3^{-|\sigma|} \cdot w)\boldsymbol{e}_1, \psi(\tau) \pm (3^{-|\tau|} \cdot w)\boldsymbol{e}_1\right) \\ : \pm \in \{-,+\} \;\&\; \sigma, \tau \in T \;\&\; lh(\sigma) = lh(\tau) + 1\Big\}\Big).$$

If $\lim_s w_s = 0$ then we have $\lim_s \Psi(T; w_s) = \Psi(T)$. Moreover, if $\{w_s : s \in \mathbb{N}\}$ is a uniformly computable sequence of rational numbers, then $\{\Psi(T; w_s) : s \in \mathbb{N}\}$ is also a uniformly computable sequence of computable closed sets. Additionally,

[1] The author is grateful to an anonymous referee for suggesting an alternative proof using a computably inseparable pair $\langle A, B\rangle$, which does not use our lemmata concerning plotting of binary trees into Euclidean plane. We sketch out the proof of the referee: Let $P \subseteq \mathbb{R}^2$ be the union of $[0,2] \times \{0\}$ and vertical lines $L_n = \{1 - 2^{-n}\} \times [-2^{-n}, 2^{-n}]$. Define Q from P as follows. If n enters A at stage s, then remove the bottom half of L_n, and the open segment $(1-2^{-n}, 1-2^{-n}+2^{-s}) \times \{0\}$. To keep Q connected, add the diagonal line segment connecting the points $(1-2^{-n}, 2^{-n})$ and $(1 - 2^{-n} + 2^{-s}, 0)$. If $n \in D$, do likewise but remove the top half of L_n and use the bottom half to keep Q connected. As in the proof of Theorem 1, we can show that the resulting set Q is a Π^0_1 dendrite, and if Q $*$-includes a computable dendrite, then we could compute a separator for A and B.

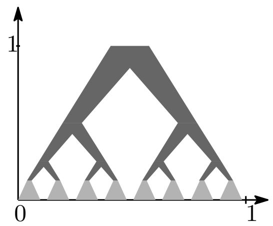

Fig. 6. The fat approximation $\Psi(T; w)$.

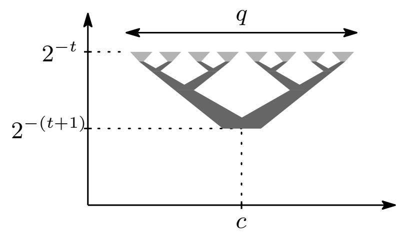

Fig. 7. The basic object $\Psi(T; w, c, t, q)$.

the set $\Psi(T; w, c, t, q)$, for a tree $T \subseteq 2^{<\mathbb{N}}$, for $w, c, q \in \mathbb{Q}$, and for $t \in \mathbb{N}$, is defined by

$$\Psi(T; w, c, t, q) = \left\{ \left\langle c + q \cdot \left(x - \frac{1}{2} \right), \frac{2-y}{2^{t+1}} \right\rangle \in \mathbb{R}^2 : \langle x, y \rangle \in \Psi(T; w) \right\}.$$

Note that $\Psi(T; w, c, t, q) \subseteq [c - q/2, c + q/2] \times [2^{-(t+1)}, 2^{-t}]$ as in Fig. 7. For $t \in \mathbb{N}$, and for $\mathrm{st}^A(t) = \min\{s : t \in A_s\}$ in the proof of Theorem 1, let $l(t) \in 2^{\mathbb{N}}$ be the leftmost path of $P_{\mathrm{st}^A(t)}$. If $\mathrm{st}^A(t)$ is undefined (i.e., $t \notin A$) then $l(t)$ is also undefined. For each $t \in \mathbb{N}$ we define $F(t) = \{\sigma \in 2^{<\mathbb{N}} : \sigma \subseteq l(t)\}$ if $l(t)$ is defined; $F(t) = P$ otherwise. Then $\{F(t) : t \in \mathbb{N}\}$ is a computable sequence of Π^0_1 subsets of $2^{<\mathbb{N}}$. Furthermore, we have $\Psi(F(t)) \cap ([0,1] \times \{0\}) \neq \emptyset$, since $F(t)$ has a path for every $t \in \mathbb{N}$. For each $t \in \mathbb{N}$, $w(t)$ is defined again as in the proof of Theorem 1. Now we define a Π^0_1 dendrite $H \subseteq \mathbb{R}^2$ as follows:

$$\begin{aligned}
H^*_t &= \Psi(F(t); w(t), 2^{-t}, t, 2^{-(t+2)}) \\
H^0_t &= (\{2^{-t} - w(t)\} \cup \{2^{-t} + w(t)\}) \times [0, 2^{-(t+1)}] \\
H^{**}_t &= (2^{-t} - w(t), 2^{-t} + w(t)) \times \{2^{-(t+1)}\} \\
H^2_t &= (2^{-t} - w(t), 2^{-t} + w(t)) \times (-1, 2^{-(t+1)}) \\
H &= \Big(\bigcup_{t \in \mathbb{N}} \left(H^*_t \cup H^0_t \setminus (H^{**}_t \cup intH^*_t) \right) \Big) \cup \Big(([-1,1] \times \{0\}) \setminus \bigcup_{t \in \mathbb{N}} H^2_t \Big).
\end{aligned}$$

Put $H_t = H^*_t \setminus (H^{**}_t \cup intH^*_t)$ (see Fig. 8). We can also show that H is a Π^0_1 dendrite in the same way as for Theorem 1.

Claim. The Π^0_1 dendrite H does not $*$-include a computable dendrite.

Let J be a computable subdendrite of H. Put $S(t) = [3 \cdot 2^{-(t+2)}, 5 \cdot 2^{-(t+2)}] \times [2^{-(t+1)}, 2^{-t}]$. Then, we note that $J(t) = J \cap S(t)$ is also a computable dendrite, since $H_t \subseteq S(t)$ and it is a dendrite. However, by Lemma 5, if $t \notin A$ then we have $J(t) \cap (\mathbb{R} \times \{2^{-t}\}) = \emptyset$. So we consider the following set:

$$C = \{t \in \mathbb{N} : J(t) \cap \left([3 \cdot 2^{-(t+2)}, 5 \cdot 2^{-(t+2)}] \times [2^{-t}, 1] \right) = \emptyset\}.$$

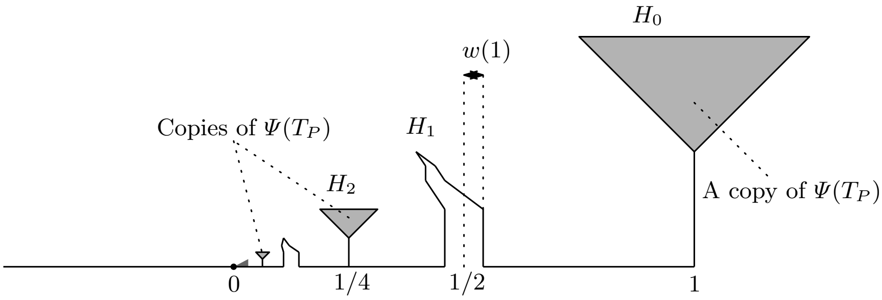

Fig. 8. The dendrite H for $0, 2, 4 \notin A$ and $1, 3 \in A$.

Since $J(t)$ is uniformly computable in t, the set C is clearly c.e., and we have $\mathbb{N} \setminus A \subseteq C$. However, if $\mathbb{N} \setminus A = C$, then this contradicts the incomputability of A. Thus, there must be infinitely many $t \in A$ such that t is enumerated into C. However, if $t \in A$ is enumerated into C, it *cuts* the dendrite H. In other words, since $J \subseteq H$ is connected, either $J \subseteq [-1, 5\cdot 2^{-(t+2)}] \times \mathbb{R}$ or $J \subseteq [3\cdot 2^{-(t+2)}, 1] \times \mathbb{R}$. Hence we must have $d_H(J, H) \geq 1$. □

Corollary 2. *There exists a nonempty Π^0_1 subset of $[0,1]^2$ which is contractible, locally contractible, and $*$-includes no connected computable closed subset.*

4 Incomputability of Dendroids

4.1 A Computable Dendroid Approximable by No Π^0_1 Dendrites

By Proposition 1, topologically, every planar dendroid $*$-includes a dendrite. However, we have no effectivization of this proposition.

Theorem 3. *Not every computable planar dendroid $*$-includes a Π^0_1 dendrite.*

Lemma 6. *There is a limit computable function f such that, for every uniformly c.e. sequence $\{U_n : n \in \mathbb{N}\}$ of nonempty c.e. sets, we have $f(n) \in U_n$ for infinitely many $n \in \mathbb{N}$.*

Proof. Let $\{V_e : e \in \mathbb{N}\}$ be an effective enumeration of all uniformly c.e. sequences $\{U_n : n \in \mathbb{N}\}$ of c.e. sets, where $(V_e)_n = U_n = \{x \in \mathbb{N} : \langle n, x\rangle \in V_e\}$. We construct a partial computable function g as follows: For each $\langle e, k\rangle \in \mathbb{N}$, wait for enumerating some element $y \in \mathbb{N}$ into $(V_e)_{\langle e,k\rangle}$. Then, define $g(\langle e, k\rangle)$ to be the first such y. If $(V_e)_n \neq \emptyset$ for each $n \in \mathbb{N}$, then $g(n)$ is contained in $(V_e)_n$ for infinitely many $n \in \mathbb{N}$. Then, let f be a limit computable total extension of g. □

Proof (Theorem 3). Pick a limit computable function $f = \lim_s f_s$ in Lemma 6. For every $t, u \in \mathbb{N}$, put $v(t, u) = 2^{-s}$ for the least s such that $f_s(t) = u$ if such s exists; $v(t, u) = 0$ otherwise. Since $\{f_s : s \in \mathbb{N}\}$ is uniformly computable, $v : \mathbb{N}^2 \to \mathbb{R}$ is computable.

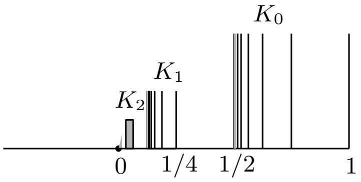

Fig. 9. The dendroid K.

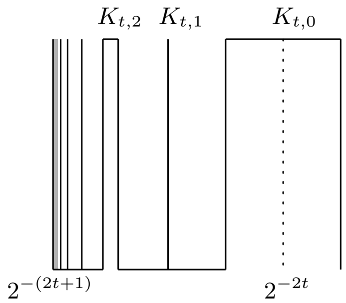

Fig. 10. The harmonic comb K_t for $f_0(t) = 0,\ f_1(t) = 0,\ f_2(t) = 2,\ \dots$

Construction. For each $t \in \mathbb{N}$, *the center position of the u-th rising of the t-th comb* is defined as $c_*(t,u) = 2^{-(2t+1)} + 2^{-(2t+u+1)}$, and *the width of the u-th rising of the t-th comb* is defined as $v_*(t,u) = v(t,u) \cdot 2^{-(2t+u+3)}$. Then, we define *the t-th harmonic comb* K_t for each $t \in \mathbb{N}$ as follows:

$$\begin{aligned}
K_t^* &= \{2^{-(2t+1)}\} \times [0, 2^{-t}] \\
K_{t,u}^0 &= \{c_*(t,u) - v_*(t,u), c_*(t,u) + v_*(t,u)\} \times [0, 2^{-t}] \\
K_{t,u}^1 &= [c_*(t,u) - v_*(t,u), c_*(t,u) + v_*(t,u)] \times \{2^{-t}\} \\
K_{t,u}^2 &= (c_*(t,u) - v_*(t,u), c_*(t,u) + v_*(t,u)) \times (-1, 2^{-t}) \\
K_t &= \left(K_t^* \cup \bigcup_{i<2} \bigcup_{u \in \mathbb{N}} K_{t,u}^i\right) \cup \left(([2^{-(2t+1)}, 2^{-2t}] \times \{0\}) \setminus \bigcup_{u \in \mathbb{N}} K_{t,u}^2\right).
\end{aligned}$$

Note that K_t is homeomorphic to the harmonic comb $\mathcal{H}$ for each $t \in \mathbb{N}$. This is because, for fixed $t \in \mathbb{N}$, since $\lim_s f_s(t)$ exists we have $v(t,u) = 0$ for almost all $u \in \mathbb{N}$. Then the desired computable dendroid is defined as follows.

$$K = ([-1,0] \times \{0\}) \cup \bigcup_{t \in \mathbb{N}} \left(([2^{-(2t+2)}, 2^{-(2t+1)}] \times \{0\}) \cup K_t\right).$$

Claim. The set K is a computable dendroid.

Clearly K is a computable closed set. To show that K is pathwise connected, we consider the following subcontinuum K_t^-, *the grip of the comb $K_{t,m}$*, for each $t \in \mathbb{N}$.

$$K_t^- = (\bigcup_{i<2} \bigcup_{v(t,u)>0} K_{t,u}^i) \cup \big(([2^{-(2t+1)}, 2^{-2t}] \times \{0\}) \setminus \bigcup_{v(t,u)>0} K_{t,u}^2\big).$$

Then $K^- = ([-1,0] \times \{0\}) \cup \bigcup_{t \in \mathbb{N}} \big(([2^{-(2t+2)}, 2^{-(2t+1)}] \times \{0\}) \cup K_t^-\big)$ has no ramification points. We claim that K^- is connected and K^- is even an arc. To show this claim, we first observe that K_t^- is an arc for any $t \in \mathbb{N}$, since $v(t,u) > 0$

occurs for finitely many $u \in \mathbb{N}$. Moreover $K_t^- \subseteq S(t)$, and $\lim_t \operatorname{diam}(S(t)) = 0$ holds. Therefore, we see that K^- is locally connected and, hence, an arc. For points $p, q \in K$, if $p, q \in K^-$ then p and q are connected by a subarc of K^-. In the case $p \in K \setminus K^-$, the point p lies on $K_{t,u}^0$ for some t, u such that $v(t, u) = 0$. If $q \in K^-$ then there is a subarc $A \subseteq K^-$ (one of whose endpoints must be $\langle c_*(t, u), 0\rangle$) such that $A \cup K_{t,u}^0$ is an arc containing p and q. In the case $q \in K \setminus K^-$, similarly we can connect p and q by an arc in K. Hence, K is pathwise connected. K is hereditarily unicoherent, since the harmonic comb is hereditarily unicoherent. Thus, K is a dendroid.

Claim. The computable dendroid K does not $*$-include a Π_1^0 dendrite.

What remains to show is that every Π_1^0 subdendrite $R \subseteq K$ satisfies the condition $d_H(R, K) \geq 1$. Let $R \subseteq K$ be a Π_1^0 dendrite. Put $S(t) = [2^{-(2t+1)}, 2^{-2t}] \times [0, 2^{-t}]$. Since R is locally connected, $R \cap S(t) = R \cap K_t$ is also locally connected for each $t \in \mathbb{N}$ and $m < 2^t$. Thus, for fixed $t \in \mathbb{N}$, let $K_{t,u}^{1*} = [c_*(t, u) - 2^{-(2t+u+3)}, c_*(t, u) + 2^{-(2t+u+3)}] \times \{2^{-t}\}$. For any continuum $R^* \subset K_t$, if $R^* \cap K_{t,u}^{1*} \neq \emptyset$ for infinitely many $u \in \mathbb{N}$, then R^* must be homeomorphic to the harmonic comb $\mathcal{H}$. Hence, R^* is not locally connected. Therefore, we have $R \cap K_{t,u}^{1*} = \emptyset$ for almost all $u \in \mathbb{N}$. Since $K_{t,u}^{1*}$ and $K_{s,v}^{1*}$ is disjoint whenever $\langle t, u\rangle \neq \langle s, v\rangle$, and since R is Π_1^0, we can effectively enumerate $U_t = \{u \in \mathbb{N} : R \cap K_{t,u}^{1*} = \emptyset\}$, i.e., $\{U_t : t \in \mathbb{N}\}$ is uniformly c.e. Moreover, U_t is cofinite for every $t \in \mathbb{N}$. Then, by our definition of $f = \lim_s f_s$ in Lemma 6, we have $f(t) \in U_t$ for infinitely many $t \in \mathbb{N}$. Note that $v(t, f(t)) > 0$ and thus the condition $f(t) \in U_t$ (i.e., $R \cap K_{t,f(t)}^{1*} = \emptyset$) implies that either $R \subseteq [-1, c_*(t, u) + v_*(t, u)] \times [0, 1]$ or $R \subseteq [c_*(t, u) - v_*(t, u), 1] \times [0, 1]$ holds. Thus we obtain the desired condition $d_H(R, K) \geq 1$. ◻

Remark 1. It is easy to see that the dendroid constructed in the proof of Theorem 3 is contractible.

Corollary 3. *There exists a nonempty contractible planar computable closed subset of* $[0, 1]^2$ *which $*$-includes no* Π_1^0 *subset which is connected and locally connected.*

4.2 A Π_1^0 Dendroid without Computable Points

Theorem 4. *Not every nonempty* Π_1^0 *planar dendroid contains a computable point.*

Proof. One can easily construct a Π_1^0 Cantor fan F containing at most one computable point $p \in F$, and such p is the unique ramification point of F. Our basic idea is to construct a topological copy of the Cantor fan F along a pathological located arc A constructed by Miller [10, Example 4.1]. We can guarantee that moving the fan F along the arc A cannot introduce new computable points. Additionally, this moving will make a ramification point p^* in a copy of F incomputable.

Fat Approximation. To archive this construction, we consider a fat approximation of a subset P of the middle third Cantor set $C \subseteq \mathbb{R}^1$, by modifying the standard construction of C. For a tree $T \subseteq 2^{<\mathbb{N}}$, put $\pi(\sigma) = 3^{-1} + 2\sum_{i<lh(\sigma)\ \&\ \sigma(i)=1} 3^{-(i+2)}$ for $\sigma \in T$, and $J(\sigma) = [\pi(\sigma) - 3^{-(lh(\sigma)+1)}, \pi(\sigma) + 2 \cdot 3^{-(lh(\sigma)+1)}]$. If a binary string σ is incomparable with a binary string τ then $J(\sigma) \cap J(\tau) = \emptyset$. We extend π to a homeomorphism π_* between Cantor space $2^{\mathbb{N}}$ and $C \cap [1/3, 2/3]$ by defining $\pi_*(f) = 3^{-1} + 2\sum_{f(i)=1} 3^{-(i+2)}$ for $f \in 2^{\mathbb{N}}$. Let $P^* \subseteq 2^{\mathbb{N}}$ be a nonempty Π^0_1 set without computable elements. Then there exists a computable tree T_P such that P^* is the set of all paths of T_P, since P^* is Π^0_1. *A fat approximation* $\{P_s : s \in \mathbb{N}\}$ *of* $P = \pi_*(P^*)$ is defined as $P_s = \bigcup\{J(\sigma) : lh(\sigma) = s\ \&\ \sigma \in T_P\}$. Then $\{P_s : s \in \mathbb{N}\}$ is a computable decreasing sequence of computable closed sets, and we have $P = \bigcap_s P_s$. Since P is a nonempty bounded closed subset of a real line $\mathbb{R}^1$, both $\min P$ and $\max P$ exist. By the same reason, both $l^-_s = \min P_s$ and $r^+_s = \max P_s$ also exist, for each $s \in \mathbb{N}$, and $\lim_s l^-_s = \min P$ and $\lim_s r^+_s = \max P$, where $\{l_s : s \in \mathbb{N}\}$ is increasing, and $\{r_s : s \in \mathbb{N}\}$ is decreasing. Let $l_s = l^-_s + 3^{-(s+1)}$ and $r_s = r^+_s - 3^{-(s+1)}$. We also set $l^*_s = l^-_s + 3^{-(s+2)}$ and $r_s = r^+_s - 3^{-(s+2)}$. Note that $l_s < r_s$, $\lim_s l_s = \min P$, and $\lim_s r_s = \max P$. Since $\min P, \max P \in P$ and P contains no computable points, $\min P$ and $\max P$ are non-computable, and so $l_s < \min P < \max P < r_s$ holds for any $s \in \mathbb{N}$. The fat approximation of P has the following remarkable property:

$$[l^-_s, l_s] \subseteq P_s,\ \ [l^-_s, l_s] \cap P = \emptyset,\ \ [r_s, r^+_s] \subseteq P_s,\ \text{ and } [r_s, r^+_s] \cap P = \emptyset.$$

To simplify the construction, we may also assume that P has the following property:

$$P = \{1 - x \in \mathbb{R} : x \in P\}$$

Because, for any Π^0_1 subset $A \subseteq C$, the Π^0_1 set $A^* = \{x/3 : x \in A\} \cup \{1 - x/3 : x \in A\} \subseteq C$ has that property.

Basic Notation. For each $i, j < 2$, for each $a, b \in \mathbb{R}^2$, and for each $q, r \in \mathbb{R}$, *the 2-cube* $\Delta_{ij}(a, b; q, r) \subseteq [a, a+q] \times [b, b+r]$ is defined as the smallest convex set containing the three points $\{(a, b), (a+q, b), (a, b+r), (a+q, b+r)\} \setminus \{(a + (1-i)q, b + (1-j)r)\}$. Namely,

$$\begin{aligned}\Delta_{ij}(a, b; q, r) = \{&\langle (-1)^i x + a + iq, (-1)^j y + b + jr\rangle \in \mathbb{R}^2\\ &: x, y \geq 0\ \&\ rx + qy \leq qr\}.\end{aligned}$$

For a set $R \subseteq \mathbb{R}^1$ and real numbers $r, y \in \mathbb{R}$, put $\Theta(R; r, y) = \{rx + y \in \mathbb{R} : x \in R\}$. Clearly $\Theta(R; r, y)$ is computably homeomorphic to R. Let four symbols $\llcorner$, $\urcorner$, $\lrcorner$, and $\ulcorner$ denote $\langle 10, 01\rangle$, $\langle 01, 10\rangle$, $\langle 00, 11\rangle$, and $\langle 11, 00\rangle$, respectively. For $v \in \{\llcorner, \urcorner, \lrcorner, \ulcorner\}$ and for any $R \subseteq [0, 1]$, $a, b \in \mathbb{R}^2$, and $q, r \in \mathbb{R}$, we define $[v](R; a, b; q, r) \subseteq [a, a+q] \times [b, b+r]$ as follows:

$$\begin{aligned}[v](R; a, b; q, r) = &\big(([a, a+q] \times \Theta(R; r, b)) \cap \Delta_{v(0)}(a, b; q, r)\big)\\ &\cup\big((\Theta(R; q, a) \times [b, b+r]) \cap \Delta_{v(1)}(a, b; q, r)\big).\end{aligned}$$

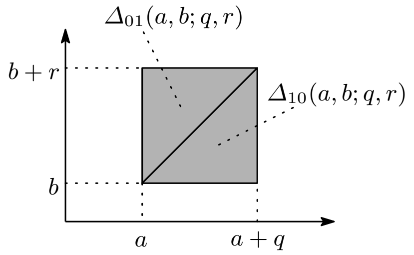

Fig. 11. The cubes $\Delta_{ij}(a,b,q,r)$.

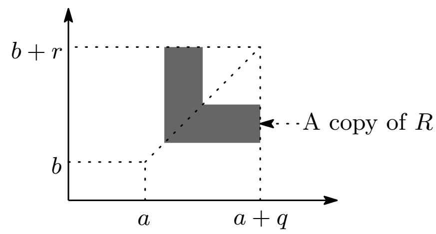

Fig. 12. $[\llcorner](R;a,b;q,r)$ for $R=[1/6,1/2]$

Sublemma 1 *$[v](P;a,b;q,r)$ is computably homeomorphic to $P\times[0,1]$. In particular, $[v](P;a,b;q,r)$ contains no computable points.* □

To simplify our argument, we use the normalization $\tilde{P}^s_t$ of P_t for $t\geq s$, that is defined by $\tilde{P}^s_t=\{(x-l^-_s)/(r^+_s-l^-_s)\in\mathbb{R}:x\in P_t\}$, for each $s\in\mathbb{N}$. Note that $\tilde{P}^s_t\subseteq[0,1]$ for $t\geq s$, and $0,1\in\tilde{P}^s_s$ holds for each $s\in\mathbb{N}$. Put $[v]^s_t([a,a+q]\times[b,b+r])=[v](\tilde{P}^s_t;a,b;q,r)$ for $t\geq s$. We also introduce the following two notions:

$$[-]^s_t([a,a+q]\times[b,b+r])=[a,a+q]\times\Theta(\tilde{P}^s_t;r,b);$$
$$[\,|\,]^s_t([a,a+q]\times[b,b+r])=\Theta(\tilde{P}^s_t;q,a)\times[b,b+r].$$

Here we code two symbols $-$ and $|$ as 0 and 1 respectively.

Sublemma 2 *$[v]^s_t([a,a+q]\times[b,b+r])\subseteq[a,a+q]\times[b,b+r]$, and $[v]^s_t([a,a+q]\times[b,b+r])$ intersects with the boundary of $[a,a+q]\times[b,b+r]$.*

Sublemma 3 *There is a computable homeomorphism between $[v]^s_t(a,b;q,r)$ and $P_t\times[0,1]$ for any $t\in\mathbb{N}$. Therefore, $\bigcap_t[v]^s_t(a,b;q,r)$ is computably homeomorphic to $P\times[0,1]$.* □

Blocks. A *block* is a set $Z\subseteq\mathbb{R}^2$ with *a bounding box* $\mathrm{Box}(Z)=[a,a+q]\times[b,b+r]$. Each $\delta\in2^2$ is called a *direction*, and directions $\langle00\rangle$, $\langle01\rangle$, $\langle10\rangle$, and $\langle11\rangle$ are also denoted by $[\leftarrow]$, $[\rightarrow]$, $[\downarrow]$, and $[\uparrow]$, respectively. For $\delta\in2^2$, $\delta^\circ=\langle\delta(0),1-\delta(0)\rangle$ is called *the reverse direction of* δ. Put $\mathrm{Line}(Z;[\leftarrow])=\{a\}\times[b,b+r]$; $\mathrm{Line}(Z;[\rightarrow])=\{a+q\}\times[b,b+r]$; $\mathrm{Line}(Z;[\downarrow])=[a,a+q]\times\{b\}$; $\mathrm{Line}(Z;[\uparrow])=[a,a+q]\times\{b+r\}$. Assume that a class $\mathcal{Z}$ of blocks is given. We introduce a relation $\overset{\delta}{\dashrightarrow}$ on $\mathcal{Z}$ for each direction δ. Fix a block $Z_{\mathrm{first}}\in\mathcal{Z}$, and we call it *the first block*. Then we declare that $\overset{[\leftarrow]}{\dashrightarrow}Z_{\mathrm{first}}$ holds. We inductively define the relation $\overset{\delta}{\dashrightarrow}$ on $\mathcal{Z}$. If $Z\overset{\delta}{\dashrightarrow}Z_0$ (resp. $Z_0\overset{\delta}{\dashrightarrow}Z$) for some Z and δ, then we also write it as $\overset{\delta}{\dashrightarrow}Z_0$ (resp. $Z_0\overset{\delta}{\dashrightarrow}$). For any two blocks Z_0 and Z_1, the relation $Z_0\overset{\delta}{\dashrightarrow}Z_1$ holds if the following three conditions are satisfied:

1. $Z_0\cap Z_1=\mathrm{Line}(Z_0;\delta)\cap Z_0=\mathrm{Line}(Z_1;\delta^\circ)\cap Z_1\neq\emptyset$.
2. $\overset{\varepsilon}{\dashrightarrow}Z_0$ has been already satisfied for some direction ε.

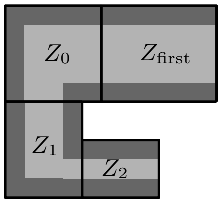

Fig. 13. $\overset{[\leftarrow]}{\dashrightarrow} Z_{\mathrm{first}} \overset{[\leftarrow]}{\dashrightarrow} Z_0 \overset{[\downarrow]}{\dashrightarrow} Z_1 \overset{[\rightarrow]}{\dashrightarrow} Z_2$.

3. $Z_1 \overset{\varepsilon}{\dashrightarrow} Z_0$ does not satisfied for any direction ε

If $Z_0 \overset{\delta}{\dashrightarrow} Z_1$ for some δ, then we say that Z_1 *is a successor of* Z_0 (Z_0 is a predecessor of Z_1), and we also write it as $Z_0 \dashrightarrow Z_1$.

We will construct a partial computable function $\mathcal{Z} : \mathbb{N}^3 \to \mathcal{A}(\mathbb{R}^2)$ with a computable function $k : \mathbb{N} \to \mathbb{N}$ and $\mathrm{dom}(\mathcal{Z}) = \{(u,i,t) \in \mathbb{N}^3 : u \leq t \;\&\; i < k(u)\}$ such that $\mathcal{Z}(u,i,t)$ is a block with a bounding box for any $(u,i,t) \in \mathrm{dom}(\mathcal{Z})$, and the block $\mathcal{Z}(u,i,t)$ is computably homeomorphic to $P_t \times [0,1]$ uniformly in (u,i,t). Here $\mathcal{A}(\mathbb{R}^2)$ is the hyperspace of all closed subsets in $\mathbb{R}^2$ with positive and negative information. For each stage t, $\mathcal{Z}_t(u) = \{\mathcal{Z}(t,u,i) : i < k(u)\}$ for each $u \leq t$ is defined. Let $\mathcal{Z}(u)$ denote the finite set $\{\lambda t.\mathcal{Z}(t,u,i) : i < k(u)\}$ of functions, for each $u \in \mathbb{N}$. The relation $\dashrightarrow$ induces a pre-ordering $\prec$ on $\bigcup_{u\in\mathbb{N}} \mathcal{Z}(u)$ as follows: $Z_0 \prec Z_1$ if there is a finite path from $Z_0(t)$ to $Z_1(t)$ on the finite directed graph $(\bigcup_{u\leq t} \mathcal{Z}_t(u), \dashrightarrow)$ at some stage $t \in \mathbb{N}$. We will assure that $\prec$ is a well-ordering of order type ω, and $Z_0 \prec Z_1$ whenever $Z_0 \in \mathcal{Z}(u)$, $Z_1 \in \mathcal{Z}(v)$, and $u < v$. In particular, for every $Z \in \bigcup_{u\in\mathbb{N}} \mathcal{Z}(u)$, the predecessor Z_{pre} of Z and the successor Z_{suc} of Z under $\prec$ are uniquely determined. If $Z_{\mathrm{pre}}(t) \overset{\delta}{\dashrightarrow} Z(t) \overset{\varepsilon}{\dashrightarrow} Z_{\mathrm{suc}}(t)$, then we say that Z *moves from δ to ε*, and that $\langle \delta, \varepsilon \rangle$ is *the direction of Z*.

Destination Point. Basically, our construction is similar to the construction by Miller [10]. Pick the standard homeomorphism ρ between $2^{\mathbb{N}}$ and the middle third Cantor set, i.e., $\rho(M) = 2\sum_{i\in M}(1/3)^{i+1}$ for $M \subseteq \mathbb{N}$, and pick a non-computable c.e. set $B \subseteq \mathbb{N}$ and put $\gamma = \rho(B)$. We will construct a Cantor fan so that the first coordinate of the unique ramification point is γ, hence the fan will have a non-computable ramification point. Let $\{B_s : s \in \mathbb{N}\}$ be a computable enumeration of B, and let n_s denote the element enumerated into B at stage s, where we may assume just one element is enumerated into B at each stage. Put $\gamma_s^{\min} = \rho(B_s)$ and $\gamma_s^{\max} = \rho(B_s \cup \{i \in \mathbb{N} : i \geq n_s\})$. Note that γ is not computable, and so $\gamma_s^{\min} \neq \gamma$ and $\gamma_s^{\max} \neq \gamma$ for any $s \in \mathbb{N}$. This means that for every $s \in \mathbb{N}$ there exists $t > s$ such that $\gamma_s^{\min} \neq \gamma_t^{\min}$ and $\gamma_s^{\max} \neq \gamma_t^{\max}$. By this observation, without loss of generality, we can assume that $\gamma_s^{\min} \neq \gamma_t^{\min}$ and $\gamma_s^{\max} \neq \gamma_t^{\max}$ whenever $s \neq t$. We can also assume $1/3 \leq \gamma_s^{\min} \leq \gamma_s^{\max} \leq 2/3$ for any $s \in \mathbb{N}$.

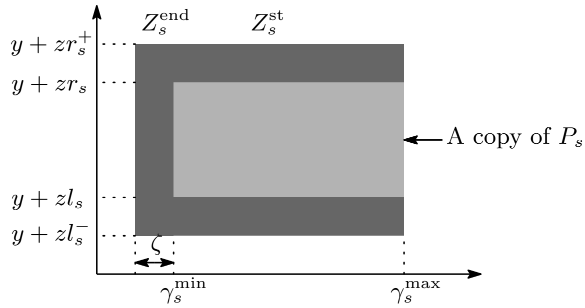

Fig. 14. The active block $Z_s^{\mathrm{st}} \cup Z_s^{\mathrm{end}}$ at stage s.

Stage 0**.** We now start to construct a Π^0_1 Cantor fan $Q = \bigcap_{s\in\mathbb{N}} Q_s$. At the first stage 0, and for each $t \geq 0$, we define the following sets:

$$Z^{\mathrm{st}}_{0,t} = [-]^s_t([\gamma_0^{\min}, \gamma_0^{\max}] \times [l_0^-, r_0^+]);\; Z_0^{\mathrm{end}} = [\gamma_0^{\min} - 1/3, \gamma_0^{\min}] \times [l_0^-, r_0^+].$$

Moreover, we set $Q_0 = Z^{\mathrm{st}}_{0,0} \cup Z_0^{\mathrm{end}}$. By our choice of P_0, actually $Q_0 = [\gamma_0^{\min} - 1/3, \gamma_0^{\max}] \times [l_0^-, r_0^+]$. $Z^{\mathrm{st}}_{0,0}$ is called *the straight block from* 2/3 *to* 1/3 *at stage* 0, and Z_0^{end} is called *the end box at stage* 0. The *bounding box* of the block Z_0^{st} is defined by $[\gamma_0^{\min}, \gamma_0^{\max}] \times [l_0^-, r_0^+]$. *The collection of* 0*-blocks at stage* t is $\mathcal{Z}_t(0) = \{Z^{\mathrm{st}}_{0,t}\}$. We declare that Z_0^{st} is the first block, and that $\overset{[\leftarrow]}{\dashrightarrow} Z_0^{\mathrm{st}}$.

Stage $s+1$**.** Inductively assume that we have already constructed the collection of u-blocks $\mathcal{Z}_t(u)$ at stage $t \geq u$ is already defined for every $u \leq s$. For any u, we think of the collection $\mathcal{Z}(u) = \{\mathcal{Z}_t(u) : t \geq u\}$ as a finite set $\{Z_i^u\}_{i<\#\mathcal{Z}_u(u)}$ of computable functions $Z_i^u : \{t \in \mathbb{N} : t \geq u\} \to \bigcup_t \mathcal{Z}_t(u)$ such that $\mathcal{Z}_t(u) = \{Z_i^u(t) : i < \#\mathcal{Z}_u(u)\}$ for each $t \geq u$. We inductively assume that the collection $\mathcal{Z}(u) = \{\mathcal{Z}_t(u) : t \geq u\}$ satisfies the following conditions:

(IH1) For each $Z \in \mathcal{Z}(u)$ and for each $t \geq v \geq u$, $Z(t) \subseteq Z(v)$.
(IH2) There is a computable function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f \upharpoonright \bigcup\bigcup_{u\leq s} \mathcal{Z}_t(u)$ is a homeomorphism between $\bigcup\bigcup_{u\leq s} \mathcal{Z}_t(u)$ and $P_t \times [0,1]$ for any $t \geq s$.
(IH3) There are $y, z, \zeta \in \mathbb{Q}$ such that the blocks $Z^{\mathrm{st}}_{s,t}$ and Z_s^{end} are constructed as follows:

$$Z^{\mathrm{st}}_{s,t} = [-]^s_t([\gamma_s^{\min}, \gamma_s^{\max}] \times [y + zl_s^-, y + zr_s^+]);$$
$$Z_s^{\mathrm{end}} = [\gamma_s^{\min} - \zeta, \gamma_s^{\min}] \times [y + zl_s^-, y + zr_s^+].$$

Here, a computable closed set Q_s, *an approximation of our Π^0_1 Cantor fan Q at stage s*, is defined by $Q_s = Z_s^{\mathrm{end}} \cup \bigcup\bigcup_{u\leq s} \mathcal{Z}_s(u)$.

Non-injured Case. First we consider the case $[\gamma_{s+1}^{\min}, \gamma_{s+1}^{\max}] \subseteq [\gamma_s^{\min}, \gamma_s^{\max}]$, i.e., this is the case that our construction is *not injured* at stage $s+1$. In this case, we construct $(s+1)$-blocks in the active block $Z_s^{\mathrm{st}} \cup Z_s^{\mathrm{end}}$. We will define $Z_t(s,i,j)$

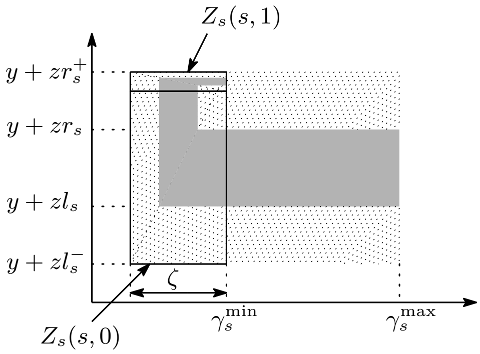

Fig. 15. The first two corner blocks $Z_s(s,0)$ and $Z_s(s,1)$.

and $\mathrm{Box}(s,i,j) = \mathrm{Box}(Z_t(s,i,j))$ for each $j < 6$. *The first two corner blocks* at stage $t \geq s+1$ are defined by:

$$\begin{aligned}
\mathrm{Box}(s,0) =& [\gamma_s^{\min} - \zeta, \gamma_s^{\min}] \times [y + zl_s^-, y + zr_s^*],\\
Z_t(s,0) =& [\llcorner]_t^s([\gamma_s^{\min} - \zeta, \gamma_s^{\min}] \times [y + zl_s^-, y + zr_s^+]) \cap \mathrm{Box}(s,0),\\
\mathrm{Box}(s,1) =& [\gamma_s^{\min} - \zeta, \gamma_s^{\min}] \times [y + zr_s^*, y + zr_s^+],\\
Z_t(s,1) =& [\ulcorner]_t^s(\mathrm{Box}(s,1)).
\end{aligned}$$

Sublemma 4 $Z_t(s,0) \cup Z_t(s,1) \subseteq Z_s^{\mathrm{end}}$ *for any* $t \geq s+1$. □

Sublemma 5 $Z_{s,t}^{\mathrm{st}} \overset{[\leftarrow]}{\dashrightarrow} Z_t(s,0) \overset{[\uparrow]}{\dashrightarrow} Z_t(s,1)$, *for any* $t \geq s+1$. □

The next block is *a straight block from* $\gamma_s^{\min}$ *to* $\gamma_{s+1}^{\max}$ which is defined as follows:

$$\begin{aligned}
\mathrm{Box}(s,2) =& [\gamma_s^{\min}, \gamma_s^{\max}] \times [y + zr_s^*, y + zr_s^+].\\
Z_t(s,2) =& [-](\mathrm{Box}_t(s,2)).
\end{aligned}$$

For given $a, b, \alpha, \beta \in \mathbb{Q}$, we can calculate $N_{0,s}(a,b;\alpha,\beta)$ and $N_{1,s}(a,b;\alpha,\beta)$ satisfying $N_{0,s}(a,b;\alpha,\beta) + N_{1,s}(a,b;\alpha,\beta) \cdot l_s^- = a + b\alpha$, and $N_{0,s}(a,b;\alpha,\beta) + N_{1,s}(a,b;\alpha,\beta) \cdot r_s^+ = a + b\beta$. Then, we put $y^\star = N_{0,s}(y,z;r_s^*,r_s^+)$, and $z^\star = N_{1,s}(y,z;r_s^*,r_s^+)$.

Sublemma 6 $\mathrm{Box}(s,2) = [\gamma_s^{\min}, \gamma_s^{\max}] \times [y^\star + z^\star l_s^-, y^\star + z^\star r_s^+]$. □

Put $\zeta^\star = (\gamma_s^{\max} - \gamma_{s+1}^{\max})/3^s$. Note that $\zeta^\star > 0$ since $\gamma_s^{\max} > \gamma_{s+1}^{\max}$. We then again define *corner blocks*.

$$\begin{aligned}
\mathrm{Box}(s,3) =& [\gamma_{s+1}^{\max}, \gamma_{s+1}^{\max} + \zeta^\star] \times [y^\star + z^\star l_s^-, y^\star + z^\star r_s^*],\\
Z_t(s,3) =& [\lrcorner]_t^s([\gamma_{s+1}^{\max}, \gamma_{s+1}^{\max} + \zeta^\star] \times [y^\star + z^\star l_s^-, y^\star + z^\star r_s^+]) \cap \mathrm{Box}(s,3),\\
\mathrm{Box}(s,4) =& [\gamma_{s+1}^{\max}, \gamma_{s+1}^{\max} + \zeta^\star] \times [y^\star + z^\star r_s^*, y^\star + z^\star r_s^+],\\
Z_t(s,4) =& [\urcorner]_t^s(\mathrm{Box}(s,4)).
\end{aligned}$$

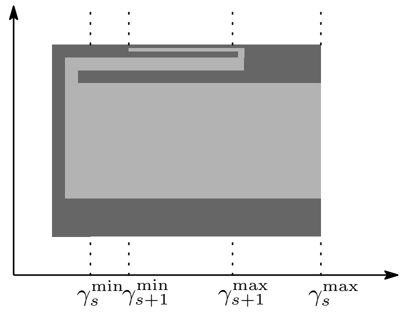

Fig. 16. $Z_s(s-1,5) \cup \bigcup \mathcal{Z}_s(s+1)$.

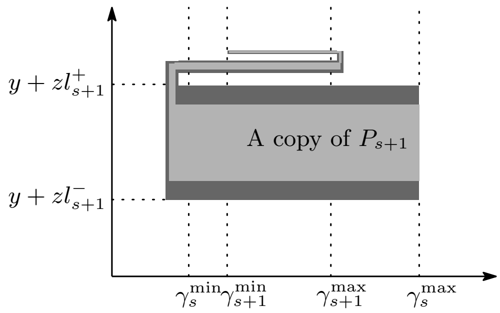

Fig. 17. $Z_{s+1}(s-1,5) \cup \bigcup \mathcal{Z}_{s+1}(s+1)$.

Next, *a straight block from $\gamma_s^{\min}$ to $\gamma_{s+1}^{\max}$* is defined as follows:

$$\begin{aligned} \mathrm{Box}(s,5) &= [\gamma_{s+1}^{\min}, \gamma_{s+1}^{\max}] \times [y^\star + z^\star r_s^*, y^\star + z^\star r_s^+], \\ Z_t(s,5) &= [-]_t^s[\mathrm{Box}(s,5)]. \end{aligned}$$

Put $y^{\star\star} = N_{0,s}(y^\star, z^\star; r_s^*, r_s^+)$, and $z^{\star\star} = N_{1,s}(y^\star, z^\star; r_s^*, r_s^+)$.

Sublemma 7 $\mathrm{Box}(s,5) = [\gamma_s^{\min}, \gamma_s^{\max}] \times [y^{\star\star} + z^{\star\star} l_s^-, y^{\star\star} + z^{\star\star} r_s^+]$. □

Put $\zeta^{\star\star} = (\gamma_{s+1}^{\min} - \gamma_s^{\min})/3^s$. Note that $\zeta^{\star\star} > 0$ since $\gamma_{s+1}^{\min} > \gamma_s^{\max}$. *The end box at stage $s+1$* is:

$$Z(s,6) = [\gamma_{s+1}^{\min} - \zeta^{\star\star}, \gamma_{s+1}^{\min}] \times [y^{\star\star} + z^{\star\star} l_s^-, y^{\star\star} + z^{\star\star} r_s^+].$$

Then put $Z_{s+1,t}^{\mathrm{st}} = Z_t(s,5)$, $Z_{s+1}^{\mathrm{st}} = Z_{s+1,s+1}^{\mathrm{st}}$, and $Z_{s+1}^{\mathrm{end}} = Z(s,6)$. *The active block at stage $s+1$* is the set $Z_{s+1,s+1}^{\mathrm{st}} \cup Z_{s+1}^{\mathrm{end}}$, and *the collection of $(s+1)$-blocks at stage t* is defined by $\mathcal{Z}_t(s+1) = \{Z_t(s,i) : i \leq 5\}$. Clearly, our definition satisfies the induction hypothesis (IH3) at stage $s+1$.

Sublemma 8 *$Z_t(s,i) \subseteq Z_v(s,i)$ for each $t \geq v \geq s+1$ and $i \leq 5$.* □

Sublemma 9 *For any $t \geq s+1$,*

$$Z_{s,t}^{\mathrm{st}} \overset{[\leftarrow]}{\dashrightarrow} Z_t(s,0) \overset{[\uparrow]}{\dashrightarrow} Z_t(s,1) \overset{[\rightarrow]}{\dashrightarrow} Z_t(s,2) \overset{[\rightarrow]}{\dashrightarrow} Z_t(s,3) \overset{[\uparrow]}{\dashrightarrow} Z_t(s,4) \overset{[\leftarrow]}{\dashrightarrow} Z_t(s,5).$$

Proof. It follows straightforwardly from the definition of these blocks $Z_t(s,i)$, and Sublemma 6 and 7. □

Sublemma 10 $\bigcup_{2 \leq i \leq 6} Z_t(s,i) \subseteq Z_s^{\mathrm{st}} \cap [\gamma_s^{\min}, \gamma_s^{\max}] \times (y + zr_s, y + zr_s^+]$. *Hence,* $\left(\bigcup_{2 \leq i \leq 6} Z_t(s,i)\right) \cap Z_{s,s+1}^{\mathrm{st}} = \emptyset$ □

Consequently, we can show the following extension property.

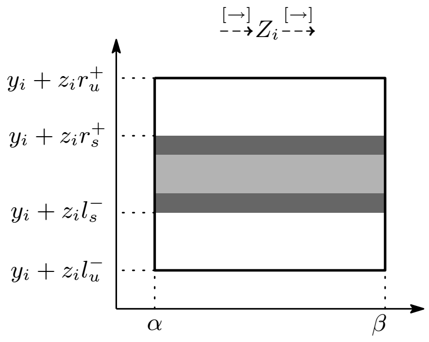

Fig. 18. The block Z_i.

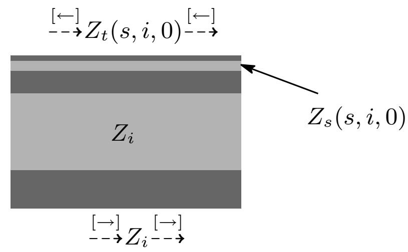

Fig. 19. The block $Z_t(s,i,0)$.

Sublemma 11 *Assume that we have a computable function $f_s : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f_s \upharpoonright \bigcup\bigcup_{u \leq s} \mathcal{Z}_t(u)$ is a computable homeomorphism between $\bigcup\bigcup_{u\leq s} \mathcal{Z}_t(u)$ and $P_t \times [1/(s+2), 1]$ for any $t \geq s$. Then we can effectively find a computable function $f_{s+1} : \mathbb{R}^2 \to \mathbb{R}^2$ extending $f_s \upharpoonright \bigcup\bigcup_{u\leq s} \mathcal{Z}_{s+1}(u)$ such that $f_{s+1} \upharpoonright \bigcup\bigcup_{u\leq s+1} \mathcal{Z}_t(u)$ is a computable homeomorphism between $\bigcup\bigcup_{u\leq s+1} \mathcal{Z}_t(u)$ and $P_t \times [1/(s+3), 1]$ for any $t \geq s+1$.*

Proof. By Sublemma 5, 9, and 10. □

By Sublemma 8 and 11, induction hypothesis (IH1) and (IH2) are satisfied. Since $Z^{\mathrm{end}}_{s+1} \cup \bigcup \mathcal{Z}_{s+1}(s+1) \subseteq Z^{\mathrm{st}}_s \cup Z^{\mathrm{end}}_s$ by Sublemma 4 and 10, and $\bigcup \mathcal{Z}_{s+1}(u) \subseteq \bigcup \mathcal{Z}_s(u)$ for each $u \leq s$, by induction hypothesis (IH1), we have the following:

$$Q_{s+1} = Z^{\mathrm{end}}_{s+1} \cup \bigcup \bigcup_{u\leq s+1} \mathcal{Z}_{s+1}(u) \subseteq Z^{\mathrm{st}}_s \cup Z^{\mathrm{end}}_s \cup \bigcup \bigcup_{u \leq s} \mathcal{Z}_s(u) \subseteq Q_s.$$

Injured Case. Secondly we consider the case that our construction *is injured*. This means that $[\gamma^{\min}_{s+1}, \gamma^{\max}_{s+1}] \not\subseteq [\gamma^{\min}_s, \gamma^{\max}_s]$. In this case, indeed, we have $[\gamma^{\min}_{s+1}, \gamma^{\max}_{s+1}] \cap [\gamma^{\min}_s, \gamma^{\max}_s] = \emptyset$. Fix the greatest stage $p \leq s$ such that $[\gamma^{\min}_{s+1}, \gamma^{\max}_{s+1}] \subseteq [\gamma^{\min}_p, \gamma^{\max}_p]$ occurs. We again, inside the end box Z^{end}_s at stage s, define corner blocks $Z_t(s,0)$ and $Z_t(s,1)$ as non-injuring stage, whereas the construction of $Z_t(s,i)$ for $i \geq 2$ differs from non-injuring stage. The end box of our construction at stage $s+1$ will turn back along all blocks belonging $\mathcal{Z}_s(u)$ for $p < u \leq s$ in the reverse ordering of $\prec$. Let $\{Z_i : i < k_{s+1}\}$ be an enumeration of all blocks in $\mathcal{Z}_s(u)$ for $p < u \leq s$, under the reverse ordering of $\prec$. In other words, Z_i is the successor block of Z_{i+1} under $\dashrightarrow$, for each $i < k_{s+1} - 1$. There are two kind of blocks; one is *a straight block*, and another is *a corner block*. We will define blocks $Z_t(s,i,j)$ for $i < k_{s+1}$ and $j < 3$. Now we check the direction $\langle \delta_i, \varepsilon_i \rangle$ of Z_i. Here, we may consistently assume that the condition $Z_0 \overset{[\leftarrow]}{\dashrightarrow}$ holds.

Subcase 1. If $\delta_i(0) = \varepsilon_i(0)$ then Z_i is a straight block. In this case, we only construct $Z_t(s,i,0)$. Since Z_i is straight, there are $y_i, z_i, \alpha, \beta \in \mathbb{Q}$ and $u \leq s$ such that, for $B_i(0) = [\alpha, \beta]$ and $B_i(1) = [y_i + z_i l_u^-, y_i + z_i r_u^+]$ such that

$\mathrm{Box}(Z_i) = B_i(\delta_i(0)) \times B_i(1-\delta_i(0))$. If $\delta_i(1) = 0$, then set $y_i^\star = N_{0,s}(y_i, z_i; l_s^-, l_s^*)$ and $z_i^\star = N_{1,s}(y_i, z_i; l_s^-, l_s^*)$. If $\delta_i(1) = 1$, then set $y_i^\star = N_{0,s}(y_i, z_i; r_s^-, r_s^+)$ and $z_i^\star = N_{1,s}(y_i, z_i; r_s^*, r_s^+)$. Then, we define $Z_t(s, i, 0)$ as the following straight block:

$$B_i^\star(0) = B_i(0); \quad B_i^\star(1) = [y_i^\star + z_i^\star l_s^-, y_i^\star + z_i^\star r_s^+];$$
$$Z_t(s, i, 0) = [\delta_i(0)]_t^s (B_i^\star(\delta_i(0)) \times B_i^\star(1 - \delta_i(0))).$$

Here, $\mathrm{Box}(Z_t(s, i, 0))$ is defined by $B_i^\star(\delta_i(0)) \times B_i^\star(1 - \delta_i(0))$.

Sublemma 12 $Z_t(s, i, 0) \subseteq Z_i$.

Proof. By our definition of $N_{0,s}$ and $N_{1,s}$, we have $B_i^\star(1) = [y_i + z_i l_s^-, y_i + z_i l_s^*]$ or $B_i^\star(1) = [y_i + z_i r_s^*, y_i + z_i r_s^+]$. $\square$

Subcase 2. If $\delta_i(0) \neq \delta_i(2)$ then Z_i is a corner block. We will construct three blocks; one corner block $Z_t(s, i, 0)$, and two straight blocks $Z_t(s, i, 1)$ and $Z_t(s, i, 2)$. We may assume that Z_i is of the following form:

$$Z_i = [e]_s^u([x_i + \zeta_i l_u^-, x_i + \zeta_i r_u^+] \times [y_i + z_i l_u^-, y_i + z_i r_u^+]),$$
$$\text{or } Z_i = [e]_s^u([x_i + \zeta_i l_u^-, x_i + \zeta_i r_u^+] \times [y_i + z_i l_u^-, y_i + z_i r_u^+])$$
$$\cap\, ([x_i + \zeta_i l_u^-, x_i + \zeta_i r_u^+] \times [y_i + z_i l_u^-, y_i + z_i r_u^*])$$

Set $z = 0$ if the former case occurs; otherwise, set $z = 1$. Let $\{p_n : n < 6\}$ be an enumeration of $\{l_u^-, l_s^-, l_s^*, r_s^*, r_s^+, r_u^+\}$ in increasing order, and let p_6 be r_u^*. First we compute the value $\texttt{rot} = 2|\varepsilon_i(0) - |\delta_i(1) - \varepsilon_i(1)|| + 1$. Note that $\texttt{rot} \in \{1, 3\}$, and, if Z_i rotates clockwise then $\texttt{rot} = 1$; and if Z_i rotates counterclockwise then $\texttt{rot} = 3$. If $\overset{[\rightarrow]}{\dashrightarrow} Z_i$ or $Z_i \overset{[\rightarrow]}{\dashrightarrow}$, then put $D(0) = 1$; otherwise put $D(0) = 3$. If $\overset{[\downarrow]}{\dashrightarrow} Z_i$ or $Z_i \overset{[\downarrow]}{\dashrightarrow}$, then put $D(1) = 1$; otherwise put $D(1) = 3$. If $\overset{[\rightarrow]}{\dashrightarrow} Z_i$ or $Z_i \overset{[\leftarrow]}{\dashrightarrow}$, then put $E(0) = 0$; otherwise put $E(0) = 5 - \mathrm{rot}$. If $\overset{[\uparrow]}{\dashrightarrow} Z_i$ or $Z_i \overset{[\downarrow]}{\dashrightarrow}$, then put $E(1) = 0$; otherwise put $E(1) = 5 - \mathrm{rot}$. Then we now define $Z_t(s, i, j)$ for $j < 3$ as follows:

$$\mathrm{Box}(s, i, 0) = [x_i + \zeta_i p_{D(0)}, x_i + \zeta_i p_{D(0)+2}] \times [y_i + z_i p_{D(1)}, y_i + z_i p_{D(1)+2}],$$
$$\mathrm{Box}(s, i, 1) = [x_i + \zeta_i p_{E(0)}, x_i + \zeta_i p_{E(0)+\texttt{rot}}] \times [y_i + z_i p_{D(1)}, y_i + z_i p_{D(1)+2}],$$
$$\mathrm{Box}(s, i, 2) = [x_i + \zeta_i p_{D(0)}, x_i + \zeta_i p_{D(0)+2}] \times [y_i + z_i p_{E(1)}, y_i + z_i p_{E(1)+\texttt{rot}+z}],$$
$$Z_t(s, i, 0) = [e]_t^s(\mathrm{Box}(s, i, 0)),$$
$$Z_t(s, i, 1) = [-]_t^s(\mathrm{Box}(s, i, 1)),$$
$$Z_t(s, i, 2) = [\,|\,]_t^s(\mathrm{Box}(s, i, 2)).$$

Intuitively, $D(0) = 1$ (resp. $D(0) = 3$) indicates that $Z_t(s, i, 0)$ passes the west (resp. the east) of Z_i; $D(1) = 1$ (resp. $D(1) = 3$) indicates that $Z_t(s, i, 0)$ passes the south (resp. the north) of Z_i; $E(0) = 0$ (resp. $E(0) = 5 - \texttt{rot}$) indicates that $Z_t(s, i, 1)$ passes the west (resp. the east) border of the bounding box of Z_i; and $E(1) = 0$ (resp. $E(1) = 5 - \texttt{rot}$) indicates that $Z_t(s, i, 2)$ passes the south (resp. the north) border of the bounding box of Z_i. Note that the corner block $Z_t(s, i, 0)$ leaves Z_i on his right, and $Z_t(s, i, 0)$ has the reverse direction to Z_i.

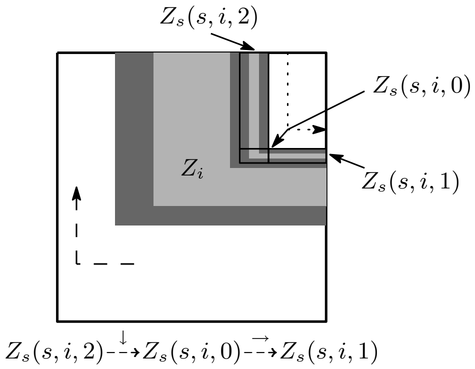

Fig. 20. rot = 1.

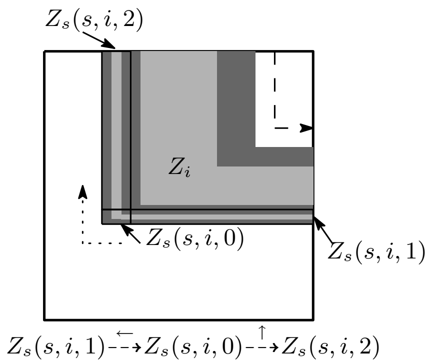

Fig. 21. rot = 3.

Sublemma 13 $Z_t(s,i,2-\delta_i(0)) \overset{\varepsilon^\circ}{\dashrightarrow} Z_t(s,i,0) \overset{\delta^\circ}{\dashrightarrow} Z_t(s,t,1+\delta_i(0))$. □

Sublemma 14 $Z_t(s,i,j) \subseteq Z_i$. □

For each $i < k_{s+1}$, we have already constructed $\mathcal{Z}_t(s+1;i) = \{Z_t(s,i,j) : j < 3\}$. All of these blocks constructed at the current stage are included in $Z_s^{\mathrm{end}} \cup \bigcup\bigcup_{p<u\leq s} \mathcal{Z}_s(u)$. Let $Z^0[i]$ (resp. $Z^1[i]$) be the $\prec$-least (resp. the $\prec$-greatest) element of $\{\lambda t. Z_t(s,i,j) : j < 3\}$. It is not hard to see that our construction ensures the following condition.

Sublemma 15 $Z_t^1[i] \dashrightarrow Z_t^0[i+1]$. □

Thus, $\bigcup_{i<k_{s+1}} \mathcal{Z}_t(s+1;i)$ is computably homeomorphic to $P_t \times [0,1]$, uniformly in $t \geq s+1$. Therefore, we can connect blocks $Z_s(s,i)$ for $i < k_{s+1}$, and we succeed to return back on the current approximation of the $\prec$-greatest p-block $Z_s(p) = Z_{p,s}^{\mathrm{st}} \in \mathcal{Z}_s(p)$. Then we construct blocks $Z_t(s,k)$ for $2 \leq k \leq 6$ on the block $Z_s(p)$. The construction is essentially similar as the non-injuring case. By induction hypothesis (IH3), we note that $Z_s(p)$ must be of the following form for some $y_p, z_p \in \mathbb{Q}$:

$$Z_s(p) = [-]_s^p([\gamma_p^{\min}, \gamma_p^{\max}] \times [y_p + z_p l_p^-, y_p + z_p r_p^+]).$$

On $Z_s(p)$, we define *a straight block from* $\gamma_p^{\min}$ *to* $\gamma_{s+1}^{\max}$ as follows:

$$Z_t(s,2) = [-]_s^p([\gamma_p^{\min}, \gamma_{s+1}^{\max}] \times [y_p + z_p r_s^*, y_p + z_p r_s^+]).$$

Here, by our assumption, $\gamma_{s+1}^{\max} < \gamma_p^{\max}$ holds since $\gamma_{s+1}^{\max} \leq \gamma_p^{\max}$. The blocks $Z_t(s,k)$ for $3 \leq k \leq 6$ are defined as in the same method as non-injuring case. The active block at stage $s+1$ is $Z_{s+1}(s,5)$, and the end box at stage $s+1$ is $Z_{s+1}(s,6)$. $(s+1)$-*blocks at stage* t are $Z_t(s,i)$ for $i < 6$, and $Z_t(s,i,j)$ for $i < k_{s+1}$ and $j < 3$ if it is constructed. $\mathcal{Z}_t(s+1)$ denotes *the collection of* $(s+1)$*-blocks at stage* t.

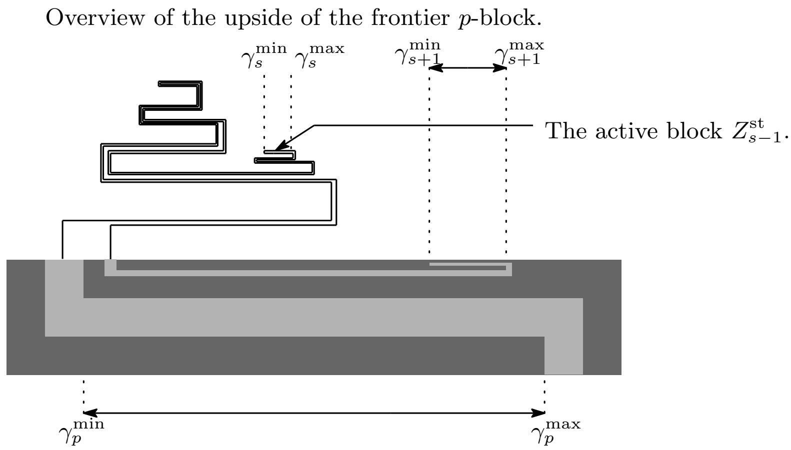

Fig. 22. Outline of our construction of the injured case.

Sublemma 16 $Z_{s+1}^{\mathrm{end}} \cup \bigcup \mathcal{Z}_{s+1}(s+1) \subseteq Z_s^{\mathrm{end}} \cup \bigcup\bigcup_{p \le u \le s} \mathcal{Z}_s(u)$. □

Thus we again have the following:

$$Q_{s+1} = Z_{s+1}^{\mathrm{end}} \cup \bigcup \bigcup_{u \le s+1} \mathcal{Z}_{s+1}(u) \subseteq Z_s^{\mathrm{st}} \cup Z_s^{\mathrm{end}} \cup \bigcup \bigcup_{u \le s} \mathcal{Z}_s(u) \subseteq Q_s.$$

Sublemma 17 *Assume that we have a computable function* $f_s : \mathbb{R}^2 \to \mathbb{R}^2$ *such that* $f_s \restriction \bigcup\bigcup_{u \le s} \mathcal{Z}_t(u)$ *is a computable homeomorphism between* $\bigcup\bigcup_{u \le s} \mathcal{Z}_t(u)$ *and* $P_t \times [1/(s+2), 1]$ *for any* $t \ge s$*. Then we can effectively find a computable function* $f_{s+1} : \mathbb{R}^2 \to \mathbb{R}^2$ *extending* $f_s \restriction \bigcup\bigcup_{u \le s} \mathcal{Z}_{s+1}(u)$ *such that* $f_{s+1} \restriction \bigcup\bigcup_{u \le s+1} \mathcal{Z}_t(u)$ *is a computable homeomorphism between* $\bigcup\bigcup_{u \le s+1} \mathcal{Z}_t(u)$ *and* $P_t \times [1/(s+3), 1]$ *for any* $t \ge s+1$*.* □

Finally we put $Q = \bigcap_{s \in \mathbb{N}} Q_s$ and $\mathcal{Z}^* = \bigcup_{u \in \mathbb{N}} \mathcal{Z}(u)$. The construction is completed.

Verification. Now we start to verify our construction.

Lemma 7. *Q is Π_1^0.* □

Sublemma 18 $\bigcap_{t \in \mathbb{N}} \bigcup_{Z \in \mathcal{Z}^*} Z_t = \bigcup_{Z \in \mathcal{Z}^*} \bigcap_{t \in \mathbb{N}} Z_t$. □

Sublemma 19 $\bigcup_{Z \in \mathcal{Z}(u)} \bigcap_{t \in \mathbb{N}} Z_t$ *is computably homeomorphic to* $[0,1] \times P$*, for each* $u \in \mathbb{N}$*.*

Proof. By the induction hypothesis (IH2). □

Sublemma 20 $\bigcup_{Z \in \mathcal{Z}^*} \bigcap_{t \in \mathbb{N}} Z_t$ *is homeomorphic to* $(0,1] \times P$*.*

Proof. By Sublemma 11 and 17. □

Lemma 8. *Q is homeomorphic to a Cantor fan.*

Proof. By Sublemma 18, there exists a real $y_0 \in \mathbb{R}$ such that the following holds:

$$Q = \left(\bigcup_{Z \in \mathcal{Z}^*} \bigcap_{t \in \mathbb{N}} Z_t \right) \cup \{\langle \gamma, y_0 \rangle\}.$$

Therefore, by Sublemma 20, Q is homeomorphic to the one-point compactification of $(0,1] \times P$. □

Lemma 9. *Q contains no computable point.*

Proof. By Sublemma 19, $\bigcup_{Z \in \mathcal{Z}^*} \bigcap_{t \in \mathbb{N}} Z_t$ contains no computable point. □

By Lemmata 7, 8, and 9, Q is the desired dendroid. □

Remark 2. Since dendroids are compact and simply connected, Theorem 4 is the solution to the question of Le Roux and Ziegler [13]. Indeed, the dendroid constructed in the proof of Theorem 4 is contractible.

Corollary 4. *Not every nonempty contractible Π^0_1 subset of $[0,1]^2$ contains a computable point.*

Table 1 expresses that which topological property determines which computability property. Here the symbols (C), (C$^-$), (B) denote the following questions:

(C) *Computability*: Whenever it is Π^0_1, is it always computable?
(C$^-$) *Almost Computability*: Whenever it is Π^0_1, is it always almost computable?
(B) *Basis Theorem*: Whenever it is Π^0_1, does it always contain a computable point?

Table 1. Does Topology determine Computability?

Topology	(C)	(C$^-$)	(B)
Sphere	YES	YES	YES
Ball	NO	?	YES
Jordan curve	YES	YES	YES
Simple curve	NO	YES	YES
Dendrite	NO	NO	?
Dendroid	NO	NO	NO

Question 1. Does every locally connected planar Π^0_1 set contain a computable point?

5 Immediate Consequences

5.1 Effective Hausdorff Dimension

For basic definition and properties of the the effective Hausdorff dimension of a point of Euclidean plane, see Lutz-Weihrauch [9]. For any $I \subseteq [0,2]$, let DIM^I denote the set of all points in $\mathbb{R}^2$ whose effective Hausdorff dimensions lie in I. Lutz-Weihrauch [9] showed that $\mathrm{DIM}^{[1,2]}$ is path-connected, but $\mathrm{DIM}^{(1,2]}$ is totally disconnected. In particular, $\mathrm{DIM}^{(1,2]}$ has no nondegenerate connected subset. It is easy to see that $\mathrm{DIM}^{(0,2]}$ has no nonempty Π^0_1 simple curve, since every Π^0_1 simple curve contains a computable point, and the effective Hausdorff dimension of each computable point is zero.

Theorem 5. $\mathrm{DIM}^{[1,2]}$ *has a nondegenerate contractible* Π^0_1 *subset.*

Proof. For any strictly increasing computable function $f : \mathbb{N} \to \mathbb{N}$ with $f(0) = 0$ and any infinite binary sequence $\alpha \in 2^{<\mathbb{N}}$, we define a function $\iota_f : 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ as follows:

$$\iota_f(\alpha)(n) = \begin{cases} \alpha(i), & \text{if } n = f(i) + i \\ \alpha(n-i-1) & \text{if } f(i) + i < n < f(i+1) + i + 1 \end{cases}$$

Intuitively, the function ι_f *inserts* the extra bit $\alpha(i)$ between $\alpha(f(i)-1)$ and $\alpha(f(i))$. For each $n \in \mathbb{N}$, put $f^{-1}(n) = \min\{s \in \mathbb{N} : f(s) \geq n\}$. By removing extra bits, we can compute the value $\alpha \upharpoonright n$ from $\iota_f(\alpha) \upharpoonright n + f^{-1}(n)$. Then, $r : 2^{\mathbb{N}} \to \mathbb{R}$ is defined as $r(\alpha) = \sum_{i \in \mathbb{N}} (\alpha(i) \cdot 2^{-(i+1)})$.

Claim. $r \circ \iota_f : 2^{\mathbb{N}} \to \mathbb{R}$ is injective.

Note that $\alpha \neq \beta$ and $r(\alpha) = r(\beta)$ hold if and only if there is a common initial segment $\sigma \in 2^{<\mathbb{N}}$ of α and β such that $\sigma 0$ and $\sigma 1$ are initial segments of α and β respectively, and that $\alpha(m) = 1$ and $\beta(m) = 0$ for any $m > lh(\sigma)$, where $lh(\sigma)$ denotes the length of σ. In this case, we say that α *sticks to* β *on* σ. If $r(\alpha) \neq r(\beta)$, then clearly $r \circ \iota_f(\alpha) \neq r \circ \iota_f(\beta)$. Assume that α sticks to β on σ. Then there are $m_0 < m_1$ such that $\iota_f(\alpha)(m_0) = \iota_f(\alpha)(m_1) = \alpha(lh(\sigma)) = 0$ and $\iota_f(\beta)(m_0) = \iota_f(\beta)(m_1) = \beta(lh(\sigma)) = 1$ by our definition of ι_f. Therefore, $\iota_f(\alpha)$ does not stick to $\iota_f(\beta)$. Hence, $r \circ \iota_f(\alpha) \neq r \circ \iota_f(\beta)$ whenever $\alpha \neq \beta$. Actually, $r \circ \iota_f : 2^{\mathbb{N}} \to \mathbb{R}$ is a computable embedding.

Then, there is a constant $c \in \mathbb{N}$ such that, for any $\alpha \in 2^{\mathbb{N}}$ and $n \in \mathbb{N}$, we have $K(\iota_f(\alpha) \upharpoonright n + f^{-1}(n)) \geq K(\alpha \upharpoonright n) - c$, where K denotes the prefix-free Kolmogorov complexity. Therefore, for any sufficiently fast-growing function $f : \mathbb{N} \to \mathbb{N}$ and any Martin-Löf random sequence $\alpha \in 2^{\mathbb{N}}$, we have the following for some constant $d \in \mathbb{N}$:

$$\frac{K(\iota_f(\alpha) \upharpoonright n + f^{-1}(n))}{n + f^{-1}(n)} \geq \frac{K(\alpha \upharpoonright n) - c}{n + f^{-1}(n)} \geq \frac{n - d}{n + f^{-1}(n)} \longrightarrow 1.$$

Hence, the effective Hausdorff dimension of $r \circ \iota_f(\alpha)$ must be 1. Thus, for any nonempty Π^0_1 set $R \subseteq 2^{\mathbb{N}}$ consisting of Martin-Löf random sequences, $\{0\} \times (r \circ \iota_f(R))$ is a Π^0_1 subset of $\mathrm{DIM}^{\{1\}}$. Let Q be the dendroid constructed from $P = r \circ \iota_f(R)$ as in the proof of Theorem 4, where we choose $\gamma = \rho(B)$ as Chaitin's halting probability Ω. For every point $\langle x_0, x_1\rangle \in Q$, the effective Hausdorff dimension of x_i for some $i < 2$ is equivalent to that of an element of P or that of Ω. Consequently, $Q \subseteq \mathrm{DIM}^{[1,2]}$. $\square$

5.2 Reverse Mathematics

Theorem 6. *For every Π^0_1 set $P \subseteq 2^{\mathbb{N}}$, there is a contractible planar Π^0_1 set Q such that Q has the same Turing upward closure as P, i.e., $\{y : (\exists x \leq_T y)\ x \in P\} = \{y : (\exists x \leq_T y)\ x \in Q\}$.*

Proof. We choose B as a c.e. set of the same degree with the leftmost path of P. Then, the dendroid Q constructed from P and B as in the proof of Theorem 4 is the desired one. $\square$

A compact Π^0_1 subset P of a computable topological space is *Muchnik complete* if every element of P computes the set of all theorems of T for some consistent complete theory T containing Peano arithmetic. By Scott Basis Theorem (see Simpson [15]), P is Muchnik complete if and only if P is nonempty and every element of P computes an element of any nonempty Π^0_1 set $Q \subseteq 2^{\mathbb{N}}$.

Corollary 5. *There is a Muchnik complete contractible planar Π^0_1 set.*

A compact Π^0_1 subset P of a computable topological space is *Medvedev complete* (see also Simpson [15]) if there is a uniform computable procedure Φ such that, for any name $x \in \mathbb{N}^{\mathbb{N}}$ of an element of P, $\Phi(x)$ is the set of all theorems of T for some consistent complete theory T containing Peano arithmetic.

Question 2. Does there exist a Medvedev complete simply connected planar Π^0_1 set? Does there exist a Medvedev complete contractible Euclidean Π^0_1 set?

Our Theorem 4 also provides a reverse mathematical consequence. For basic notation for Reverse Mathematics, see Simpson [14]. Let RCA_0 denote the subsystem of second order arithmetic consisting of $I\Sigma^0_1$ (Robinson arithmetic with induction for Σ^0_1 formulas) and Δ^0_1-CA (comprehension for Δ^0_1 formulas). Over RCA_0, we say that a sequence $(B_i)_{i \in \mathbb{N}}$ of open rational balls is *flat* if there is a homeomorphism between $\bigcup_{i<n} B_i$ and the open square $(0,1)^2$ for any $n \in \mathbb{N}$. It is easy to see that RCA_0 proves that every flat cover of $[0,1]$ has a finite subcover.

Theorem 7. *The following are equivalent over RCA_0.*

1. *Weak König's Lemma: every infinite binary tree has an infinite path.*
2. *Every open cover of $[0,1]$ has a finite subcover.*
3. *Every flat open cover of $[0,1]^2$ has a finite subcover.*

Proof. The equivalence of the item (1) and (2) is well-known. It is not hard to see that RCA_0 proves the existence of the sequence $\{Q_s\}_{s\in\mathbb{N}}$ as in our construction of the dendroid Q in Theorem 4, by formalizing our proof in Theorem 4 in RCA_0. Here we may assume that $\{Q_s\}_{s\in\mathbb{N}}$ is constructed from the set of all infinite paths of a given infinite binary tree $T \subseteq 2^{<\mathbb{N}}$, and a c.e. complete set $B \subseteq \mathbb{N}$. Note that $\bigcup_{s<t}([0,1]^2 \setminus Q_s)$ does not cover $[0,1]^2$ for every $t \in \mathbb{N}$. Over RCA_0, there is a flat sequence $\{[0,1]^2 \setminus Q_s^*\}_{s\in\mathbb{N}}$ of open rational balls such that $\bigcap_{s<t} Q_s^* \supseteq \bigcap_{s<t} Q_s$ for any $t \in \mathbb{N}$, and that an open rational ball U is removed from some Q_s^* if and only if an open rational ball U is removed from some Q_u. However, if T has no infinite path, then Q has no element. In other words, $\{[0,1]^2 \setminus Q_s^*\}_{s\in\mathbb{N}}$ covers $[0,1]^2$. $\square$

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