Inside the Muchnik Degrees II: The Degree Structures induced by the Arithmetical Hierarchy of Countably Continuous Functions

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Abstract

It is known that infinitely many Medvedev degrees exist inside the Muchnik degree of any nontrivial Π^0_1 subset of Cantor space. We shed light on the fine structures inside these Muchnik degrees related to learnability and piecewise computability. As for nonempty Π^0_1 subsets of Cantor space, we show the existence of a finite-Δ^0_2-piecewise degree containing infinitely many finite-$(\Pi^0_1)_2$-piecewise degrees, and a finite-$(\Pi^0_2)_2$-piecewise degree containing infinitely many finite-Δ^0_2-piecewise degrees (where $(\Pi^0_n)_2$ denotes the difference of two Π^0_n sets), whereas the greatest degrees in these three "finite-Γ-piecewise" degree structures coincide. Moreover, as for nonempty Π^0_1 subsets of Cantor space, we also show that every nonzero finite-$(\Pi^0_1)_2$-piecewise degree includes infinitely many Medvedev (i.e., one-piecewise) degrees, every nonzero countable-Δ^0_2-piecewise degree includes infinitely many finite-piecewise degrees, every nonzero finite-$(\Pi^0_2)_2$-countable-Δ^0_2-piecewise degree includes infinitely many countable-Δ^0_2-piecewise degrees, and every nonzero Muchnik (i.e., countable-Π^0_2-piecewise) degree includes infinitely many finite-$(\Pi^0_2)_2$-countable-Δ^0_2-piecewise degrees. Indeed, we show that any nonzero Medvedev degree and nonzero countable-Δ^0_2-piecewise degree of a nonempty Π^0_1 subset of Cantor space have the strong anticupping properties. Finally, we obtain an elementary difference between the Medvedev (Muchnik) degree structure and the finite-Γ-piecewise degree structure of all subsets of Baire space by showing that none of the finite-Γ-piecewise structures are Brouwerian, where Γ is any of the Wadge classes mentioned above.

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1. Summary

1.1. Introduction

This paper is a continuation of Higuchi-Kihara [29]. Our objective in this paper is to investigate the degree structures induced by intermediate notions between the Medvedev reduction (uniformly computable function) and Muchnik reduction (nonuniformly computable function). We will shed light on a hidden, but extremely deep, structure inside the Muchnik degree of each Π^0_1 subset of Cantor space.

In 1963, Albert Muchnik [46] introduced the notion of Muchnik reduction as a partial function on Baire space that is decomposable into countably many computable functions. Such a reduction is also called a *countably computable* function, *σ-computable* function, or *nonuniformly computable* function. The notion of Muchnik reduction has been a powerful tool for clarifying the noncomputability structure of the Π^0_1 subsets of Cantor space [57–59, 61]. Muchnik reductions have been classified in Part I [29] by introducing the notion of piecewise computability.

Remarkably, many descriptive set theorists have recently focused their attention on the concept of *piecewise definability* of functions on Polish spaces, in association with the Baire hierarchy of Borel measurable functions (see [43, 44, 55]). Roughly speaking, if $\mathbf{\Gamma}$ is a pointclass (in the Borel hierarchy) and $\mathbf{\Lambda}$ is a class of functions (in the Baire hierarchy), a function is said to be $\mathbf{\Gamma}$-piecewise $\mathbf{\Lambda}$ if it is decomposable into countably many $\mathbf{\Lambda}$-functions with $\mathbf{\Gamma}$ domains. If $\mathbf{\Gamma}$ is the class of all closed sets and $\mathbf{\Lambda}$ is the class of all continuous functions, it is simply called *piecewise continuous* (see for instance [32, 36, 45, 50]). The notion of piecewise continuity is known to be equivalent to the $\mathbf{\Delta}^0_2$-measurability [32]. If $\mathbf{\Gamma}$ is the class of all sets and $\mathbf{\Lambda}$ is the class of all continuous functions, it is also called *countably continuous* [44] or *σ-continuous* [54]. Nikolai Luzin was the first to investigate the notion of countable-continuity, and today, many researchers have studied this concept, in particular, with an important dichotomy theorem (see [51, 64]).

Our concepts introduced in Part I [29], such as *Δ^0_2-piecewise computability*, are indeed the lightface versions of piecewise definability. This notion is also known to be equivalent to the effective Δ^0_2-measurability [50]. See also [5, 19, 38] for more information on effective Borel measurability.

To gain a deeper understanding of piecewise definability, we investigate the Medvedev- and Muchnik-like degree structures induced by piecewise computable notions. This also helps us to understand the notion of relative learnability since we have observed a close relationship between lightface piecewise definability and algorithmic learning in Part I [29].

In Part II, we restrict our attention to the local substructures consisting of the degrees of all Π^0_1 subsets of Cantor space. This indicates that we consider the relative piecewise computably (or learnably) solvability of *computably-refutable problems*. When a scientist attempts to verify a statement P, his verification will be algorithmically refuted whenever it is incorrect. This *falsifiability principle* holds only when P is represented as a Π^0_1 subset of a space. Therefore, the restriction to the Π^0_1 sets can be regarded as an analogy of *Popperian learning* [11] because of the falsifiability principle.

From this perspective, the universe of the Π^0_1 sets is expected to be a good playground of Learning Theory [31].

The restriction to the Π^0_1 subsets of Cantor space $2^{\mathbb{N}}$ is also motivated by several other arguments. First, many mathematical problems can be represented as Π^0_1 subsets of certain topological spaces (see Cenzer and Remmel [15]). The Π^0_1 sets in such spaces have become important notions in many branches of Computability Theory, such as *Recursive Mathematics* [23], *Reverse Mathematics* [60], *Computable Analysis* [65], *Effective Randomness* [21, 48], and *Effective Descriptive Set Theory* [42]. For these reasons, degree structures on Π^0_1 subsets of Cantor space $2^{\mathbb{N}}$ are widely studied from the viewpoint of *Computability Theory* and *Reverse Mathematics*.

In particular, many theorems have been proposed on the algebraic structure of the Medvedev degrees of Π^0_1 subsets of Cantor space, such as density [13], embeddability of distributive lattices [3], join-reducibility [2], meet-irreducibility [1], noncuppability [12], non-Brouwerian property [28], decidability [16], and undecidability [56] (see also [30, 57–59, 61] for other properties on the Medvedev and Muchnik degree structures). The Π^0_1 sets have also been a key notion (under the name of *closed choice*) in the study of the structure of the Weihrauch degrees, which is an extension of the Medvedev degrees (see [6–8]).

Among other results, Cenzer and Hinman [13] showed that the Medvedev degrees of the Π^0_1 subsets of Cantor space are dense, and Simpson [57] questioned whether the Muchnik degrees of Π^0_1 subsets of Cantor space are also dense. However, this question remains unanswered. We have limited knowledge of the Muchnik degree structure of the Π^0_1 sets because the Muchnik reductions are very difficult to control. What we know is that as shown by Simpson-Slaman [62] and Cole-Simpson [17], there are infinitely many Medvedev degrees in the Muchnik degree of any nontrivial Π^0_1 subsets of Cantor space. Now, it is necessary to clarify the internal structure of the Muchnik degrees. In Part II, we apply the disjunction operations introduced in Part I [29] to understand the inner structures of the Muchnik degrees induced by various notions of piecewise computability.

1.2. Results

In Part I [29], the notions of piecewise computability and the induced degree structures are introduced. Our objective in Part II is to study the interaction among the structures $\mathcal{P}/\mathcal{F}$ of $\mathcal{F}$-degrees of nonempty Π^0_1 subsets of Cantor space for notions $\mathcal{F}$ of piecewise computability listed as follows.

- $\mathrm{dec}^{<\omega}_{\mathrm{p}}[\Pi^0_1]$ also denotes the set of all partial functions on $\mathbb{N}^{\mathbb{N}}$ that are decomposable into finitely many partial computable functions with Π^0_1 domains.
- $\mathrm{dec}^{<\omega}_{\mathrm{d}}[\Pi^0_1]$ denotes the set of all partial functions on $\mathbb{N}^{\mathbb{N}}$ that are decomposable into finitely many partial computable functions with $(\Pi^0_1)_2$ domains, where a $(\Pi^0_1)_2$ set is the difference of two Π^0_1 sets.
- $\mathrm{dec}^{<\omega}_{\mathrm{p}}[\Delta^0_2]$ denotes the set of all partial functions on $\mathbb{N}^{\mathbb{N}}$ that are decomposable into finitely many partial computable functions with Δ^0_2 domains.

- $\mathrm{dec}_{\mathrm{p}}^{\omega}[\Delta^0_2]$ denotes the set of all partial functions on $\mathbb{N}^{\mathbb{N}}$ that are decomposable into countably many partial computable functions with Δ^0_2 domains.
- $\mathrm{dec}_{\mathrm{d}}^{<\omega}[\Pi^0_2]$ denotes the set of all partial functions on $\mathbb{N}^{\mathbb{N}}$ that are decomposable into finitely many partial computable functions with $(\Pi^0_2)_2$ domains.
- $\mathrm{dec}_{\mathrm{d}}^{<\omega}[\Pi^0_2]\mathrm{dec}_{\mathrm{p}}^{\omega}[\Delta^0_2]$ denotes the set of all partial functions on $\mathbb{N}^{\mathbb{N}}$ that are decomposable into finitely many partial Δ^0_2-piecewise computable functions with $(\Pi^0_2)_2$ domains, where a $(\Pi^0_2)_2$ set is the difference of two Π^0_2 sets.
- $\mathrm{dec}_{\mathrm{p}}^{\omega}[\Pi^0_2]$ denotes the set of all partial functions on $\mathbb{N}^{\mathbb{N}}$ that are decomposable into countably many partial computable functions with Π^0_2 domains.

The relationship among these notions is summarized as follows.

$$\mathcal{P}/\mathrm{dec}_{\mathrm{p}}^{<\omega}[\Pi^0_1] \;-\; \mathcal{P}/\mathrm{dec}_{\mathrm{d}}^{<\omega}[\Pi^0_1] \;-\; \mathcal{P}/\mathrm{dec}_{\mathrm{p}}^{<\omega}[\Delta^0_2] \begin{array}{c} \nearrow \; \mathcal{P}/\mathrm{dec}_{\mathrm{d}}^{<\omega}[\Pi^0_2] \; \searrow \\ \searrow \; \mathcal{P}/\mathrm{dec}_{\mathrm{p}}^{\omega}[\Delta^0_2] \; \nearrow \end{array} \mathcal{P}/\mathrm{dec}_{\mathrm{d}}^{<\omega}[\Pi^0_2]\mathrm{dec}_{\mathrm{p}}^{\omega}[\Delta^0_2] \;-\; \mathcal{P}/\mathrm{dec}_{\mathrm{p}}^{\omega}[\Pi^0_2]$$

In Part I [29], we observed that these degree structure are exactly those induced by the $(\alpha,\beta|\gamma)$-computability.

- $[\mathfrak{C}_T]^1_1$ denotes the set of all partial computable functions on $\mathbb{N}^{\mathbb{N}}$.
- $[\mathfrak{C}_T]^1_{<\omega}$ denotes the set of all partial functions on $\mathbb{N}^{\mathbb{N}}$ learnable with bounded mind changes.
- $[\mathfrak{C}_T]^1_{\omega|<\omega}$ denotes the set of all partial functions on $\mathbb{N}^{\mathbb{N}}$ learnable with bounded errors.
- $[\mathfrak{C}_T]^1_{\omega}$ denotes the set of all partial learnable functions on $\mathbb{N}^{\mathbb{N}}$.
- $[\mathfrak{C}_T]^{<\omega}_1$ denotes the set of all partial k-wise computable functions on $\mathbb{N}^{\mathbb{N}}$ for some $k \in \mathbb{N}$.
- $[\mathfrak{C}_T]^{<\omega}_{\omega}$ denotes the set of all partial functions on $\mathbb{N}^{\mathbb{N}}$ learnable by a team.
- $[\mathfrak{C}_T]^{\omega}_1$ denotes the set of all partial nonuniformly computable functions on $\mathbb{N}^{\mathbb{N}}$ (i.e., all functions f satisfying $f(x) \leq_T x$ for any $x \in \mathrm{dom}(f)$).

As in Part I [29], each degree structure $\mathcal{P}/[\mathfrak{C}_T]^{\alpha}_{\beta|\gamma}$ is abbreviated as $\mathcal{P}^{\alpha}_{\beta|\gamma}$. Then, we have the following relationship among these notions.

$$\mathcal{P}^1_1 \;-\; \mathcal{P}^1_{<\omega} \;-\; \mathcal{P}^1_{\omega|<\omega} \begin{array}{c} \nearrow \; \mathcal{P}^{<\omega}_1 \; \searrow \\ \searrow \; \mathcal{P}^1_{\omega} \; \nearrow \end{array} \mathcal{P}^{<\omega}_{\omega} \;-\; \mathcal{P}^{\omega}_1$$

We will see that all of the above inclusions are proper. Beyond the properness of these inclusions, there are four LEVELs signifying the differences between two classes $\mathfrak{F}$ and $\mathfrak{G}$ of partial functions on $\mathbb{N}^{\mathbb{N}}$ (lying between $[\mathfrak{C}_T]^1_1$ and $[\mathfrak{C}_T]^{\omega}_1$) listed as follows.

1. There is a function $\Gamma \in \mathfrak{F} \setminus \mathfrak{G}$.
2. There are sets $X, Y \subseteq \mathbb{N}^{\mathbb{N}}$ such that $\mathfrak{F}$ has a function $\Gamma_{\mathfrak{F}} : X \to Y$, but $\mathfrak{G}$ has *no* function $\Gamma_{\mathfrak{G}} : X \to Y$.
3. There are Π^0_1 sets $X, Y \subseteq 2^{\mathbb{N}}$ such that $\mathfrak{F}$ has a function $\Gamma_{\mathfrak{F}} : X \to Y$, but $\Gamma_{\mathfrak{G}}$ has *no* function $\Gamma_{\mathfrak{G}} : X \to Y$.
4. For every special Π^0_1 set $Y \subseteq 2^{\mathbb{N}}$, there is a Π^0_1 set $X \subseteq 2^{\mathbb{N}}$ such that $\mathfrak{F}$ has a function $\Gamma_{\mathfrak{F}} : X \to Y$, but $\mathfrak{G}$ has *no* function $\Gamma_{\mathfrak{G}} : X \to Y$.

The LEVEL 1 separation just represents $\mathfrak{F} \not\subseteq \mathfrak{G}$. Clearly, $4 \to 3 \to 2 \to 1$. Note that the LEVEL 2 separation holds for *no* Σ^0_1 sets $X, Y \subseteq \mathbb{N}^{\mathbb{N}}$, since Π^0_1 is the first level in the arithmetical hierarchy which can define a nonempty set $S \subseteq \mathbb{N}^{\mathbb{N}}$ without computable element. Such a Π^0_1 set is called *special*, i.e., a subset of Baire space is special if it is nonempty and contains no computable points. As mentioned before, Simpson-Slaman [62] (see Cole-Simpson [17]) showed that the LEVEL 4 separation holds between $[\mathfrak{C}_T]^1_1$ and $[\mathfrak{C}_T]^\omega_1$, that is, every nonzero Muchnik degree $\mathbf{a} \in \mathcal{P}^\omega_1$ contains infinitely many Medvedev degrees $\mathbf{b} \in \mathcal{P}^1_1$.

In section 2, we use the consistent two-tape disjunction operations on Π^0_1 subsets of Cantor space introduced in Part I [29] to obtain LEVEL 3 separation results.

- $\triangledown_n$ is the disjunction operation on Π^0_1 sets induced by the two-tape Brouwer-Heyting-Kolmogorov-interpretation with mind-changes $< n$.
- $\triangledown_\omega$ is the disjunction operation on Π^0_1 sets induced by the two-tape Brouwer-Heyting-Kolmogorov-interpretation with finitely many mind-changes.
- $\triangledown_\infty$ is the disjunction operation on Π^0_1 sets induced by the two-tape Brouwer-Heyting-Kolmogorov-interpretation permitting unbounded mind-changes.

By using these operations, we obtain the LEVEL 3 separation results for $[\mathfrak{C}_T]^1_1$, $[\mathfrak{C}_T]^1_{<\omega}$, $[\mathfrak{C}_T]^1_{\omega|<\omega}$, and $[\mathfrak{C}_T]^{<\omega}_1$. We show that there exist Π^0_1 sets $P, Q \subseteq 2^{\mathbb{N}}$ such that all of the following conditions are satisfied.

1. (a) There is *no* computable function $\Gamma^1_1 : P\triangledown_2 Q \to P\triangledown_1 Q$;
   (b) There is a function $\Gamma^1_{<\omega} : P\triangledown_2 Q \to P\triangledown_1 Q$ learnable with bounded mind-changes.
2. (a) There is *no* function $\Gamma^1_{<\omega} : P\triangledown_\omega Q \to P\triangledown_1 Q$ learnable with bounded mind-changes;
   (b) There is a function $\Gamma^1_{\omega|<\omega} : P\triangledown_\omega Q \to P\triangledown_1 Q$ learnable with bounded errors.
3. (a) There is *no* function $\Gamma^1_{\omega|<\omega} : P\triangledown_\infty Q \to P\triangledown_1 Q$ learnable with bounded errors;
   (b) There is a 2-wise computable function $\Gamma^{<\omega}_1 : P\triangledown_\infty Q \to P\triangledown_1 Q$.

The above conditions also suggest how does degrees of difficulty of our disjunction operations behave.

In contrast to the above results, in section 3, we will see that the hierarchy between $[\mathfrak{C}_T]^1_{<\omega}$ and $[\mathfrak{C}_T]^{<\omega}_1$ collapses for *homogeneous* Π^0_1 subsets of Cantor space $2^{\mathbb{N}}$. In other words, the LEVEL 4 separations *fail* for $[\mathfrak{C}_T]^1_{<\omega}$, $[\mathfrak{C}_T]^1_{\omega|<\omega}$, and $[\mathfrak{C}_T]^{<\omega}_1$. For other classes, is the LEVEL 4 separation successful?

To archive the LEVEL 4 separations, we use dynamic disjunction operations developed in Part I [29].

1. The concatenation $P \mapsto P^\frown P$ of two Π^0_1 sets $P \subseteq 2^{\mathbb{N}}$ indicates the mass problem "solve P by a learning proof process with mind-change-bound 2".
2. Every iterated concatenation along a well-founded tree indicates a learning proof process with an ordinal bounded mind changes.
3. The hyperconcatenation $P \mapsto P \blacktriangledown P$ of two Π^0_1 sets $P \subseteq 2^{\mathbb{N}}$ is defined as the iterated concatenation of P along the corresponding ill-founded tree of P.

These operations turn out to be extremely useful to establish the LEVEL 4 separation results. Some of these results will be proved by applying priority argument *inside* some learning proof model of P.

1. The LEVEL 4 separation succeeds for $[\mathfrak{C}_T]^1_1$ and $[\mathfrak{C}_T]^1_{<\omega}$, via the map $P \mapsto P^\frown P$.
2. The LEVEL 4 separation succeeds for $[\mathfrak{C}_T]^{<\omega}_1$ and $[\mathfrak{C}_T]^1_\omega$, via the map

$$P \mapsto \bigcup_{m \in \mathbb{N}} (P^\frown P^\frown \dots (m \text{ times}) \dots {}^\frown P^\frown P).$$

3. The LEVEL 4 separation succeeds for $[\mathfrak{C}_T]^1_\omega$ and $[\mathfrak{C}_T]^{<\omega}_\omega$, via the map $P \mapsto P \blacktriangledown P$.
4. The LEVEL 4 separation succeeds for $[\mathfrak{C}_T]^{<\omega}_\omega$ and $[\mathfrak{C}_T]^\omega_1$, via the map $P \mapsto \widehat{\mathrm{Deg}}(P)$, where $\widehat{\mathrm{Deg}}(P)$ denotes the Turing upward closure of P.

The method that we use to show the first and the third items also implies that any nonzero $\mathbf{a} \in \mathcal{P}^1_1$ and $\mathbf{a} \in \mathcal{P}^1_\omega$ have *the strong anticupping property*, i.e., for every nonzero $\mathbf{a} \in \mathcal{P}$, there is a nonzero $\mathbf{b} \in \mathcal{P}$ below $\mathbf{a}$ such that $\mathbf{a} \leq \mathbf{b} \vee \mathbf{c}$ implies $\mathbf{a} \leq \mathbf{c}$. Indeed, these strong anticupping results are established via concatenation $^\frown$ and hyperconcatenation $\blacktriangledown$.

1. $\mathcal{P}^1_1 \models (\forall \mathbf{a}, \mathbf{c})\ (\mathbf{a} \leq (\mathbf{a}^\frown \mathbf{a}) \vee \mathbf{c} \ \rightarrow\ \mathbf{a} \leq \mathbf{c})$.
2. $\mathcal{P}^1_\omega \models (\forall \mathbf{a}, \mathbf{c})\ (\mathbf{a} \leq (\mathbf{a} \blacktriangledown \mathbf{a}) \vee \mathbf{c} \ \rightarrow\ \mathbf{a} \leq \mathbf{c})$.

In section 5, we apply our results to sharpen Jockusch's theorem [33] and Simpson's Embedding Lemma [58]. Jockusch showed the following nonuniform computability result for DNR_k, the set of all k-valued *diagonally noncomputable functions*.

1. There is *no* (uniformly) computable function $\Gamma^1_1 : \mathsf{DNR}_3 \to \mathsf{DNR}_2$.
2. There is a nonuniformly computable function $\Gamma^\omega_1 : \mathsf{DNR}_3 \to \mathsf{DNR}_2$.

This result will be sharpened by using our learnability notions as follows.

1. There is *no* learnable function $\Gamma^1_\omega : \mathsf{DNR}_3 \to \mathsf{DNR}_2$.
2. There is *no* k-wise computable function $\Gamma^{<\omega}_1 : \mathsf{DNR}_3 \to \mathsf{DNR}_2$ for $k \in \mathbb{N}$.
3. There is a (uniformly) computable function $\Gamma^1_1 : \mathsf{DNR}_3 \to \mathsf{DNR}_2 \blacktriangledown \mathsf{DNR}_2$. Hence, there is a function $\Gamma^{<\omega}_\omega : \mathsf{DNR}_3 \to \mathsf{DNR}_2$ learnable by a team of two learners.

Finally, we employ concatenation and hyperconcatenation operations to show that neither $\mathcal{D}^1_{<\omega}$ nor $\mathcal{D}^{<\omega}_1$ nor $\mathcal{D}^{<\omega}_\omega$ are Brouwerian. Hence, these degree structures are not elementarily equivalent to the Medvedev (Muchnik) degree structure.

For any sets X and Y, for convenience, we say that f *is a function from X to Y* (written $f : X \to Y$) if the domain $\mathrm{dom}(f)$ of f includes X, and the image of X under f is included in Y. We also use the notation $f :\subseteq X \to Y$ to denote that f is a partial function from X to Y, i.e., the image of $\mathrm{dom}(f) \cap X$ under f is included in Y.

For basic terminology in Computability Theory, see Soare [63]. For $\sigma \in \mathbb{N}^{<\mathbb{N}}$, we let $|\sigma|$ denote the length of σ. For $\sigma \in \mathbb{N}^{<\mathbb{N}}$ and $f \in \mathbb{N}^{<\mathbb{N}} \cup \mathbb{N}^{\mathbb{N}}$, we say that σ is *an initial segment* of f (denoted by $\sigma \subset f$) if $\sigma(n) = f(n)$ for each $n < |\sigma|$. Moreover, $f \upharpoonright n$ denotes the unique initial segment of f of length n. let σ^- denote an immediate predecessor node of σ, i.e. $\sigma^- = \sigma \upharpoonright (|\sigma| - 1)$. We also define $[\sigma] = \{f \in \mathbb{N}^{\mathbb{N}} : f \supset \sigma\}$. A *tree* is a subset of $\mathbb{N}^{<\mathbb{N}}$ closed under taking initial segments. For any tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$, we also let $[T]$ be the set of all infinite paths of T, i.e., f belongs to $[T]$ if $f \upharpoonright n$ belongs to T for each $n \in \mathbb{N}$. A node $\sigma \in T$ is *extendible* if $[T] \cap [\sigma] \neq \emptyset$. Let T^{ext} denote the set of all extendible nodes of T. We say that $\sigma \in T$ is *a leaf* or *a dead end* if there is no $\tau \in T$ with $\tau \supsetneq \sigma$.

For any set X, the tree $X^{<\mathbb{N}}$ of finite words on X forms a monoid under concatenation $^\frown$. Here *the concatenation of σ and τ* is defined by $(\sigma^\frown\tau)(n) = \sigma(n)$ for $n < |\sigma|$ and $(\sigma^\frown\tau)(|\sigma| + n) = \tau(n)$ for $n < |\tau|$. We use symbols $^\frown$ and $\bigsqcap$ for the operation on this monoid, where $\bigsqcap_{i \leq n} \sigma_i$ denotes $\sigma_0{}^\frown\sigma_1{}^\frown \dots {}^\frown\sigma_n$. To avoid confusion, the symbols $\times$ and $\prod$ are only used for a product of sets. We often consider the following three left monoid actions of $X^{<\mathbb{N}}$: The first one is the set $X^{\mathbb{N}}$ of infinite words on X with an operation $^\frown : X^{<\mathbb{N}} \times X^{\mathbb{N}} \to X^{\mathbb{N}}$; $(\sigma^\frown f)(n) = \sigma(n)$ for $n < |\sigma|$ and $(\sigma^\frown f)(|\sigma| + n) = f(n)$ for $n \in \mathbb{N}$. The second one is the set $\mathcal{T}(X)$ of subtrees $T \subseteq X^{<\mathbb{N}}$ with an operation $^\frown : X^{<\mathbb{N}} \times \mathcal{T}(X) \to \mathcal{T}(X)$; $\sigma^\frown T = \{\sigma^\frown\tau : \tau \in T\}$. The third one is the power set $\mathcal{P}(X^{\mathbb{N}})$ of $X^{\mathbb{N}}$ with an operation $^\frown : X^{<\mathbb{N}} \times \mathcal{P}(X^{\mathbb{N}}) \to \mathcal{P}(X^{\mathbb{N}})$; $\sigma^\frown P = \{\sigma^\frown f : f \in P\}$.

We say that a set $P \subseteq \mathbb{N}^{\mathbb{N}}$ is Π^0_1 if there is a computable relation R such that $P = \{f \in \mathbb{N}^{\mathbb{N}} : (\forall n)R(n, f)\}$ holds. Equivalently, $P = [T_P]$ for some computable tree $T_P \subseteq \mathbb{N}^{\mathbb{N}}$. Let $\{\Phi_e\}_{e\in\mathbb{N}}$ be an effective enumeration of all Turing functionals (all partial computable functions[1]) on $\mathbb{N}^{\mathbb{N}}$. Then the e-th Π^0_1 subset of $2^{\mathbb{N}}$ is defined by $P_e = \{f \in 2^{\mathbb{N}} : \Phi_e(f; 0) \uparrow\}$. Note that $\{P_e\}_{e\in\mathbb{N}}$ is an effective enumeration of all Π^0_1 subsets of Cantor space $2^{\mathbb{N}}$. If (an index e of) a Π^0_1 set $P_e \subseteq 2^{\mathbb{N}}$ is given, then $T_e = \{\sigma \in 2^{<\mathbb{N}} : \Phi_e(\sigma; 0) \uparrow\}$ is called *the corresponding tree for P_e*. Here $\Phi(\sigma; n)$ for $\sigma \in \mathbb{N}^{<\mathbb{N}}$ and $n \in \mathbb{N}$ denotes the computation of Φ with an oracle σ, an input n, and step $|\sigma|$. Whenever a Π^0_1 set P is given, we assume that an index e of P is also given. If $P \subseteq 2^{\mathbb{N}}$ is Π^0_1, then the corresponding tree $T_P \subseteq 2^{<\mathbb{N}}$ of P is computable, and $[T_P] = P$. Moreover, the set L_P of all leaves of the computable tree T_P is also computable. We also say that a sequence of $\{P_i\}_{i\in I}$ of Π^0_1 subsets of a space X is *computable* or *uniform* if the set $\{(i, f) \in I \times X : f \in P_i\}$ is again a Π^0_1 subset of the product space $I \times X$. A set $P \subseteq \mathbb{N}^{\mathbb{N}}$ is *special* if P is nonempty and P has no computable member. For $f, g \in \mathbb{N}^{\mathbb{N}}$, $f \oplus g$ is defined by $(f \oplus g)(2n) = f(n)$ and $(f \oplus g)(2n+1) = g(n)$ for each $n \in \mathbb{N}$. For $P, Q \subseteq \mathbb{N}^{\mathbb{N}}$, put $P \oplus Q = (\langle 0\rangle^\frown P) \cup (\langle 1\rangle^\frown Q)$ and $P \otimes Q = \{f \oplus g : f \in P \ \& \ g \in Q\}$.

[1]In some contexts, a function Φ is called partial computable if it can be extended to some Φ_e. In this paper, we identify each partial computable function with such a Φ_e.

1.4. Notations from Part I

1.4.1. Functions

Every partial function $\Psi :\subseteq \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ is called a *learner*. In Part I [29, Proposition 1], it is shown that we may assume that Ψ is total, and we fix an effective enumeration $\{\Psi_e\}_{e\in\mathbb{N}}$ of all learners. For any string $\sigma \in \mathbb{N}^{<\mathbb{N}}$, the set of *mind-change locations of a learner* Ψ *on the informant* σ is defined by

$$\mathtt{mcl}_\Psi(\sigma) = \{n < |\sigma| : \Psi(\sigma \upharpoonright n+1) \neq \Psi(\sigma \upharpoonright n)\}.$$

We also define $\mathtt{mcl}_\Psi(f) = \bigcup_{n\in\mathbb{N}} \mathtt{mcl}_\Psi(f \upharpoonright n)$ for any $f \in \mathbb{N}^\mathbb{N}$. Then, $\#\mathtt{mcl}_\Psi(f)$ denotes the *number of times that the learner* Ψ *changes her/his mind on the informant* f. Moreover, the set of *indices predicted by a learner* Ψ *on the informant* σ is defined by

$$\mathtt{indx}_\Psi(\sigma) = \{\Psi(\sigma \upharpoonright n) : n \leq |\sigma|\}.$$

We also define $\mathtt{indx}_\Psi(f) = \bigcup_{n\in\mathbb{N}} \mathtt{indx}_\Psi(f \upharpoonright n)$ for any $f \in \mathbb{N}^\mathbb{N}$. We say that *a partial function* $\Gamma :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ *is identified by a learner* Ψ *on* $g \in \mathbb{N}^\mathbb{N}$ if $\lim_n \Psi_e(g \upharpoonright n)$ converges, and $\Phi_{\lim_n \Psi_e(g\upharpoonright n)}(g) = \Gamma(g)$. We also say that a partial function Γ is identified by a learner Ψ if it is identified by Ψ on every $g \in \mathrm{dom}(\Gamma)$. In Part I [29, Definition 2], we introduced the seven notions of $(\alpha,\beta|\gamma)$-computability for a partial function $\Gamma :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ listed as follows:

1. Γ is $(1,1)$-*computable* if it is computable.
2. Γ is $(1,<\omega)$-*computable* if it is identified by a learner Ψ with $\sup\{\#\mathtt{mcl}_\Psi(g) : g \in \mathrm{dom}(\Gamma)\} < \omega$.
3. Γ is $(1,\omega|<\omega)$-*computable* if it is identified by a learner Ψ with $\sup\{\#\mathtt{indx}_\Psi(g) : g \in \mathrm{dom}(\Gamma)\} < \omega$.
4. Γ is $(1,\omega)$-*computable* if it is identified by a learner.
5. Γ is $(<\omega,1)$-*computable* if there is $b \in \mathbb{N}$ such that for every $g \in \mathrm{dom}(\Gamma)$, $\Gamma(g) = \Phi_e(g)$ for some $e < b$.
6. Γ is $(<\omega,\omega)$-*computable* if there is $b \in \mathbb{N}$ such that for every $g \in \mathrm{dom}(\Gamma)$, Γ is identified by Ψ_e for some $e < b$ on g.
7. Γ is $(\omega,1)$-*computable* if it is nonuniformly computable, i.e., $\Gamma(g) \leq_T g$ for every $g \in \mathrm{dom}(\Gamma)$.

Let $[\mathfrak{C}_T]^\alpha_\beta$ (resp. $[\mathfrak{C}_T]^\alpha_{\beta|\gamma}$) denote the set of all (α,β)-computable (resp. $(\alpha,\beta|\gamma)$-computable) functions. If $\mathcal{F}$ be a monoid consisting of partial functions under composition, $\mathcal{P}(\mathbb{N}^\mathbb{N})$ is preordered by the relation $P \leq_\mathcal{F} Q$ indicating the existence of a function $\Gamma \in \mathcal{F}$ from Q into P, that is, $P \leq_\mathcal{F} Q$ if and only if there is a partial function $\Gamma :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that $\Gamma \in \mathcal{F}$ and $\Gamma(g) \in P$ for every $g \in Q$. Let $\mathcal{D}/\mathcal{F}$ and $\mathcal{P}/\mathcal{F}$ denote the quotient sets $\mathcal{P}(\mathbb{N}^\mathbb{N})/\equiv_\mathcal{F}$ and $\Pi^0_1(2^\mathbb{N})/\equiv_\mathcal{F}$, respectively. Here, $\Pi^0_1(2^\mathbb{N})$ denotes the set of all nonempty Π^0_1 subsets of $2^\mathbb{N}$. For $P \in \mathcal{P}(\mathbb{N}^\mathbb{N})$, the equivalence class $\{Q \subseteq \mathbb{N}^\mathbb{N} : Q \equiv_\mathcal{F} P\} \in \mathcal{D}/\mathcal{F}$ is called *the* $\mathcal{F}$*-degree* of P. If $\mathcal{F} = [\mathfrak{C}_T]^\alpha_{\beta|\gamma}$ for some $\alpha,\beta,\gamma \in \{1,<\omega,\omega\}$, we write $\leq^\alpha_{\beta|\gamma}$, $\mathcal{D}^\alpha_{\beta|\gamma}$, and $\mathcal{P}^\alpha_{\beta|\gamma}$ instead of $\leq_\mathcal{F}$, $\mathcal{D}/\mathcal{F}$ and $\mathcal{P}/\mathcal{F}$. The preorderings $\leq^1_1$ and $\leq^\omega_1$ are equivalent to the Medvedev reducibility [41] and the Muchnik reducibility [46], respectively.

In Part I [29, Theorem 26 and Proposition 27], we showed the following equivalences:

$$\mathcal{P}^1_{<\omega} = \mathcal{P}/\mathrm{dec}^{<\omega}_{\mathrm{d}}[\Pi^0_1] \qquad \mathcal{P}^1_{\omega|<\omega} = \mathcal{P}/\mathrm{dec}^{<\omega}_{\mathrm{p}}[\Delta^0_2] \qquad \mathcal{P}^1_{\omega} = \mathcal{P}/\mathrm{dec}^{\omega}_{\mathrm{p}}[\Delta^0_2]$$
$$\mathcal{P}^{<\omega}_1 = \mathcal{P}/\mathrm{dec}^{<\omega}_{\mathrm{d}}[\Pi^0_2] \qquad \mathcal{P}^{<\omega}_{\omega} = \mathcal{P}/\mathrm{dec}^{<\omega}_{\mathrm{d}}[\Pi^0_2]\mathrm{dec}^{\omega}_{\mathrm{p}}[\Delta^0_2] \qquad \mathcal{P}^{\omega}_1 = \mathcal{P}/\mathrm{dec}^{\omega}_{\mathrm{p}}[\Pi^0_2]$$

Here, for a pointclass Λ, a function $\Gamma :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is *finite (countable, resp.) Λ-piecewise computable* if there is a finite Λ-cover $\{X_i\}_{i<\omega}$ (a uniform Γ-cover $\{X_i\}_{i\in\omega}$, resp.) of $\mathrm{dom}(f)$ such that $\Gamma \upharpoonright X_i$ is computable for any $i \in \mathbb{N}$, and the set of all finite (countable, resp.) Λ-piecewise computable functions is denoted by $\mathrm{dec}^{<\omega}_{\mathrm{p}}[\Lambda]$ ($\mathrm{dec}^{\omega}_{\mathrm{p}}[\Lambda]$). We denote by $\mathrm{dec}^{<\omega}_{\mathrm{d}}[\Pi^0_n]$ the set of all finite Π^0_n-layerwise computable function (see Part I [29, Section 2.5]), which is equivalent to $\mathrm{dec}^{<\omega}_{\mathrm{p}}[(\Pi^0_n)_2]$, where $(\Pi^0_n)_2$ is the complexity of the differences of two Π^0_n sets.

This observation allows us to think of each degree structure $\mathcal{P}^{\alpha}_{\beta|\gamma}$ as a piecewise-degree structure in the following sense.

1. $\mathcal{P}^1_1$ is the Medvedev degrees of Π^0_1 sets.
2. $\mathcal{P}^1_{<\omega}$ is the finite-$(\Pi^0_1)_2$-piecewise degrees of Π^0_1 sets.
3. $\mathcal{P}^1_{\omega|<\omega}$ is the finite-Δ^0_2-piecewise degrees of Π^0_1 sets.
4. $\mathcal{P}^1_{\omega}$ is the countable-Δ^0_2-piecewise degrees of Π^0_1 sets.
5. $\mathcal{P}^{<\omega}_1$ is the finite-$(\Pi^0_2)_2$-piecewise degrees of Π^0_1 sets.
6. $\mathcal{P}^{<\omega}_{\omega}$ is the finite-$(\Pi^0_2)_2$-countable-Δ^0_2-piecewise degrees of Π^0_1 sets.
7. $\mathcal{P}^{\omega}_1$ is the Muchnik degrees (or equivalently, the countable-Π^0_2-piecewise degrees) of Π^0_1 sets.

1.4.2. Sets

To define the disjunction operations in Part I [29, Definition 29], we introduced some auxiliary notions. Let $I \subseteq \mathbb{N}$ be a set. Fix $\sigma \in (I \times \mathbb{N})^{<\mathbb{N}}$, and $i \in I$. Then *the i-th projection of σ* is inductively defined as follows.

$$\mathtt{pr}_i(\langle\rangle) = \langle\rangle, \qquad \mathtt{pr}_i(\sigma) = \begin{cases} \mathtt{pr}_i(\sigma^-)^\frown n, & \text{if } \sigma = \sigma^{-\frown}\langle(i,n)\rangle, \\ \mathtt{pr}_i(\sigma^-), & \text{otherwise.} \end{cases}$$

Moreover, *the number of times of mind-changes of (the process reconstructed from a record) $\sigma \in (I \times \mathbb{N})^{<\mathbb{N}}$* is given by

$$\mathtt{mc}(\sigma) = \#\{n < |\sigma| - 1 : (\sigma(n))_0 \neq (\sigma(n+1))_0\}.$$

Here, for $x = (x_0, x_1) \in I \times \mathbb{N}$, the first (second, resp.) coordinate x_0 (x_1, resp.) is denoted by $(x)_0$ ($(x)_1$, resp.). Furthermore, for $f \in (I \times \mathbb{N})^{\mathbb{N}}$, we define $\mathtt{pr}_i(f) = \bigcup_{n\in\mathbb{N}} \mathtt{pr}_i(f \upharpoonright n)$ for each $i \in I$, and $\mathtt{mc}(f) = \lim_n \mathtt{mc}(f \upharpoonright n)$, where if the limit does not exist, we write $\mathtt{mc}(f) = \infty$.

In Part I [29, Definition 33, 36 and 55], we introduced the disjunction operations. Fix a collection $\{P_i\}_{i\in I}$ of subsets of Baire space $\mathbb{N}^{\mathbb{N}}$.

1. $[\![\bigvee_{i\in I} P_i]\!]_{\mathsf{Int}} = \{f \in (I \times \mathbb{N})^{\mathbb{N}} : ((\exists i \in I)\ \mathtt{pr}_i(f) \in P_i) \ \&\ \mathtt{mc}(f) = 0\}$.

2. $[\![\bigvee_{i\in I} P_i]\!]_{\mathsf{LCM[n]}} = \{f \in (I\times\mathbb{N})^{\mathbb{N}} : ((\exists i \in I)\ \mathtt{pr}_i(f) \in P_i)\ \&\ \mathtt{mc}(f) < n\}$.
3. $[\![\bigvee_{i\in I} P_i]\!]_{\mathsf{CL}} = \{f \in (I\times\mathbb{N})^{\mathbb{N}} : (\exists i \in I)\ \mathtt{pr}_i(f) \in P_i\}$.

As in Part I, we use the notation $\mathtt{write}(i,\sigma)$ for any $i \in \mathbb{N}$ and $\sigma \in \mathbb{N}^{<\mathbb{N}}$.

$$\mathtt{write}(i,\sigma) = i^{|\sigma|} \oplus \sigma = \langle (i,\sigma(0)),(i,\sigma(1)),(i,\sigma(2)),\dots,(i,\sigma(|\sigma|-1))\rangle.$$

This string indicates the *instruction to write the string σ on the i-th tape* in the one/two-tape model. We also use the notation $\mathtt{write}(i,f) = \bigcup_{n\in\mathbb{N}} \mathtt{write}(i, f\upharpoonright n) = i^{\mathbb{N}} \oplus f$ for any $f \in \mathbb{N}^{\mathbb{N}}$.

In Part II, we are mostly interested in the degree structures of Π^0_1 subsets of $2^{\mathbb{N}}$. As mentioned in Part I [29], the consistent disjunction operations are useful to study such local degree structures. *The consistency set* $\mathrm{Con}(T_i)_{i\in I}$ *for a collection* $\{T_i\}_{i\in I}$ *of trees* is defined as follows.

$$\mathrm{Con}(T_i)_{i\in I} = \{f \in (I\times\mathbb{N})^{\mathbb{N}} : (\forall i \in I)(\forall n \in \mathbb{N})\ \mathtt{pr}_i(f\upharpoonright n) \in T_i\}.$$

Then we use the following modified definitions. Fix a collection $\{P_i\}_{i\in I}$ of Π^0_1 subsets of Baire space $\mathbb{N}^{\mathbb{N}}$ and $n \in \omega \cup \{\omega\}$.

1. $\left[\bigtriangledown_n\right]_{i\in I} P_i = [\![\bigvee_{i\in I} P_i]\!]_{\mathsf{LCM}[n]} \cap \mathrm{Con}(T_{P_i})_{i\in I}$.
2. $[\bigtriangledown_\infty]_{i\in I} P_i = [\![\bigvee_{i\in I} P_i]\!]_{\mathsf{CL}} \cap \mathrm{Con}(T_{P_i})_{i\in I}$.

Here T_{P_i} is the corresponding tree for P_i for every $i \in I$. If $i \in \{0,1\}$, then we simply write $P_0 \triangledown_n P_1$, $P_0 \triangledown_\omega P_1$, and $P_0 \triangledown_\infty P_1$ for these notions. In Part II, we use the following notion.

Definition 1. Pick any $* \in \mathbb{N}\cup\{\omega\}\cup\{\infty\}$. For each disjunctive notions $\triangledown_*$ and collection $\{P_i\}_{i\in I}$ of subsets of $\mathbb{N}^{\mathbb{N}}$, fix the corresponding tree $T_{P_i} \subseteq \mathbb{N}^{<\mathbb{N}}$ of P_i for every $i \in I$ and we may also associate a tree T_* with (the closure of) $P_0 \triangledown_* P_1$. Then *the heart of* $P_0 \triangledown_* P_1$ is defined by $T_*^{\heartsuit} = \{\sigma \in T_* : (\forall i \in I)\ \mathtt{pr}_i(\sigma) \in T^{ext}_{P_i}\}$.

Note that every $\sigma \in T_*^{\heartsuit}$ is extendible in T_*, since $T_*^{\heartsuit} \subseteq \{\sigma \in T_* : (\exists i \in I)\ \mathtt{pr}_i(\sigma) \in T^{ext}_{P_i}\}$.

Let L_P denote the set of all leaves of the corresponding tree for a nonempty Π^0_1 set P (where recall that such a tree is assumed to be uniquely determined when an index of P is given). Then *the (non-commutative) concatenation of P and Q* is defined as follows.

$$P^\frown Q = P \cup \bigcup_{\rho\in L_P} \rho^\frown Q.$$

We also write $T_P{}^\frown T_Q$ for the corresponding tree of $P^\frown Q$. Moreover, the commutative concatenation $P \triangledown Q$ is defined as $(P^\frown Q)\oplus(Q^\frown P)$. Let P and $\{Q_n\}_{n\in\mathbb{N}}$ be computable collection of Π^0_1 subsets of $2^{\mathbb{N}}$, and let ρ_n denote the length-lexicographically n-th leaf of the corresponding computable tree of P. Then, we define the *infinitary concatenation* and *recursive meet* [3] as follows:

$$P^\frown\{Q_i\}_{i\in\mathbb{N}} = P \cup \bigcup_n \rho_n{}^\frown Q_n, \qquad \bigoplus_{\overrightarrow{i\in\mathbb{N}}} Q_i = \mathsf{CPA}^\frown\{Q_i\}_{i\in\mathbb{N}}.$$

Here, CPA is a Medvedev complete set, which consists of all *complete consistent extensions of Peano Arithmetic*. The Medvedev completeness of CPA ensures that for any nonempty Π^0_1 subset $P \subseteq 2^{\mathbb{N}}$, a computable function $\Phi : \mathsf{CPA} \to P$ exists.

In Part I, we studied the disjunction and concatenation operations along graphs. For nonempty Π^0_1 subsets P and Q of $2^{\mathbb{N}}$, the *hyperconcatenation* $Q\blacktriangledown P$ *of* Q *and* P is defined by the iterated concatenation of P's along the ill-founded tree T_Q, that is,

$$Q\blacktriangledown P = \left[\bigcup_{\tau \in T_Q}\left(\prod_{i<|\tau|} T_P{}^\frown\langle\tau(i)\rangle\right)^\frown T_P\right].$$

Note that, after writing this paper, Kihara [37] gave effective topological interpretations of some of these constructions.

Remark. Recall from Section 1.3 that a corresponding tree of a Π^0_1 set is assumed to be uniquely determined when *an index of the* Π^0_1 *set is given*. Indeed, most of our above definitions obviously depend on our choice of indices (hence, corresponding trees) of given Π^0_1 sets, that is, most of operations introduced above are defined on subtrees of $\mathbb{N}^{<\mathbb{N}}$ rather than subsets of $\mathbb{N}^{\mathbb{N}}$. Although there is no effective well-defined map from the Π^0_1 sets into the indices, it does not really matter what we chose, if we only focus on the degree-theoretic behavior. Formally, the reader should replace the words "for any (there exists a) Π^0_1 set" in this paper with "for any (there exitsts an) *index of* a Π^0_1 set", or simply, the reader may suppose the definition of "a Π^0_1 set" to mean a structure $\mathcal{P}$ consisting of a pair of a Π^0_1 set P and its index e (or equivalently, its corresponding tree T_P). We will frequently use index-dependent definitions in order to simplify our notations, but in each case, one can easily ensure that it cause no problems at all.

Contents

2. Degrees of Difficulty of Disjunctions

The main objective in this section is to establish LEVEL 3 separation results among our classes of nonuniformly computable functions by using disjunction operations introduced in Part I [29, Sections 3 and 5]. We have already seen the following inequalities for Π^0_1 subsets $P, Q \subseteq 2^{\mathbb{N}}$ in Part I [29, Section 5.1].

$$P \oplus Q \geq^1_1 P \cup Q \geq^1_1 P \triangledown Q \geq^1_1 P \triangledown_\omega Q \geq^1_1 P \triangledown_\infty Q.$$

As observed in Part I [29, Section 4], these binary disjunctions are closely related to the reducibilities $\leq^1_1$, $\leq^{<\omega}_{tt,1}$, $\leq^1_{<\omega}$, $\leq^1_{\omega|<\omega}$, and $\leq^{<\omega}_1$, respectively. We employ rather exotic Π^0_1 sets constructed by Jockusch and Soare to separate the strength of these disjunctions. We say that a set $A \subseteq \mathbb{N}^{\mathbb{N}}$ is *an antichain* if it is an antichain with respect to the Turing reducibility $\leq_T$. In other words, f is Turing incomparable with g, for any two distinct elements $f, g \in A$. A nonempty closed set $A \subseteq \mathbb{N}^{\mathbb{N}}$ is *perfect* if it has no isolated point.

Theorem 2 (Jockusch-Soare [35])**.** *There exists a perfect* Π^0_1 *antichain in* $2^{\mathbb{N}}$.

A stronger condition is sometimes required. For a set $P \subseteq \mathbb{N}^{\mathbb{N}}$ and an element $g \in \mathbb{N}^{\mathbb{N}}$, let $P^{\leq_T g}$ denote the set of all element of P which are Turing reducible to g. Then, a set $A \subseteq \mathbb{N}^{\mathbb{N}}$ is antichain if and only if $A^{\leq_T g} = \{g\}$ for every $g \in A$. A set $P \subseteq \mathbb{N}^{\mathbb{N}}$ is *independent* if $P^{\leq_T \bigoplus D} = D$ for every finite subset $D \subset P$.

Theorem 3 (see Binns-Simpson [3])**.** *There exists a perfect independent* Π^0_1 *subset of* $2^{\mathbb{N}}$.

On the other hand, in Section 3.1, we will see that our hierarchy of disjunctions collapses for homogeneous sets, which may be regarded as an opposite notion to antichains and independent sets.

2.1. The Disjunction $\oplus$ versus the Disjunction $\cup$

We first separate the strength of the coproduct (the intuitionistic disjunction) $\oplus$ and the union (the classical one-tape disjunction) $\cup$. This automatically establish the LEVEL 3 separation result between $[\mathfrak{C}_T]^1_1$ and $[\mathfrak{C}_{tt}]^{<\omega}_1$. Recall that a set $P \subseteq \mathbb{N}^\mathbb{N}$ is *special* if it is nonempty and it contains no computable points.

Lemma 4. *Let P_0, P_1 be Π^0_1 subsets of $2^\mathbb{N}$, and let Q be a special Π^0_1 subset of $2^\mathbb{N}$. Assume that there exist $f \in P_0$ and $g \in P_1$ with $Q^{\leq_T f\oplus g} = Q^{\leq_T f} \cup Q^{\leq_T g}$ such that $Q^{\leq_T f}$ and $Q^{\leq_T g}$ are separated by open sets. Then $Q \not\leq^1_1 (P_0 \otimes 2^\mathbb{N}) \cup (2^\mathbb{N} \otimes P_1)$.*

Proof. Suppose that $Q \leq^1_1 (P_0 \otimes 2^\mathbb{N}) \cup (2^\mathbb{N} \otimes P_1)$ via a computable functional Φ. Then $f \oplus g \in (P_0 \otimes 2^\mathbb{N}) \cup (2^\mathbb{N} \otimes P_1)$. By our choice of f and g, $\Phi(f \oplus g)$ must belong to $Q^{\leq_T f\oplus g} = Q^{\leq_T f} \cup Q^{\leq_T g}$. By our assumption, $Q^{\leq_T f}$ and $Q^{\leq_T g}$ are separated by two disjoint open sets $U, V \subseteq 2^\mathbb{N}$. That is, $Q^{\leq_T f} \subseteq U$, $Q^{\leq_T g} \subseteq V$, and $U \cap V = \emptyset$. Therefore, either $\Phi(f \oplus g) \in Q \cap U$ or $\Phi(f \oplus g) \in Q \cap V$ holds. In any case, there exists an open neighborhood $[\sigma] \ni \Phi(f \oplus g)$ such that $[\sigma] \subseteq U$ or $[\sigma] \subseteq V$. Without loss of generality, we can assume $[\sigma] \subseteq U$. We pick initial segments $\tau_0 \subset f$ and $\tau_1 \subset g$ with $\Phi(\tau_0 \oplus \tau_1) \supseteq \sigma$. Then $(\tau_0{}^\frown 0^\mathbb{N}) \oplus g \in (P_0 \otimes 2^\mathbb{N}) \cup (2^\mathbb{N} \otimes P_1)$, and it is Turing equivalent to g. However this is impossible because $\Phi(\tau_0{}^\frown 0^\mathbb{N} \oplus g) \in [\sigma]$, and $[\sigma] \cap Q^{\leq_T g} \subseteq U \cap Q^{\leq_T g} = \emptyset$. □

Corollary 5. 1. *There are Π^0_1 sets $P, Q \subseteq 2^\mathbb{N}$ such that $P \cup Q <^1_1 P \oplus Q$.*
2. *There are Π^0_1 sets $P, Q \subseteq 2^\mathbb{N}$ such that $P \equiv^{<\omega}_{tt,1} Q$ and $P <^1_1 Q$.*

Proof. (1) Let R be a perfect independent Π^0_1 subset of $2^\mathbb{N}$. Set $P = 2^\mathbb{N} \otimes R$ and $Q = R \otimes 2^\mathbb{N}$. Note that $P \oplus Q \equiv^1_1 R$. Pick $f, g \in R$ such that $f \neq g$. Then $R^{\leq_T f} = \{f\}$, $R^{\leq_T g} = \{g\}$, and $R^{\leq_T f\oplus g} = R^{\leq_T f} \sqcup R^{\leq_T g} = \{f, g\}$. Since $2^\mathbb{N}$ is Hausdorff, two points f and g are separated by open sets. Thus, $P \oplus Q \equiv^1_1 R \not\leq^1_1 P \cup Q$ by Lemma 4. (2) $P \oplus Q \equiv^{<\omega}_{tt,1} P \cup Q <^1_1 P \oplus Q$. □

Remark. One can adopt the unit interval $[0, 1]$ as our whole space instead of Cantor space $2^\mathbb{N}$. Then, $P_0 \dagger P_1 := (P_0 \times [0, 1]) \cup ([0, 1] \times P_1)$ is connected as a topological space. If $P_0 \subseteq [0, 1]$ is homeomorphic to Cantor space, then the connected space $P_0 \dagger P_0$ is sometimes called *the Cantor tartan*. The above proof shows that every perfect independent Π^0_1 set $R \subseteq [0, 1]$ is not $(1, 1)$-reducible to the obtained tartan $R \dagger R$, while these sets are $(< \omega, 1)$-*tt*-equivalent. Note that the tartan plays an important role on the constructive study of Brouwer's fixed point theorem (see [10]).

2.2. The Disjunction $\cup$ versus the Disjunction $\triangledown$

We next separate the strength of the union $\cup$ and the concatenation (the LCM disjunction with mind-change-bound 2) $\triangledown$. Moreover, we also see the LEVEL 3 separation between $[\mathfrak{C}_{tt}]^{<\omega}_1$ and $[\mathfrak{C}_T]^1_{<\omega}$.

Lemma 6. *Let P_0, P_1 be Π^0_1 subsets of $2^\mathbb{N}$, and let Q be a special Π^0_1 subset of $2^\mathbb{N}$. Assume that there exist $f \in P_0$ and $g \in P_1$ such that any $h \in Q^{\leq_T f}$ and $Q^{\leq_T g}$ are separated by open sets. Then $Q \not\leq^1_1 P_0{}^\frown P_1$.*

Proof. Suppose that $Q \leq^1_1 P_0{}^\frown P_1$ via a computable functional Φ. By our choice of $f \in P_0 \subseteq P_0{}^\frown P_1$, there must exist an open set $U \subseteq 2^{\mathbb{N}}$ such that $\Phi(f) \in Q \cap U$ and $Q^{\leq_T g} \cap U = \emptyset$. Since U is open there exists a clopen neighborhood $[\sigma] \ni \Phi(f)$ such that $[\sigma] \cap Q \subseteq U$. We pick an initial segment $\tau \subset f$ with $\Phi(\tau) \supseteq \sigma$. Since $f \in P_0$ holds, we have that $\tau \in T_{P_0}$, and we pick $\rho \in L_{P_0}$ extending τ. Then $\rho^\frown g \in P_0{}^\frown P_1$, and $\rho^\frown g$ is Turing equivalent to g. So, if $Q \leq^1_1 P_0{}^\frown P_1$ via Φ, then $\Phi(\rho^\frown g)$ must belong to $Q^{\leq_T g}$. However this is impossible because $\Phi(\rho^\frown g) \in [\sigma]$, and $[\sigma] \cap Q^{\leq_T g} \subseteq U \cap Q^{\leq_T g} = \emptyset$. □

Corollary 7. *There are Π^0_1 sets $P, Q \subseteq 2^{\mathbb{N}}$ such that $P^\frown Q <^1_1 P \cup Q <^1_1 P \oplus Q$.*

Proof. Assume that R be a perfect Π^0_1 antichain of $2^{\mathbb{N}}$. Set $P = 2^{\mathbb{N}} \otimes R$ and $Q = R \otimes 2^{\mathbb{N}}$. Pick $f, g \in R$ such that $f \neq g$. Then $R^{\leq_T f} = \{f\}$ and $R^{\leq_T g} = \{g\}$ since R is antichain. Therefore, $(P \cup Q)^{\leq_T X} \subseteq (\{X\} \otimes 2^{\mathbb{N}}) \cup (2^{\mathbb{N}} \otimes \{X\})$ for each $X \in \{f, g\}$. For $h = h_0 \oplus h_1 \in (P \cup Q)^{\leq_T f}$, we have $h_0 \neq g$ and $h_1 \neq g$. Thus, $h \notin (2^{\mathbb{N}} \otimes \{g\}) \cup (\{g\} \otimes 2^{\mathbb{N}})$, and note that $(2^{\mathbb{N}} \otimes \{g\}) \cup (\{g\} \otimes 2^{\mathbb{N}})$ is closed. Then, there is an open neighborhood $U \subseteq 2^{\mathbb{N}}$ such that $h \in U$ and $U \cap (P \cup Q)^{\leq_T g} = \emptyset$, since $P \cup Q$ is regular, and $(P \cup Q)^{\leq_T g} \subseteq (2^{\mathbb{N}} \otimes \{g\}) \cup (\{g\} \otimes 2^{\mathbb{N}})$. Namely, any $h \in (P \cup Q)^{\leq_T f}$ and $(P \cup Q)^{\leq_T g}$ are separated by some open set. Consequently, by Lemma 6, we have $P \cup Q \not\leq^1_1 P^\frown Q$. □

One can establish another separation result for the concatenation. Recall from [12] that a closed set $P \subseteq \mathbb{N}^{\mathbb{N}}$ is *immune* if T^{ext}_P contains no infinite c.e. subset. In [12] it is shown that the class of non-immune Π^0_1 subsets of Cantor space is downward closed in the Medvedev degrees $\mathcal{P}^1_1$. This property also holds in a coarser degree structure. In Part I [29, Section 2.4] we have seen that $\mathcal{P}^{<\omega}_{tt,1}$ is an intermediate structure between $\mathcal{P}^1_1$ and $\mathcal{P}^1_{<\omega}$.

Lemma 8. *Let P and Q be Π^0_1 subsets of $2^{\mathbb{N}}$. If P is not immune, and $Q \leq^{<\omega}_{tt,1} P$, then Q is not immune.*

Proof. Let V be an infinite c.e. subset of T^{ext}_P. Assume that $Q \leq^{<\omega}_{tt,1} P$ holds via n truth-table functionals $\{\Gamma_i\}_{i<n}$. Note that every functional Γ_i can be viewed as a computable monotone function from $2^{<\omega}$ into $2^{<\omega}$. Let V_k be the c.e. set $V \cap \bigcap_{i<k} \Gamma^{-1}_i[2^{<\omega} \setminus T^{ext}_P]$ for each $k \leq n$. By our assumption, V_n is finite, since otherwise the tree generated from V has an infinite path f such that $\Phi_i(f) \notin P$ for every $i < n$. Let k be the least number such that V_{k+1} is finite. Then, $\Gamma_k[V_k]$ is an infinite c.e. set, and $\Gamma_k[V_k]$ is included in T^{ext}_P except for finite elements. □

Corollary 9. *There are Π^0_1 sets $P, Q \subseteq 2^{\mathbb{N}}$ such that $Q <^{<\omega}_{tt,1} P \equiv^1_{<\omega} Q$.*

Proof. Let P be an immune Π^0_1 subset of $2^{\mathbb{N}}$. Put $Q = P^\frown P$. As seen in Part I [29, Section 4], we have $Q \leq^1_1 P \equiv^1_{<\omega} Q$. Then, Q is not immune since T^{ext}_Q includes an infinite computable subset T_P. Hence, $P \not\leq^{<\omega}_{tt,1} Q$ by Proposition 8. □

We have introduced two concatenation operations $^\frown$ and $\triangledown$, while there are several other concatenation-like operations (see Duparc [22]). For Π^0_1 sets P and Q, let $P^{\rightarrow}Q$ and $P^{\sqcap}Q$ denote $[\{\sigma^\frown\sharp^\frown\tau : \sigma \in T_P \;\&\; \tau \in T_Q\}]$ and $[\{\sigma^\frown\tau : \sigma \in T_P \;\&\; \tau \in T_Q\}]$, respectively. (Note that these definitions are also index-dependent, and recall that the final remark in Section 1.4.2.) As seen in Part I [29, Proposition 53], we have $P^\frown Q \equiv^1_1 P^{\rightarrow}Q$. However, there is a $(1, 1)$-difference between $P^\frown Q$ and $P^{\sqcap}Q$.

Proposition 10. *There are* Π^0_1 *sets* $P, Q \subseteq 2^{\mathbb{N}}$ *such that* $P^{\sqcap}Q <^1_1 P^{\frown}Q$.

Proof. It is easy to see that $P^{\sqcap}Q \leq^1_1 P^{\rightarrow}Q$ for any Π^0_1 sets $P, Q \subseteq \mathbb{N}^{\mathbb{N}}$. Let $R \subseteq 2^{\mathbb{N}}$ be a Π^0_1 antichain. Then we divide R into four parts, P_0, P_1, P_2, and P_3. Put $P = P_3$, and $Q = (\langle 0,1\rangle^{\frown}P_2{}^{\frown}P_0) \cup (\langle 1\rangle^{\frown}P_2{}^{\frown}P_1)$. Without loss of generality, we may assume that $\langle 0\rangle \in T_P$. Suppose that $P^{\rightarrow}Q \leq^1_1 P^{\sqcap}Q$ via a computable function Φ. Choose $g \in P_2$. Then we have $\langle 0,1\rangle^{\frown}g \in P^{\sqcap}Q$. Therefore, $\Phi(\langle 0,1\rangle^{\frown}g) \in P^{\rightarrow}Q$ must contain $\sharp$, since $P = P_3$ has no element computable in $g \in P_2$. Thus, there is $n \in \mathbb{N}$ such that $\Phi(\langle 0,1\rangle^{\frown}(g \upharpoonright n))$ contains $\langle \sharp, i\rangle$ as a substring for some $i < 2$. Fix such i. Then, $\Phi(\langle 0,1\rangle^{\frown}(g \upharpoonright n)) \in P^{\rightarrow}(Q \cap [\langle i\rangle])$. We extend $g \upharpoonright n$ to some leaf ρ of P_2. Choose $h_k \in P_k$ for each $k < 2$. Then, $\langle 0,1\rangle^{\frown}\rho^{\frown}h_0 \in Q \subseteq P^{\sqcap}Q$, and $\langle 0,1\rangle^{\frown}\rho^{\frown}h_1 \in \langle 0\rangle^{\frown}Q \subseteq P^{\sqcap}Q$. Thus, $\Phi(\langle 0,1\rangle^{\frown}\rho^{\frown}h_k)$ must belongs to $P^{\rightarrow}(Q \cap [\langle i\rangle])$, for each $k < 2$. However $P^{\rightarrow}(Q \cap [\langle i\rangle])$ has no element computable in $\langle 0,1\rangle^{\frown}\rho^{\frown}h_{1-i}$. A contradiction. □

Proposition 11. $P^{\sqcap}Q \equiv^1_\omega P^{\frown}Q$ *holds for every* Π^0_1 *sets* $P, Q \subseteq \mathbb{N}^{\mathbb{N}}$.

Proof. It suffices to show that $P^{\rightarrow}Q \leq^1_\omega P^{\sqcap}Q$. Given $f \in P^{\sqcap}Q$, our learner Ψ first guesses that f is also a correct solution to $P^{\rightarrow}Q$. If $f \upharpoonright n \notin T_P$ happens, we know that $(f \upharpoonright m)^{\frown}\sharp^{\frown}f^{\leftarrow m} \in P^{\rightarrow}Q$ for some $m \leq n$, where note that $f = (f \upharpoonright m)^{\frown}f^{\leftarrow m}$ holds for each $m \in \mathbb{N}$. Thus, the learner Ψ can guess a correct number $m \leq n$ such that $(f \upharpoonright m)^{\frown}\sharp^{\frown}f^{\leftarrow m} \in P^{\rightarrow}Q$ with at most n mind-changes. □

2.3. The Disjunction $\triangledown$ *versus the Disjunction* $\triangledown_\omega$

Let Ψ be a learner (i.e., a total computable function $\Psi : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$). A point $\alpha \in \mathbb{N}^{\mathbb{N}}$ is said to be *an m-changing point of* Ψ if $\#\mathtt{mcl}_\Psi(\alpha) \geq m$. Then, the set of all m-changing points[2] of Ψ is denoted by $\mathtt{mc}_\Psi(\geq m)$. A point $\alpha \in \mathbb{N}^{\mathbb{N}}$ is *anti-Popperian* with respect to Ψ if $\lim_n \Psi(\alpha \upharpoonright n)$ converges, but $\Phi_{\lim_n \Psi(\alpha\upharpoonright n)}(\alpha)$ is partial[3]. The set of all anti-Popperian points of Ψ is denoted by AP_Ψ.

Remark (Trichotomy)**.** Let Γ be a $(1,\omega)$-computable function identified by a learner Ψ, and let P be any subset of Baire space $\mathbb{N}^{\mathbb{N}}$. Then $\mathbb{N}^{\mathbb{N}} \setminus \Gamma^{-1}(P)$ is divided into the following three parts: the set $\Gamma^{-1}(\mathbb{N}^{\mathbb{N}} \setminus P)$; the Σ^0_2 set AP_Ψ; and the Π^0_2 set $\bigcap_{m\in\mathbb{N}} \mathtt{mc}_\Psi(\geq m)$.

We say that P_0 and P_1 are *everywhere* $(\omega,1)$*-incomparable* if $P_0 \cap [\sigma_0]$ is Muchnik incomparable with $P_1 \cap [\sigma_1]$ (that is, $P_i \cap [\sigma_i] \not\leq^\omega_1 P_{1-i} \cap [\sigma_{1-i}]$ for each $i < 2$) whenever $[\sigma_i] \cap P_i \neq \emptyset$ for each $i < 2$.

---

[2]The set of m-changing points is closedly related to the m-th derived set obtained from the notion of discontinuity levels ([19, 26, 27, 40]). See also Part I [29, Section 5.3] for more information on the relationship between the notion of mind-changes and the level of discontinuity.

[3]In the sense of the identification in the limit [24], the learner Ψ is said to be Popperian if $\Phi_{\Psi(\sigma)}(\emptyset)$ is total for every $\sigma \in \mathbb{N}^{<\mathbb{N}}$ such that $\Psi(\sigma)$ is defined. This definition indicates that, given any sequence $\alpha \in \mathbb{N}^{\mathbb{N}}$, if the learner makes an incorrect guess $\Phi_{\Psi(\alpha\upharpoonright s)}(\emptyset) \neq \alpha$ at stage s, the leaner will eventually find his mistake $\Phi_{\Psi(\alpha\upharpoonright s)}(\emptyset; n) \downarrow\neq \alpha(n)$. In our context, the learner shall be called Popperian if given any falsifiable (i.e., Π^0_1) mass problem Q and any sequence $\alpha \in \mathbb{N}^{\mathbb{N}}$, the incorrectness $\Phi_{\Psi(\alpha\upharpoonright s)}(\alpha) \notin Q$ implies $\Phi_{\Psi(\alpha\upharpoonright s)}(\alpha) \upharpoonright n \downarrow\notin T_Q$ for some $n \in \mathbb{N}$. Every anti-Popperial point of Ψ witnesses that Ψ is not Popperian.

Theorem 12. *Let P_0, P_1 be everywhere $(\omega,1)$-incomparable Π^0_1 subsets of $2^{\mathbb{N}}$, and ρ be any binary string. For any $(1,\omega)$-computable function Γ identified by a learner Ψ, the closure of $\mathtt{mc}_\Psi(\geq m) \cup \Gamma^{-1}(\mathbb{N}^{\mathbb{N}} \setminus P_0 \oplus P_1) \cup \mathrm{AP}_\Psi$ includes $\rho^\frown(P_0 \triangledown_n P_1)^\heartsuit$ with respect to the relative topology on $\rho^\frown(P_0 \triangledown_{m+n} P_1)^\heartsuit$ (as a subspace of Baire space $\mathbb{N}^{\mathbb{N}}$).*

Proof. Fix a string $\rho^\frown\tau_0$ which is extendible in the heart of $\rho^\frown(P_0 \triangledown_n P_1)$. Then, $\mathtt{pr}_i(\tau_0)$ must be extendible in P_i. Fix $f_i \in P_i \cap [\mathtt{pr}_i(\tau_0)]$ witnessing $P_{1-i} \not\leq^\omega_1 P_i$ for each $i < 2$, i.e., P_{1-i} contains no f_i-computable element. Such f_i exists, by everywhere $(\omega,1)$-incomparability. Assume that $f_i = \mathtt{pr}_i(\tau_0)^\frown f^*_i$ for each $i < 2$ and that the last declaration along τ_0 is j_0, i.e., $\tau_0 = \tau_0^{-\frown}(j_0, k)$ for some $k < 2$. Then we can proceed the following actions.

- Extend τ_0 to $g_0 = \tau_0{}^\frown\mathtt{write}(j_0, f^*_{j_0}) \in \rho^\frown(P_0 \triangledown_n P_1)$.
- Wait for the least $s_0 > |\tau_0|$ such that $\Phi_{\Psi(g_0 \upharpoonright s_0)}(g_0 \upharpoonright s_0; 0) = j_0$.
- Extend $g_0 \upharpoonright s_0$ to $g_1 = (g_0 \upharpoonright s_0)^\frown\mathtt{write}(j_1, f^*_{j_1}) \in \rho^\frown(P_0 \triangledown_{n+1} P_1)$, where $j_1 = 1 - j_0$.
- Wait for the least $s_1 > s_0$ such that $\Phi_{\Psi(g_1 \upharpoonright s_1)}(g_1 \upharpoonright s_1; 0) = 1 - j_0$.

If both s_0 and s_1 are defined, then this action forces the learner Ψ to change his mind. In other words, $g_1 \in \mathtt{mc}_\Psi(\geq 1)$. Assume that s_l is undefined for some $l < 2$ Note that $g_l \equiv_T f_{j_l}$, since $\mathtt{pr}_{j_l}(g_l) = f_{j_l}$ and $\mathtt{pr}_{1-j_l}(g_l)$ is finite, for each $l < 2$. Therefore, $\Gamma(g_l) \notin (1 - j_l)^\frown P_{1-j_l}$ since P_{1-j_l} has no g_l-computable element. In this case, $g_l \in \Gamma^{-1}(\mathbb{N}^{\mathbb{N}} \setminus P_0 \oplus P_1)$. Hence, in $\rho^\frown(P_0 \triangledown_{n+1} P_1)^\heartsuit$, the closure of $\mathtt{mc}_\Psi(\geq 1) \cup \Gamma^{-1}(\mathbb{N}^{\mathbb{N}} \setminus P_0 \oplus P_1) \cup \mathrm{AP}_\Psi$ includes $\rho^\frown(P_0 \triangledown_n P_1)^\heartsuit$. By iterating this procedure, in $\rho^\frown(P_0 \triangledown_{m+n} P_1)^\heartsuit$, we can easily see that the closure of $\mathtt{mc}_\Psi(\geq m) \cup \Gamma^{-1}(\mathbb{N}^{\mathbb{N}} \setminus P_0 \oplus P_1) \cup \mathrm{AP}_\Psi$ includes $\rho^\frown(P_0 \triangledown_n P_1)^\heartsuit$. □

Corollary 13.

1. *There exists Π^0_1 sets $P, Q \subseteq 2^{\mathbb{N}}$ such that $P \triangledown_\omega Q <^1_{<\omega} P \triangledown Q$.*
2. *There exists Π^0_1 sets $P, Q \subseteq 2^{\mathbb{N}}$ such that $P \equiv^1_{\omega|<\omega} Q$ and $P <^1_{<\omega} Q$.*

Proof. (1) Let P be a perfect Π^0_1 antichain in $2^{\mathbb{N}}$ of Theorem 2. Fix a clopen set C such that $P_0 = P \cap C \neq \emptyset$, and $P_1 = P \setminus C \neq \emptyset$. Then every $f \in P_0$ and $g \in P_1$ are Turing incomparable. Therefore, P_0 and P_1 are everywhere $(\omega,1)$-incomparable. Let ρ_n denote the n-th leaf of the tree T_{CPA} of a Medvedev complete Π^0_1 set $\mathsf{CPA} \subseteq 2^{\mathbb{N}}$. Fix a $(1,m)$-computable function Γ identified by a learner Ψ. By Theorem 12, $\rho_{m+1}{}^\frown(P_0 \triangledown_{m+1} P_1)$ intersects with $\mathtt{mc}_\Psi(\geq m+1) \cup \Gamma^{-1}(\omega^\omega \setminus P_0 \oplus P_1)$. Thus, $P_0 \oplus P_1 \not\leq^1_{<\omega} \bigoplus\overrightarrow{_n}(P_0 \triangledown_n P_1)$. Additionally, we easily have $P_0 \triangledown_\omega P_1 \leq^1_{<\omega} \bigoplus\overrightarrow{_n}(P_0 \triangledown_n P_1)$. (2) $P = \bigoplus\overrightarrow{_n}(P_0 \triangledown_n P_1)$ and $Q = P_0 \oplus P_1$ are Π^0_1. □

2.4. The Disjunction $\triangledown_\omega$ versus the Disjunction $\triangledown_\infty$

By the similar argument, we can separate the strength of the concatenation $\triangledown_\omega$ and the classical disjunction $\triangledown_\infty$.

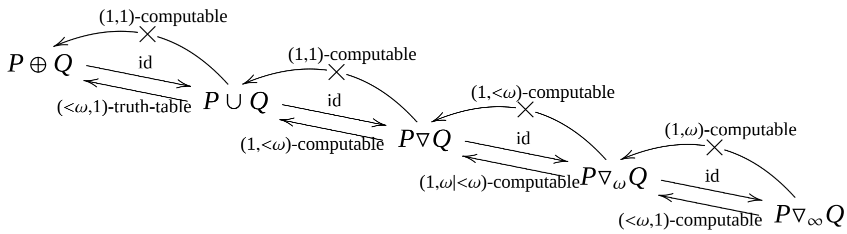

Figure 1: The two-tape (bounded-errors) model of disjunctions for independent Π^0_1 sets $P, Q \subseteq 2^{\mathbb{N}}$.

Theorem 14. *Let P_0, P_1 be everywhere $(\omega, 1)$-incomparable Π^0_1 subsets of $2^{\mathbb{N}}$. For any $(1,\omega)$-computable function Γ, the complement of $\Gamma^{-1}(P_0 \oplus P_1)$ is dense in $(P_0 \triangledown_\infty P_1)^\heartsuit$ (as a subspace of Baire space $\mathbb{N}^{\mathbb{N}}$).*

Proof. Fix a learner Ψ which identifies the $(1,\omega)$-computable function Γ. Fix any clopen set $[\tau]$ intersecting with the heart of $(P_0 \triangledown_\infty P_1)$. Assume that $[\tau] \cap (P_0 \triangledown_\infty P_1)^\heartsuit$ contains no element of $\Gamma^{-1}(\mathbb{N}^{\mathbb{N}} \setminus P_0 \oplus P_1) \cup \mathrm{AP}_\Psi$. By Theorem 12, $\mathtt{mc}_\Psi(\geq n)$ is dense and open in the heart of $(P_0 \triangledown_\infty P_1) \cap [\tau] = \tau^\frown((P_0 \cap [\mathtt{pr}_0(\tau)]) \triangledown_\infty (P_1 \cap [\mathtt{pr}_1(\tau)]))$. As $[\tau] \cap (P_0 \triangledown_\infty P_1)^\heartsuit$ is Π^0_1, the intersection $\bigcap_{n \in \mathbb{N}} \mathtt{mc}_\Psi(\geq n)$ is dense in $[\tau] \cap (P_0 \triangledown_\infty P_1)^\heartsuit$, by Baire Category Theorem. Hence, $\mathbb{N}^{\mathbb{N}} \setminus \Gamma^{-1}(P_0 \oplus P_1)$ intersects with any nonempty clopen set $[\tau]$ with $[\tau] \cap (P_0 \triangledown_\infty P_1)^\heartsuit \neq \emptyset$. □

Corollary 15.

1. *There exist Π^0_1 sets $P, Q \subseteq 2^{\mathbb{N}}$ such that $P \equiv^{<\omega}_1 Q$ holds but $Q \not\leq^1_\omega P$ holds.*
2. *There exist Π^0_1 sets $P, Q \subseteq 2^{\mathbb{N}}$ such that $P \equiv^{<\omega}_\omega Q$ holds but $P <^1_\omega Q$ holds.*

Proof. Let P be a perfect Π^0_1 antichain in $2^{\mathbb{N}}$ of Theorem 2. Fix a clopen set C such that $P_0 = P \cap C \neq \emptyset$, and $P_1 = P \setminus C \neq \emptyset$. Then P_0 and P_1 are everywhere $(\omega, 1)$-incomparable. Fix a $(1,\omega)$-computable function Γ identified by a learner Ψ. By Theorem 14, $\mathbb{N}^{\mathbb{N}} \setminus (P_0 \oplus P_1)$ is dense in $(P_0 \triangledown_\infty P_1)^\heartsuit$. For Π^0_1 sets $P_0, P_1 \subseteq 2^{\mathbb{N}}$, both $P_0 \triangledown_\infty P_1$ and $P_0 \oplus P_1$ are Π^0_1, and $P_0 \triangledown_\omega P_1 \leq^1_1 P_0 \oplus P_1$. □

2.5. The Disjunction $\bigoplus$ versus the Disjunction $\triangledown_\infty$

By the similar argument, we can separate infinitary disjunctions. A sequence $\{x_i\}_{i \in \mathbb{N}}$ of elements of $\mathbb{N}^{\mathbb{N}}$ is *Turing independent* if x_i is not computable in $\bigoplus_{j \neq i} x_j$ for each $i \in \mathbb{N}$. A collection $\{P_i\}_{i \in I}$ of subsets of $\mathbb{N}^{\mathbb{N}}$ is *pairwise everywhere independent* if, for any collection $\{[\sigma_i]\}_{i \in I}$ of clopen sets with $P_i \cap [\sigma_i] \neq \emptyset$ for each $i \in I$, there is a choice $\{x_i\}_{i \in I} \in \prod_{i \in I}(P_i \cap [\sigma_i])$ such that P_i has no element computable in $\bigoplus_{j \in I \setminus \{i\}} x_j$ for each $i \in I$.

Theorem 16. *Let $\{P_i\}_{i < 2^t}$ be a pairwise everywhere independent collection of Π^0_1 subsets of $2^{\mathbb{N}}$, and let ρ be any binary string. For any (t,ω)-computable function Γ, the complement of $\Gamma^{-1}(P_0 \oplus \cdots \oplus P_{2^t - 1})$ is dense in the heart of $\rho^\frown(P_0 \triangledown_\infty \ldots \triangledown_\infty P_{2^t - 1})$ (as* *a subspace of Baire space $\mathbb{N}^{\mathbb{N}}$). Indeed, for any nonempty interval I in the heart of $\rho^\frown(P_0\triangledown_\infty\dots\triangledown_\infty P_{2^t-1})$, there is $g\in\rho^\frown(P_0\triangledown_\infty\dots\triangledown_\infty P_{2^t-1})^\heartsuit\cap I\setminus\Gamma^{-1}(P_0\oplus\dots\oplus P_{2^t-1})$ which is computable in some $g^*\in\bigotimes_{k<2^t-1}P_k$.*

Proof. Assume that the (t,ω)-computable function Γ is identified by a team $\{\Psi_i\}_{i<t}$ of learners. Fix a string $\rho^\frown\tau_0$ which is extendible in the heart of $\rho^\frown(P_0\triangledown_\infty\dots\triangledown_\infty P_{2^t-1})$. Then, $\mathtt{pr}_i(\tau_0)$ must be extendible in P_i for each $i<2^t$. Fix $\{f_i\}_{i<2^t}\in\prod_{i<2^t}(P_i\cap[\mathtt{pr}_i(\tau_0)])$ witnessing the independence of $\{P_i\}_{i<2^t}$, i.e., P_i contains no $\bigoplus_{j\neq i}f_j$-computable element. Assume that $f_i=\mathtt{pr}_i(\tau_0)^\frown f_i^*$ for each $i<2^t$ and that the last declaration along τ_0 is $j_0<2^t$, i.e., $\tau_0=\tau_0^{-\frown}(j_0,k)$ for some $k<2$. Fix a computable function δ mapping $j<2^t$ to a unique binary string $\delta(j)$ satisfying $j=\sum_{e=0}^{t-1}2^e\cdot\delta(j;e)$. Let E^e_k denote the set $\{j<2^t:\delta(j;e)=k\}$. Then we can proceed the following actions.

- Extend τ_0 to $g_0=\tau_0{}^\frown\mathtt{write}(j_0,f^*_{j_0})\in\rho^\frown(P_0\triangledown_\infty\dots\triangledown_\infty P_{2^t-1})$.
- Wait for the least $s_0>|\tau_0|$ such that $\Phi_{\Psi_e(g_0\upharpoonright s_0)}(g_0\upharpoonright s_0;0)\in E^e_{\delta(j_0;e)}$ for some $e<2^t$.
- If such s_0 exists, then enumerate all such $e<2^t$ into an auxiliary set Ch_0, and define $\delta(j_1)$ as follows:

$$\delta(j_1;e)=\begin{cases}\delta(j_0;e) & \text{if } e\notin\mathrm{Ch}_0,\\ 1-\delta(j_0;e) & \text{if } e\in\mathrm{Ch}_0.\end{cases}$$

- Extend $g_0\upharpoonright s_0$ to $g_1=(g_0\upharpoonright s_0)^\frown\mathtt{write}(j_1,f^*_{j_1})\in\rho^\frown(P_0\triangledown_\infty\dots\triangledown_\infty P_{2^t-1})$, where $j_1=\sum_{e=0}^{t-1}2^e\cdot\delta(j_1;e)$.

These actions force each learner Ψ_e with $e\in\mathrm{Ch}_0$ to change his mind whenever the learner Ψ_e want to have an element of $\bigoplus_{i<2^t}P_i$. Fix $u\in\mathbb{N}$. Assume that j_u, g_u, s_u, and Ch_u has been already defined, and the following induction hypothesis at stage u is satisfied.

- $\mathtt{pr}_e(g_u)\subseteq f_e$ for any $e<2^t$, hence, $g_u\in\rho^\frown(P_0\triangledown_\infty\dots\triangledown_\infty P_{2^t-1})^\heartsuit\cap[\rho^\frown\tau_0]$.
- $\{s_v\}_{v\leq u}$ is strict increasing, and $\mathrm{Ch}_u\neq\emptyset$.
- For each $e\in\mathrm{Ch}_u$, if $\Phi_{\Psi_e(g_u\upharpoonright s_u)}(g_u\upharpoonright s_u;0)$ converges to some value $k<2^t$, then $k\in E^e_{\delta(j_u;e)}$

It is easy to see that $u=0$ satisfies the induction hypothesis. At stage $u+1\in\mathbb{N}$, we proceeds the following actions.

- Define $\delta(j_{u+1})$ as follows:

$$\delta(j_{u+1};e)=\begin{cases}\delta(j_u;e) & \text{if } e\notin\mathrm{Ch}_u,\\ 1-\delta(j_u;e) & \text{if } e\in\mathrm{Ch}_u.\end{cases}$$

- Extend $g_u\upharpoonright s_u$ to $g_{u+1}=(g_u\upharpoonright s_u)^\frown\mathtt{write}(j_{u+1},f^{(u+1)}_{j_{u+1}})$, where $j_{u+1}=\sum_{e=0}^{t-1}2^e\cdot\delta(j_{u+1};e)$, and f^{u+1} satisfies $\mathtt{pr}_{j_{u+1}}(g_{u+1})=\mathtt{pr}_{j_{u+1}}(g_u\upharpoonright s_u)^\frown f^{(u+1)}_{j_{u+1}}=\rho^\frown f_{j_{u+1}}$.

- Wait for the least $s_{u+1} > s_u$ such that $\Phi_{\Psi_e(g_{u+1}\upharpoonright s_{u+1})}(g_{u+1} \upharpoonright s_{u+1}; 0) \in E^e_{\delta(j_{u+1};e)}$ for some $e < 2^t$.
- If such s_{u+1} exists, then enumerate all such $e < 2^t$ into Ch_{u+1},

By our action, it is easy to see that $u+1$ satisfies the induction hypothesis. As the set $\rho^\frown(P_0\triangledown_\infty \dots \triangledown_\infty P_{2^t-1})^\heartsuit$ is closed (with respect to the Baire topology) and $\{s_u\}_{u\in\mathbb{N}}$ is strictly increasing, the sequence $\{g_u\}_{u\in\mathbb{N}}$ converges to some $g \in \rho^\frown(P_0\triangledown_\infty \dots \triangledown_\infty P_{2^t-1})^\heartsuit$. Let $I(g) \subseteq 2^t$ be the set of all $e < 2^t$ such that $\mathtt{pr}_e(g)$ is total.

Claim. $g \leq_T \bigoplus_{e\in I(g)} f_e$.

Note that $g = g[f_0, \dots, f_{2^t-1}]$ is effectively constructed uniformly in a given collection $\{f_k\}_{k<2^t}$. In other words, there is a (uniformly) computable function Θ mapping $\{f_k\}_{k<2^t}$ to $\Theta(\{f_k\}_{k<2^t}) = g = g[f_0, \dots, f_{2^t-1}]$. Then, it is easy to see that the function Θ maps $\{f_e\}_{e\in I(g)} \cup \{\mathtt{pr}_e(g)^\frown 0^\mathbb{N}\}_{e\in 2^t\setminus I(g)}$ to g. Hence, $g \leq_T \bigoplus_{e\in I(g)} f_e \oplus \bigoplus_{e\in 2^t\setminus I(g)} \mathtt{pr}_e(g)^\frown 0^\mathbb{N}$. Therefore, $g \leq_T \bigoplus_{e\in I(g)} f_e$ as desired, since $\mathtt{pr}_e(g)^\frown 0^\mathbb{N}$ is computable for any $e \in 2^t \setminus I(g)$.

Let Γ_e denote the $(1,\omega)$-computable function identified by Ψ_e, that is, $\Gamma_e(\alpha) = \Phi_{\lim_n \Psi(\alpha\upharpoonright n)}(\alpha)$ for any $\alpha \in \mathbb{N}^\mathbb{N}$. We consider the following two cases.

Case 1 ($e \in \mathrm{Ch}_u$ for finitely many $u \in \mathbb{N}$). Fix u such that $e \notin \mathrm{Ch}_v$ for any $v > u$. For each $v > u$, $\Phi_{\Psi_e(g\upharpoonright s_u)}(g \upharpoonright s_u; 0)$ does not converges to an element of $E^e_{\delta(j_v;e)} = E^e_{\delta(j_u;e)}$. By our definition, for each $k \notin E^e_{\delta(j_u;e)}$, $\mathtt{pr}_k(g) \subset \rho^\frown f_k$ is finite. By previous claim, $g \leq_T \bigoplus_{e\neq k} f_e$. Thus, by independence, P_k has no g-computable element. If $\Phi_{\Psi_e(g\upharpoonright s_u)}(g \upharpoonright s_u; 0)\uparrow$ for any $u \in \mathbb{N}$, then $g \in \mathrm{AP}_{\Psi_e}$. If $\lim_n \Psi_e(g \upharpoonright n)$ does not converge, then $g \in \bigcap_{m\in\mathbb{N}} \mathtt{mc}_{\Psi_e}(\geq m)$. Otherwise, $\Phi_{\lim_n \Psi_e(g\upharpoonright n)}(g; 0)$ converges to some value $k \notin E^e_{\delta(j_u;e)}$. As $\Phi_{\lim_n \Psi_e(g\upharpoonright n)}(g)$ is g-computable, we see $\Phi_{\lim_n \Psi_e(g\upharpoonright n)}(g) \notin k^\frown P_k$. Consequently, $g \in \mathbb{N}^\mathbb{N} \setminus \Gamma_e^{-1}(\bigoplus_{k<2^t} P_k)$.

Case 2 ($e \in \mathrm{Ch}_u$ for infinitely many $u \in \mathbb{N}$). We enumerate an infinite increasing sequence $\{u[n]\}_{n\in\mathbb{N}}$, where $u[n]$ is the n-th element such that $e \in \mathrm{Ch}_u$. As $e \in \mathrm{Ch}_{u[n]}$, we have $\Phi_{\Psi_e(g\upharpoonright u[n])}(g \upharpoonright u[n]; 0) \in E^e_{\delta(j_{u[n]};e)}$. By our action, $\delta(j_{u[n+1]}; e) = \delta(j_{u[n]+1}; e) \neq \delta(j_{u[n]}; e)$. This implies $E^e_{\delta(j_{u[n+1]};e)} \cap E^e_{\delta(j_{u[n]};e)} = \emptyset$. However, we must have $\Phi_{\Psi_e(g\upharpoonright u[n+1])}(g \upharpoonright u[n+1]; 0) \in E^e_{\delta(j_{u[n+1]};e)}$, since $e \in \mathrm{Ch}_{u[n+1]}$. This forces the learner Ψ_e to change his mind. By iterating this procedure, we eventually obtain $g \in \bigcap_{m\in\mathbb{N}} \mathtt{mc}_{\Psi_e}(\geq m)$.

Consequently, $g \in \bigcap_{e\in\mathbb{N}}(\mathbb{N}^\mathbb{N} \setminus \Gamma_e^{-1}(\bigoplus_{k<2^t} P_k))$. Thus, $g \in \mathbb{N}^\mathbb{N} \setminus \Gamma_e^{-1}(\bigoplus_{k<2^t} P_k)$. For any τ_0 such that $\rho^\frown\tau_0$ which is extendible in the heart of $\rho^\frown(P_0\triangledown_\infty \dots \triangledown_\infty P_{2^t-1})$, we can construct such g extending τ_0. Therefore, $\mathbb{N}^\mathbb{N} \setminus \Gamma_e^{-1}(\bigoplus_{k<2^t} P_k)$ intersects any nonempty interval in $\rho^\frown(P_0\triangledown_\infty \dots \triangledown_\infty P_{2^t-1})^\heartsuit$. In other words, $\mathbb{N}^\mathbb{N} \setminus \Gamma_e^{-1}(P_0 \oplus \dots \oplus P_{2^t-1})$ is dense in $\rho^\frown(P_0\triangledown_\infty \dots \triangledown_\infty P_{2^t-1})^\heartsuit$ as desired. □

The following theorem by Jockusch-Soare [35, Theorem 4.1] is important.

Theorem 17 (Jockusch-Soare [35])**.** *There is a computable sequence $\{\prod_n P^i_n\}_{i\in\mathbb{N}}$ of nonempty homogeneous Π^0_1 subsets of $2^\mathbb{N}$ such that $\{x_i\}_{i\in\mathbb{N}}$ is Turing independent for any choice $x_i \in \prod_n P^i_n$, $i \in \mathbb{N}$.*

Clearly any such Π^0_1 set contains no element of a PA degree, a Turing degree of a complete consistent extension of Peano Arithmetic. Accordingly, every element of such a Π^0_1 set computes no element of a Medvedev complete Π^0_1 set CPA.

Corollary 18. *There are Π^0_1 sets $P_n \subseteq 2^{\mathbb{N}}$, $n \in \mathbb{N}$, such that $\bigoplus\overrightarrow{_t}(P_0 \triangledown_\infty \dots \triangledown_\infty P_t) <^{<\omega}_{\omega} \bigoplus\overrightarrow{_t} P_t$.*

Proof. Fix the computable sequence $\{P_i\}_{i\in\mathbb{N}}$ of Theorem 17. Then $\{P_i\}_{i\in\mathbb{N}}$ is pairwise everywhere independent. Assume that $\bigoplus\overrightarrow{_t} P_t \leq^{<\omega}_{\omega} \bigoplus\overrightarrow{_t}(P_0 \triangledown_\infty \dots \triangledown_\infty P_t)$ via a (t,ω)-computable function Γ. Let ρ_n denote the n-th leaf of the tree T_{CPA} of a Medvedev complete Π^0_1 subset of $2^{\mathbb{N}}$. By Theorem 16, $\Gamma^{-1}(\bigcup_{k<2^t} \rho_k{}^\frown P_k)$ is dense in the heart of $\rho_{2^t}{}^\frown(P_0 \triangledown_\infty \dots \triangledown_\infty P_{2^t-1})$. In particular, there is $g \in \rho_{2^t}{}^\frown(P_0 \triangledown_\infty \dots \triangledown_\infty P_{2^t-1})$ such that $\Gamma(g) \notin \bigcup_{k<2^t} \rho_k{}^\frown P_k$ which is computable in some $g^* \leq_T \bigotimes_{k<2^t} P_k$. By our choice of $\{P_i\}_{i\in\mathbb{N}}$, $\Gamma(g)$ computes no element of $\bigcup_{k\geq 2^t} P_k \cup \mathsf{CPA}$. Thus, $\Gamma(g) \notin \bigoplus\overrightarrow{_t} P_t$. □

Corollary 19.

1. *There exists a computable sequence $\{P_n\}_{n\in\mathbb{N}}$ of Π^0_1 subsets of Cantor space $2^{\mathbb{N}}$, such that the condition $[\bigtriangledown_\infty]_n P_n <^{<\omega}_{\omega} \bigoplus_n P_n$ is satisfied.*
2. *There exist Π^0_1 sets $P, Q \subseteq 2^{\mathbb{N}}$ such that $P \equiv^{\omega}_1 Q$ holds but $P <^{<\omega}_{\omega} Q$ holds.*

Proof. (1) By Corollary 18. the condition $[\bigtriangledown_\infty]_n P_n \leq^1_1 \bigoplus\overrightarrow{_t}(P_0 \triangledown_\infty \dots \triangledown_\infty P_t) <^{<\omega}_{\omega} \bigoplus\overrightarrow{_t} P_t \leq^1_1 \bigoplus_n P_n$ is satisfied. (2) Put $P = [\bigtriangledown_\infty]_n P_n \leq^1_1 \bigoplus\overrightarrow{_t}(P_0 \triangledown_\infty \dots \triangledown_\infty P_t)$ and $Q = \bigoplus\overrightarrow{_t} P_t$. Then P and Q are $(1,1)$-equivalent to Π^0_1 subsets as seen in Part I [29, Section 5.2]. By Theorem 18, $P <^{<\omega}_{\omega} Q$, and $Q \equiv^{\omega}_1 P$ as seen in Part I [29, Sections 4 and 5.2]. □

3. Contiguous Degrees and Dynamic Infinitary Disjunctions

3.1. When the Hierarchy Collapses

We have already observed the following hierarchy, for pairwise independent Π^0_1 sets $P, Q \subseteq 2^{\mathbb{N}}$.

$$P \oplus Q >^1_1 P \cup Q >^1_1 P \triangledown Q >^1_{<\omega} P \triangledown_\omega Q >^1_{\omega|<\omega} P \triangledown_\infty Q \equiv^{<\omega}_1 P \oplus Q.$$

Homogeneity is an opposite notion of antichain (and independence). Recall that $S \subseteq \mathbb{N}^{\mathbb{N}}$ is *homogeneous* if $S = \prod_n S_n$ for some $S_n \subseteq \mathbb{N}$, $n \in \mathbb{N}$. Every antichain is degree-non-isomorphic everywhere. On the other hand, every homogeneous set S is *degree-isomorphic everywhere*, that is to say, $S \cap C$ is degree-isomorphic to $S \cap D$ for any clopen sets $C, D \subseteq \mathbb{N}^{\mathbb{N}}$ with $S \cap C \neq \emptyset$ and with $S \cap D \neq \emptyset$[4].

The next observation is that every finite-piecewise computable method of solving a homogeneous Π^0_1 mass problem can be refined by a finite-$(\Pi^0_1)_2$-piecewise computable method. That is to say, our hierarchy between $\leq^1_{<\omega}$ and $\leq^{<\omega}_1$ collapses for homogeneous Π^0_1 sets, modulo the $(1, <\omega)$-equivalence.

[4] An anonymous referee pointed out that the notion of degree-isomorphic everywhere is related to the notion of *fractal* in the study of Weihrauch degrees [9, 53]. The (reverse) lattice embedding d of the Medvedev degrees into the Weihrauch degrees has the property that a subset P of Baire space is degree-isomorphic everywhere if and only if $d(P)$ is a fractal.

Theorem 20. *For every homogeneous Π^0_1 set $S \subseteq \mathbb{N}^{\mathbb{N}}$ and for any set $Q \subseteq \mathbb{N}^{\mathbb{N}}$, if $S \leq^{<\omega}_1 Q$ then $S \leq^1_{<\omega} Q$.*

Proof. Let $S = \prod_x F_x$ for some Π^0_1 sets $F_x \subseteq \mathbb{N}$. Assume $S \leq^{<\omega}_1 Q$ via the bound b. That is, for every $g \in Q$ there exists an index $e < b$ such that $\Phi_e(g) \in S$. Let us begin defining a learner Ψ who changes his mind at most finitely often. Fix $g \in Q$. The learner Ψ first sets $A_0 = \{e \in \mathbb{N} : e < b\}$. By our assumption, we have $\Phi_e(g) \in S$ for some $e \in A_0$. Then the learner Ψ challenges to predict the solution algorithm $e < b$ such that $\Phi_e(g) \in S$ by using an observation $g \in Q$. He begins the 1-st challenge. On *the* $(s+1)$*-th challenge of* Ψ, inductively assume that, the learner have already defined a set $A_s \subseteq A_0$. Let v be a stage at which the $s+1$-th challenge of Ψ on g begins. In this challenge, the learner Ψ uses the two following computable functionals Γ and Δ.

- For a given argument x, $\Gamma(x, s+1)$ outputs the least $\langle e(x), t(x)\rangle$ such that $e(x) \in A_s$ and $\Phi_{e(x)}(g \upharpoonright t(x); x)\downarrow$ if such $\langle e(x), t(x)\rangle$ exists.
- If $\Gamma(x, s+1) = \langle e(x), t(x)\rangle$, then $\Delta(g; x, s+1) = \Phi_{e(x)}(g \upharpoonright t(x); x)$.

Set $\Delta_{s+1}(g; x) = \Delta(g; x, s+1)$. Clearly, an index $d(s+1)$ of Δ_{s+1} is calculated from $s+1$. Then the learner $\Psi(g \upharpoonright v)$ outputs $d(s+1)$ on the $(s+1)$-th challenge. Hence $\Phi_{\Psi(g\upharpoonright v)}(g; x) = \Phi_{d(s+1)}(g; x) = \Phi_{e(x)}(g \upharpoonright t(x); x)$ for any x. He does not change his mind until the beginning stage v' of the next challenge, i.e., $\Phi_{\Psi(g\upharpoonright v'')}(g) = \Phi_{\Psi(g\upharpoonright k)}(g)$ for $k \leq v'' < v'$. *The next challenge* might begin when it turns out that Ψ's prediction on his $(s+1)$-th challenge is incorrect, namely:

- $\Phi_{\Psi(g\upharpoonright v)}(g \upharpoonright u) \upharpoonright n \notin T_{S,u}$ for some $n < u$ at some stage $u > v$.

Here T_S is a corresponding computable tree of S. For each $x \in \mathbb{N}$, fix a decreasing approximation $\{F_{x,s}\}_{s\in\mathbb{N}}$ of a Π^0_1 set $F_x \subseteq \{0,1\}$, uniformly in x. In this case, there exists $x < n$ such that the following condition holds.

$$\Phi_{\Psi(g\upharpoonright v)}(g \upharpoonright u; x) = \Delta_{s+1}(g; x) = \Phi_{e(x)}(g; x) \notin F_{x,s}.$$

For such a least x, the learner removes $e(x)$ from A_s, that is, let $A_{s+1} = A_s \setminus \{e(x)\}$. If $A_{s+1} \neq \emptyset$ then the learner Ψ begins *the* $(s+2)$*-th challenge* at the current stage u. The construction of the learner Ψ is completed. An important point of this construction is that the learner never uses an index rejected on some challenge. This makes the prediction on $g \in Q$ of the learner Ψ converge.

Claim. Ψ changes his mind at most b times.

Whenever Ψ changes, A_s must decrease. However $\#A_0 = b$.

Claim. For every $g \in Q$ it holds that $\Phi_{\lim_s \Psi(g\upharpoonright s)}(g) \in S$.

For $g \in Q$, let $B^g \subseteq A_0$ be the set of all $e \in A_0$ such that $\Phi_e(g) \in S = \prod_x F_x$. By the definition of A_0, clearly B^g is not empty. Moreover, $B^g \subseteq \bigcap_s A_s$ holds, since e is removed from $\bigcap_s A_s$ only when $\Phi_e(g; x) \notin F_x$ for some x. Thus, $\Phi_{\Psi(g\upharpoonright v)}(g) : \mathbb{N} \to \mathbb{N}$ is total for every stage v. This means that, if $\Phi_{\Psi(g\upharpoonright v)}(g) \notin S$, then the learner Ψ will know his mistake at some stage u, i.e., $\Phi_{\Psi(g\upharpoonright v)}(g \upharpoonright u; x) \notin F_{x,u}$ for some $x < u$. Then some index is removed from $\bigcap_s A_s$. However, this occurs at most b times. Thus, $\Phi_{\lim_s \Psi(g\upharpoonright s)}(g) \in S$. □

Let $\alpha, \beta, \gamma \in \{1, <\omega, \omega\}$. We say that a $(\alpha, \beta|\gamma)$-degree **a** of a nonempty Π^0_1 subset of $2^{\mathbb{N}}$ is $(\alpha, \beta|\gamma)$-*complete* if $\mathbf{b} \leq \mathbf{a}$ for every $(\alpha, \beta|\gamma)$-degree **b** of a nonempty Π^0_1 subset of $2^{\mathbb{N}}$. If a Π^0_1 set $P \subseteq 2^{\mathbb{N}}$ has an $(\alpha, \beta|\gamma)$-complete $(\alpha, \beta|\gamma)$-degree, then it is also called $(\alpha, \beta|\gamma)$-*complete*.

Corollary 21. *A Π^0_1 subset of $2^{\mathbb{N}}$ is $(1, <\omega)$-complete if and only if it is $(1, \omega| <\omega)$-complete if and only if it is $(<\omega, 1)$-complete.*

Proof. Let DNR_2 denote the set of all two-valued diagonally noncomputable functions, where a function $f : \mathbb{N} \to 2$ is *diagonally noncomputable* if $f(e) \neq \Phi_e(e)$ for any index e. This set is clearly homogeneous, and Π^0_1. Moreover, it is $(1, 1)$-complete (hence $(\alpha, \beta|\gamma)$-complete for any $\alpha, \beta, \gamma \in \{1, <\omega, \omega\}$). Therefore, we can apply Theorem 20 with $S = \mathsf{DNR}_2$. □

Corollary 22. *There are Π^0_1 sets $P, Q \subseteq 2^{\mathbb{N}}$ such that $P \oplus Q \equiv^1_{<\omega} P \triangledown_\infty Q$. Indeed, if P is homogeneous and $Q \equiv^1_1 P$, then $P \oplus Q \equiv^1_{<\omega} P \triangledown_\infty Q$ is satisfied.*

Proof. Let P be any homogeneous Π^0_1 subset of $2^{\mathbb{N}}$. Then $P \oplus P$ is also homogeneous. As seen in Part I [29, Section 4], there is a $(2, 1)$-computable function from $P \triangledown_\infty P$ to $P \oplus P$, hence $P \oplus P \leq^{<\omega}_1 P \triangledown_\infty P$. Thus, by Theorem 20, $P \oplus P \leq^1_{<\omega} P \triangledown_\infty P$. Recall from Part I [29, Proposition 38] that $Q \equiv^1_1 P$ implies $P \triangledown_\infty P \equiv^1_1 P \triangledown_\infty Q$. Hence, $Q \equiv^1_1 P$ implies $P \oplus Q \equiv^1_1 P \oplus P \leq^1_{<\omega} P \triangledown_\infty P \equiv^1_1 P \triangledown_\infty Q$. □

It is natural to ask whether our hierarchy of disjunctive notions for homogeneous Π^0_1 sets also collapses *modulo the* $(1, 1)$-*equivalence*. The answer is *negative*. We say that a homogeneous set $\prod_n F_n$ is *computably bounded* if there is a computable function $l : \mathbb{N} \to \mathbb{N}$ such that $F_n \subseteq \{0, \dots, l(n)\}$ for any $n \in \mathbb{N}$. Clearly, every homogeneous subset of Cantor space $2^{\mathbb{N}}$ is computably bounded. Cenzer-Kihara-Weber-Wu [12] introduced the notion of immunity for closed sets. A closed subset P of Cantor space $2^{\mathbb{N}}$ is *immune* if T^{ext}_P has no infinite computable subset.

Theorem 23. *Let $P \subseteq 2^{\mathbb{N}}$ be a non-immune Π^0_1 set, and $S_0, S_1, \dots, S_m \subseteq \mathbb{N}^{\mathbb{N}}$ be special computably bounded homogeneous Π^0_1 sets. Then $\bigcup_{i \leq m} S_i \not\leq^1_1 P$.*

Proof. Let V_0 be an infinite c.e. subtree of T^{ext}_P. Assume that $\bigcup_{i \leq m} \prod_n F^i_n \leq^1_1 P$ via a computable functional Φ, where, for each $i < m$, $\{F^i_n\}_{n \in \omega}$ is a uniformly Π^0_1 sequence of subsets of $\{0, 1, \dots l_i\}$. Let S^{ext}_i denotes the corresponding Π^0_1 tree of $\prod_n F^i_n$, and let $L_i = \{\rho : (\exists \tau \in S^{ext}_i)(\exists i)\ \rho = \tau^\frown \langle i \rangle \notin S^{ext}_i\}$, for each i. Note that L_i *differs* from the set of leaves of the corresponding *computable* tree of $\prod_n F^i_n$. We first consider the set $L^\Phi_i = \{\rho \in L_i : (\exists \sigma \in V_i)\ \Phi(\sigma) \supseteq \rho\}$, where V_i for $0 < i \leq m$ will be defined in the below construction. Note that L^Φ_0 is computably enumerable. There are three cases:

1. L^Φ_0 is infinite;
2. L^Φ_0 is finite, hence $\Phi([V_0])$ is a subset of $\prod_n F^0_n$;
3. otherwise.

(Case 1): For any n, there exists $\rho \in L^\Phi_0$ of height $> n + 1$, and $\rho(n) \in F^0_n$. From any computable enumeration of L^Φ_0 we can calculate a computable path of $\prod_n F^0_n$. This contradicts the specialness of $S_0 = \prod_n F^0_n$.

(Case 2): There exists a finite number k such that, for every string $\sigma \in V_0$ of height $> k$, $\Phi(\sigma)$ belongs to S_i^{ext}. This also contradicts the specialness of $S_0 = \prod_n F_n^0$.

(Case 3): There exists infinitely many strings $\sigma \in V_0$ such that $\Phi(\sigma)$ extends some string of L_0^Φ. Since L_0^Φ is finite, by the pigeon hole principle, there exists $\rho_0 \in L_0^\Phi$ such that $\Phi(\sigma)$ extends ρ for infinitely many $\sigma \in V_0$. Fix such ρ_0, and let $V_1 = \{\sigma \in V_0 : \Phi(\sigma) \supseteq \rho\}$. Then the downward closure of V_1 is an infinite c.e. subtree of T_P^{ext}, and $\Phi([V_1]) \cap S_0 = \emptyset$.

By iterating this procedure, we win the either of the cases 1 or 2 for some $i \leq m$. The reason is that, if the case 3 occurs for j, then V_{j+1} is defined as an infinite c.e. subtree of T_P^{ext} such that $\Phi([V_1]) \cap (\bigcup_{i\leq j} S_i) = \emptyset$. Since $\bigcup_{i\leq m} \prod_n F_n^i \leq_1^1 P \leq [V_m]$, i.e., $\Phi([V_m]) \subseteq \bigcup_{i\leq m} S_i$, the case 3 does not occur for m. □

Corollary 24. *Let P, Q be any nonempty Π_1^0 subsets of $2^{\mathbb{N}}$, and S, T be special computably bounded homogeneous Π_1^0 sets. Then $S \cup T \not\leq_1^1 P^\frown Q$.*

Proof. Clearly $P^\frown Q$ is not immune. Thus, Theorem 23 implies $S \cup T \not\leq_1^1 P^\frown Q$. □

To understand degrees of difficulty of disjunctive notions, and to discover new *easier* (possibly infinitary) disjunctive notions, it is interesting to discuss *contiguous degrees*.

Definition 25. Let $(\alpha,\beta,\gamma), (\alpha^*,\beta^*,\gamma^*) \in \{1, <\omega, \omega\}^3$, and assume that $\leq_{\beta|\gamma}^{\alpha}$ is not finer than $\leq_{\beta^*|\gamma^*}^{\alpha^*}$. An $(\alpha,\beta|\gamma)$-degree $\mathbf{a}_{\beta|\gamma}^{\alpha}$ is $(\alpha^*,\beta^*|\gamma^*)$-*contiguous* if $\mathbf{a}_{\beta|\gamma}^{\alpha}$ contains at most one $(\alpha^*,\beta^*|\gamma^*)$-degree, that is to say, for any representatives $A, B \in \mathbf{a}_{\beta|\gamma}^{\alpha}$, we have that A is $(\alpha^*,\beta^*|\gamma^*)$-equivalent to B.

Corollary 26.

1. *There is a* $(1,<\omega)$*-contiguous* $(<\omega,1)$*-degree of* Π_1^0 *sets of* $2^{\mathbb{N}}$.
2. *Every* $(1,<\omega)$*-degree which contains a homogeneous* Π_1^0 *set or a* Π_1^0 *antichain is not* $(1,1)$*-contiguous.*
3. *Every* $(1,\omega|<\omega)$*-degree of* Π_1^0 *antichains is not* $(1,<\omega)$*-contiguous.*
4. *Every* $(<\omega,1)$*-degree of* Π_1^0 *antichains is not* $(1,\omega)$*-contiguous (hence, is not* $(1,\omega|<\omega)$*-contiguous).*

Proof. (1) This follows from Theorem 20.

(2) If $\mathbf{d}$ is a $(1,<\omega)$-degree of a homogeneous Π_1^0 set S, then $\mathbf{d}$ contains S and $S \triangledown S$, since $S \equiv_{<\omega}^1 S \triangledown S$. However, $S \triangledown S <_1^1 S \cup S = S$ by Corollary 24. If $\mathbf{d}$ is a $(1,<\omega)$-degree of a Π_1^0 antichain P, then $\mathbf{d}$ contains $(P \times 2^{\mathbb{N}}) \cup (2^{\mathbb{N}} \times P)$ and $P \triangledown P$, since $P \equiv_{<\omega}^1 (P \times 2^{\mathbb{N}}) \cup (2^{\mathbb{N}} \times P)$. However, $P \triangledown P <_1^1 (P \times 2^{\mathbb{N}}) \cup (2^{\mathbb{N}} \times P)$ holds by Lemma 6.

(3) Note that, for any Π_1^0 set P and any clopen set C, it holds that $(P \cap C) \oplus (P \setminus C) \equiv_1^1 P$. Let $\mathbf{d}$ be a $(1,\omega|<\omega)$-degree of a Π_1^0 antichain P. Fix a clopen set C such that $P_0 = P \cap C \neq \emptyset$, and $P_1 = P \setminus C \neq \emptyset$. Then $\mathbf{d}$ contains $P_0 \oplus P_1$ and $\overrightarrow{\bigoplus}_n (P_0 \triangledown_n P_1)$, since $P_0 \oplus P_1 \equiv_{\omega|<\omega}^1 \overrightarrow{\bigoplus}_n (P_0 \triangledown_n P_1)$. However, $\overrightarrow{\bigoplus}_n (P_0 \triangledown_n P_1) <_{<\omega}^1 P_0 \oplus P_1$ holds by Corollary 13.

(4) Let $\mathbf{d}$ be a $(<\omega,1)$-degree of a Π_1^0 antichain P. Fix a clopen set C such that $P_0 = P \cap C \neq \emptyset$, and $P_1 = P \setminus C \neq \emptyset$. Then $\mathbf{d}$ contains $P_0 \oplus P_1$ and $P_0 \triangledown_\omega P_1$, since $P_0 \oplus P_1 \equiv_1^{<\omega} P_0 \triangledown_\omega P_1$. However, $P_0 \triangledown_\omega P_1 <_l P_0 \oplus P_1$ holds by Corollary 15. □

We next show the non-existence of nonzero $(1,1)$-contiguous $(1,<\omega)$-degree, that is, we will see the LEVEL 4 separation between $[\mathfrak{C}_T]^1_1$ and $[\mathfrak{C}_T]^1_{<\omega}$. Indeed, we show the strong anti-cupping result for $(1,1)$-degrees inside every nonzero $(1,<\omega)$-degree via the concatenation operation. The following theorem is one of the most important and nontrivial results in this paper.

Theorem 27. *For any nonempty Π^0_1 sets $P,Q\subseteq 2^{\mathbb{N}}$, $Q^\frown P$ does not $(1,1)$-cup to P. That is to say, for any $R\subseteq\mathbb{N}^{\mathbb{N}}$, if $P\leq^1_1 (Q^\frown P)\otimes R$ then $P\leq^1_1 R$.*

Proof. We first note that P and Q may be assumed to be special. If P is not special, the assertion is trivial. If Q has a computable element, then $Q^\frown P$ also has a computable element. In this case, $(Q^\frown P)\otimes R\equiv^1_1 R$, and then the assertion is obvious. Therefore, we may assume that Q is special. Let T_P and T_Q be corresponding trees of P and Q, and let L_P and L_Q denote all leaves of T_P and T_Q, respectively. Note that T_Q is infinite since Q is special. For a tree $T\subseteq 2^{<\mathbb{N}}$ and $g\in\mathbb{N}^{\mathbb{N}}$, we write $T\otimes\{g\}$ for $\{\sigma\oplus\tau:\sigma\in T\ \&\ \tau\subset g\ \&\ |\sigma|=|\tau|\}$. For computable trees S and T, we also write $S^\frown T$ for $S\cup\bigcup_{\rho\in L_S}\rho^\frown T$, where L_S denotes the set of all leaves of S.

Assume $P\leq^1_1 (Q^\frown P)\otimes R$ via a computable functional Φ. We need to construct a computable functional Ψ witnessing $P\leq^1_1 R$. Fix $g\in R$. Then we will find a g-c.e. tree $D^g\subseteq T_P$ without dead ends. To this end, we inductively construct a uniformly g-computable sequences $\{D^g_i\}_{i\in\omega},\{E^g_i\}_{i\in\omega}$ of g-computable trees, as follows.

$$E^g_0=T_Q\otimes\{g\};\qquad D^g_0=\Phi(E^g_0).$$
$$E^g_{i+1}=(T_Q{}^\frown D^g_i)\otimes\{g\};\qquad D^g_{i+1}=\Phi(E^g_{i+1}).$$

Here $\Phi(E^g_i)$ denotes the image of E^g_i under a functional Φ, namely, $\Phi(E^g_i)=\{\tau\subseteq 2^{<\mathbb{N}}:(\exists\sigma\in E^g_i)\ \tau\subseteq\Phi(\sigma)\}$. Finally, we define a g-c.e. tree $D^g=\bigcup_n D^g_n$. Now, we let W be the tree $T_Q{}^\frown T_P$, and then we observe $[W]=Q^\frown P$ and $T_Q\subseteq W^{ext}$.

Lemma 28. *For any i, $D^g_i\subseteq T^{ext}_P$ and $E^g_i\subseteq W^{ext}\otimes\{g\}$.*

Proof. This lemma is proved by induction. First, our assumption $T_Q\subseteq W^{ext}$ ensures $E^g_0=T_Q\otimes\{g\}\subseteq W^{ext}\otimes\{g\}$, and we also have $D^g_0=\Phi(E^g_0)\subseteq T^{ext}_P$ since $\Phi((Q^\frown P)\otimes R)\subseteq[T_P]$ implies $\Phi(W^{ext}\otimes\{g\})\subseteq T^{ext}_P$ for $g\in R$. Assume the lemma holds for each $j\leq i$. We now show that the lemma also holds for $i+1$. By assumption, $T_Q{}^\frown D^g_i\subseteq T_Q{}^\frown T^{ext}_P=W^{ext}$. So by definition of E^g_{i+1}, we get $E^g_{i+1}\subseteq W^{ext}\otimes\{g\}$. Furthermore, we observe $D^g_{i+1}=\Phi(E^g_{i+1})\subseteq\Phi(W^{ext}\otimes\{g\})\subseteq T^{ext}_P$. ☐

Lemma 29. *There is a computable function Γ mapping each $g\in R$ to a g-computable sequence $\Gamma(g)=\{D^g_n\}_{n\in\omega}$ of g-computable trees.*

Proof. Clearly E^g_0 is computable in g, and $D^g_i\mapsto E^g_{i+1}$ is uniformly g-computable. Therefore, it suffices to show that we can construct D^g_i from E^g_i by a uniformly g-computable way. Our proof is essentially an effectivization of the classical fact saying that the continuous image of a compact space is compact (see also [49]).

Assume that $E^g_i\subseteq 2^{<\mathbb{N}}\otimes\{g\}$ is given. For each $\sigma\in 2^{\mathbb{N}}$, if $\sigma\oplus(g\upharpoonright|\sigma|)\in E^g_i$, then put $l(\sigma)=|\Phi(\sigma\oplus(g\upharpoonright|\sigma|))|$. If $\sigma\oplus(g\upharpoonright|\sigma|)\notin E^g_i$, then put $l(\sigma)=\infty$. Note that

$l : 2^{<\mathbb{N}} \to \mathbb{N} \cup \{\infty\}$ is g-computable, since the notation $\Phi(\sigma)$ just means the computation of Φ restricted to step $|\sigma|$ with the oracle σ. By Lemma 28, $\lim_n l(f \upharpoonright n) = \infty$ for any $f \in 2^{\mathbb{N}}$. Because, for any f with $f \oplus g \in [E_i^g]$, we have $\Phi(f \oplus g) \in [\Phi(E_i^g)] \subseteq \Phi([W] \otimes \{g\}) \subseteq \Phi((Q^\frown P) \otimes R) \subseteq P$, hence, $f \oplus g \in \mathrm{dom}(\Phi)$. Therefore, by compactness, for each $n \in \mathbb{N}$, there is $h_n \in \mathbb{N}$ such that $l(\sigma) \geq n$ for each $\sigma \in 2^{<\mathbb{N}}$ of length h_n. We can compute $h_i^g(n) = h_n$ with the oracle g, since l is g-computable. Here, we can compute a g-computable index of h_i^g from an index of E_i^g, uniformly in $i \in \mathbb{N}$ and $g \in \mathbb{N}^{\mathbb{N}}$. Thus, the relation $\tau \in D_i^g$ is equivalent to the g-computable condition that $\tau \subseteq \Phi(\sigma)$ for some $\sigma \in E_i^g$ of length $h_i^g(|\tau|)$, uniformly in $i \in \mathbb{N}$ and $g \in \mathbb{N}^{\mathbb{N}}$. Formally, the set $\{(\tau, i, g) \in 2^{<\mathbb{N}} \times \mathbb{N} \times \mathbb{N}^{\mathbb{N}} : \tau \in D_i^g\}$ is computable. □

Define L_{D_i} as the set of all leaves of the tree D_i^g, and define L_{E_i} as the set of all leaves of the tree E_i^g for each i.

Lemma 30. *Let X be D or E, and i be any natural number. For any $\rho \in L_{X_i}^g$, there are infinitely many nodes $\tau \in L_{X_{i+1}}^g$ which are extensions of ρ.*

Proof. This lemma is proved by induction. First we pick $\rho \in L_{E_0} = L_Q \otimes \{g\} = \{\sigma \oplus \tau : \sigma \in L_Q \ \&\ \tau \subset g \ \&\ |\sigma| = |\tau|\}$. We note that T_P is an infinite tree since P is special. By using our assumption $P \leq_1^1 (Q^\frown P) \otimes R$ via Φ and the property $[T_Q] \otimes \{g\} \subseteq (Q^\frown P) \otimes R$, the tree $D_0^g = \Phi(E_0^g)$ has a path, i.e., it is infinite. By definition, we have $E_1^g = T_P{}^\frown D_0^g \supseteq \rho^\frown D_0^g$, and so E_1^g has infinitely many extensions of ρ. Now, we assume this lemma for E and any $j \leq i$. For a given $\rho \in L_{D_i^g}$, there is a node $\sigma \in E_i^g$ such that $\Phi(\sigma) = \rho$ by our definition of $D_i^g = \Phi(E_i^g)$. Note that we have $\Phi(\sigma^*) = \rho$ for every $\sigma^* \in E_i^g$ extending such a σ, since $\Phi(\sigma^*) \in D_i^g$ extends $\Phi(\sigma) = \rho$ while ρ is a leaf of the tree D_i^g. Therefore, without loss of generality, we can pick σ as a leaf of E_i^g.

By induction hypothesis, σ has infinitely many extensions in E_{i+1}^g. By Lemma 28, we know $E_{i+1}^g \subseteq W^{ext} \otimes \{g\}$. This implies that $\Phi(E_{i+1}^g(\supseteq \sigma))$ must be infinite whenever $E_{i+1}^g(\supseteq \sigma)$ is infinite, where $E(\supseteq \sigma)$ denotes the set of all nodes in a tree E extending σ. We now remark that, for any $\sigma' \in \Phi(E_{i+1}^g(\supseteq \sigma))$, $\Phi(\sigma') \supseteq \Phi(\sigma) = \rho$. Thus, $\Phi(E_{i+1}^g(\supseteq \sigma))$ gives infinitely many extensions of ρ, and our definition $D_{i+1}^g = \Phi(E_{i+1}^g)$ clearly implies the lemma for D and i. Now, we will show the lemma for E and $i+1$. By our definition of $E_{i+1}^g = (T_Q{}^\frown D_i^g) \otimes \{g\}$, every $\rho \in L_{E_{i+1}}$ must be of form $\rho = (\sigma^\frown \tau) \oplus (g \upharpoonright |\sigma^\frown \tau|)$ for some $\sigma \in L_Q$ and $\tau \in L_{D_i}$. So if $\tau \in L_{D_i}$ has infinitely many extensions in $L_{D_{i+1}}$ then $\rho = (\sigma^\frown \tau) \oplus (g \upharpoonright |\sigma^\frown \tau|)$ has infinitely many extensions in $L_{E_{i+2}}$. Thus, we have established the lemma for E and $i + 1$. Now, the lemma follows by induction. □

As a consequence of the previous lemma, D^g turns out to be an infinite g-c.e. subtree of T_P without dead ends for any $g \in P$. Hence, we can compute a path through D^g uniformly in g as follows.

Lemma 31. *D^g has a g-computable path of T_P uniformly in $g \in R$.*

Proof. The set of all infinite paths through a c.e. tree of $\mathbb{N}^{<\mathbb{N}}$ without dead ends is also called a *c.e. closed* or *overt* ([49]) subset of $\mathbb{N}^{\mathbb{N}}$. If a nonempty set is c.e. closed, then one can easily find its computable element in a uniform way. □

Then we define a computable functional Ψ as $\Psi(g) = \Delta(\bigcup_n \Gamma(g))$ for any $g \in \mathbb{N}^{\mathbb{N}}$. This witnesses $P \leq^1_1 R$ as desired. □

Corollary 32. *For every special Π^0_1 set $P \subseteq 2^{\mathbb{N}}$, there exists a Π^0_1 set $Q \subseteq 2^{\mathbb{N}}$ such that $Q <^1_1 P$ and $Q \equiv^1_{<\omega} P$.*

Proof. By Theorem 27, we have $P^\frown P <^1_1 P \equiv^1_{<\omega} P$. □

Definition 33. Fix $\alpha,\beta,\gamma \in \{1, <\omega, \omega\}$. An $(\alpha,\beta|\gamma)$-degree $\mathbf{a} \in \mathcal{P}^\alpha_{\beta|\gamma}$ has *the strong anticupping property* if there is a nonzero $(\alpha,\beta|\gamma)$-degree $\mathbf{b} \in \mathcal{P}^\alpha_{\beta|\gamma}$ such that, for any $(\alpha,\beta|\gamma)$-degree $\mathbf{c}$, if $\mathbf{a} \leq \mathbf{b} \vee \mathbf{c}$, then $\mathbf{a} \leq \mathbf{c}$.

Corollary 34. *Every nonzero $\mathbf{a} \in \mathcal{P}^1_1$ has the strong anticupping property.*

Proof. Fix $P \in \mathbf{a}$. Let $\mathbf{b}$ be the $(1,1)$-degree of $P^\frown P$. Then, by Theorem 27, for any $(1,1)$-degree $\mathbf{c}$, if $\mathbf{a} \leq \mathbf{b} \vee \mathbf{c}$, then $\mathbf{a} \leq \mathbf{c}$. □

For Π^0_1 sets, if P and Q are disjoint, then $P \oplus Q$ is equivalent to $P \cup Q$ modulo the $(1,1)$-equivalence, since $\mathcal{P}^1_1 = \mathcal{P}/\mathrm{dec}^{<\omega}_{\mathrm{p}}[\Pi^0_1]$. However, if P and Q are not Π^0_1, the above claim is false, in general.

Proposition 35. *For any special Π^0_1 set $P \subseteq 2^{\mathbb{N}}$, there exists a $(\Pi^0_1)_2$ set $Q \subseteq 2^{\mathbb{N}}$ such that $P \cap Q = \emptyset$ but $P \cup Q <^1_1 P \oplus Q$.*

Proof. Put $Q = (P^\frown P) \setminus P$. For any $g \in Q$, there is a leaf $\rho \in L_P$ such that $\rho \subset g$. So we wait for such a leaf $\rho \in L_P$. Then $g^{\leftarrow|\rho|}$ belongs to P. Hence, $P \leq^1_1 Q$. Thus, we have $P \leq^1_1 P \oplus Q$, while $P \cup Q = P^\frown P <^1_1 P$ by Theorem 27. □

Definition 36. The operation $\triangledown : \mathcal{P}(\mathbb{N}^{\mathbb{N}}) \to \mathcal{P}(\mathbb{N}^{\mathbb{N}})$ is defined as the map sending P to $P^\triangledown = \mathsf{CPA}^\frown P$, where CPA denotes the set of all complete consistent extensions of Peano Arithmetic, and it is a $(1,1)$-complete Π^0_1 subset of $2^{\mathbb{N}}$.

By the previous theorem, the derived set $P^\triangledown$ does not $(1,1)$-cup to P whenever P is Π^0_1. In particular, we have $P^\triangledown <^1_1 P$. Recall from Part I [29, Proposition 38] that the operator $\triangledown : \mathcal{P}^1_1 \to \mathcal{P}^1_1$ introduced by $(\deg^1_1(P))^\triangledown = \deg^1_1(P^\triangledown)$ is well-defined. Moreover, $\mathcal{P}^1_1(\leq \mathbf{1}^\triangledown) = \{\mathbf{a} \in \mathcal{P}^1_1 : \mathbf{a} \leq \mathbf{1}^\triangledown\}$ is a principal prime ideal consisting of tree-immune-free Medvedev degrees [12]. Here, recall that a Π^0_1 set $P \subseteq 2^{\mathbb{N}}$ is tree-immune if T^{ext}_P contains no infinite computable subtree. Then, we also observe the following.

Proposition 37. *Fix Π^0_1 sets $P_0, P_1, Q_0, Q_1 \subseteq 2^{\mathbb{N}}$, and assume that $P_0{}^\frown P_1 \leq^1_1 Q_0{}^\frown Q_1$. Then, either $P_0 \leq^1_1 Q_0$ or $P_1 \leq^1_1 Q_1$ holds. Moreover, if P_0 is tree-immune and Q_0 is nonempty, then $P_1 \leq^1_1 Q_1$.*

Proof. Assume that $P_0{}^\frown P_1 \leq^1_1 Q_0{}^\frown Q_1$ via a computable function Φ. If $\Phi(\rho) \in T^{ext}_{P_0}$ for any leaf $\rho \in L_{Q_0}$, then $\Phi(g) \in [T_{P_0}]$ for any $g \in [Q_0]$, i.e., $P_0 \leq^1_1 Q_0$. If $\Phi(\rho) \notin T^{\mathrm{ext}}_{P_0}$ for some leaf $\rho \in L_{Q_0}$, then there are only finitely many strings of T_{P_0} extending $\Phi(\rho)$. Thus, $[T_{P_0}{}^\frown T_{P_1}] \cap [\Phi(\rho)]$ is essentially a sum of finitely many P_1's, hence it is $(1,1)$-equivalent to P_1. Since a computable functional Φ maps $\rho^\frown Q_1$ to the above class, obviously, $P_1 \leq^1_1 Q_1$. If P_0 is tree-immune, then $\Phi(\rho) \notin T^{\mathrm{ext}}_{P_0}$ for some leaf $\rho \in L_{Q_0}$, since otherwise the image of T_{Q_0} under Φ is included in T_{P_0}, and clearly it is infinite and computable. Therefore, we must have $P_1 \leq^1_1 Q_1$. □

Corollary 38. *The operator* $\triangledown : \mathbf{a} \mapsto \mathbf{a}^{\triangledown}$ *is injective. Hence,* $\triangledown$ *provides an order-preserving self-embedding of the* $(1,1)$*-degrees* $\mathcal{P}^1_1$ *of nonempty* Π^0_1 *subsets of* $2^{\mathbb{N}}$.

Proof. By Cenzer-Kihara-Weber-Wu [12], CPA is tree-immune. Therefore, by Proposition 37, $\mathsf{CPA}^\frown Q = Q^{\triangledown} \leq^1_1 P^{\triangledown} = \mathsf{CPA}^\frown P$ implies $Q \leq^1_1 P$. □

It is natural to ask whether the image of $\mathcal{P}^1_1$ under the operator is exactly $\mathcal{P}^1_1(\leq \mathbf{1}^{\triangledown})$. Unfortunately, it turns out to be false.

Proposition 39. *There exists a non-tree-immune* Π^0_1 *set* $Q \subseteq 2^{\mathbb{N}}$ *such that no nonempty* Π^0_1 *sets* $P_0, P_1 \subseteq 2^{\mathbb{N}}$ *satisfy* $Q \equiv^1_1 P_0{}^\frown P_1$. *In particular, the operator* $\triangledown : \mathcal{P}^1_1 \to \mathcal{P}^1_1(\leq \mathbf{1}^{\triangledown})$ *is not surjective.*

Proof. Let $\{Q_n\}_{n\in\mathbb{N}}$ be a computable sequence of nonempty Π^0_1 subsets of $2^{\mathbb{N}}$ such that $\bigoplus_{n\in\mathbb{N}} Q_n$ forms a Turing antichain. Define $Q = Q_0{}^\frown\{Q_{n+1}\}_{n\in\mathbb{N}}$. Suppose that there exist nonempty Π^0_1 sets $P_0, P_1 \subseteq 2^{\mathbb{N}}$ with $Q \equiv^1_1 P_0{}^\frown P_1$. Choose computable functions $\Phi : Q \to P_0{}^\frown P_1$ and $\Psi : P_0{}^\frown P_1 \to Q$. Since $\{Q_n\}_{n\in\mathbb{N}}$ forms a Turing antichain, $\Psi \circ \Phi$ is an identity function on Q. Consider two cases.

The first case is that $\Phi(Q) \subseteq P_0$. In this case, $\Psi(P_0) = Q$ since $\Psi \circ \Phi$ is identity. Thus, every string in T^{ext}_Q is extended by some string in $\Psi(T_{P_0})$. Moreover, he condition $T_{P_0} \subseteq T^{ext}_{P_0{}^\frown P_1}$ implies $\Psi(T_{P_0}) \subseteq T^{ext}_Q$. Therefore, $\Psi(T_{P_0}) = T^{ext}_Q$. Hence T^{ext}_Q is a computable tree without leaves. But this is impossible since Q contains no computable elements.

The second case is that $\Phi(Q) \not\subseteq P_0$, that is, there exists $f \in Q$ such that $\Phi(f) \in \rho^\frown P_1$, where ρ is a leaf of T_{P_0}. We have $f \equiv_T \Phi(f)$ since $\Psi \circ \Phi$ is identity. Note that we may assume that $f = \rho_k{}^\frown f_k$ for some leaf $\rho_k \in T_{Q_0}$ and $f_k \in Q_k$, since even if $f \in Q_0$ the string $\Phi(f \upharpoonright n)$ extends ρ for sufficiently large n, and replace f with a string extending $f \upharpoonright n$ which is removed from Q_0. On the one hand, f is the only element in Q computable in f. On the other hand, every $\sigma \in T_{P_0}$ always extends to an element of P which is Turing equivalent to f. Thus, for every $\sigma \in T_{P_0}$, the string $\Phi(\sigma)$ must be compatible with ρ_k. Hence, $\Psi(P) \subseteq \rho_k{}^\frown Q_k$. This contradicts the property that $\Psi \circ \Phi(Q) = Q$. □

Let $\mathcal{O}$ denote Kleene's system of ordinal notations (see [52]). As usual, this system involves a representation $|\cdot|_{\mathcal{O}} :\subseteq \mathbb{N} \to \omega_1^{CK}$ of computable ordinals with a Π^1_1 domain $\mathrm{dom}(|\cdot|) = \mathcal{O}$, where $|0|_{\mathcal{O}} = 0$, $|2^a|_{\mathcal{O}} = |a|_{\mathcal{O}}+1$, and $|3\cdot 5^e|_{\mathcal{O}} = \sup_n |\Phi_e(n)|_{\mathcal{O}}$ if $\Phi_e : \mathbb{N} \to \mathbb{N}$ is total and strictly increasing. Recall from Part I [29, Definition 62] that $P^{(a)}$ is the a-th derivative of P, i.e., the a-th iterated concatenation starting from P, for every $a \in \mathcal{O}$.

Proposition 40. *For any special* Π^0_1 *set* $P \subseteq 2^{\mathbb{N}}$*, if* $a, b \in \mathcal{O}$ *and* $a <_{\mathcal{O}} b$*, then* $P^{(b)}$ *does not* $(1,1)$*-cup to* $P^{(a)}$*, i.e., for any set* $R \subseteq \mathbb{N}^{\mathbb{N}}$*, if* $P^{(a)} \leq^1_1 P^{(b)} \otimes R$ *then* $P^{(a)} \leq^1_1 R$.

Proof. The assumption $a <_{\mathcal{O}} b$ implies $2^a \leq_{\mathcal{O}} b$. Therefore, we have $P^{(b)} \leq^1_1 P^{(2^a)}$. By Theorem 27, $P^{(2^a)}$ does not $(1,1)$-cup to $P^{(a)}$. Thus, $P^{(b)}$ does not $(1,1)$-cup to $P^{(a)}$. □

Fix any notation $\mathtt{omega} \in \mathcal{O}$ such that $|\Phi_{\mathtt{omega}}(n)|_{\mathcal{O}} = n$ for each $n \in \mathbb{N}$. Note that $|\mathtt{omega}|_{\mathcal{O}} = \omega$.

Proposition 41. *Let* P *be a special* Π^0_1 *subset of* $2^{\mathbb{N}}$. *For any* Π^0_1 *set* $R \subseteq 2^{\mathbb{N}}$*, if* $P \leq^1_{<\omega} P^{(\mathtt{omega})} \otimes R$*, then* $P \leq^1_{<\omega} R$.

Proof. As seen in Part I [29, Section 2.4], for every Π^0_1 sets $P, Q \subseteq 2^{\mathbb{N}}$, $P \leq^1_{<\omega}$ implies $P \leq^1_{tt,<\omega} Q$. Since $P^{(\text{omega})} \otimes R$ is Π^0_1, $P \leq^1_{<\omega} P^{(\text{omega})} \otimes R$ implies $P \leq^1_{tt,<\omega} P^{(\text{omega})} \otimes R$, and then there is a $(1, n)$-truth-table function $\Gamma : P^{(\text{omega})} \otimes R \to P$ for some $n \in \mathbb{N}$. In particular, $\Gamma : (\rho_{n+1}{}^\frown P^{(\Phi_{\text{omega}}(n+1))}) \otimes R \to P$, where ρ_{n+1} is the $(n+1)$-th leaf of T_P. By modifying Γ, we can easily construct a $(1, n)$-truth-table function $\Theta : P^{(n+1)} \otimes R \to P$.

Assume that Θ is $(1, n)$-truth-table via n many total computable functions $\Theta_0, \dots, \Theta_{n-1}$. We define a computable function $\gamma : n \times 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ as follows. If $\Theta_m(\sigma) \in T_P$, then put $\gamma(m, \sigma) = \Theta_m(\sigma)$. If $\Theta_m(\sigma) \supsetneq \rho$ for some $\rho \in L_P$, then we define $\gamma(m, \sigma)$ to be such ρ. Let $z(\sigma) = \min\{m < n : \Theta_m(\sigma) \in T_P\}$. Then, for $\sigma \in 2^{<\mathbb{N}}$, the value $\Phi(\sigma)$ is defined by $\bigsqcap_{m \leq z(\sigma)} \gamma(m, \sigma)$. Then Φ ensures that $P^{(n)} \leq^1_1 P^{(n+1)} \otimes R$. By Theorem 27, we have $P^{(n)} \leq^1_1 R$. Consequently, $P \leq^1_{<\omega} R$. □

Corollary 42. *For every $a \in \mathcal{O}$ there exists a computable function g such that, for any Π^0_1 index e, if P_e is special then the following properties hold.*

1. *$P_{g(e,b)} <^1_1 P_{g(e,c)}$ holds for every $c <_{\mathcal{O}} b <_{\mathcal{O}} a$, indeed, $P_{g(e,b)}$ does not $(1, 1)$-cup to $P_{g(e,c)}$.*
2. *$P_{g(e,b)} \equiv^1_\omega P_{g(e,c)}$ for every $b, c <_{\mathcal{O}} a$.*

Proof. Let $g(e, b)$ be an index of $P^{(b)}_e$. Then, the desired conditions follow from Proposition 40. □

For any reducibility notion r, and any ordered set $(I, \leq_I)$, a sequence $\{\mathbf{a}_i\}_{i \in I}$ of r-degrees is *r-noncupping* if, for any $i <_I j$, the condition $\mathbf{a}_i \leq_r \mathbf{b}$ must be satisfied whenever $\mathbf{a}_i \leq_r \mathbf{a}_j \vee \mathbf{b}$, for any r-degree $\mathbf{b}$. In particular, any r-noncupping sequence is strictly decreasing, in the sense of r-degrees.

Corollary 43. *For any nonzero $(1, \omega)$-degree $\mathbf{a} \in \mathcal{P}^1_\omega$, there is a $(1, 1)$-noncupping computable sequence of $(1, 1)$-degrees inside $\mathbf{a}$ of arbitrary length $\alpha < \omega^{CK}_1$.* □

3.3. Infinitary Disjunctions along the Straight Line

We next see the LEVEL 4 separation between $[\mathfrak{C}_T]^1_{\omega|<\omega}$ and $[\mathfrak{C}_T]^1_\omega$. Indeed, we show the non-existence of a $(<\omega, 1)$-contiguous $(1, \omega)$-degree. We introduce the LCM disjunctions of $\{P_i\}_{i \in \mathbb{N}}$ as $\bigtriangledown_{n \in \mathbb{N}} P_n = \bigcup_{n \in \mathbb{N}} (P_0{}^\frown \dots {}^\frown P_n)$. This is a straightforward infinitary iteration of the concatenations. If $P_n = P$ for all $n \in \mathbb{N}$, we write $\bigtriangledown P$ instead of $\bigtriangledown_n P_n$.

Proposition 44. *Let $\{P_i\}_{i \in \mathbb{N}}$ be a computable collection of nonempty Π^0_1 subsets of $2^{\mathbb{N}}$. Then $\bigtriangledown_n P_n$ is $(1, 1)$-equivalent to a dense Σ^0_2 set in Cantor space $2^{\mathbb{N}}$.*

Proof. Let S denote the set $\{g \in \{0, 1, \sharp\}^{\mathbb{N}} : (\exists n \in \mathbb{N})\ (\mathtt{count}(g) = n\ \&\ \mathtt{tail}(g) \in P_n)\}$, where $\mathtt{count}(g) = \#\{n \in \mathbb{N} : g(n) = \sharp\}$. Then, S is clearly a Σ^0_2 subset of $\{0, 1, \sharp\}^{\mathbb{N}}$, and it is easy to see $S \equiv^1_1 \bigtriangledown_n P_n$. For any $\sigma \in \{0, 1, \sharp\}^{<\mathbb{N}}$, we have $\sigma^\frown \langle \sharp \rangle^\frown h \in S$ for any $h \in P_{\mathtt{count}(\sigma)+1}$. Thus, S intersects with any clopen set. □

Example 45. Let MLR denote the set of all Martin-Löf random reals. Then $\mathsf{MLR} \equiv^1_1 \bigtriangledown P$ for any nonempty Π^0_1 set $P \subseteq \mathsf{MLR}$, by Kučera-Gács Theorem (see [48]), while $\mathsf{MLR} <^1_1 P$ for any Π^0_1 set $P \subseteq \mathsf{MLR}$ as follows.

Proposition 46 (Lewis-Shore-Sorbi [39]). *No somewhere dense set in Baire space $(1,1)$-cup to a closed set in Baire space. In other words, for any somewhere dense set $D \subseteq \mathbb{N}^{\mathbb{N}}$, any closed set $C \subseteq \mathbb{N}^{\mathbb{N}}$, and any set $R \subseteq \mathbb{N}^{\mathbb{N}}$, if $C \leq^1_1 D \otimes R$ then $C \leq^1_1 R$.* □

Proposition 47. *For any somewhere dense set $D \subseteq \mathbb{N}^{\mathbb{N}}$ and any special closed set $C \subseteq \mathbb{N}^{\mathbb{N}}$, we have $C \not\leq^{<\omega}_1 D$.*

Proof. If $\{D_i\}_{i<b}$ is a finite partition of D, then $\bigcup_{i<b} \mathrm{Cl}_{\mathbb{N}^{\mathbb{N}}}(D_i) = \mathrm{Cl}_{\mathbb{N}^{\mathbb{N}}}(D)$, where the topological closure of D, in the standard Baire topology on $\mathbb{N}^{\mathbb{N}}$, is denoted by $\mathrm{Cl}_{\mathbb{N}^{\mathbb{N}}}(D)$. To show the claim, for every $x \in \mathrm{Cl}_{\mathbb{N}^{\mathbb{N}}}(D)$ we have a sequence $\{x_k\}_{k\in\mathbb{N}} \subseteq D$ converging to x. By pigeonhole principle, there is $i < b$ such that there are infinitely many k such that $x_k \in D_i$. For such i, clearly $x \in \mathrm{Cl}_{\mathbb{N}^{\mathbb{N}}}(D_i)$. However, since the somewhere density of D implies that $\mathrm{Cl}_{\mathbb{N}^{\mathbb{N}}}(D)$ contains some clopen set, and hence $\mathrm{Cl}_{\mathbb{N}^{\mathbb{N}}}(D_i)$ contains a computable element r for some i. Additionally, $\mathrm{Cl}_{\mathbb{N}^{\mathbb{N}}}(C) = C$ since C is closed. If $C \leq^{<\omega}_1 D \otimes R$, then there is a finite partition $\{D_i\}_{i<b}$ of D such that $C \leq^1_1 D_i$ via a computable function $\Phi_{e(i)}$. Fix i such that $\mathrm{Cl}_{\mathbb{N}^{\mathbb{N}}}(D_i)$ contains a computable element. Therefore, $C = \mathrm{Cl}_{\mathbb{N}^{\mathbb{N}}}(C) \leq^1_1 \mathrm{Cl}_{\mathbb{N}^{\mathbb{N}}}(D_i) \supseteq \{r\}$ via $\Phi^f_{e(i)}$. Hence, C contains a computable element. □

Especially, if P is a special Π^0_1 set, then there is no nonzero $(<\omega,1)$-degree of Π^0_1 subsets of $2^{\mathbb{N}}$ below the $(<\omega,1)$-degree of $\bigtriangledown P$. We will see that the set $\bigtriangledown P$ has a stronger property.

Theorem 48. *Let P be any Π^0_1 subset of $2^{\mathbb{N}}$. Then, for every special Π^0_1 set $Q \subseteq 2^{\mathbb{N}}$, there exists a Π^0_1 set $\widehat{P} \subseteq \bigtriangledown P$ such that $Q \not\leq^1_{<\omega} \widehat{P}$.*

The n-th bounded learner will be diagonalized above the n-th leaf of the spine T_P, where note that $P = [T_P] \subseteq \bigtriangledown P$. To make a desired Π^0_1 set inside the Σ^0_2 set $\bigtriangledown P$, we need to specify upper bounds of mind-changes to diagonalize all bounded learners. Unfortunately, we cannot give a computable sequence of such upper bounds. However, our *finite injury* construction will specify such upper bounds by a left-c.e. way, which will be called a timekeeper.

Definition 49. A sequence $\langle t_n\rangle_{n\in\mathbb{N}}$ of finite strings is *a timekeeper* if there is a uniformly c.e. collection of finite sets, $\{V_n\}_{n\in\mathbb{N}}$, such that, for any $n \in \mathbb{N}$, $|t_n| = |V_n|$ and $t_n(i)$ is given as the stage at which the i-th element is enumerated into V_n, for each $i < |t_n|$.

Definition 50. For a finite string $\tau \in \mathbb{N}^{<\mathbb{N}}$, *the τ-delayed* $(|\tau|+1)$*-derivative* $P^{(\tau)}$ is inductively defined as follows:

$$P^{(\tau\upharpoonright 0)} = P; \qquad P^{(\tau\upharpoonright i+1)} = \bigcup\{\sigma^\frown P : \sigma \in L_{P^{(\tau\upharpoonright i)}} \;\&\; |\sigma| \geq \tau(i)\} \text{ for each } i < |\tau|.$$

Proposition 51. *If $\tau(m) = 0$ for each $m < |\tau|$, then $P^{(\tau)} = P^{(|\tau|+1)}$.*

Proof. Straightforward from the definition. □

Lemma 52. *For any timekeeper $\langle t_n\rangle_{n\in\mathbb{N}}$, the following conditions hold.*

1. $P^{(t_n)} \subseteq P^{(|t_n|+1)}$. *Hence, $P^\frown\{P^{(t_n)}\}_{n\in\mathbb{N}} \subset \bigtriangledown P$.*

2. $P^{(t_n)}$ *is* Π^0_1*, uniformly in n. Hence,* $P^\frown\{P^{(t_n)}\}_{n\in\mathbb{N}}$ *is* Π^0_1*.*

Proof. (1) Straightforward. (2) We construct a computable tree $T^{(t_n)}$ corresponding to $P^{(t_n)}$. Each $\sigma \in 2^{\mathbb{N}}$ can be represented as $\sigma = \rho_0{}^\frown\rho_1{}^\frown\dots{}^\frown\rho_k{}^\frown\tau$, where $\rho_m \in L_P$ for any $m \le k$, and $\langle\rangle \ne \tau \in T_P$. Then $\sigma \in T^{(t_n)}$ if and only if $t_n(k)$ holds by stage $|\rho_0{}^\frown\rho_1{}^\frown\dots{}^\frown\rho_k|$. Then $T^{(t_n)}$ is a computable tree, and clearly $P^{(t_n)} = [T^{(t_n)}]$. □

Remark. The delayed derivative construction is useful to bound the complexity of the set, since the recursive meet $P^\frown\{P^{(|t_n|+1)}\}_{n\in\mathbb{N}}$ of the standard derivatives along a timekeeper $\{t_n\}_{n\in\mathbb{N}}$ is only assured to be $\Pi^{0,\emptyset'}_1$.

Proof of Theorem 48. Let Q be a special Π^0_1 set, and P be a given Π^0_1 set. By a uniformly computable procedure, from P, we will construct a timekeeper $\{t_n\}_{n\in\mathbb{N}}$. The desired class $\widehat{P}$ will be given by $\widehat{P} = P^\frown\{P^{(t_n)}\}_{n\in\mathbb{N}}$.

Requirements. We need to ensure, for all $n \in \mathbb{N}$, the following:

$$R_n \ : \ Q \le^1_{<\omega} \widehat{P} \text{ via } n \ \rightarrow \ (\exists\Delta_n)\ \Delta_n \in Q.$$

Here, Δ_n ranges over computable elements of $2^{\mathbb{N}}$.

Action of an R_n-strategy. Fix an effective enumeration $\{\rho_n : n \in \mathbb{N}\}$ of all leaves of T_P. An R_n-strategy uses nodes extending the n-th leaf ρ_n of T_P, and it constructs a finite sequence $t_n[s]$, a sequence $\tau_n[s]$ of strings, and a computable function Δ_n. For any n, put $t_n[0] = \langle\rangle$, and $\tau_n[0] = \rho_n$ at stage 0. *An R_n-strategy acts at stage $s+1$ if the* following condition holds:

$$(\exists\rho \in T^s_P)(\exists e < n)\ \Phi_e(\tau_n[s]^\frown\rho) \in T_Q \ \&\ \Phi_e(\tau_n[s]^\frown\rho) \supsetneq \Phi_e(\tau_n[s]).$$

If an R_n-strategy acts at stage $s+1$ then, for a witness $\rho \in T^s_P$, we pick $\rho^* \in L_P$ extending ρ. Then let us define $\tau_n[s+1] = \tau_n[s]^\frown\rho^*$, $t_n[s+1] = t_n[s]^\frown\langle|\tau_n[s+1]|\rangle$, and $\Delta_{e,n} \upharpoonright l = \Phi_e(\tau_n[s+1])$, where l is the length of $\Phi_e(\tau_n[s+1])$. Otherwise, $t_n[s+1] = t_n[s]$, $\tau_i[s+1] = \tau_i[s]$. Note that the mapping $(n,m) \mapsto \tau_n(m)$ is partial computable. At the end of the construction, set $t_n = \bigcup_s t_n[s]$. As mentioned above, $\widehat{P}$ is defined by $\widehat{P} = P^\frown\{P^{(t_n)}\}_{n\in\mathbb{N}}$.

Claim. An R_n-strategy acts at most finitely often for each n.

Clearly $\tau_n = \bigcup_s \tau_n[s]$ is a computable string. If R_n acts infinitely often, then $\Delta_{e,n} = \Phi_e(\tau_n) \in Q$ for some $e < n$ by our choice of τ_n. Since $\Phi_e(\tau_n)$ is computable, Q contains a computable element. However, this contradicts our assumption that Q is special. Therefore, we concludes the claim. As a corollary, $\langle t_n\rangle_{n\in\mathbb{N}}$ is a timekeeper.

Claim. $P \not\le^1_{<\omega} \widehat{P}$.

Let $\tau_n = \bigcup_s \tau_n[s]$. By induction we show that $\tau_n \in \rho_n{}^\frown T^{ext}_{P^{(t_n)}}$. First we have the following observation:

$$\tau_n[0] = \rho_n \in \rho_n{}^\frown T^{ext}_P = \rho_n{}^\frown T^{ext}_{P^{(t_n\upharpoonright 0)}} \subseteq \rho_n{}^\frown T^{ext}_{P^{(t_n[0])}}.$$

Assume $\tau_n[s] \in \rho_n{}^\frown T^{ext}_{P^{(t_n[s])}}$. If $\tau_n[s+1] = \tau_n[s]^\frown\rho^*$ for $\rho^* \in L_P$ then $t_n[s+1] = t_n[s]^\frown\langle|\tau_n[s+1]|\rangle$. In particular $\tau_n[s+1] \in \rho_n{}^\frown L_{P^{(t_n[s+1]\upharpoonright|t_n[s]|)}}$ and $|\tau_n[s+1]| \ge t_n[s+$

$1](|t_n[s]|)$. Hence, by the definition of $P^{(t_n[s+1])}$, it is easy to see that $\tau_n[s+1]^\frown P \subseteq \rho_n{}^\frown P^{(t_n[s+1])}$. Thus, $\tau_n[s+1] \in \rho_n{}^\frown T^{ext}_{P^{(t_n[s+1])}}$. So we obtain $\tau_n \in \rho_n{}^\frown T^{ext}_{P^{(t_n)}}$ and by our construction of τ_n there is no $\rho \in P$ and $e < n$ such that $\Phi_e(\tau_n{}^\frown\rho) \supsetneq \Phi_e(\tau_n)$. Since $\Phi_e(\tau_n)$ is a finite string, for any $g \in \rho_n{}^\frown T^{ext}_{P^{(t_n)}} \subset \hat{P}$ extending τ_n, $\Phi_e(g)$ is also a finite string. Consequently, this g witnesses that $P \not\leq^1_{<\omega} \widehat{P}$. □

Corollary 53. 1. *For every special Π^0_1 set $P \subseteq 2^{\mathbb{N}}$, we have $\bigtriangledown P <^{<\omega}_1 P \equiv^1_\omega \bigtriangledown P$.*
2. *For every special Π^0_1 set $P \subseteq 2^{\mathbb{N}}$ there exists a Π^0_1 set $Q \subseteq 2^{\mathbb{N}}$ with $Q <^{<\omega}_1 P \equiv^1_\omega Q$.*

Proof. By applying Theorem 48 to $Q = P$, we have $P \not\leq^1_{<\omega} \widehat{P} \geq^1_{<\omega} \bigtriangledown P$. Moreover, $P \oplus \widehat{P} <^{<\omega}_1 P \equiv^1_\omega P \oplus \widehat{P}$. □

3.4. *Infinitary Disjunctions along ill-Founded Trees*

We next show the LEVEL 4 separation between $[\mathfrak{C}_T]^1_\omega$ and $[\mathfrak{C}_T]^{<\omega}_\omega$. The following theorem concerning the hyperconcatenation $\blacktriangledown$ and the $(1,\omega)$-reducibility $\leq^1_\omega$ is a counterpart of Theorem 27 concerning the concatenation $\triangledown$ and the $(1,1)$-reducibility $\leq^1_1$.

Theorem 54. *For every special Π^0_1 sets $P, Q \subseteq 2^{\mathbb{N}}$, and for any R, if $P \leq^1_\omega (Q\blacktriangledown P) \otimes R$ then $P \leq^1_\omega R$ holds.*

Proof. Let $T_\blacktriangledown$ denote the corresponding computable tree for $Q\blacktriangledown P$. *The heart of $T_\blacktriangledown$*, $T^\heartsuit_\blacktriangledown$, is the set of all strings $\gamma \in T_\blacktriangledown$ such that $\gamma \subseteq \bigsqcap_{i<n}(\sigma_i{}^\frown\langle\tau(i)\rangle)$ for some $\{\sigma_i\}_{i<n} \subseteq L_P$, and $\tau \in T^{ext}_Q$. If γ is precisely of the form $\bigsqcap_{i<n}(\sigma_i{}^\frown\langle\tau(i)\rangle)$, then γ is called a *quasi-root* of $T^\heartsuit_\blacktriangledown$.

Lemma 55. *The heart $T^\heartsuit_\blacktriangledown$ is a Π^0_1 subtree of $T_\blacktriangledown$. Moreover, the complexity of the set of all quasi-roots of $T^\heartsuit_\blacktriangledown$ is also Π^0_1.*

Proof. The first assertion is trivial. For the second assertion, by an effective way, every string $\sigma \in 2^{<\mathbb{N}}$ is uniquely decomposed into $\sigma_0, m_0, \sigma_1, m_1, \dots, \sigma_n, m_n, \rho$ such that $\sigma = (\bigsqcap_{i<n}(\sigma_i{}^\frown m_i))^\frown\rho$ and $\{\sigma_i\}_{i\leq n} \subseteq L_P$. Recall from Part I [29, Definition 70] that $\langle\sigma_0, \sigma_1, \dots, \sigma_n, \rho\rangle$ is written as $\mathtt{cut}(\sigma)$, $\langle m_0, m_1, \dots, m_n\rangle$ is written as $\mathtt{walk}(\sigma)$, and ρ is written as $\mathtt{tail}^{\mathtt{cut}}(\sigma)$. Clearly, one can effectively determine whether $\mathtt{tail}^{\mathtt{cut}}(\sigma) = \langle\rangle$ or not. Now, σ is a quasi-root of $T^\heartsuit_\blacktriangledown$ if and only if $\sigma \in T^\heartsuit_\blacktriangledown$ and $\mathtt{tail}^{\mathtt{cut}}(\sigma) = \langle\rangle$. □

Now we assume $P \leq^1_\omega (Q\blacktriangledown P) \otimes R$ via a learner Ψ. To show the theorem it is needed to construct a new learner Δ witnessing $P \leq^1_\omega R$. Fix $g \in R$.

Lemma 56. *There exists a string $\rho \in T^\heartsuit_\blacktriangledown$ such that, for every $\tau \in T^\heartsuit_\blacktriangledown$ extending ρ, we have $\Psi(\rho \oplus (g \upharpoonright |\rho|)) = \Psi(\gamma)$ for any γ with $\rho \oplus (g \upharpoonright |\rho|) \subseteq \gamma \subseteq \tau \oplus (g \upharpoonright |\tau|)$.*

Proof. If Lemma 56 were false, we could inductively define an increasing sequence $\{\tau_i\}_{i\in\omega}$ of strings. First let $\tau_0 = \langle\rangle$, and τ_{i+1} be the least $\tau \supsetneq \tau_i$ such that $\tau \in T^\heartsuit_\blacktriangledown$ and $\Psi(\tau \oplus (g \upharpoonright (|\tau|+i))) \neq \Psi(\rho \oplus (g \upharpoonright (|\rho|+j)))$ for some $i, j < 2$. Since $\bigcup_i \tau_i \in Q\blacktriangledown P$, clearly $(\bigcup_i \tau_i) \oplus g \in (Q\blacktriangledown P) \otimes R$. However, based on the observation $(\bigcup_i \tau_i) \oplus g$, the learner Ψ changes his mind infinitely often. This means that his prediction $\lim_n \Psi((\bigcup_i \tau_i) \oplus g)$ diverges. This contradicts our assumption that $P \leq^1_\omega (Q\blacktriangledown P) \otimes R$ via the learner Ψ. Thus, our claim is verified. □

Lemma 56 can be seen as an analogy of an observation of Blum-Blum [4] in the theory of inductive inference for total computable functions on $\mathbb{N}$. Such ρ is sometimes called *a locking sequence*.

Lemma 57. *There exist an effective procedure $\Theta : \mathbb{N}^{\mathbb{N}} \times 2^{<\mathbb{N}} \times 2 \to \mathbb{N}^{\mathbb{N}}$ and a Π^0_1 condition φ such that, for any $g \in Q$, $\varphi(g, \rho, m)$ holds for some $\rho \in 2^{<\mathbb{N}}$, and $m < 2$, and that for any $\rho \in 2^{<\mathbb{N}}$ and $m \in \mathbb{N}$, if $\varphi(g, \rho, m)$ holds, then $\Theta(g, \rho, m) \in P$.*

Proof. The desired condition $\varphi(g, \rho, m)$ is given by the conjunction of the following three conditions.

1. ρ is a quasi-root of $T^{\heartsuit}_{\blacktriangledown}$.
2. $\tau^\frown\langle m\rangle \in T^{ext}_Q$.
3. $\Psi(\rho \oplus (g \upharpoonright |\rho|)) = \Psi(\gamma)$ for any $\gamma \in (\rho^\frown T_P{}^\frown\langle m\rangle^\frown T_P) \otimes \{g\}$.

By Lemma 55, the first condition is Π^0_1. The second condition is clearly Π^0_1. Since Ψ is total computable, the last condition is also Π^0_1. Consequently, φ is Π^0_1. We first show that $\varphi(g, \rho, m)$ holds for some $\rho \in 2^{<\mathbb{N}}$ and $m \in \mathbb{N}$. Let $\rho \in T^{\heartsuit}_{\blacktriangledown}$ be a locking sequence in Lemma 56, which forces Ψ to stop changing the mind. Without loss of generality, we can assume that ρ satisfies the condition (1). Since $\tau \in T^{ext}_Q$, there exists $m \in \omega$ such that $\tau^\frown\langle m\rangle \in T^{ext}_Q$, and this m satisfies the condition (2). From conditions (1) and (2), we conclude that $\rho^\frown P^\frown\langle m\rangle^\frown P = (\rho^\frown P) \cup (\rho^\frown \bigcup_{\sigma \in L_P} \sigma^\frown\langle m\rangle^\frown P) \subseteq T^{\heartsuit}_{\blacktriangledown}$, and so condition (3) is satisfied. Since we assume that $P \leq^1_\omega (Q \blacktriangledown P) \otimes \{g\}$ via the learner Ψ, if $\varphi(g, \rho, m)$ is satisfied, then the following holds.

$$P \leq^1_1 (\rho^\frown P^\frown\langle m\rangle^\frown P) \otimes \{g\} \text{ via } \Phi_{\Psi(g \upharpoonright |\rho| \oplus \rho)}.$$

Our proof process in Theorem 27 is effective with respect to g, m, and an index of $\Phi_{\Psi(g \upharpoonright |\rho| \oplus \rho)}$ which are calculated from g, ρ, and an index of Ψ. To see this, recall our proof in Theorem 27. Define $V^m_P = T_P \cup \{\rho^\frown\langle m\rangle : \rho \in L_P\}$.

$$E^{g,\rho,m}_0 = V^m_P \otimes \{g\}; \qquad D^{g,\rho,m}_0 = \Phi_{\Psi(g \upharpoonright |\rho| \oplus \rho)}(E^{g,\rho,m}_0).$$
$$E^{g,\rho,m}_{i+1} = (V^{m\frown}_P D^{g,\rho,m}_i) \otimes \{g\}; \qquad D^{g,\rho,m}_{i+1} = \Phi_{\Psi(g \upharpoonright |\rho| \oplus \rho)}(E^{g,\rho,m}_{i+1}).$$

Then, as in the proof of Theorem 27, $D^{g,\rho,m} = \bigcup_{i \in \mathbb{N}} D^{g,\rho,m}_{i+1}$ is a subtree of V_P, and it has no dead ends. Moreover, this construction is clearly c.e. uniformly in g, ρ, and m. Therefore, we can effectively choose an element $\Theta(g, \rho, m) \in [D^{g,\rho,m}] \subseteq P$, uniformly in g, ρ, and m. □

Now, a procedure to get $P \leq^1_\omega R$ is follows. For given $g \in Q$, on *the i-th challenge* of a learner Δ, the learner Δ chooses the lexicographically i-th least pair $\langle \rho, m\rangle \in 2^{<\mathbb{N}} \times \mathbb{N}$, and Δ calculates an index $e(\rho, m)$ of the computable functional $g \mapsto \Theta(g, \rho, m)$, that is to say, $\Delta(g \upharpoonright s) = e(\rho, m)$ at the current stage s. At each stage in the i-th challenge, the learner Δ tests whether the Π^0_1 condition $\varphi(g, \rho, m)$ is refuted. When $\varphi(g, \rho, m)$ is refuted, Δ changes his mind, and goes to the $(i + 1)$-th challenge. Clearly $\lim_s \Delta(g \upharpoonright s)$ converges, and $\Phi_{\lim_s \Delta(g \upharpoonright s)}(g) \in P$ holds. □

Corollary 58. *For every special Π^0_1 set $P \subseteq 2^{\mathbb{N}}$ there exists a Π^0_1 set $Q \subseteq 2^{\mathbb{N}}$ with $Q <^1_\omega P \equiv^{<\omega}_\omega Q$.*

Proof. By Theorem 54, if $P \leq^1_\omega (P \blacktriangledown P) \otimes 2^{\mathbb{N}} \equiv^1_1 P \blacktriangledown P$, then $P \leq^1_\omega 2^{\mathbb{N}}$, i.e., P contains a computable element. As P is special, we must have $P \not\leq^1_\omega P \blacktriangledown P$. As seen in Part I [29, Section 4], $P \leq^{<\omega}_\omega P \blacktriangledown P$. Therefore, for $Q = P \blacktriangledown P$, we have $Q <^1_\omega P \equiv^{<\omega}_\omega Q$. □

Corollary 59. *Every nonzero* $\mathbf{a} \in \mathcal{P}^1_\omega$ *has the strong anticupping property.*

Proof. Fix $P \in \mathbf{a}$. Let $\mathbf{b}$ be the $(1, \omega)$-degree of $P \blacktriangledown P$. Then, by Theorem 54, for any $(1, \omega)$-degree $\mathbf{c}$, if $\mathbf{a} \leq \mathbf{b} \vee \mathbf{c}$, then $\mathbf{a} \leq \mathbf{c}$. □

The primary motivation of the second author behind introducing the notions of learnability reduction was to attack an open problem on Π^0_1 subsets of $2^{\mathbb{N}}$. The problem (see Simpson [57]) is whether the Muchnik degrees ($(\omega, 1)$-degrees) of Π^0_1 classes are dense. Cenzer-Hinman [13] showed that the Medvedev degrees ($(1, 1)$-degrees) of Π^0_1 classes are dense. One can easily apply their priority construction to prove densities of $(1, <\omega)$-degrees and $(<\omega, 1)$-degrees. The reason is that the arithmetical complexity of $A^\alpha_\beta = \{(i, j) \in \mathbb{N}^2 : P_i \leq^\alpha_\beta P_j\}$ is Σ^0_3 for $(\alpha, \beta) \in \{(1, 1), (1, <\omega), (<\omega, 1)\}$, where $\{P_e\}_{e \in \mathbb{N}}$ is an effective enumeration of all Π^0_1 subsets of $2^{\mathbb{N}}$. It enables us to use a priority argument directly. However, for other reductions (α, β), the complexity of A^α_β seems to be Π^1_1. For instance, Cole-Simpson [17] showed that $\{\langle i, j\rangle : P_i \leq^\omega_1 P_j\}$ is Π^1_1-complete. This observation hinders us from using priority arguments. Hence it seems to be a hard task to prove densities of such (α, β)-degrees. Nevertheless, our disjunctive notions turn out to be useful to obtain some partial results.

Theorem 60 (Weak Density)**.** *For nonempty* Π^0_1 *sets* $P, Q \subseteq 2^{\mathbb{N}}$*, if* $P <^1_\omega Q$ *and* $P <^{<\omega}_\omega Q$ *then there exists a* Π^0_1 *set* $R \subseteq 2^{\mathbb{N}}$ *such that* $P <^1_\omega R <^1_\omega Q$*.*

Proof. Assume $P <^1_\omega Q$ and $P <^{<\omega}_\omega Q$. Let $R = (Q \blacktriangledown Q) \otimes P$. Then $P \leq^1_\omega R \leq^1_\omega Q$. Moreover $Q \not\leq^1_\omega P$ implies $Q \not\leq^1_\omega R = (Q \blacktriangledown Q) \otimes P$, by non-cupping property of $\blacktriangledown$. On the other hand, $R = (Q \blacktriangledown Q) \otimes P \not\leq^1_\omega P$ since $Q \blacktriangledown Q \equiv^{<\omega}_\omega Q \not\leq^{<\omega}_\omega P$. Consequently, $P <^1_\omega R = (Q \blacktriangledown Q) \otimes P <^1_\omega Q$. □

One can introduce a transfinite iteration $P^{\blacktriangledown(a)}$ of hyperconcatenation along $a \in \mathcal{O}$ (see also the nested tape model introduced in Part I [29, Section 5.6]).

Proposition 61. *For any special* Π^0_1 *set* $P \subseteq 2^{\mathbb{N}}$*, if* $a, b \in \mathcal{O}$ *and* $a <_{\mathcal{O}} b$*, then* $P^{\blacktriangledown(b)}$ *does not* $(1, \omega)$*-cup to* $P^{\blacktriangledown(a)}$*, i.e., for any set* $R \subseteq \mathbb{N}^{\mathbb{N}}$*, if* $P^{\blacktriangledown(a)} \leq^1_\omega P^{\blacktriangledown(b)} \otimes R$ *then* $P^{\blacktriangledown(a)} \leq^1_\omega R$*.*

Proof. The assumption $a <_{\mathcal{O}} b$ implies $2^a \leq_{\mathcal{O}} b$. Therefore, we have $P^{\blacktriangledown(b)} \leq^1_\omega P^{\blacktriangledown(2^a)}$. By Theorem 54, $P^{\blacktriangledown(2^a)}$ does not $(1, \omega)$-cup to $P^{\blacktriangledown(a)}$. Thus, $P^{\blacktriangledown(b)}$ does not $(1, \omega)$-cup to $P^{\blacktriangledown(a)}$. □

Fix again any notation $\texttt{omega} \in \mathcal{O}$ such that $|\Phi_{\texttt{omega}}(n)|_{\mathcal{O}} = n$ for each $n \in \mathbb{N}$. Recall from Part I that a learner Ψ is *eventually-Popperian* if, for every $f \in \mathbb{N}^{\mathbb{N}}$, $\Phi_{\lim_s \Psi(f \upharpoonright s)}(f)$ is total whenever $\lim_s \Psi(f \upharpoonright s)$ converges.

Proposition 62. *Let* P *be a special* Π^0_1 *subset of* $2^{\mathbb{N}}$*. For any set* $R \subseteq \mathbb{N}^{\mathbb{N}}$*, if* $P \leq^{<\omega}_\omega P^{\blacktriangledown(\texttt{omega})} \otimes R$ *by a team of eventually-Popperian learners, then* $P \leq^{<\omega}_\omega R$*.*

Proof. If $P \leq_\omega^{<\omega} P^{\blacktriangledown(\mathtt{omega})} \otimes R$ via a team of eventually-Popperian learners, then this reduction is also witnessed by a team of n eventually-Popperian learners, for some $n \in \mathbb{N}$. In particular, by modifying this reduction, we can easily construct a team of n eventually-Popperian learners witnessing $P \leq_\omega^{<\omega} P^{\blacktriangledown(n+1)} \otimes R$. In this case, it is not hard to show $P^{\blacktriangledown(n)} \leq_1^1 P^{\blacktriangledown(n+1)} \otimes R$. By Theorem 54, $P^{\blacktriangledown(n)} \leq_\omega^1 R$. Hence, $P \leq_\omega^{<\omega} R$ is witnessed by a team of n learners, as seen in Part I [29, Proposition 75]. □

Corollary 63. *For every $a \in O$ there exists a computable function g such that, for any Π_1^0 index e, if P_e is special then the following properties hold.*

1. $P_{g(e,b)} <_\omega^1 P_{g(e,c)}$ *holds for every $c <_O b <_O a$, indeed, $P_{g(e,b)}$ does not $(1,\omega)$-cup to $P_{g(e,c)}$.*
2. $P_{g(e,b)} \equiv_1^\omega P_{g(e,c)}$ *for every $b, c <_O a$.*

Proof. Let $g(e,b)$ be an index of $P_e^{\blacktriangledown(b)}$. Then the desired conditions follow from Proposition 61. □

Corollary 64. *For any nonzero $(\omega,1)$-degree $\mathbf{a} \in \mathcal{P}_1^\omega$, there is a $(1,\omega)$-noncupping computable sequence of $(1,\omega)$-degrees inside $\mathbf{a}$ of arbitrary length $\alpha < \omega_1^{CK}$.* □

3.5. Infinitary Disjunctions along Infinite Complete Graphs

The following is the last LEVEL 4 separation result, which reveals a difference between $[\mathfrak{C}_T]_\omega^{<\omega}$ and $[\mathfrak{C}_T]_1^\omega$.

Theorem 65. *For every special Π_1^0 set $P, Q \subseteq 2^\mathbb{N}$ there exists a Π_1^0 set $\widehat{P} \subseteq 2^\mathbb{N}$ such that $Q \not\leq_\omega^{<\omega} \widehat{P}$ and $\widehat{P}$ is $(\omega,1)$-equivalent to P.*

Proof. We construct a Π_1^0 set $P \subseteq 2^\mathbb{N}$ by priority argument with infinitely many requirements $\{\mathcal{P}_e, \mathcal{G}_e\}_{e\in\mathbb{N}}$. Each preservation ($\mathcal{P}_e$-)strategy will injure our coding ($\mathcal{G}$-)strategy of P into $\widehat{P}$ infinitely often. In other words, for each $\mathcal{P}_e$-requirement, $\widehat{P}$ contains an element $f_e^\star$ which is a counterpart of each $f \in P$, but each $f_e^\star$ has infinitely many noises. Indeed, to satisfy the $\mathcal{P}$-requirements, we need to ensure that there is no uniformly team-learnable way to extract the information of $f \in P$ from its code $f_e^\star \in \widehat{P}$. Nevertheless, the global ($\mathcal{G}$-)requirement must guarantee that $f \in P$ is computable in $f_e^\star \in \widehat{P}$ via a non-uniform way. Let $\{\Psi_i^e\}_{i<b(e)}$ be the e-th team of learners, where $b = b(e)$ is the number of members of the e-th team.

Requirements. It suffices to construct a Π_1^0 set $\widehat{P} \subseteq 2^\mathbb{N}$ satisfying the following requirements.

$$\mathcal{P}_e : (\exists g_e \in \widehat{P})(\forall i < b) \left(\lim_s \Psi_i^e(g_e \upharpoonright s)\downarrow \to \ \Phi_{\lim_s \Psi_i^e(g_e\upharpoonright s)}(g_e) \notin Q\right).$$
$$\mathcal{G}_e : (\forall f \in P)\ f \leq_T f_e^\star.$$

Here, the desired Π_1^0 set $\widehat{P} \subseteq 2^\mathbb{N}$ will be of form $P \cup \{f_e^\star : e \in \mathbb{N}\ \&\ f \in P\}$.

Construction. We will construct a computable sequence of computable trees $\{T_s\}_{s\in\mathbb{N}}$, and a computable sequence of natural numbers $\{h_s\}_{s\in\mathbb{N}}$. The desired set $\widehat{P}$ is defined as $[\bigcup_s T_s]$, and h_s is called *active height at stage s*. We will ensure that the tree T_s consists of strings of length $\leq h_s$. The strategy for the $\mathcal{P}_e$-requirement acts on some string extending the e-th leaf ρ_e of T_{CPA}.

We will inductively define a string $\gamma_e(\alpha, s) \in T_s$ extending ρ_e for each $s \in \mathbb{N}$ and $\alpha \in T_P$ of height $\leq s$. The map $\alpha \mapsto \lim_s \gamma_e(\alpha, s)$ restricted to T_P^{ext} will provide a tree-isomorphism between T_P^{ext} and $(\bigcup_s T_s)^{ext}$, i.e., $\widehat{P} \cap [\rho_e]$ will be constructed as the set of all infinite paths of the tree generated by $\{\lim_s \gamma_e(\alpha, s) : \alpha \in T_P\}$. In other words, $f_e^\star$ is defined by $\bigcup_{\alpha\subset f} \lim_s \gamma_e(\alpha, s)$, and each string $\gamma_e(\alpha, s)$ is an approximation of $g_e \in \widehat{P}$ witnessing to satisfy the $\mathcal{P}_e$ requirements.

We will also define a finite set $M_e(\alpha, s) \subseteq b$ for each $s \in \mathbb{N}$ and $\alpha \in T_P$ of height $\leq s$. Intuitively, $M_e(\alpha, s)$ contains any index of the learner who have been already changed his mind $|\alpha|$ times along any string extending α of length s, and the string $\gamma_e(\alpha, s)$ also plays the role of an active node for learners in $M_e(\alpha, s)$. To satisfy the $\mathcal{P}_e$-requirement, each learner in $M_e(\alpha, s)$ can act on $\gamma_e(\alpha, s)$ at stage $s + 1$, and then he extends $\gamma_e(\alpha, s)$ to some new string $\gamma_e(\alpha, s + 1)$ of length h_s, and *injures* all constructions of $\gamma_e(\beta, s + 1)$ for $\beta \supsetneq \alpha$. We assume that, for any $\alpha \in T_P$ of length s, $\{M_e(\beta, s)\}_{\beta\subseteq\alpha}$ is a partition of $\{i \in \mathbb{N} : i < b\}$.

Stage 0. At first, put $T_s = \{\langle\rangle\}$, $h_s = 0$, $M_e(\langle\rangle, 0) = \{i \in \mathbb{N} : i < b\}$, and $\gamma_e(\langle\rangle, 0) = \rho_e$.

Stage $s + 1$**.** At the beginning of each stage $s + 1$, assume that T_s and h_s are given, and that $M_e(\beta, s)$ and $\gamma_e(\beta, s)$ have been already defined for each $s \in \mathbb{N}$ and $\beta \in T_P$ of height $\leq s$. For each $i, e \in \mathbb{N}$ and each $\tau \in 2^{\mathbb{N}}$, *the length-of-agreement function* $l_e^i(\tau)$ is the maximal $l \in \mathbb{N}$ such that $\Phi_{\Psi_i^e(\tau)}(\tau; x)\downarrow$ for each $x < l$, and $\Phi_{\Psi_i^e(\tau)}(\tau) \in T_Q$.

Fix a string $\alpha \in T_P$ of length s, and then each i belongs to some $M_e(\beta, s)$ for $\beta \subseteq \alpha$. In this case, the learner Ψ_i^e can act on $\gamma_e(\beta, s)$. Then, we say that *the learner* Ψ_i^e *requires attention along* α *at stage* $s + 1$ if there exists $\tau \in T_s$ of length h_s extending $\gamma_e(\beta, s)$ such that either of the following conditions are satisfied.

1. Ψ_i^e *changes on* $(\gamma_e(\beta, s), \tau]$, i.e., there is a string σ such that $\gamma_e(\beta, s) \subsetneq \sigma \subseteq \tau$ and $\Psi_i^e(\sigma^-) \neq \Psi_i^e(\sigma)$.
2. or, $l_e^i(\tau) > \max\{l_e^i(\sigma) : \sigma \subseteq \gamma_e(\beta, s)\}$.

Let R_s be the set of all $\alpha \in T_P$ of length s such that some learner requires attention along α at stage $s + 1$. For $\alpha \in R_s$, let $m(\alpha)$ be the least m such that there is a string $\beta \subseteq \alpha$ of length m and an index $i \in M_e(\beta, s)$ such that Ψ_i^e requires attention along α at stage $s + 1$. That is to say, some learner Ψ_i^e who has already changed his mind $m(\alpha)$ times requires attention.

Claim. For any $\alpha, \beta \in R_s$, we have that $\alpha \upharpoonright m(\alpha) = \beta \upharpoonright m(\beta)$ holds or $\alpha \upharpoonright m(\alpha)$ is incomparable with $\beta \upharpoonright m(\beta)$.

Put $R_s^* = \{\alpha \upharpoonright m(\alpha) : \alpha \in R_s\}$. Then, for $\beta \in R_s^*$, let $i(\beta)$ be the least $i \in M(m(\alpha), s)$ such that Ψ_i^e requires attention along some $\alpha \supseteq \beta$ of length s at stage $s + 1$. For $\beta \in R_s^*$, we say that $\Psi_{i(\beta)}^e$ *acts at stage* $s + 1$. Moreover, for $\beta \in R_s^*$, let $\tau(\beta)$ be the lexicographically least string $\tau \in T_s$ of length h_s extending $\gamma_e(\beta, s)$ such that τ

witnesses that the learner $\Psi^e_{i(\beta)}$ requires attention along some $\alpha \supseteq \beta$ of length s at stage $s+1$. Then $R^{**}_s \subseteq R^*_s$ is defined as the set of all $\beta \in R^*_s$ such that $\Psi^e_{i(\beta)}$ changes on $(\gamma_e(\beta, s), \tau(\beta)]$.

For each $\beta \in R^{**}_s$, put $M_e(\beta, s+1) = M_e(\beta, s) \setminus \{i(\beta)\}$, and put $M_e(\beta^\frown i, s+1) = M_e(\beta, s) \cup \{i(\beta)\}$ for $\beta^\frown i \in T_P$. For any $\beta \notin R^{**}_s$, put $M_e(\beta, s+1) = M_e(\beta, s)$. For each $\beta \in R^*_s$, if $\beta^\frown\sigma \in T_P$ is length $\leq s$ for some $\sigma \in 2^{<\mathbb{N}}$, then put $\gamma_e(\beta^\frown\sigma, s+1) = \tau(\beta)^\frown\sigma$. If $\alpha \in T_P$ of length $\leq s$ has no substring $\beta \in R^*_s$, then put $\gamma_e(\alpha, s+1) = \gamma_e(\alpha, s)$. For each $\alpha \in T_P$ of length s, if $|\gamma_e(\alpha, s+1)| < h_s$ then pick the lexicographically least node $\gamma^*_e(\alpha, s+1) \in T_s$ such that $|\gamma^*_e(\alpha, s+1)| = h_s$ and $\gamma^*_e(\alpha, s+1) \supseteq \gamma_e(\alpha, s+1)$. Otherwise put $\gamma^*_e(\alpha, s+1) = \gamma_e(\alpha, s+1)$. Then, for each $\alpha^\frown i \in T_P$ of length s, put $\gamma_e(\alpha^\frown i, s+1) = \gamma^*_e(\alpha, s+1)^\frown i$. Put $h_{s+1} = \max\{|\gamma_e(\alpha, s+1)| : \alpha \in T_P \ \&\ |\alpha| = s+1\}$. Then we define the approximation of $\widehat{P}$ at stage $s+1$ as follows.

$$T_{s+1} = T_s \cup \{\sigma \subseteq \gamma_e(\alpha, s+1)^\frown 0^{h_{s+1} - |\gamma_e(\alpha, s+1)|} : \alpha \in T_P \ \&\ |\alpha| = s+1 \ \&\ e \in \mathbb{N}\}.$$

Finally, we set $\widehat{P} = [\bigcup_{s \in \mathbb{N}} T_s]$. Clearly, $\widehat{P}$ is a nonempty Π^0_1 subset of $2^{\mathbb{N}}$.

Lemma 66. *$\lim_s \gamma_e(\alpha, s)$ converges for any $e \in \mathbb{N}$ and $\alpha \in T_P$.*

Proof. Note that $\gamma_e(\alpha, s)$ is incomparable with $\gamma_e(\beta, s)$ whenever α is incomparable with β. Therefore, $\gamma_e(\alpha, s)$ changes only when some learner in $M_e(\beta, s)$ acts for some $\beta \subseteq \alpha$. Assume that $\gamma_e(\alpha, s)$ changes infinitely often. Then there is $\beta \subseteq \alpha$, $t \in \mathbb{N}$ and $i \in M_e(\beta, t)$ such that $i \in M_e(\beta, s)$ for any $s \geq t$, and $\Psi^e_{i(\beta)}$ acts infinitely often. However, by our construction, $g^\alpha_e = \lim_s \gamma_e(\alpha, s)$ is computable. Additionally, since $i \in M_e(\beta, s)$ for any $s \geq t$, $\lim_n \Psi^e_{i(\beta)}(g^\alpha_e \upharpoonright n)$ exists, and $\Phi_{\lim_n \Psi^e_{i(\beta)}(g^\alpha_e \upharpoonright n)}(g^\alpha_e) \in Q$. This contradicts our assumption that Q is special. ◻

For $f \in P$, put $f^\star_e = \bigcup_{\alpha \subset f} \lim_s \gamma_e(\alpha, s)$. By this lemma, such $f^\star_e$ exists, and we observe that $\widehat{P}$ can be represented as $\widehat{P} = P \cup \{f^\star_e : e \in \mathbb{N} \ \&\ f \in P\}$. For each $e \in \mathbb{N}$ and $\alpha \in T_P$, we pick $t(e, \alpha) \in \mathbb{N}$ such that $\gamma_e(\alpha, s) = \gamma_e(\alpha, t)$ for any $s, t \geq t(e, \alpha)$.

Lemma 67. *The $\mathcal{P}$-requirements are satisfied.*

Proof. Assume that $P \leq^{<\omega}_\omega \widehat{P}$ via the e-th team $\{\Psi_i\}_{i<b}$ of learners. Then, for any $f \in P$, there is $i < b$ such that $\lim_n \Psi_i(f^\star_e \upharpoonright n)$ exists and $\Phi_{\lim_n \Psi_i(f^\star_e \upharpoonright n)}(f^\star_e) \in Q$. Since $\lim_n \Psi_i(f^\star_e \upharpoonright n)$ exists, there exists $\alpha \subset f$ such that $i \in M_e(\alpha, t(e, \alpha))$. However, by the previous claim, no learner in $\bigcup_{\beta \subseteq \alpha} M_e(\beta, t(e, \alpha))$ requires attention after stage $t(e, \alpha)$. This implies $\lim_n l^i_e(f^\star_e \upharpoonright n) < \infty$. In other words, $\Phi_{\lim_n \Psi_i(f^\star_e \upharpoonright n)}(f^\star_e) \notin Q$. This contradicts our assumption. ◻

Lemma 68. *The $\mathcal{G}$-requirements are satisfied.*

Proof. It suffices to show that $f \leq_T f^\star_e$ for any $e \in \mathbb{N}$ and $f \in P$. Assume that $\{\Psi_i\}_{i<b}$ is the e-th team of learners. Let $H_e(f)$ denote the set of all $i < b$ such that $\lim_n \Psi_i(f^\star_e \upharpoonright n)$ converges. By our construction and the first claim, if $i \in H_e(f)$ then $i \in M_e(\alpha_i, t(e, \alpha_i))$ for some $\alpha_i \subset f$. If $i \notin H_e(f)$ then for any $\alpha \subset f$ there exists s such that $i \in M_e(\alpha, s)$. Set $l = \max_{i \in H_e(f)} |\alpha_i|$, and $u = \max_{i \in H_e(f)} t(e, \alpha_i)$. For $n > l$, to compute $f(n)$, we wait for stage $v(n) > u$ such that, for every $i \notin H_e(f)$, $i \in M_e(f \upharpoonright m, v(n))$ for some $m \geq n+1$.

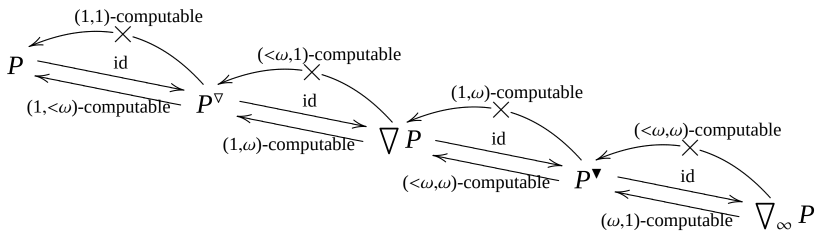

Figure 2: The dynamic proof model for a special Π^0_1 set $P \subseteq 2^{\mathbb{N}}$.

By our construction, it is easy to see that we can extract $f(n)$ from $\gamma_e(f \upharpoonright n+1, v(n))$, by a uniformly computable procedure in n. ◻

Thus, we have $Q \not\leq^{<\omega}_{\omega} \widehat{P}$ by Lemma 67, and $P \subseteq \widehat{P} \subseteq \widehat{\mathrm{Deg}}(P)$ by Lemma 68. Thus, $\widehat{P}$ is a Π^0_1 set satisfying $Q \not\leq^{<\omega}_{\omega} \widehat{P} \equiv^{\omega}_1 P$. This concludes the proof. ◻

Corollary 69. *For any nonempty Π^0_1 sets $P, Q \subseteq 2^{\mathbb{N}}$, if $Q \leq^{<\omega}_{\omega} \widehat{\mathrm{Deg}}(P)$ then Q contains a computable element.*

Proof. Assume that $Q \leq^{<\omega}_{\omega} \widehat{\mathrm{Deg}}(P)$ is satisfied. Suppose that Q has no computable element. Then, for $P, Q \subseteq 2^{\mathbb{N}}$, we obtain $Q \not\leq^{<\omega}_{\omega} \widehat{P} \equiv^{\omega}_1 P$ by Theorem 65. Note that the condition $\widehat{P} \equiv^{\omega}_1 P$ implies $\widehat{P} \subseteq \widehat{\mathrm{Deg}}(P)$. Then, $Q \leq^{<\omega}_{\omega} \widehat{\mathrm{Deg}}(P) \leq^1_1 \widehat{P}$. It involves a contradiction. ◻

4. Applications and Questions

4.1. Diagonally Noncomputable Functions

A total function $f : \mathbb{N} \to \mathbb{N}$ is a *k-valued diagonally noncomputable function* if $f(n) < k$ for any $n \in \mathbb{N}$ and $f(e) \neq \Phi_e(e)$ whenever $\Phi_e(e)$ converges. Let DNR_k denote the set of all k-valued diagonally noncomputable functions. Jockusch [33] showed that every DNR_k function computes a DNR_2 function. However, he also noted that there is no uniformly computable algorithm finding a DNR_2 function from any DNR_k function.

Theorem 70 (Jockusch [33])**.**

1. $\mathsf{DNR}_k >^1_1 \mathsf{DNR}_{k+1}$ *for any* $k \in \mathbb{N}$.
2. $\mathsf{DNR}_2 \equiv^{\omega}_1 \mathsf{DNR}_k$ *for any* $k \in \mathbb{N}$.

Proposition 71.

1. *If a $(1, \omega)$-degree $\mathbf{d}^1_{\omega}$ of subsets of $\mathbb{N}^{\mathbb{N}}$ contains a $(1,1)$-degree $\mathbf{h}^1_1$ of homogeneous sets, then $\mathbf{h}^1_1$ is the greatest $(1,1)$-degree inside $\mathbf{d}^1_{\omega}$.*
2. *If an $(<\omega, 1)$-degree $\mathbf{d}^{<\omega}_1$ of Π^0_1 subsets of $2^{\mathbb{N}}$ contains a $(1, <\omega)$-degree $\mathbf{h}^1_{<\omega}$ of homogeneous Π^0_1 sets, then $\mathbf{h}^1_{<\omega}$ is the least $(1, <\omega)$-degree inside $\mathbf{d}^{<\omega}_1$.*

3. *Every* $(<\omega,1)$*-degree of* Π^0_1 *subsets of* $2^{\mathbb{N}}$ *contains at most one* $(1,1)$*-degree of homogeneous* Π^0_1 *sets.*

Proof. For the item 1, we can see that, for any $P \subseteq \mathbb{N}^{\mathbb{N}}$ and any closed set $Q \subseteq \mathbb{N}^{\mathbb{N}}$, if $P \leq^1_\omega Q$ then there is a node σ such that $Q \cap [\sigma]$ is nonempty and $P \leq^1_1 Q \cap [\sigma]$. That is, σ is a locking sequence. If Q is homogeneous, then $P \leq^1_1 Q \equiv^1_1 Q \cap [\sigma]$. The item 2 follows from Theorem 20. By combining the item 1 and 2, we see that every $(<\omega,1)$-degree of Π^0_1 subsets of $2^{\mathbb{N}}$ contains at most one $(1,<\omega)$-degree of homogeneous Π^0_1 sets which contains at most one $(1,1)$-degree of homogeneous Π^0_1 sets. □

Corollary 72. $\mathsf{DNR}_3 <^{<\omega}_1 \mathsf{DNR}_2$*, and* $\mathsf{DNR}_3 <^1_\omega \mathsf{DNR}_2$.

Proof. By Jockusch [33], we have $\mathsf{DNR}_3 <^1_1 \mathsf{DNR}_2$. Thus, Proposition 71 implies the desired condition. □

By analyzing Jockusch's proof [33] of the Muchnik equivalence of DNR_2 and DNR_k for any $k \geq 2$, we can directly establish the $(<\omega,\omega)$-equivalence of DNR_2 and DNR_k for any $k \geq 2$. However, one may find that Jockusch's proof [33] is essentially based on the Σ^0_2 law of excluded middle. Therefore, the fine analysis of this proof structure establishes the following theorem.

Theorem 73. $\mathsf{DNR}_k \blacktriangledown \mathsf{DNR}_k <^1_1 \mathsf{DNR}_{k^2}$ *for any* k.

Proof. As Jockusch [33], fix a computable function $z : \mathbb{N}^2 \to \mathbb{N}$ such that $\Phi_{z(v,u)}(z(v,u)) = \langle \Phi_v(v), \Phi_u(u)\rangle$ for any $v,u \in \mathbb{N}$. Note that every $g \in (k^2)^{\mathbb{N}}$ determines two functions $g_0 \in k^{\mathbb{N}}$ and $g_1 \in k^{\mathbb{N}}$ such that $g(n) = \langle g_0(n), g_1(n)\rangle$ for any $n \in \mathbb{N}$. We define a uniform sequence $\{\Gamma_v\}_{v\in\mathbb{N}}, \Delta$ of computable functions as $\Gamma_v(g;u) = g_1(z(v,u))$, and $\Delta(g;v) = g_0(z(v,u_v))$, where $u_v = \min\{u \in \mathbb{N} : g_1(z(v,u)) = \Phi_u(u)\downarrow\}$. Fix $g \in \mathsf{DNR}_{k^2}$. Since $\langle g_0(z(v,u)), g_1(z(v,u))\rangle = g(z(v,u)) \neq \langle \Phi_v(v), \Phi_u(u)\rangle$, either $g_0(z(v,u)) \neq \Phi_v(v)$ or $g_1(z(v,u)) \neq \Phi_u(u)$ holds for any $v,u \in \mathbb{N}$. We consider the following Σ^0_2 sentence:

$$(\exists v)(\forall u)\ (\Phi_u(u)\downarrow \to\ g_1(z(v,u)) \neq \Phi_u(u)).$$

Let $\theta(g,v)$ denote the Π^0_1 sentence $(\forall u)\ (\Phi_u(u)\downarrow \to\ g_1(z(v,u)) \neq \Phi_u(u))$. If $\theta(g,v)$ holds, then $\Gamma_v(g;u) = g_1(z(v,u)) \neq \Phi_u(u)$ for any $u \in \mathbb{N}$. Hence, $\Gamma_v(g) \in \mathsf{DNR}_k$. If $\neg\theta(g,v)$ holds, then u_v is defined. Therefore, $\Delta(g;v) = g_0(z(v,u_v))\downarrow \neq \Phi_v(v)$, since $g_1(z(v,u_v)) = \Phi_{u_v}(u_v)\downarrow$. Thus, $\Delta(g;v)$ is extendible to a function in DNR_k. This procedure shows that there is a function $\Gamma : \mathsf{DNR}_{k^2} \to \mathsf{DNR}_k$ that is computable strictly along Π^0_1 sets $\{S_v\}_{v\in\mathbb{N}}$ via Δ and $\{\Gamma_v\}_{v\in\mathbb{N}}$, where $S_v = \{g : \theta(g,v)\}$. Consequently, $\mathsf{DNR}_k \blacktriangledown \mathsf{DNR}_k \leq^1_1 \mathsf{DNR}_{k^2}$ by Part I [29, Theorem 46].

To see $\mathsf{DNR}_k \blacktriangledown \mathsf{DNR}_k \not\geq^1_1 \mathsf{DNR}_{k^2}$, we note that $\mathsf{DNR}_k \blacktriangledown \mathsf{DNR}_k$ is not tree-immune. By Cenzer-Kihara-Weber-Wu [12], $\mathsf{DNR}_k \blacktriangledown \mathsf{DNR}_k$ does not cup to the generalized separating class DNR_{k^2}. □

Corollary 74. $\mathsf{DNR}_k \equiv^{<\omega}_\omega \mathsf{DNR}_2$ *for any* $k \geq 2$*. Indeed, for any* $k \in \mathbb{N}$*, the direction* $\mathsf{DNR}_k \leq^{<\omega}_\omega \mathsf{DNR}_{k^2}$ *is witnessed by a team of a confident learner and a eventually-Popperian learner. In particular,* $\mathsf{DNR}_k \equiv^\omega_1 \mathsf{DNR}_2$ *for any* $k \geq 2$.

Proof. As seen in Part I [29, Proposition 75], $P \leq^{<\omega}_{\omega} P\blacktriangledown P$ is witnessed by a team of a confident learner and a eventually-Popperian learner. Thus, Theorem 73 implies the desired condition. □

Corollary 75. *There is an* $(<\omega,\omega)$*-degree which contains infinitely many* $(1,1)$*-degrees of homogeneous* Π^0_1 *sets.*

Proof. By Corollary 74, the $(<\omega,\omega)$-degree of DNR_2 contains DNR_k for any $k \in \mathbb{N}$, while $\mathsf{DNR}_k \not\equiv^1_1 \mathsf{DNR}_l$ for $k \neq l$. □

4.2. Simpson's Embedding Lemma

For a pointclass Γ in a space X, we say that $\mathrm{SEL}(\Gamma, X)$ *holds for* (α,β)*-degrees* holds for (α,β)-degrees if, for every Γ set $S \subseteq X$ and for every nonempty Π^0_1 set $Q \subseteq 2^{\mathbb{N}}$, there exists a Π^0_1 set $P \subseteq 2^{\mathbb{N}}$ such that $P \equiv^{\alpha}_{\beta} S \cup Q$. Jockusch-Soare [34] indicates that $\mathrm{SEL}(\Pi^0_2, \mathbb{N}^{\mathbb{N}})$ holds for $(\omega,1)$-degrees, and points out that $\mathrm{SEL}(\Pi^0_3, 2^{\mathbb{N}})$ does not hold for $(\omega,1)$-degrees, since the set of all noncomputable elements in $2^{\mathbb{N}}$ is Π^0_3. Simpson's Embedding Lemma [58] determines the limit of $\mathrm{SEL}(\Gamma, X)$ for $(\omega,1)$-degrees.

Theorem 76 (Simpson [58])**.** $\mathrm{SEL}(\Sigma^0_3, \mathbb{N}^{\mathbb{N}})$ *holds for* $(\omega,1)$*-degrees.* □

Theorem 77 (Simpson's Embedding Lemma for other degree structures)**.**

1. $\mathrm{SEL}(\Sigma^0_2, 2^{\mathbb{N}})$ *does not hold for* $(<\omega,1)$*-degrees.*
2. $\mathrm{SEL}(\Sigma^0_2, 2^{\mathbb{N}})$ *holds for* $(1,\omega)$*-degrees.*
3. $\mathrm{SEL}(\Pi^0_2, 2^{\mathbb{N}})$ *does not hold for* $(1,\omega)$*-degrees.*
4. $\mathrm{SEL}(\Pi^0_2, \mathbb{N}^{\mathbb{N}})$ *holds for* $(<\omega,\omega)$*-degrees.*
5. $\mathrm{SEL}(\Sigma^0_3, 2^{\mathbb{N}})$ *does not hold for* $(<\omega,\omega)$*-degrees.*

Proof. (1) For any Π^0_1 set $P \subseteq 2^{\mathbb{N}}$, we note that $\bigtriangledown P \subseteq 2^{\mathbb{N}}$ is Σ^0_2. By Theorem 48, there is no Π^0_1 set $2^{\mathbb{N}}$ which is $(<\omega,1)$-below $\bigtriangledown P$. In particular, there is no Π^0_1 set $2^{\mathbb{N}}$ which is $(<\omega,1)$-equivalent to $P \cup \bigtriangledown P = \bigtriangledown P$.

(2) For a given Σ^0_2 set $S \subseteq 2^{\mathbb{N}}$, there is a computable increasing sequence $\{P_i\}_{i\in\mathbb{N}}$ of Π^0_1 classes such that $S = \bigcup_{i\in\mathbb{N}} P_i$. We need to show $\bigcup_{i\in\mathbb{N}} P_i \equiv^1_\omega \bigoplus_{i\in\mathbb{N}} P_i$, since $\bigoplus_{i\in\mathbb{N}} P_i$ is $(1,<\omega)$-equivalent to the Π^0_1 class $\bigoplus^{\rightarrow}_{i} P_i$. Then, it is easy to see $\bigcup_i P_i \leq^1_1 \bigoplus_i P_i$. For given $f \in \bigcup_i P_i$, from each initial segment $f \upharpoonright n$, a learner Ψ guesses an index of a computable function $\Phi_{\Psi(f\upharpoonright n)}(g) = i^\frown g$ for the least number i such that $f \upharpoonright n \in T_{P_i}$ but $f \upharpoonright n \notin T_{P_{i-1}}$. For any $f \in \bigcup_i P_i$, for the least i such that $f \in P_i \setminus P_{i-1}$, $\lim_n \Psi(f \upharpoonright n)$ converges to an index of $\Phi_{\lim_n \Psi(f\upharpoonright n)}(g) = i^\frown g$. Thus, $\Phi_{\lim_n \Psi(f\upharpoonright n)}(g) \in i^\frown P_i$. Consequently, $S = \bigcup_i P_i \leq^1_\omega \bigoplus^{\rightarrow}_{i} P_i$.

(3) Fix any special Π^0_1 set $P \subseteq 2^{\mathbb{N}}$. By Jockusch-Soare [34], there is a noncomputable Σ^0_1 set $A \subseteq \mathbb{N}$ such that P has no A-computable element. Then $\{A\} \subseteq 2^{\mathbb{N}}$ is a Π^0_2 singleton, since A is Σ^0_1. Therefore, $P \oplus \{A\}$ is Π^0_2. It suffices to show that there is no Π^0_1 set $Q \subseteq 2^{\mathbb{N}}$ such that $Q \equiv^1_\omega P \oplus \{A\}$. Assume that $Q \equiv^1_\omega P \oplus \{A\}$ is satisfied for some Π^0_1 set $Q \subseteq 2^{\mathbb{N}}$. Then Q must have an A-computable element $\alpha \in Q$. Fix a learner Ψ witnessing $P \oplus \{A\} \leq^1_\omega Q$. Then, we have $\Phi_{\lim_n \Psi(\alpha\upharpoonright n)}(\alpha) = 1^\frown A$, since P has no element computable in $\alpha \leq_T A$. We wait for $s \in \mathbb{N}$ such that $\Psi(\alpha \upharpoonright t) = \Psi(\alpha \upharpoonright s)$ for any $t \geq s$. Then, fix $u \geq s$ with $\Phi_{\Psi(\alpha\upharpoonright u)}(\alpha \upharpoonright u; 0) \downarrow= 1$. Consider the Π^0_1 set

$Q^* = \{f \in Q \cap [\alpha \upharpoonright u] : (\forall v \geq u)\ \Psi(f \upharpoonright v) = \Psi(f \upharpoonright u)\}$. Then, for any $f \in Q^*$, $\Phi_{\lim_s \Psi(f \upharpoonright s)}(f) = \Phi_{\Psi(\alpha \upharpoonright u)}(f)$ must extends $\langle 1 \rangle$. Thus, $\{1^\frown A\} \leq^1_1 Q^*$ via the computable function $\Phi_{\Psi(\alpha \upharpoonright u)}$. Since Q^* is special Π^0_1 subset of $2^\mathbb{N}$, this implies the computability of $1^\frown A$ which contradicts our choice of A.

(4) Fix a Π^0_2 set $S \subseteq \mathbb{N}^\mathbb{N}$. As Simpson's proof, there is a Π^0_1 set $\widehat{S} \subseteq \mathbb{N}^\mathbb{N}$ such that $S \equiv^1_1 \widehat{S}$. We can find a Π^0_1 set $\widehat{P} \subseteq \widehat{S} \blacktriangledown Q$ such that $\widehat{P} \leq^1_1 S \cup Q$, and $\widehat{P}$ is computably homeomorphic to a Π^0_1 set $P \subseteq 2^\mathbb{N}$. Since $S \cup Q \leq^{<\omega}_\omega \widehat{S} \blacktriangledown Q$, we have $S \cup Q \equiv^{<\omega}_\omega P$.

(5) For every Π^0_1 set $P \subseteq 2^\mathbb{N}$, the Turing upward closure $\widehat{\mathrm{Deg}}(P) = \{g \in 2^\mathbb{N} : (\exists f \in P)\ f \leq_T g\}$ of P is Σ^0_3, and $\widehat{\mathrm{Deg}}(P)$ has the least $(<\omega, \omega)$-degree inside $\deg^\omega_1(P)$. By Theorem 65, there is no Π^0_1 subset of $2^\mathbb{N}$ which is $(<\omega, \omega)$-equivalent to $\widehat{\mathrm{Deg}}(P)$. □

4.3. Weihrauch Degrees

The notion of piecewise computability could be interpreted as the computability relative to the principle of excluded middle in a certain sense. Indeed, in Part I [29, Section 6], we have characterized the notions of piecewise computability as the computability relative to nonconstructive principles in the context of Weihrauch degrees. Thus, one can rephrase our separation results in the context of Weihrauch degrees as follows.

Theorem 78. *The symbols P, Q, and R range over all special Π^0_1 subset of $2^\mathbb{N}$, and X ranges over all subsets of $\mathbb{N}^\mathbb{N}$.*

1. *There are P and $Q \leq^1_1 P$ such that $P \leq_{\Sigma^0_1\text{-LLPO}} Q$ but $P \not\leq^1_1 Q$.*
2. *There are P and $Q \leq^1_1 P$ such that $P \leq_{\Sigma^0_1\text{-LEM}} Q$ but $P \not\leq_{\Sigma^0_1\text{-LLPO}} Q$.*
3. *For every P, there exists $Q \leq^1_1 P$ such that $P \leq_{\Sigma^0_1\text{-LEM}} Q$, whereas, for every X, if $P \leq_{\Sigma^0_1\text{-DNE}} Q \otimes X$ then $P \leq^1_1 X$.*
4. *There are P and $Q \leq^1_1 P$ such that $P \leq_{\Delta^0_2\text{-LEM}} Q$ but $P \not\leq_{\Sigma^0_1\text{-LEM}} Q$.*
5. *There are P and $Q \leq^1_1 P$ such that $P \leq_{\Sigma^0_2\text{-LLPO}} Q$ but $P \not\leq_{\Delta^0_2\text{-LEM}} Q$.*
6. *There is P such that, for every Q, if $P \leq_{\Sigma^0_2\text{-LLPO}} Q$, then $P \leq_{\Sigma^0_1\text{-LEM}} Q$.*
7. *For every P and R, there exists $Q \leq^1_1 P$ such that $P \leq_{\Sigma^0_2\text{-DNE}} Q$ but $R \not\leq_{\Sigma^0_2\text{-LLPO}} Q$.*
8. *For every P, there exists $Q \leq^1_1 P$ such that $P \leq_{\Sigma^0_2\text{-LEM}} Q$, whereas, for every X, if $P \leq_{\Sigma^0_2\text{-DNE}} Q \otimes X$ then $P \leq_{\Sigma^0_2\text{-DNE}} X$.*
9. *For every P and R, there exists $Q \leq P$ such that $P \leq_{\Sigma^0_3\text{-DNE}} Q$ but $R \not\leq_{\Sigma^0_2\text{-LEM}} Q$.*

Proof. See Part I [29, Section 6] for the definitions of partial multivalued functions and their characterizations.

(1) By Corollary 5. (2) By Corollary 9. (3) By Corollary 32. (4) By Corollary 13 (2). (5) By Corollary 15 (1). (6) By Theorem 20. (7) By Corollary 53 (2). (8) By Corollary 58. (9) By Theorem 65. □

Definition 79 (Mylatz [47]). The Σ^0_1 *lessor limited principle of omniscience with* (m/k) *wrong answers*, $\Sigma^0_1\text{-LLPO}_{m/k}$, is the following multi-valued function.

$$\Sigma^0_1\text{-LLPO}_{m/k} :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows k, \qquad x \mapsto \{l < k : (\forall n \in \mathbb{N})\ x(kn + l) = 0\}.$$

Here, $\mathrm{dom}(\Sigma^0_1\text{-LLPO}_{m/k}) = \{x \in \mathbb{N}^\mathbb{N} : x(n) \neq 0, \text{ for at most } m \text{ many } n \in \mathbb{N}\}$.

Remark. It is well-known that the parallelization of Σ^0_1-$\mathsf{LLPO}_{1/2}$ is equivalent to Weak König's Lemma, WKL (hence, is Weihrauch equivalent to the closed choice for Cantor space, $\mathsf{C}_{2^{\mathbb{N}}}$).

Definition 80.

1. (Cenzer-Hinman [14]) A set $P \subseteq k^{\mathbb{N}}$ is (m,k)*-separating* if $P = \prod_{n\in\mathbb{N}} F_n$ for some uniform sequence $\{F_n\}_{n\in\mathbb{N}}$ of Π^0_1 sets $F_n \subseteq k$, where $\#(k \setminus F_n) \leq m$ for any $n \in \mathbb{N}$.
2. A function $f : \mathbb{N}^m \to k$ is *k-valued m-diagonally noncomputable in* $\alpha \in \mathbb{N}^{\mathbb{N}}$ if the value $f(\langle e_0,\dots,e_{m-1}\rangle)$ does not belong to $\{\Phi_{e_i}(\alpha;\langle e_0,\dots,e_{m-1}\rangle) : i < m\}$ for each argument $\langle e_0,\dots,e_{m-1}\rangle \in \mathbb{N}^m$. By $\mathsf{DNR}_{m/k}(\alpha)$, we denote the set of all k-valued functions which are m-diagonally noncomputable in α.
3. The (m/k) *diagonally noncomputable operation* $\mathsf{DNR}_{m/k} : \mathbb{N}^{\mathbb{N}} \rightrightarrows k^{\mathbb{N}}$ is the multi-valued function mapping $\alpha \in \mathbb{N}^{\mathbb{N}}$ to $\mathsf{DNR}_{m/k}(\alpha)$.

Remark. Clearly $\mathsf{DNR}_{m/k}(\emptyset)$ is (m,k)-separating. The structure of Medvedev degrees of (m,k)-separating sets have been studied by Cenzer-Hinman [14]. Diagonally noncomputable functions are extensively studied in connection with *algorithmic randomness*, for example, see Greenberg-Miller [25].

Proposition 81. $\mathsf{DNR}_{m/k}$ *is Weihrauch equivalent to* Σ^0_1-$\widehat{\mathsf{LLPO}}_{m/k}$.

Proof. To see Σ^0_1-$\widehat{\mathsf{LLPO}}_{m/k} \leq_W \mathsf{DNR}_{m/k}$, for given $(x_i : i \in \mathbb{N})$, let e^i_t be an $\bigoplus_{i\in\mathbb{N}} x_i$-computable index of an algorithm, for any argument, which returns l at stage s if $l \in L_{s+1} \setminus L_s$ and $\#L_s = t$, where $L_s = \{l^* < k : (\exists n)\ kn + l^* < s\ \&\ x_i(kn + l^*) \neq 0\}$. Clearly, $\{e^i_t : i \in \mathbb{N}\ \&\ t < m\}$ is computable uniformly in $\bigoplus_{i\in\mathbb{N}} x_i$. For any $f \in \mathsf{DNR}_{m/k}(\bigoplus_{i\in\mathbb{N}} x_i)$, the function $i \mapsto f(\langle e^i_0,\dots,e^i_{m-1}\rangle)$ belongs to Σ^0_1-$\widehat{\mathsf{LLPO}}_{m/k}(\langle x_i : i \in \mathbb{N}\rangle)$. Conversely, for given $x \in \mathbb{N}^{\mathbb{N}}$, for the i-th m-tuple $\langle e_0,\dots,e_{m-1}\rangle \in \mathbb{N}^m$, we set $x_i(ks + l) = 1$ if $\Phi_{e_t}(\langle e_0,\dots,e_{m-1}\rangle)$ converges to $l < k$ at stage $s \in \mathbb{N}$ for some $t < m$, and otherwise we set $x_i(ks + l) = 0$. Clearly $\{x_i : i \in \mathbb{N}\}$ is uniformly computable in x. Then, for any $\langle l_i : i \in \mathbb{N}\rangle \in \Sigma^0_1$-$\widehat{\mathsf{LLPO}}_{m/k}(\langle x_i : i \in \mathbb{N}\rangle) \subseteq k^{\mathbb{N}}$, we have $l_i \notin \{\Phi_{e_t}(\langle e_0,\dots,e_{m-1}\rangle) : t < m\}$ by our construction. Hence, the k-valued function $i \mapsto l_i$ is m-diagonally noncomputable in x. □

Recall from Part I [29, Section 6] that $\star$ is the operation on Weihrauch degrees such that is defined by $F \star G = \max\{F^* \circ G^* : F^* \leq_W F\ \&\ G^* \leq_W G\}$. See [53] for more information on $\star$.

Corollary 82. *Let* $k \geq 2$ *be any natural number.*

1. Σ^0_1-$\widehat{\mathsf{LLPO}}_{1/k} \not\leq_W \Sigma^0_2$-$\mathsf{DNE} \star \Sigma^0_1$-$\widehat{\mathsf{LLPO}}_{1/k+1}$.
2. Σ^0_1-$\widehat{\mathsf{LLPO}}_{1/k} \not\leq_W \Sigma^0_2$-$\mathsf{LLPO} \star \Sigma^0_1$-$\widehat{\mathsf{LLPO}}_{1/k+1}$.
3. Σ^0_1-$\widehat{\mathsf{LLPO}}_{1/k} \leq_W \Sigma^0_2$-$\mathsf{LEM} \star \Sigma^0_1$-$\widehat{\mathsf{LLPO}}_{1/k+1}$.

Proof. By Corollary 72 and Proposition 81, the item (1) and (2) are satisfied. It is not hard to show the item (3) by analyzing Theorem 73. □

Remark. By combining the results from Cenzer-Hinman [14] and our previous results, we can actually show the following.

1. $\Sigma^0_1\text{-}\widehat{\mathsf{LLPO}}_{n/l} \not\leq_W \Sigma^0_2\text{-}\mathsf{DNE} \star \Sigma^0_1\text{-}\widehat{\mathsf{LLPO}}_{m/k}$, whenever $0 < n < l < \lceil k/m \rceil$.
2. $\Sigma^0_1\text{-}\widehat{\mathsf{LLPO}}_{n/l} \not\leq_W \Sigma^0_2\text{-}\mathsf{LLPO} \star \Sigma^0_1\text{-}\widehat{\mathsf{LLPO}}_{m/k}$, whenever $0 < n < l < \lceil k/m \rceil$.
3. $\Sigma^0_1\text{-}\widehat{\mathsf{LLPO}}_{n/l} \leq_W \Sigma^0_2\text{-}\mathsf{LEM} \star \Sigma^0_1\text{-}\widehat{\mathsf{LLPO}}_{m/k}$, whenever $0 < n < l$ and $0 < m < k$.

These results suggest, within some constructive setting, that the Σ^0_2 law of excluded middle is sufficient to show the formula $\Sigma^0_1\text{-}\widehat{\mathsf{LLPO}}_{m/k} \to \Sigma^0_1\text{-}\widehat{\mathsf{LLPO}}_{n/l}$, whereas neither the Σ^0_2 double negation elimination nor the Σ^0_2 lessor limited principle of omniscience is sufficient.

Corollary 83. *$\mathsf{DNR}_2 \leq_{\Sigma^0_2\text{-}\mathsf{LEM}} \mathsf{DNR}_3$; $\mathsf{DNR}_2 \not\leq_{\Sigma^0_2\text{-}\mathsf{LLPO}} \mathsf{DNR}_3$; $\mathsf{DNR}_2 \not\leq_{\Sigma^0_2\text{-}\mathsf{DNE}} \mathsf{DNR}_3$; $\mathsf{MLR} \leq_{\Sigma^0_2\text{-}\mathsf{LEM}} \mathsf{DNR}_3$; and $\mathsf{MLR} \not\leq_{\Sigma^0_2\text{-}\mathsf{DNE}} \mathsf{DNR}_3$. Here, MLR denotes the set of all Martin-Löf random reals.*

Proof. For the first three statements, see Corollary 72 and Theorem 73. It is easy to see that $\mathsf{MLR} \leq^1_1 \mathsf{DNR}_2 \leq_{\Sigma^0_2\text{-}\mathsf{LEM}} \mathsf{DNR}_3$. It is shown by Downey-Greenberg-Jockusch-Millans [20] that $\mathsf{MLR} \not\leq^1_1 \mathsf{DNR}_3$. By homogeneity of DNR_3 and Proposition 71, we have $\mathsf{MLR} \not\leq_{\Sigma^0_2\text{-}\mathsf{DNE}} \mathsf{DNR}_3$. □

4.4. Some Intermediate Lattices are Not Brouwerian

Recall from Medvedev's Theorem [41], Muchnik's Theorem [46], and Part I [29, Proposition 16] that the degree structures $\mathcal{D}^1_1$, $\mathcal{D}^1_\omega$, and $\mathcal{D}^\omega_1$ are Browerian. Indeed, we have already observed that one can generate $\mathcal{D}^1_\omega$ from a logical principle so called the Σ^0_2-*double negation elimination*. Though $\mathcal{D}^1_{<\omega}$, $\mathcal{D}^1_{\omega|<\omega}$ and $\mathcal{D}^{<\omega}_1$ are also generated from certain logical principles over $\mathcal{D}^1_1$ as seen before, surprisingly, these degree structures are *not* Brouwerian.

Theorem 84. *The degree structures $\mathcal{D}^1_{<\omega}$, $\mathcal{D}^1_{\omega|<\omega}$, $\mathcal{D}^{<\omega}_1$, $\mathcal{P}^1_{<\omega}$, $\mathcal{P}^1_{\omega|<\omega}$, and $\mathcal{P}^{<\omega}_1$ are not Brouwerian.*

Put $\mathcal{A}(P,Q) = \{R \subseteq \mathbb{N}^\mathbb{N} : Q \leq^1_{<\omega} P \otimes R\}$, and $\mathcal{B}(P,Q) = \{R \subseteq \mathbb{N}^\mathbb{N} : Q \leq^{<\omega}_1 P \otimes R\}$. Note that $\mathcal{A}(P,Q) \subseteq \mathcal{B}(P,Q)$. Then we show the following lemma.

Lemma 85. *There are Π^0_1 sets $P, Q \subseteq 2^\mathbb{N}$, and a collection $\{Z_e\}_{e\in\mathbb{N}}$ of Π^0_1 subsets of $2^\mathbb{N}$ such that $Z_e \in \mathcal{A}(P,Q)$, and that, for every $R \in \mathcal{B}(P,Q)$, we have $R \not\leq^\omega_1 Z_e$ for some $e \in \mathbb{N}$.*

Proof. By Theorem 17, we have a collection $\{S_i\}_{i\in\mathbb{N}}$ of nonempty Π^0_1 subsets of $2^\mathbb{N}$ such that $x_k \not\leq_T \bigoplus_{j\neq k} x_j$ for any choice $x_i \in S_i$, $i \in \mathbb{N}$. Consider the following sets.

$$P = \mathsf{CPA}^\frown\{S_{\langle e,0\rangle}{}^\frown S_{\langle e,1\rangle}{}^\frown \dots {}^\frown S_{\langle e,e\rangle}\}_{e\in\mathbb{N}}, \quad Z_e = S_{\langle e,e+1\rangle},$$

$$Q = \mathsf{CPA}^\frown\{Q_n\}_{n\in\mathbb{N}}, \quad \text{where } Q_{\langle e,i\rangle} = \begin{cases} S_{\langle e,i\rangle} \otimes Z_e, & \text{if } i \leq e, \\ (P \setminus [\rho_e]) \otimes Z_e, & \text{if } i = e+1, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Here, ρ_e is the e-th leaf of the corresponding computable tree T_{CPA} for CPA. To see $Z_e \in \mathcal{A}(P, Q)$, choose an element $f \oplus g \in P \otimes Z_e$. If $f \upharpoonright n \in T_{\mathsf{CPA}}$ or $f \upharpoonright n$ extends a leaf except ρ_e, our learner $\Psi((f \upharpoonright n) \oplus g)$ guesses an index of the identity function. If $f \upharpoonright n$ extends ρ_e, then Ψ first guesses $\Phi_{\Psi((f\upharpoonright n)\oplus g)}(f \oplus g) = (f^{\leftarrow|\rho_e|}) \oplus g$. By continuing this guessing procedure, if $f \upharpoonright n$ is of the form $\rho_e{}^\frown\tau^0{}^\frown\tau^1{}^\frown \dots {}^\frown\tau^i{}^\frown\tau$ such that τ^j is a leaf of $S_{\langle e,j\rangle}$ for each $j \leq i$, and τ does not extend a leaf of $S_{\langle e,j+1\rangle}$, then Ψ guesses $\Phi_{\Psi((f\upharpoonright n)\oplus g)}(f \oplus g) = (f^{\leftarrow(|\rho_e|+|\tau^0|+\dots+|\tau^i|)}) \oplus g$. Note that $i < e$, since $f \in P$. It is easy to see that $Q \leq^1_{<\omega} P \otimes Z_e$ via the learner Ψ, where $\#\{n \in \mathbb{N} : \Psi((f \oplus g) \upharpoonright n + 1) \neq \Psi((f \oplus g) \upharpoonright n)\} \leq e + 1$. Therefore, $Z_e \in \mathcal{A}(P, Q)$.

Fix $R \in \mathcal{B}(P, Q)$. As $Q \leq^{<\omega}_1 P \otimes R$, there is $b \in \mathbb{N}$ such that, for every $f \oplus g \in P \otimes R$, we must have $\Phi_e(f \oplus g) \in Q$ for some $e < b$. Suppose for the sake of contradiction that $R \leq^{\omega}_1 Z_{b+1}$. Then, for any $h \in Z_{b+1}$, we have $g \in R$ with $g \leq_T h$. Pick $f_0 \in \rho_{b+1}{}^\frown S_{\langle b+1,0\rangle} \subset P \cap [\rho_{b+1}]$. Since $R \in \mathcal{B}(P, Q)$, there is $e_0 < b$ such that $\Phi_{e_0}(f_0 \oplus g) \in Q$. By our choice of $\{S_n\}_{n\in\mathbb{N}}$ and the property $g \leq_T h \in Z_{b+1} = S_{\langle b+1,b+2\rangle}$, if $e \neq b + 1$ or $i \neq 0$, then $Q_{\langle e,i\rangle}$ has no $(f_0 \oplus g)$-computable element. Therefore, $\Phi_{e_0}(f_0 \oplus g)$ have to extend $\rho_{\langle b+1,0\rangle}$. Take an initial segment $\sigma_0 \subset f_0$ determining $\Phi_{e_0}(\sigma_0 \oplus g) \supseteq \rho_{\langle b+1,0\rangle}$. Extend σ_0 to a leaf τ^0 of $S_{b+1,0}$, and choose $f_1 \in \rho^\frown\tau^0{}^\frown S_{b+1,1} \subset P$. Again we have $e_1 < b$ such that $\Phi_{e_1}(f_1 \oplus g) \in Q$. As before, $\Phi_{e_1}(f_1 \oplus g)$ have to extend $\rho_{\langle b+1,1\rangle}$. However, $\rho_{\langle b+1,1\rangle}$ is incomparable with $\rho_{\langle b+1,0\rangle}$. Hence, we have $e_1 \neq e_0$. Again take an initial segment $\sigma_1 \subset f_1$ extending σ_0 and determining $\Phi_{e_1}(\sigma_1 \oplus g) \supseteq \rho_{\langle b+1,1\rangle}$. By iterating this procedure, we see that R requires at least $b + 1$ many indices e_i. This contradicts our assumption. Therefore, $R \not\leq^{<\omega}_1 Z_{b+1}$. □

Proof of Theorem 84. Let P, Q, and $\{Z_e\}_{e\in\mathbb{N}}$ be Π^0_1 sets in 85. Fix $(\alpha,\beta) \in \{(1, <\omega), (1, \omega| < \omega), (< \omega, 1)\}$. To see $\mathcal{D}^\alpha_\beta$ is not Brouwerian, it suffices to show that there is no (α,β)-least R satisfying $Q \leq^\alpha_\beta P \otimes R$. If R satisfies $Q \leq^\alpha_\beta P \otimes R$, then clearly $R \in \mathcal{B}(P, Q)$ since $\leq^\alpha_\beta$ is stronger than or equals to $\leq^{<\omega}_1$. Then, $R \not\leq^\alpha_\beta Z_e$ for some $e \in \mathbb{N}$. Moreover, $Z_e \in \mathcal{A}(P, Q)$ implies $Q \leq^\alpha_\beta P \otimes Z_e$, since $\leq^\alpha_\beta$ is weaker than or equals to $\leq^1_{<\omega}$. Hence R is not such a smallest set. By the same argument, it is easy to see that $\mathcal{P}^\alpha_\beta$ is not Brouwerian, since Z_e is Π^0_1. □

Theorem 86. *$\mathcal{D}^{<\omega}_\omega$ and $\mathcal{P}^{<\omega}_\omega$ are not Brouwrian. Moreover, the order structures induced by $(\mathcal{P}(\mathbb{N}^\mathbb{N}), \leq_{\Sigma^0_2\text{-}\mathsf{LEM}})$ and (the set of all nonempty Π^0_1 subsets of $2^\mathbb{N}, \leq_{\Sigma^0_2\text{-}\mathsf{LEM}}$) are not Brouwrian.*

Lemma 87. *Let $\{S_i\}_{i\leq n}$ be a collection of Π^0_1 subsets of $2^\mathbb{N}$ with the property for each $i \leq n$ that $\bigcup_{k\neq i} S_k$ has no element computable in $x_i \in S_i$. Then, there is no (n, ω)-computable function from $\blacktriangledown_{i\leq n} S_i$ to $\bigoplus_{i\leq n} S_i$.*

Proof. Assume the existence of an (n, ω)-computable function from $\blacktriangledown_{i\leq n} S_i$ to $\bigoplus_{i\leq n} S_i$ which is identified by n many learners $\{\Psi_i\}_{i<n}$. Let F_i be a partial (n, ω)-computable function identified by Ψ, i.e., $F_i(x) = \Phi_{\lim_n \Psi(x\upharpoonright n)}(x)$. Note that $\blacktriangledown_{i\leq n} S_i \subseteq \bigcup_{i<n} \mathrm{dom}(F_i)$. For each $i < n$, put $D_i = \mathrm{dom}(F_i) \cap F_i^{-1}(\bigoplus_{i\leq n} S_i)$. Let T_{S_i} denote the corresponding tree for S_i, for each $i \leq n$. Define $S^\heartsuit_E$ for each $E \subseteq n + 1$ to be the set of all infinite paths through the following tree T_E.

$$T_E = \blacktriangledown_{\sigma\in T_0^E}\left(\blacktriangledown_{\sigma\in T_1^E}\left(\dots\left(\blacktriangledown_{\sigma\in T_{n-1}^E}[T_n^E]\right)\dots\right)\right).$$

$$\text{Here, } T_i^E = \begin{cases} T_{S_i}^{ext}, & \text{if } i \in E, \\ \text{some finite subtree of } T_{S_i}^{ext}, & \text{otherwise.} \end{cases}$$

Here, the choice of "some finite subtree of $T_{S_i}^{ext}$" depends on the context, and is implicitly determined when E is defined. For each $E \subseteq n+1$, clearly $S_E^\heartsuit$ is a closed subset of $\blacktriangledown_{i\leq n}S_i$. Divide $S_{n+1}^\heartsuit$ into $n+1$ many parts $\{S_i^*\}_{i\leq n}$, where $S_{n+1}^\heartsuit$ is equal to $\bigcup_{i\leq n}S_i^*$, and each S_i^* is degree-isomorphic to S_i.

For each $i \leq n$, check whether there is a string σ extendible in $S_{n+1}^\heartsuit$ such that $S_{n+1}^\heartsuit \cap D_i \cap [\sigma]$ is contained in S_j^* for some $j \leq n$. If yes, for such a least $i \leq n$, choose a witness $\sigma_0 = \sigma$, and put $A_0 = \{i\}$, and $B_0 = \{j\}$. Then, for such $j \in B_0$, "some finite subtree of $T_{S_j}^{ext}$" is choosen as the set of all strings η used in σ_0 as a part of $T_{S_j}^{ext}$ in the sense of the definition of $\blacktriangledown_{i\leq n}S_i$, or successors of such η in $T_{S_j}^{ext}$. Note that σ_0 is also extendible in $S_{(n+1)\setminus\{j\}}^\heartsuit$. Inductively, for some $s < n$ assume that σ_s, A_s, and B_s has been already defined. For each $i \notin A_s$, check whether there is a string $\sigma \supseteq \sigma_s$ extendible in $S_{(n+1)\setminus B_s}^\heartsuit$ such that $S_{(n+1)\setminus B_s}^\heartsuit \cap D_i \cap [\sigma]$ is contained in S_j^* for some $j \notin B_s$. If yes, for such a least $i \notin A_s$, choose a witness $\sigma_0 = \sigma$, and put $A_{s+1} = A_s \cup \{i\}$, and $B_{s+1} = B_s \cup \{j\}$. As before, for such $j \in B_{s+1}$, "some finite subtree of $T_{S_j}^{ext}$" is choosen as the set of all strings η used in σ_{s+1} as a part of $T_{S_j}^{ext}$, or successors of such η in $T_{S_j}^{ext}$. Note that σ_{s+1} is also extendible in $S_{(n+1)\setminus B_{s+1}}^\heartsuit$. If no such $i \notin A_s$ exists, finish our construction of σ, A, and B. Then, put $A = A_s$, $B = B_s$, and define σ^* to be the last witness σ_s.

Put $A^- = n \setminus A$ and $B^- = (n+1) \setminus B$. Note that $\#A^- + 1 = \#B^-$, since $\#A = \#B$. Therefore, B contains at least one element. By our assumption, for any $x \in S_{B^-}^\heartsuit \cap [\sigma^*] \neq \emptyset$, we must have $F_i(x) \in \bigoplus_{i\leq n} S_i$ for some $i \in A^-$. Thus, A^- is nonempty.

Fix a sequence $\alpha \in (A^-)^{\mathbb{N}}$ such that, for each $i \in A^-$, there are infinitely many $n \in \mathbb{N}$ such that $\alpha(n) = i$. First set $\tau_0 = \sigma^*$. Inductively assume that $\tau_s \supseteq \sigma^*$ has been already defined. By our definition of σ^*, A and B, if ξ extends σ^*, then the set $S_{B^-}^\heartsuit \cap D_{\alpha(s)} \cap [\xi]$ intersects with S_j^* for at least two $j \in B^-$. Therefore, we can choose $x \in S_{B^-}^\heartsuit \cap D_{\alpha(s)} \cap [\tau_s] \cap S_j^* \neq \emptyset$ for some $j \in B^-$. Then, $F_{\alpha(s)}(x; 0) = j$, by our assumption of $\{S_i\}_{i\leq n}$. Find a string τ_s^* such that $\tau_s \subseteq \tau_s^* \subset x$ and $F_{\alpha(s)}(\tau_s^*; 0) = j$. Again, we can choose $x^* \in S_{B^-}^\heartsuit \cap D_{\alpha(s)} \cap [\tau_s^*] \cap S_k^* \neq \emptyset$ for some $k \in B^- \setminus \{j\}$. Then, we must have $F_{\alpha(s)}(x^*; 0) = k \neq j$. Let τ_{s+1} be a string such that $\tau_s^* \subseteq \tau_{s+1} \subset x^*$ and $F_{\alpha(s)}(\tau_{s+1}; 0) = k$. Therefore, between τ_s and τ_{s+1}, the learner $\Psi_{\alpha(s)}$ changes his mind.

Define $y = \bigcup_s \tau_s$. Then y is contained in $S_{B^-}^\heartsuit$, since $S_{B^-}^\heartsuit$ is closed. However, for each $i \in A^-$, by our construction of y, the value $F_i(y)$ does not converge. Moreover, for each $i \notin A^-$, by our definition of A, B, and $\sigma^* \subset y$, even if $F_i(y)$ converges, $F_i(y) \notin \bigoplus_{i\leq n} S_i$. Consequently, there is no (n,ω)-computable function from $\blacktriangledown_{i\leq n}S_i \supset S_{B^-}^\heartsuit$ to $\bigoplus_{i\leq n} S_i$ as desired. □

Put $\mathcal{J}(P,Q) = \{R \subseteq \mathbb{N}^{\mathbb{N}} : Q \leq_{\Sigma_2^0\text{-LEM}} P \otimes R\}$, and $\mathcal{K}(P,Q) = \{R \subseteq \mathbb{N}^{\mathbb{N}} : Q \leq_\omega^{<\omega} P \otimes R\}$. Note that $\mathcal{J}(P,Q) \subseteq \mathcal{K}(P,Q)$. Then we show the following lemma.

Lemma 88. *There are Π^0_1 sets $P, Q \subseteq 2^{\mathbb{N}}$, and a collection $\{Z_e\}_{e\in\mathbb{N}}$ of Π^0_1 subsets of $2^{\mathbb{N}}$ such that $Z_e \in \mathcal{J}(P, Q)$, and that, for every $R \in \mathcal{K}(P, Q)$, we have $R \not\leq^{\omega}_1 Z_e$ for some $e \in \mathbb{N}$.*

Proof. By Theorem 17, we have a collection $\{S_i\}_{i\in\mathbb{N}}$ of nonempty Π^0_1 subsets of $2^{\mathbb{N}}$ such that $x_k \not\leq_T \bigoplus_{j\neq k} x_j$ for any choice $x_i \in S_i$, $i \in \mathbb{N}$. Consider the following sets.

$$P = \mathsf{CPA}^\frown\{S_{\langle e,0\rangle}\blacktriangledown S_{\langle e,1\rangle}\blacktriangledown \dots \blacktriangledown S_{\langle e,e\rangle}\}_{e\in\mathbb{N}}, \quad Z_e = S_{\langle e,e+1\rangle},$$

$$Q = \mathsf{CPA}^\frown\{Q_n\}_{n\in\mathbb{N}}, \quad \text{where } Q_{\langle e,i\rangle} = \begin{cases} S_{\langle e,i\rangle} \otimes Z_e, & \text{if } i \leq e, \\ (P \setminus [\rho_e]) \otimes Z_e, & \text{if } i = e+1, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Here, ρ_e is the e-th leaf of the corresponding computable tree T_{CPA} for CPA. To see $Z_e \in \mathcal{J}(P, Q)$, for $f \oplus g \in P \otimes Z_e$, by using Σ^0_1-LEM, check whether f does no extend ρ_e. If no, outputs $\rho_{e,e+1}{}^\frown(f \oplus g)$. If f extends ρ_e, it is not hard to see that an finite iteration of Σ^0_2-LEM can divide $(\rho_e{}^\frown S_{\langle e,0\rangle}\blacktriangledown S_{\langle e,1\rangle}\blacktriangledown \dots \blacktriangledown S_{\langle e,e\rangle}) \otimes Z_e$ into $\{S_{\langle e,i\rangle} \otimes Z_e\}_{i\leq e}$.

Fix $R \in \mathcal{K}(P, Q)$. As $Q \leq^{\omega}_{<\omega} P \otimes R$, some (b, ω)-computable function F maps $P \otimes R$ into Q. Suppose for the sake of contradiction that $R \leq^{\omega}_1 Z_b$. Then, for any $h \in Z_b$, we have $g \in R$ with $g \leq_T h$. Then, F maps $(P \cap [\rho_b]) \otimes \{g\}$ into $Q \cap (\bigcup_{i\leq b} \rho_{\langle b,i\rangle})$ by our choice of $\{S_n\}_{n\in\mathbb{N}}$. Note that $(P \cap [\rho_b]) \otimes \{g\} \equiv^1_1 (\blacktriangledown_{i\leq b} S_{\langle e,i\rangle}) \otimes \{g\}$, and $Q \cap (\bigcup_{i\leq b} \rho_{\langle b,i\rangle}) \equiv^1_1 (\bigoplus_{i\leq b} S_{\langle e,i\rangle}) \otimes Z_e$. Therefore, by Lemma 87, F is not (b, ω)-computable. □

Proof of Theorem 86. Let P, Q, and $\{Z_e\}_{e\in\mathbb{N}}$ be Π^0_1 sets in 88. Then, by the same argument as in the proof of Theorem 84, it is not hard to show the desired statement. □

Corollary 89. *If $(\alpha, \beta) \in \{(1, 1), (1, \omega), (\omega, 1)\}$, and $(\gamma, \delta) \in \{(1, <\omega), (1, \omega| <\omega), (<\omega, 1), (<\omega, \omega)\}$, then, there is an elementary difference between $\mathcal{D}^{\alpha}_{\beta}$ and $\mathcal{D}^{\gamma}_{\delta}$, in the language of partial orderings $\{\leq\}$.*

Proof. Recall that the degree structures $\mathcal{D}^1_1$, $\mathcal{D}^1_{\omega}$, and $\mathcal{D}^{\omega}_1$ are Browerian, i.e., they satisfy the following elementary formula ψ in the language of partial orders.

$$\psi \equiv (\forall p, q)(\exists r)(\forall s)\ (p \leq q \vee r \ \&\ (p \leq q \vee s \ \rightarrow\ r \leq s)).$$

Here, the supremum $\vee$ is first-order definable in the language of partial orders. On the other hand, by Theorem 84 and 86, $\mathcal{D}^1_{<\omega}$, $\mathcal{D}_{\omega|<\omega}$, $\mathcal{D}^{<\omega}_1$, and $\mathcal{D}^{<\omega}_{\omega}$ are not Brouwerian, i.e., they satisfy $\neg\psi$. □

4.5. Open Questions

Question 90 (Small Questions)**.**

1. *Determine the intermediate logic corresponding the degree structure $\mathcal{D}^1_{\omega}$, where recall that $\mathcal{D}^1_1$ and $\mathcal{D}^{\omega}_1$ are exactly Jankov's Logic.*
2. *Does there exist Π^0_1 sets $P, Q \subseteq 2^{\mathbb{N}}$ with $P \leq^1_{\omega} Q$ such that there is no $|a|$-bounded learnable function $\Gamma : Q \to P$ for any $a \in O$? For a Π^0_1 set $\widehat{P}$ in Theorem 48, does there exist a function $\Gamma : \widehat{P} \to P$ $(1, \omega)$-computable via an $|a|$-bounded learner for some notation $a \in O$?*

3. *Does there exist a pair of special Π^0_1 sets $P, Q \subseteq 2^{\mathbb{N}}$ with a function $\Gamma : Q\blacktriangledown P \to Q \oplus P$ (or $\Gamma : Q\triangledown_\infty \bigtriangledown P \to Q \oplus P$) which is learnable by a team of confident learners (or a team of eventually-Popperian learners)?*
4. *Let P_0, P_1, Q_0, and Q_1 be Π^0_1 subsets of $2^{\mathbb{N}}$ with $Q_0 \leq^1_\omega Q_1$ and $P_0 \leq^1_\omega P_1$. Then, does $[\![P_0 \vee Q_0]\!]_{\Sigma^0_2} \leq^1_\omega [\![P_1 \vee Q_1]\!]_{\Sigma^0_2}$ hold? Moreover, if $Q_0 \leq^1_\omega Q_1$ is witnessed by an eventually Lipschitz learner, then does $P_0\blacktriangledown Q_0 \leq^1_\omega P_1\blacktriangledown Q_1$ hold?*
5. *Compare the reducibility $\leq^\omega_{tt,1}$ and other reducibility notions (e.g., $\leq^{<\omega}_{tt,1}$, $\leq^1_{<\omega}$, $\leq^1_{\omega|<\omega}$, $\leq^{<\omega}_1$ and $\leq^1_\omega$) for Π^0_1 subsets of Cantor space $2^{\mathbb{N}}$.*

Question 91 (Big Questions)**.**

1. *Are there elementary differences between any two different degree structures $\mathcal{D}^\alpha_{\beta|\gamma}$ and $\mathcal{D}^{\alpha'}_{\beta'|\gamma'}$ ($\mathcal{P}^\alpha_{\beta|\gamma}$ and $\mathcal{P}^{\alpha'}_{\beta'|\gamma'}$)?*
2. *Is the commutative concatenation $\triangledown$ first-order definable in the structure $\mathcal{D}^1_1$ or $\mathcal{P}^1_1$?*
3. *Is each local degree structure $\mathcal{P}^\alpha_{\beta|\gamma}$ first-order definable in the global degree structure $\mathcal{D}^\alpha_{\beta|\gamma}$?*
4. *Is the structure $\mathcal{P}^1_\omega$ dense?*
5. *Investigate properties of $(\alpha,\beta|\gamma)$-degrees* **a** *assuring the existence of* **b** $>$ **a** *with the same $(\alpha',\beta'|\gamma')$-degree as* **a**.
6. *Investigate the nested nested model, the nested nested nested model, and so on.*
7. *Does there exist a natural intermediate notion between $(<\omega,\omega)$-computability (team-learnability) and $(\omega,1)$-computability (nonuniform computability) on Π^0_1 sets?*
8. *(Ishihara) Define a uniform (non-adhoc) interpretation (such as the Kleene realizability interpretation) translating each intuitionistic arithmetical sentence (e.g., $(\neg\neg\exists n\forall m A(n,m)) \to (\exists n\forall m A(n,m))$) into a partial multi-valued function (e.g., Σ^0_2-DNE $:\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$).*

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