Inside the Muchnik Degrees I: Discontinuity, Learnability and Constructivism

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Abstract

Every computable function has to be continuous. To develop computability theory of discontinuous functions, we study low levels of the arithmetical hierarchy of nonuniformly computable functions on Baire space. First, we classify nonuniformly computable functions on Baire space from the viewpoint of learning theory and piecewise computability. For instance, we show that mind-change-bounded-learnability is equivalent to finite $(\Pi^0_1)_2$-piecewise computability (where $(\Pi^0_1)_2$ denotes the difference of two Π^0_1 sets), error-bounded-learnability is equivalent to finite Δ^0_2-piecewise computability, and learnability is equivalent to countable Π^0_1-piecewise computability (equivalently, countable Σ^0_2-piecewise computability). Second, we introduce disjunction-like operations such as the coproduct based on BHK-like interpretations, and then, we see that these operations induce Galois connections between the Medvedev degree structure and associated Medvedev/Muchnik-like degree structures. Finally, we interpret these results in the context of the Weihrauch degrees and Wadge-like games.

Keywords: computable analysis, limit computable mathematics, identification in the limit, Medvedev degree, Weihrauch degree
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1. Summary

1.1. Introduction

Imagine the floor function, a real function that takes the integer part of an input. Although it seems easy to draw a rough graph of the floor function, it is *not* computable with respect to the standard real number representation [82], because computability automatically induces topological continuity. One way to study the floor function in

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computability theory is to "*computabilize*" it by changing the representation/topology of the real space (see, for instance, [84]). However, it is also important to enhance our knowledge of the noncomputability/discontinuity level of such seemingly computable functions without changing representation/topology. Our main objective is to study low levels of the arithmetical/Baire hierarchy of functions on Baire space from the viewpoint of approximate computability/continuity and piecewise computability/continuity.

We postulate that a *nearly computable* function shall be, at the very least, *nonuniformly computable*, where a function f is said to be nonuniformly computable if for every input x, there exists an algorithm Ψ_x that computes $f(x)$ using x as an oracle, where we do not require the map $x \mapsto \Psi_x$ to be computable. The notion of nonuniform computability naturally arises in Computable Analysis [12, 88]. However, of course, most nonuniformly computable discontinuous functions are far from being computable. Then, what type of discontinuous functions are recognized as being nearly computable? A nearly computable/continuous function has to be approximated using computable/continuous functions. For instance, a Baire function appears to be *dynamically approximated* by a sequence of continuous functions and a piecewise continuous (σ-continuous) function appears to be *statically approximated* by countably many continuous functions.

There have been many challenges [15, 83–88] in developing computability theory of (nonuniformly computable) discontinuous functions using the notion of *learnability* (dynamical-approximation) and *piecewise computability* (statical-approximation). Indeed, one can show the equivalence of effective learnability and Π^0_1-piecewise computability: the class of functions that are computable with finitely many mind changes is exactly the class of functions that are decomposable into countably many computable functions with Π^0_1 domains. In this paper, we introduce various concepts of dynamic-approximability, and then, we characterize these concepts as static-approximability.

Now, we focus our attention on the concepts lying between (uniform) computability and nonuniform computability. In 1950-60th, Medvedev [51] and Muchnik [54] introduced the degree structure induced by uniform and nonuniform computability to formulate semantics for the intuitionistic propositional calculus based on Kolmogorov's idea of interpreting each proposition as a problem. The degree structure induced by the Medvedev (Muchnik) reduction forms a Brouwer algebra (the dual of a Heyting algebra), where the (intuitionistic) disjunction is interpreted as the coproduct of subsets of Baire space.

Our objective is to reveal the hidden relationship between the hierarchy of nonuniformly computable functions and the hierarchy of disjunction operations. When a certain suitable disjunction-like operation such as the coproduct is introduced, we will see that one can recover the associated degree structure from the disjunction operation. As a consequence, we may understand the noncomputability feature of functions by observing the degree-theoretic behavior of associated disjunction operations. This phenomenon can be explained by using the terminology of Galois connections or adjoint functors. For instance, one can introduce a disjunction operation on Baire space using the limit-BHK interpretation of *Limit Computable Mathematics* [31] (abbreviated as LCM), a type of constructive mathematics based on Learning Theory, whose positive arithmetical fragment is characterized as Heyting arithmetic with the recursive ω-rule and the Σ^0_1 law of excluded middle [6, 78]. Then, the "limit-BHK disjunction" includes all the information about the reducibility notion induced by learnable functions on Baire space.

Furthermore, in this paper, we introduce more complicated disjunction-like operations using BHK-like interpretations represented as "dynamic proof models" or "nested models". For instance, a dynamic disjunction along a well-founded tree realizes the concept of learnability with ordinal-bounded mind changes, and a dynamic disjunction along an ill-founded tree realizes the concept of decomposability into countably many computable functions along a Σ^0_2 formula.

We also interpret these results in the context of the Weihrauch degrees and Wadge-like games. We introduce a partial interpretation of nonconstructive principles including LLPO and LPO in the Weihrauch degrees and characterize the noncomputability/discontinuity level of nearly computable functions using these principles.

1.2. Results

In section 2, we introduce the notion of $(\alpha,\beta|\gamma)$-computability for partial functions on $\mathbb{N}^\mathbb{N}$, for each ordinal $\alpha,\beta,\gamma \leq \omega$. Then, the notion of $(\alpha,\beta|\gamma)$-computability induces just seven classes closed under composition.

- $[\mathfrak{C}_T]^1_1$ denotes the set of all partial computable functions on $\mathbb{N}^\mathbb{N}$.
- $[\mathfrak{C}_T]^1_{<\omega}$ denotes the set of all partial functions on $\mathbb{N}^\mathbb{N}$ learnable with bounded mind changes.
- $[\mathfrak{C}_T]^1_{\omega|<\omega}$ denotes the set of all partial functions on $\mathbb{N}^\mathbb{N}$ learnable with bounded errors.
- $[\mathfrak{C}_T]^1_\omega$ denotes the set of all partial learnable functions on $\mathbb{N}^\mathbb{N}$.
- $[\mathfrak{C}_T]^{<\omega}_1$ denotes the set of all partial k-wise computable functions on $\mathbb{N}^\mathbb{N}$ for some $k \in \mathbb{N}$.
- $[\mathfrak{C}_T]^{<\omega}_\omega$ denotes the set of all partial functions on $\mathbb{N}^\mathbb{N}$ learnable by a team.
- $[\mathfrak{C}_T]^\omega_1$ denotes the set of all partial nonuniformly computable functions on $\mathbb{N}^\mathbb{N}$ (i.e., all functions f satisfying $f(x) \leq_T x$ for any $x \in \mathrm{dom}(f)$).

We will see that the following inclusions hold.

$$\begin{array}{ccccccccc} & & & & \subset & [\mathfrak{C}_T]^{<\omega}_1 & \subset & & \\ [\mathfrak{C}_T]^1_1 & \subset & [\mathfrak{C}_T]^1_{<\omega} & \subset [\mathfrak{C}_T]^1_{\omega|<\omega} & & & & [\mathfrak{C}_T]^{<\omega}_\omega & \subset [\mathfrak{C}_T]^\omega_1 \\ & & & & \subset & [\mathfrak{C}_T]^1_\omega & \subset & & \end{array}$$

These notions are characterized as the following piecewise computability notions, respectively.

- $\mathrm{dec}^1_\mathrm{p}[-]$ also denotes the set of all partial computable functions on $\mathbb{N}^\mathbb{N}$.
- $\mathrm{dec}^{<\omega}_\mathrm{d}[\Pi^0_1]$ denotes the set of all partial functions on $\mathbb{N}^\mathbb{N}$ that are decomposable into finitely many partial computable functions with $(\Pi^0_1)_2$ domains, where a $(\Pi^0_1)_2$ set is the difference of two Π^0_1 sets.

- $\mathrm{dec}_{\mathrm{p}}^{<\omega}[\Delta^0_2]$ denotes the set of all partial functions on $\mathbb{N}^{\mathbb{N}}$ that are decomposable into finitely many partial computable functions with Δ^0_2 domains.
- $\mathrm{dec}_{\mathrm{p}}^{\omega}[\Pi^0_1]$ denotes the set of all partial functions on $\mathbb{N}^{\mathbb{N}}$ that are decomposable into countably many partial computable functions with Π^0_1 domains.
- $\mathrm{dec}_{\mathrm{p}}^{<\omega}[-]$ denotes the set of all partial functions on $\mathbb{N}^{\mathbb{N}}$ that are decomposable into finitely many partial computable functions.
- $\mathrm{dec}_{\mathrm{p}}^{<\omega}\mathrm{dec}_{\mathrm{p}}^{\omega}[\Pi^0_1]$ denotes the set of all partial functions on $\mathbb{N}^{\mathbb{N}}$ that are decomposable into finitely many partial Π^0_1-piecewise computable functions.
- $\mathrm{dec}_{\mathrm{p}}^{\omega}[-]$ denotes the set of all partial functions on $\mathbb{N}^{\mathbb{N}}$ that are decomposable into countably many partial computable functions.

$$\begin{array}{ccccc} & & \mathrm{dec}_{\mathrm{p}}^{<\omega}[-] & & \\ \mathrm{dec}_{\mathrm{p}}^{1}[-] \subset \mathrm{dec}_{\mathrm{d}}^{<\omega}[\Pi^0_1] \subset \mathrm{dec}_{\mathrm{p}}^{<\omega}[\Delta^0_2] & \begin{array}{c}\subset\\ \subset\end{array} & & \begin{array}{c}\subset\\ \subset\end{array} & \mathrm{dec}_{\mathrm{p}}^{<\omega}\mathrm{dec}_{\mathrm{p}}^{\omega}[\Pi^0_1] \subset \mathrm{dec}_{\mathrm{p}}^{\omega}[-] \\ & & \mathrm{dec}_{\mathrm{p}}^{\omega}[\Pi^0_1] & & \end{array}$$

In Section 3, we formalize the disjunction operations. Medvedev interpreted the intuitionistic disjunction as the coproduct (direct sum) $\oplus : \mathcal{P}(\mathbb{N}^{\mathbb{N}}) \times \mathcal{P}(\mathbb{N}^{\mathbb{N}}) \to \mathcal{P}(\mathbb{N}^{\mathbb{N}})$. We will introduce the following disjunction operations $[\![\cdot \vee \cdot]\!]^*_* : \mathcal{P}(\mathbb{N}^{\mathbb{N}}) \times \mathcal{P}(\mathbb{N}^{\mathbb{N}}) \to \mathcal{P}(\mathbb{N}^{\mathbb{N}})$:

- $[\![\cdot \vee \cdot]\!]^3_{\mathsf{LCM[n]}}$ is the disjunction operation on $\mathcal{P}(\mathbb{N}^{\mathbb{N}})$ induced by the backtrack BHK-interpretation with mind-changes $< n$.
- $[\![\cdot \vee \cdot]\!]^2_{\mathsf{LCM}}$ is the disjunction operation on $\mathcal{P}(\mathbb{N}^{\mathbb{N}})$ induced by the two-tape BHK-interpretation with finitely many mind-changes.
- $[\![\cdot \vee \cdot]\!]^3_{\mathsf{LCM}}$ is the disjunction operation on $\mathcal{P}(\mathbb{N}^{\mathbb{N}})$ induced by the backtrack BHK-interpretation with finitely many mind-changes.
- $[\![\cdot \vee \cdot]\!]^2_{\mathsf{CL}}$ is the disjunction operation on $\mathcal{P}(\mathbb{N}^{\mathbb{N}})$ induced by the two-tape BHK-interpretation permitting unbounded mind-changes.

Then, the direct sum $\oplus$ is characterized as the LCM disjunction without mind-changes $[\![\cdot \vee \cdot]\!]^3_{\mathsf{LCM[1]}}$. In section 5, we also introduce more complicated disjunction operations, which will play key roles in Part II.

In section 4, we study the interaction between the disjunction operations and the learnable/piecewise computable functions. We will construct new operations by iterating the disjunction operations introduced in Section 3 in the following way:

$$\begin{array}{ccccc} & & \bigoplus_{m\in\mathbb{N}}[\![\bigvee^{(m)} P]\!]^2_{\mathsf{CL}} & & \\ P \geq \bigoplus_{m\in\mathbb{N}}[\![P \vee P]\!]^3_{\mathsf{LCM}[m]} \geq \bigoplus_{m\in\mathbb{N}}[\![\bigvee^{(m)} P]\!]^2_{\mathsf{LCM}} & \begin{array}{c}\gneqq\\ \gneqq\end{array} & & \begin{array}{c}\gneqq\\ \gneqq\end{array} & \bigoplus_{m\in\mathbb{N}}[\![\bigvee^{(m)}[\![P \vee P]\!]^3_{\mathsf{LCM}}]\!]^2_{\mathsf{CL}} \geq \bigcup_{m\in\mathbb{N}}[\![\bigvee^{(m)} P]\!]^2_{\mathsf{CL}} \\ & & [\![P \vee P]\!]^3_{\mathsf{LCM}} & & \end{array}$$

Every such operation induces a functor from the associated Medvedev/Muchnik-like degree structure to the Medvedev degree structure. The main result is that every such functor is left adjoint to the canonical map from the Medvedev degree structure onto the associated degree structure.

In section 6, we will see that how our classes of nonuniformly computable functions relate to the arithmetical hierarchy of non-intuitionistic principles such as *the law of excluded middle* (LEM), *the lessor limited principle of omniscience* or *de Morgan's law* (LLPO), and *the double negation elimination* (DNE). The arithmetical hierarchy of non-intuitionistic principles is illustrated as follows:

$$\mathsf{HA} \;—\; \Sigma^0_1\text{-}\mathsf{LEM} \;—\; \Delta^0_2\text{-}\mathsf{LEM} \;\begin{matrix} \diagup\; \Sigma^0_2\text{-}\mathsf{LLPO} \;\diagdown \\ \diagdown\; \Sigma^0_2\text{-}\mathsf{DNE} \;\diagup \end{matrix}\; \Sigma^0_2\text{-}\mathsf{LEM} \;—\; \Sigma^0_3\text{-}\mathsf{DNE}$$

Here, Γ-LEM represents the sentence $\varphi \vee \neg\varphi$ for Γ-sentences φ; Γ-LLPO represents the sentence $\neg(\varphi \wedge \psi) \rightarrow \neg\varphi \vee \neg\psi$ for Γ-sentences φ, ψ; and Γ-DNE represents the sentence $\neg\neg\varphi \rightarrow \varphi$ for Γ-sentences φ. We interpret these principles as partial multi-valued functions on $\mathbb{N}^{\mathbb{N}}$, and then we characterize our notions of nonuniform computability by using these principles in the context of the Weihrauch degrees. We also characterize our notions by Wadge-like games.

1.3. Notations and Conventions

For any sets X and Y, we say that f *is a function from X to Y* (written $f : X \rightarrow Y$) if the domain dom(f) of f includes X, and the range range(f) of f is included in Y. We also use the notation $f :\subseteq X \rightarrow Y$ to denote that f is a partial function from X to Y, i.e., the domain dom(f) of f is included in X, and the range rng(f) of f is also included in Y.

For basic terminology in Computability Theory, see Soare [73]. For $\sigma \in \mathbb{N}^{<\mathbb{N}}$, we let $|\sigma|$ denote the length of σ. For $\sigma \in \mathbb{N}^{<\mathbb{N}}$ and $f \in \mathbb{N}^{<\mathbb{N}} \cup \mathbb{N}^{\mathbb{N}}$, we say that σ is *an initial segment* of f (denoted by $\sigma \subset f$) if $\sigma(n) = f(n)$ for each $n < |\sigma|$. Moreover, $f \upharpoonright n$ denotes the unique initial segment of f of length n. Let σ^- denote an immediate predecessor node of σ, i.e. $\sigma^- = \sigma \upharpoonright (|\sigma| - 1)$. We also define $[\sigma] = \{f \in \mathbb{N}^{\mathbb{N}} : f \supset \sigma\}$. A *tree* is a subset of $\mathbb{N}^{<\mathbb{N}}$ closed under taking initial segments. For any tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$, we also let $[T]$ be the set of all infinite paths of T, i.e., f belongs to $[T]$ if $f \upharpoonright n$ belongs to T for each $n \in \mathbb{N}$. A node $\sigma \in T$ is *extendible* if $[T] \cap [\sigma] \neq \emptyset$. Let T^{ext} denote the set of all extendible nodes of T. We say that $\sigma \in T$ is *a leaf* or *a dead end* if there is no $\tau \in T$ with $\tau \supsetneq \sigma$.

For any set X, the tree $X^{<\mathbb{N}}$ of finite words on X forms a monoid under concatenation $\frown$. Here *the concatenation of σ and τ* is defined by $(\sigma\frown\tau)(n) = \sigma(n)$ for $n < |\sigma|$ and $(\sigma\frown\tau)(|\sigma| + n) = \tau(n)$ for $n < |\tau|$. We use symbols $\frown$ and $\bigsqcap$ for the operation on this monoid, where $\bigsqcap_{i \leq n} \sigma_i$ denotes $\sigma_0\frown\sigma_1\frown\dots\frown\sigma_n$. To avoid confusion, the symbols $\times$ and $\prod$ are only used for a product of sets. We often consider the following three left monoid actions of $X^{<\mathbb{N}}$: The first one is the set $X^{\mathbb{N}}$ of infinite words on X with an operation $\frown : X^{<\mathbb{N}} \times X^{\mathbb{N}} \rightarrow X^{\mathbb{N}}$; $(\sigma\frown f)(n) = \sigma(n)$ for $n < |\sigma|$ and $(\sigma\frown f)(|\sigma| + n) = f(n)$ for $n \in \mathbb{N}$. The second one is the set $\mathcal{T}(X)$ of subtrees $T \subseteq X^{<\mathbb{N}}$ with an operation $\frown : X^{<\mathbb{N}} \times \mathcal{T}(X) \rightarrow \mathcal{T}(X)$; $\sigma\frown T = \{\sigma\frown\tau : \tau \in T\}$. The third one is the power set $\mathcal{P}(X^{\mathbb{N}})$ of $X^{\mathbb{N}}$ with an operation $\frown : X^{<\mathbb{N}} \times \mathcal{P}(X^{\mathbb{N}}) \rightarrow \mathcal{P}(X^{\mathbb{N}})$; $\sigma\frown P = \{\sigma\frown f : f \in P\}$.

We say that a set $P \subseteq \mathbb{N}^{\mathbb{N}}$ is Π^0_1 if there is a computable relation R such that $P = \{f \in \mathbb{N}^{\mathbb{N}} : (\forall n)R(n, f)\}$ holds. Equivalently, $P = [T_P]$ for some computable tree $T_P \subseteq \mathbb{N}^{<\mathbb{N}}$.

Let $\{\Phi_e\}_{e\in\mathbb{N}}$ be an effective enumeration of all Turing functionals (all partial computable functions[1]) on $\mathbb{N}^\mathbb{N}$. Then the e-th Π^0_1 subset of $2^\mathbb{N}$ is defined by $P_e = \{f \in 2^\mathbb{N} : \Phi_e(f;0)\uparrow\}$. Note that $\{P_e\}_{e\in\mathbb{N}}$ is an effective enumeration of all Π^0_1 subsets of Cantor space $2^\mathbb{N}$. If (an index e of) a Π^0_1 set $P_e \subseteq 2^\mathbb{N}$ is given, then $T_e = \{\sigma \in 2^{<\mathbb{N}} : \Phi_e(\sigma;0)\uparrow\}$ is called *the corresponding tree for* P_e. Here $\Phi(\sigma;n)$ for $\sigma \in \mathbb{N}^{<\mathbb{N}}$ and $n \in \mathbb{N}$ denotes the computation of Φ with an oracle σ, an input n, and step $|\sigma|$. Whenever a Π^0_1 set P is given, we assume that an index e of P is also given. If $P \subseteq 2^\mathbb{N}$ is Π^0_1, then the corresponding tree $T_P \subseteq 2^{<\mathbb{N}}$ of P is computable, and $[T_P] = P$. Moreover, the set L_P of all leaves of the computable tree T_P is also computable. We also say that a sequence of $\{P_i\}_{i\in I}$ of Π^0_1 subsets of a space X is *computable* or *uniform* if the set $\{(i,f) \in I \times X : f \in P_i\}$ is again a Π^0_1 subset of the product space $I \times X$. A set $P \subseteq \mathbb{N}^\mathbb{N}$ is *special* if P is nonempty and P has no computable member. For $f,g \in \mathbb{N}^\mathbb{N}$, $f \oplus g$ is defined by $(f\oplus g)(2n) = f(n)$ and $(f\oplus g)(2n+1) = g(n)$ for each $n \in \mathbb{N}$. For $P,Q \subseteq \mathbb{N}^\mathbb{N}$, put $P \oplus Q = (\langle 0\rangle^\frown P) \cup (\langle 1\rangle^\frown Q)$ and $P \otimes Q = \{f \oplus g : f \in P \;\&\; g \in Q\}$.

Contents

[1]In some contexts, a function Φ is called partial computable if it can be extended to some Φ_e. In this paper, we identify each partial computable function with such a Φ_e.

2. Nonuniformly Computable Discontinuous Functions

2.1. Piecewise Computable Functions

Our main objective in the paper is to study the intermediate notions of *(uniform) computability* and *nonuniform computability*. The concept of nonuniform computability can be rephrased as *countable computability*, i.e., partial functions that are decomposable into countably many computable functions. One can expect that the class of nonuniformly computable functions is classified on the basis of the least cardinality and least complexity of the decomposition (see also Pauly [60]). For instance, if a partial function $\Gamma :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is decomposable into k many computable functions, we say that it is *k-wise computable* or $(k,1)$-*computable*, and if Γ is decomposable into countably many (finitely many, resp.) computable functions with uniformly Λ-definable domains, we say that it is *countable (finite, resp.)* Λ-*piecewise computable*, where Λ is a lightface pointclass.

An important subclass of the piecewise computable functions consists of partial functions that are identifiable in the limit ([29]). The relationship between the computability with *trial-and-error* (limit computability or effective learnability) and the subhierarchy of the level Δ^0_2 has been common knowledge among recursion theorists since the last fifty years or so (see also Shoenfield [67], Gold [29], Putnam [62], and Ershov [27]). A basic observation (see Theorem 26) regarding the concept of type-two learnability (see also de Brecht-Yamamoto [24, 25]) is that a partial function on $\mathbb{N}^{\mathbb{N}}$ is Π^0_1-piecewise computable if and only if it is identifiable in the limit or learnable in the following sense: a partial function $\Gamma :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ will be called *learnable* or $(1,\omega)$-*computable* if there is a computable function $\Psi :\subseteq \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ such that $\Phi_{\lim_{n\to\infty}\Psi(f\upharpoonright n)}(f) = \Gamma(f)$ for every $f \in \mathrm{dom}(\Gamma)$, where recall that $\{\Phi_e\}_{e\in\mathbb{N}}$ is a fixed enumeration of all partial computable functions. Such a Ψ is called a *learner*.

We say that partial function $\widehat{\Psi} :\subseteq \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ dominates $\Psi :\subseteq \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ as a learner if $\lim_s \widehat{\Psi}(f \upharpoonright s)$ converges to $\lim_s \Psi(f \upharpoonright s)$ whenever $\lim_s \Psi(f \upharpoonright s)$ converges. We say that $\{\Psi_e\}_{e\in\mathbb{N}}$ enumerates all learners if every partial function $\Psi :\subseteq \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ is dominated by some Ψ_e as a learner. To get a nice enumeration of all learners, we first check the following proposition.

Proposition 1. *There is an effective enumeration $\{\Psi_e\}_{e\in\mathbb{N}}$ of all learners that consists of total functions $\Psi_e : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$.*

Proof. For the e-th partial computable function $\varphi_e :\subseteq \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ and an index k, we effectively define a total computable function $\Psi_{\langle e,k\rangle} : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ that dominates φ_e as a learner. We define Φ by $\Phi(\langle\rangle) = k$ and $\Phi(\sigma) = \varphi_e(\sigma)$ for all nonempty strings σ. Given $\sigma \in \mathbb{N}^{<\mathbb{N}}$, put $\sigma^* = \max\{\tau \subseteq \sigma : \Phi(\tau)\downarrow$ by stage $|\sigma|\}$. Then define $\Psi_{\langle e,k\rangle}(\sigma) = \Phi(\sigma^*)$ for every $\sigma \in \mathbb{N}^{<\mathbb{N}}$. If $\lim_s \Phi_e(f \upharpoonright s)$ converges then clearly $\lim_s \Psi_{\langle e,k\rangle}(f \upharpoonright s)$ also converges to the same value. Hence, $\Psi_{\langle e,k\rangle}$ dominates φ_e. □

The set $\{\Psi_e\}_{e\in\mathbb{N}}$ in Proposition 1 is referred as *the effective enumeration of all learners*, and Ψ_e is called *the e-th learner*.

Remark. We urge the reader not to confuse the notions $\Psi(\sigma)$ and $\Phi(\sigma)$ for a learner Ψ and a computable function Φ (on $\mathbb{N}^{\mathbb{N}}$). In the former case, $\Psi(\sigma)$ simply denotes the output (the inference) of the learner Ψ based on the current input σ. In the latter case, however, we use σ as an initial segment of some oracle information, and so really $\Phi(\sigma)$ denotes a string $\langle\Phi(\sigma; 0), \Phi(\sigma; 1), \Phi(\sigma; 2), \dots\rangle$.

Notation. Let $\Psi : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ be a learner. For any string $\sigma \in \mathbb{N}^{<\mathbb{N}}$, the set of *mind-change locations of the learner* Ψ *on the informant* σ (denoted by $\mathtt{mcl}_\Psi(\sigma)$) is defined by

$$\mathtt{mcl}_\Psi(\sigma) = \{n < |\sigma| : \Psi(\sigma \upharpoonright n + 1) \neq \Psi(\sigma \upharpoonright n)\}.$$

We also define $\mathtt{mcl}_\Psi(f) = \bigcup_{n\in\mathbb{N}} \mathtt{mcl}_\Psi(f \upharpoonright n)$ for any $f \in \mathbb{N}^{\mathbb{N}}$. Then, $\#\mathtt{mcl}_\Psi(f)$ denotes the *number of times that the learner* Ψ *changes her/his mind on the informant* f. Moreover, the set of *indices predicted by the learner* Ψ *on the informant* σ (denoted by $\mathtt{indx}_\Psi(\sigma)$) is defined by

$$\mathtt{indx}_\Psi(\sigma) = \{\Psi(\sigma \upharpoonright n) : n \leq |\sigma|\}.$$

We also define $\mathtt{indx}_\Psi(f) = \bigcup_{n\in\mathbb{N}} \mathtt{indx}_\Psi(f \upharpoonright n)$ for any $f \in \mathbb{N}^{\mathbb{N}}$.

We now introduce various subclasses of nonuniformly computable functions on $\mathbb{N}^{\mathbb{N}}$ based on Learning Theory.

Definition 2. Let D be a subset of Baire space $\mathbb{N}^{\mathbb{N}}$, and $\alpha, \beta, \gamma \leq \omega$ be ordinals. A function $\Gamma : D \to \mathbb{N}^{\mathbb{N}}$ is $(\alpha, \beta|\gamma)$*-computable* if there is a set $I \subseteq \mathbb{N}$ of cardinality α such that, for any $g \in D$, there is an index $e \in I$ satisfying the following three conditions.

1. (Learnability) $\lim_n \Psi_e(g \upharpoonright n)$ converges, and $\Phi_{\lim_n \Psi_e(g\upharpoonright n)}(g) = \Gamma(g)$.
2. (Mind-Change Condition) $\#\mathtt{mcl}_{\Psi_e}(g) = \#\{n \in \mathbb{N} : \Psi_e(g \upharpoonright n + 1) \neq \Psi_e(g \upharpoonright n)\} < \beta$.
3. (Error Condition) $\#\mathtt{indx}_{\Psi_e}(g) = \#\{\Psi_e(g \upharpoonright n) : n \in \mathbb{N}\} \leq \gamma$.

If $\gamma = \omega$, then we simply say that Γ is (α, β)*-computable* for $(\alpha, \beta|\gamma)$-computable function Γ. Let $[\mathfrak{C}_T]^\alpha_\beta$ (resp. $[\mathfrak{C}_T]^\alpha_{\beta|\gamma}$) denote the set of all (α, β)-computable (resp. $(\alpha, \beta|\gamma)$-computable) functions. Hereafter, the symbol $< \omega$ will be used in referring to "some natural number n". For instance, Γ is said to be $(< \omega, 2| < \omega)$-computable if there are $a, c \in \mathbb{N}$ such that it is $(a, 2|c)$-computable.

Table 1: Seven Classes of Nonuniformly Computable Functions

$[\mathfrak{C}_T]^1_1$	(Uniformly) computable
$[\mathfrak{C}_T]^1_{<\omega}$	Learnable with bounded mind changes
$[\mathfrak{C}_T]^1_{\omega\|<\omega}$	Learnable with bounded errors
$[\mathfrak{C}_T]^1_{\omega}$	Learnable
$[\mathfrak{C}_T]^{<\omega}_1$	k-wise computable for some $k \in \omega$
$[\mathfrak{C}_T]^{<\omega}_{\omega}$	Learnable by a team
$[\mathfrak{C}_T]^{\omega}_1$	Nonuniformly computable

$$[\mathfrak{C}_T]^1_1 \subseteq [\mathfrak{C}_T]^1_{<\omega} \subseteq [\mathfrak{C}_T]^1_{\omega|<\omega} \begin{array}{c} \subsetneq \quad [\mathfrak{C}_T]^{<\omega}_1 = [\mathfrak{C}_T]^{<\omega}_{\omega|<\omega} \quad \subsetneq \\ \\ \subsetneq \quad [\mathfrak{C}_T]^1_{\omega} \quad \subsetneq \end{array} [\mathfrak{C}_T]^{<\omega}_{\omega} \subseteq [\mathfrak{C}_T]^{\omega}_1 = [\mathfrak{C}_T]^{\omega}_{\omega}$$

Table 2: Seven monoids of nonuniformly computable functions

Remark. Some of $(\alpha,\beta|\gamma)$-computability notions are related to learnability notions: Every $(1,<\omega)$-computable function is *learnable with bounded mind-changes*; every $(1,\omega|<\omega)$-computable function is *learnable with bounded errors*; every $(1,\omega)$-computable function is *learnable*; every $(<\omega,1)$-computable function is *k-wise computable*; and every $(<\omega,\omega)$-computable function is *team-learnable*. The concept of learnability in the context of real number computation has been studied by several researchers including Chadzelek-Hotz [21], Ziegler [85, 86], and de Brecht-Yamamoto [24, 25]. The notion of mind-change is also related to the level of discontinuity studied by several researchers, for instance, Hertling [33], and Hemmerling [32]. See also Section 5.3 for more information on the relationship between the notion of mind-changes and the level of discontinuity. The notion of k-wise computability has been also studied by, for example, Pauly [60] and Ziegler [88].

We first mention the topological interpretation of the learnability. For a sequence $\{\sigma_n\}_{n\in\mathbb{N}} \in (\mathbb{N}^{<\mathbb{N}})^{\mathbb{N}}$ of strings, $\lim_n \sigma_n$ is defined by $(\lim_n \sigma_n)(m) = \lim_n(\sigma_n(m))$. If $\lim_n \sigma_n : \mathbb{N} \to \mathbb{N}$ is total, say $\lim_n \sigma_n = h \in \mathbb{N}^{\mathbb{N}}$, then we say that $\lim_n \sigma_n \in \mathbb{N}^{\mathbb{N}}$ *converges to* h.

Proposition 3. *Fix an ordinal $\alpha \leq \omega$. A partial function $\Gamma :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is $(1,\alpha)$-computable if and only if there is a total computable function $\psi : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ such that* $\lim_n \psi(g \upharpoonright n)$ *converges to* $\Gamma(g)$*, and* $\#\{n \in \mathbb{N} : \psi(g \upharpoonright n+1) \not\supseteq \psi(g \upharpoonright n)\} < \alpha$*, for any* $g \in \mathrm{dom}(\Gamma)$.

Proof. Assume that Γ is $(1,\alpha)$-computable via a learner Ψ. We put $\psi(\sigma) = \Phi_{\Psi(\sigma)}(\sigma)$ for each $\sigma \in \mathbb{N}^{<\mathbb{N}}$. Then the condition $\#\mathtt{mcl}_\Psi(g) < \alpha$ implies $\#\{n \in \mathbb{N} : \psi(g \upharpoonright n+1) \not\supseteq \psi(g \upharpoonright n)\} < \alpha$, for any $g \in \mathrm{dom}(\Gamma)$. Because if $\Psi(g \upharpoonright n+1) = \Psi(g \upharpoonright n)$, then $\psi(g \upharpoonright n) = \Phi_{\Psi(g\upharpoonright n)}(g \upharpoonright n) \subseteq \Phi_{\Psi(g\upharpoonright n)}(g \upharpoonright n+1) = \psi(g \upharpoonright n+1)$. Thus, clearly, $\lim_n \psi(g \upharpoonright n)$ converges to $\Phi_{\lim_n \Psi(g\upharpoonright n)}(g) = \Gamma(g)$.

Assume that $\Gamma(g) = \lim_n \psi(g \upharpoonright n)$ for any $g \in \mathrm{dom}(\Gamma)$ for some ψ satisfying the condition in Proposition 3. We define a computable function $\Phi_{e(\sigma)} : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ for each $\sigma \in \mathbb{N}^{<\mathbb{N}}$. For any $g \in \mathbb{N}^\mathbb{N}$, put $\Phi_{e(\sigma)}(g; n) = \psi(g \upharpoonright s)(n)$ for each $n \in \mathbb{N}$, where $s \geq |\sigma|$ is the least number such that $\psi(g \upharpoonright s)(n)$ is defined. Clearly, $\Phi_{e(\sigma)}$ is partial computable, and indeed, we can compute an index $e(\sigma)$ of $\Phi_{e(\sigma)}$ uniformly in $\sigma \in \mathbb{N}^{<\mathbb{N}}$. Then, we define a learner Ψ inductively. Put $\Psi(\langle\rangle) = e(\langle\rangle)$. Fix $\sigma \in \mathbb{N}^{<\mathbb{N}}$, and assume that $\Psi(\sigma^-)$ has already been defined. If $\psi(\sigma) \supseteq \psi(\sigma^-)$, then set $\Psi(\sigma) = \Psi(\sigma^-)$. If $\psi(\sigma) \not\supseteq \psi(\sigma^-)$, then set $\Psi(\sigma) = e(\sigma)$. Clearly, the condition $\#\{n \in \mathbb{N} : \psi(g \upharpoonright n+1) \not\supseteq \psi(g \upharpoonright n)\} < \alpha$ implies $\#\mathtt{mcl}_\Psi(g) < \alpha$, for any $g \in \mathrm{dom}(\Gamma)$. In particular, $\lim_n \Psi(g \upharpoonright n)$ converges to some index $e(\sigma)$ for any $g \in \mathrm{dom}(\Gamma)$. Hence, $\Phi_{\lim_n \Psi(g\upharpoonright n)}(g) = \bigcup_{n\geq|\sigma|} \psi(g \upharpoonright n) = \lim_{n\in\mathbb{N}} \psi(g \upharpoonright n) = \Gamma(g)$, since $\{\psi(g \upharpoonright n)\}_{n\geq|\sigma|}$ is an increasing sequence of strings. □

Corollary 4 (de Brecht-Yamamoto [24])**.** *A partial function $\Gamma :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is $(1,\omega)$-computable if and only if there is a computable sequence $\{\Gamma_n\}_{n\in\mathbb{N}}$ of partial computable functions which converges pointwise to Γ on* $\mathrm{dom}(\Gamma)$ *with respect to the discrete topology on $\mathbb{N}^\mathbb{N}$.*

Proof. By Proposition 3. □

2.2. Seven Classes of Nonuniformly Computable Functions

We first check several basic properties of $(\alpha,\beta|\gamma)$-computability to show the following theorem stating that the classes obtained from Definition 2 closed under composition are exactly the classes listed in Table 1.

Theorem 5. $\{[\mathfrak{C}_T]^\alpha_{\beta|\gamma} : \alpha,\beta,\gamma \in \mathbb{N} \cup \{<\omega,\omega\}\}$ *contains just seven monoids,* $[\mathfrak{C}_T]^1_1$, $[\mathfrak{C}_T]^1_{<\omega}$, $[\mathfrak{C}_T]^1_{\omega|<\omega}$, $[\mathfrak{C}_T]^{<\omega}_1$, $[\mathfrak{C}_T]^1_\omega$, $[\mathfrak{C}_T]^{<\omega}_\omega$, *and* $[\mathfrak{C}_T]^\omega_1$.

Proposition 6. *Let Γ be a partial function on Baire space $\mathbb{N}^\mathbb{N}$.*

1. *If Γ is $(\alpha_0,\beta_0|\gamma_0)$-computable, $\alpha_0 \leq \alpha_1$, $\beta_0 \leq \beta_1$, and $\gamma_0 \leq \gamma_1$, then Γ is $(\alpha_1,\beta_1|\gamma_1)$-computable.*
2. *Γ is $(\alpha,1)$-computable if and only if Γ is $(\alpha,\beta|1)$-computable.*
3. *Γ is (α,β)-computable if and only if Γ is $(\alpha,\beta|\beta)$-computable.*
4. *Γ is $(1,1)$-computable if and only if Γ is computable.*
5. *Γ is $(\omega,1)$-computable if and only if Γ is (ω,ω)-computable if and only if Γ is nonuniformly computable, i.e., $\Gamma(g) \leq_T g$ for any $g \in$* $\mathrm{dom}(\Gamma)$*, where recall that $\leq_T$ denotes the Turing reducibility.*

Proof. The items (1) and (2) easily follow from the definitions. The item (3) follows from $\#\mathtt{indx}_\Psi(g) - 1 \leq \#\mathtt{mcl}_\Psi(g)$.

(4) If Γ is computable via Φ_e, then Γ is $(1,1)$-computable via the singleton $\{i(e)\}$, where $\Psi_{i(e)}(\sigma) = e$ for any $\sigma \in \mathbb{N}^{<\mathbb{N}}$. Assume that Γ is $(1,1)$-computable via a singleton $\{e\}$. Then $\Psi_e(\sigma) = \Psi_e(\langle\rangle)$ for any σ extendible to an element of $\mathrm{dom}(\Gamma)$, since $\#\mathtt{mcl}_{\Psi_e}(g) = 0$ for any $g \in \mathrm{dom}(\Gamma)$. Therefore, Γ is computable via $\Phi_{\Psi_e(\langle\rangle)}$.

(5) If Γ is nonuniformly computable, then Γ is $(\omega,1)$-computable via $\{i(e)\}_{e\in\mathbb{N}}$, where $\Psi_{i(e)}(\sigma) = e$ for any $\sigma \in \mathbb{N}^{<\mathbb{N}}$. Assume that Γ is (ω,ω)-computable via I. For any $g \in \mathrm{dom}(\Gamma)$, there is $e \in I$ such that $\lim_n \Psi_e(g \upharpoonright n)$ converges to some value $p \in \mathbb{N}$, and $\Phi_p(g) = \Gamma(g)$. Thus, $\Gamma(g) \leq_T g$ via Φ_p. □

Proposition 7. *For each $m, n \in \mathbb{N}$, every $(m, \omega|n)$-computable function is $(m \cdot n, 1)$-computable.*

Proof. Assume that $\Gamma : D \to \mathbb{N}^{\mathbb{N}}$ is $(m, \omega|n)$-computable with m-learners $\{\Psi^e\}_{e<m}$ with n-errors. Now, we define an algorithm Φ^e_k for any $e < m$ and $k < n$, and we ensure the following property:

$$(\forall g \in D)(\exists e < m)(k < n)\ \Phi^e_k(g) = \Gamma(g).$$

The algorithm Φ^e_k proceeds as follows for g. Recall that $\mathtt{indx}_{\Psi^e}(g)$ represents the set of all indices occurring in hypothesis of the learner Ψ^e. We have an effective enumeration $d^e_0(g), d^e_1(g), \dots$ of all indices contained in $\mathtt{indx}_{\Psi^e}(g)$ uniformly in g. Then, we set $\Phi^e_k(g) = \Phi_{d^e_k(g)}(g)$ if $d^e_k(g)$ is defined. For any $g \in D$, there is $e < m$ such that $\lim_s \Psi^e(g \upharpoonright s)$ converges to some correct computation d of $\Gamma(g)$, i.e., $\Phi_d(g) = \Gamma(g)$. Since $\#\mathtt{indx}_{\Psi^e}(g) < n$, we have $d^e_k(g) = d$ for some $k < n$. Thus, for any $g \in D$, there are $e < m$ and $k < n$ such that $\Phi^e_k(g) = \Gamma(g)$. Therefore, if i^e_k is an index of Φ^e_k for each $e < m$ and $k < n$, then Γ is $(m \cdot n, 1)$-computable via an upper bound $\max\{i^e_k : e < m \ \&\ k < n\}$. □

Corollary 8. *Γ is $(<\omega, \omega|<\omega)$-computable if and only if Γ is $(<\omega, 1)$-computable.*

Proof. Every $(<\omega, \omega|<\omega)$-computable function Γ is $(m, \omega|n)$-computable for some $m, n < \omega$. Therefore, by Proposition 7, Γ is $(m \cdot n, 1)$-computable. In particular, Γ is $(<\omega, 1)$-computable, since $m \cdot n < \omega$. □

Proposition 9. *For each $i < 2$, let Γ_i be a partial $(\alpha_i, \beta_i|\gamma_i)$-computable function on Baire space $\mathbb{N}^{\mathbb{N}}$, where $\alpha_i, \beta_i, \gamma_i \leq \omega$ are ordinals. Then $\Gamma_1 \circ \Gamma_0$ is $(\alpha_0 * \alpha_1, \beta_0 * \beta_1|\gamma_0 * \gamma_1)$-computable, where $*$ is the multiplication as the cardinals, or equivalently, $\kappa * \lambda = \min\{\kappa \cdot \lambda, \omega\}$ for ordinals $\kappa, \lambda \leq \omega$.*

Proof. For each $i < 2$, since Γ_i is $(\alpha_i, \beta_i|\gamma_i)$-computable, there is a collection of learners, $\{\Psi^i_j\}_{j<\alpha_i}$ and a cover $\{U^i_j\}_{j<\alpha_i}$ of $\mathrm{dom}(\Gamma_i)$ such that $\Gamma_i(f) = \Phi_{\lim_n \Psi^i_j(f \upharpoonright n)}(f \upharpoonright n)$ and $\#\mathtt{mcl}_{\Psi^i_j}(f) < \beta_i$ and $\#\mathtt{indx}_{\Psi^i_j}(f) < \gamma_i$, for any $j < \alpha_i$ and $f \in U^i_j$. Fix $j < \alpha_0$ and $k < \alpha_1$. Then $\Psi^*_{j,k}(\sigma)$ is defined as follows. Let $J(\sigma)$ be the longest interval $[r, |\sigma|)$ satisfying $\Psi^0_j(\sigma \upharpoonright r) = \Psi^0_j(\sigma)$, and define $J^+(\sigma) = J(\sigma) \setminus \{r\}$. If $\#(\mathtt{mcl}_{\Psi^1_k} \cap J^+(\sigma)) < \beta_1$ and $\#(\mathtt{indx}_{\Psi^1_k} \cap J(\sigma)) < \gamma_1$, then put $\Psi^*_{j,k}(\sigma) = \Psi^1_k(\Phi_{\Psi^0_j(\sigma)}(\sigma))$. Otherwise, put $\Psi^*_{j,k}(\sigma) = \Psi^*_{j,k}(\sigma^-)$. For given σ, we compute an index $\Psi_{j,k}(\sigma)$, where $\Phi_{\Psi_{j,k}(\sigma)}(f) = \Phi_{\Psi^*_{j,k}(\sigma))}(\Phi_{\Psi^0_j(\sigma)}(f))$ for any f.

Note that $f \in \mathrm{dom}(\Gamma_1 \circ \Gamma_0)$ if and only if $f \in \mathrm{dom}(\Gamma_0)$ and $\Gamma_0(f) \in \mathrm{dom}(\Gamma_1)$. Therefore, for such f, there are $j < \alpha_0$ and $k < \alpha_1$ such that $f \in U^0_j$ and $\Gamma_0(f) \in U^1_k$. Assume that $f \in \mathrm{dom}(\Gamma_1 \circ \Gamma_0) \cap U^0_j$ and $\Gamma_0(f) \in U^1_k$. It is easy to see that $\Psi^*_{j,k}$ is computable, $\#\mathtt{mlc}_{\Psi^*_{j,k}}(f) < \beta_0 * \beta_1$ and $\#\mathtt{indx}_{\Psi^*_{j,k}}(f) < \gamma_0 * \gamma_1$. Moreover, there exist s and e_0 such that $\Psi^0_j(f \upharpoonright t) = \Psi^0_j(f \upharpoonright s) = e_0$ for any $t \geq s$. Fix such s. Since $\Phi_{e_0}(f) = \Gamma_0(f) \in U^1_k$, for any $t \geq s$, $\#(\mathtt{mcl}_{\Psi^1_k} \cap J^+(f \upharpoonright t)) < \beta_1$ and $\#(\mathtt{indx}_{\Psi^1_k} \cap J(f \upharpoonright t)) < \gamma_1$, since $J(f \upharpoonright t) = J(f \upharpoonright s)$ and by our choice of Ψ^1_k. Therefore, $\lim_n \Psi^*_{j,k}(f \upharpoonright n)$ converges to $\lim_n \Psi^1_k(\Gamma_0(f \upharpoonright n))$. However, there exist $u \geq s$ and e_1

such that $\Psi^1_k(\Gamma_0(f\upharpoonright v)) = \Psi^1_k(\Gamma_0(f\upharpoonright u)) = e_1$ for any $v \geq u$, since $\{\Gamma_0(f\upharpoonright u)\}_{u\geq s}$ is an increasing sequence of strings and $\Gamma_0(f) \in \mathrm{dom}(\Gamma_1)$. Here $\Phi_{e_1}(\Gamma_0(f)) = \Gamma_1(\Gamma_0(f))$. Thus,

$$\Phi_{\lim_n \Psi_{j,k}(f\upharpoonright n)}(f) = \Phi_{\lim_n \Psi^*_{j,k}(f\upharpoonright n)}(\Phi_{\lim_n \Psi^0_j(f\upharpoonright n)}(f)) = \Phi_{\lim_n \Psi^1_k(\Gamma_0(f)\upharpoonright n)}(\Gamma_0(f)) = \Gamma_1(\Gamma_0(f)).$$

Consequently, $\Gamma_1 \circ \Gamma_0$ is $(\alpha_0 * \alpha_1, \beta_0 * \beta_1 | \gamma_0 * \gamma_1)$-computable, via $\{\Psi_{j,k}\}_{j<\alpha_0, k<\alpha_1}$. □

Corollary 10. $[\mathfrak{C}_T]^\alpha_{\beta|\gamma}$ *forms a monoid under composition, for any* $\alpha, \beta, \gamma \in \{1, <\omega, \omega\}$.

Proof. Straightforward from Proposition 9. □

Proposition 11. $[\mathfrak{C}_T]^1_{<\omega}$ *is the smallest monoid including* $[\mathfrak{C}_T]^1_2$*;* $[\mathfrak{C}_T]^1_{\omega|<\omega}$ *is the smallest monoid including* $[\mathfrak{C}_T]^1_{\omega|2}$*.* $[\mathfrak{C}_T]^{<\omega}_1$ *is the smallest monoid including* $[\mathfrak{C}_T]^2_1$*;* $[\mathfrak{C}_T]^{<\omega}_\omega$ *is the smallest monoid including* $[\mathfrak{C}_T]^2_\omega$*.*

Proof. The first result is known, and indeed, it has also been proved in Mylatz's PhD thesis [56], but we also give a proof here for the sake of completeness. We first show that every $(1, n+1)$-computable function Γ can be represented as $\Gamma = \Gamma_1 \circ \Gamma_0$ for some $(1, n)$-computable function Γ_0 and $(1, 2)$-computable function Γ_1. Let Ψ be a learner for Γ. We define a learner Ψ_0 for Γ_0 and a learner Ψ_1 for Γ_1. For a given string $\sigma \in \mathbb{N}^{<\mathbb{N}}$, let $\sigma^* \subseteq \sigma$ be the longest initial segment of σ satisfying $\#\mathtt{mcl}_\Psi(\sigma^*) < n$. Then, on σ, the learner Ψ_0 guesses an index of the partial computable function $g \mapsto g \oplus \Phi_{\Psi(\sigma^*)}(g)$, i.e., $\Gamma_0(g) = \Phi_{\Psi_0(\sigma)}(g) = g \oplus \Phi_{\Psi(\sigma^*)}(g)$ for any $g \in \mathbb{N}^\mathbb{N}$. Note that $\#\mathtt{mcl}_{\Psi_0}(g) < n$ for any $g \in \mathbb{N}^\mathbb{N}$. Therefore, Γ_0 is $(1, n)$-computable. For $\sigma \oplus \tau \in \mathbb{N}^\mathbb{N}$, if $\sigma^* = \sigma$ then the learner Ψ_1 guesses an index of the partial computable function $g \oplus h \mapsto h$. If $\sigma^* \neq \sigma$, then Ψ_1 guesses an index of the partial computable function $g \oplus h \mapsto \Phi_{\Psi(\sigma)}(g)$, i.e., $\Phi_{\Psi_1(\sigma\oplus\tau)}(g \oplus h) = \Phi_{\Psi(\sigma)}(g)$. Since Γ is $(1, n+1)$-computable, and by the definition of σ^*, it is easy to see that Γ_1 is $(1, 2)$-computable. For $g \in \mathbb{N}^\mathbb{N}$, if $\#\mathtt{mcl}_\Psi(g) < n$, then

$$\Gamma_1(\Gamma_0(g)) = \Gamma_1(g \oplus \Gamma(g)) = \Gamma(g).$$

If $\#\mathtt{mcl}_\Psi(g) = n$, then

$$\Gamma_1(\Gamma_0(g)) = \Gamma_1(g \oplus \Phi_{\Psi(g^*)}(g)) = \Gamma(g).$$

Consequently, $\Gamma = \Gamma_1 \circ \Gamma_0$ as desired.

We next show that every $(1, \omega|n+1)$-computable function Γ can be represented as $\Gamma = \Gamma_1 \circ \Gamma_0$ for some $(1, \omega|n)$-computable function Γ_0 and $(1, \omega|2)$-computable function Γ_1. Assume that Ψ is a learner for Γ, and we enumerate $\#\mathtt{indx}_\Psi(\sigma)$ as $\{i^\sigma_m\}_{m\leq|\sigma|}$. Here, if $m < n$ then Ψ guesses i^σ_m before Ψ guesses i^σ_n on some initial segment of σ. Note that, if $\sigma \subseteq \tau$ and i^σ_m is defined, then $i^\sigma_m = i^\tau_m$. On $\sigma \in \mathbb{N}^{<\mathbb{N}}$, if $\Psi(\sigma) \neq i^\sigma_n$, then Ψ_0 guesses an index of the partial computable function $g \mapsto g \oplus \Phi_{\Psi(\sigma)}(g)$. Otherwise, Ψ_0 guesses an index of the partial computable function $g \mapsto g \oplus \Phi_{i^\sigma_0}(g)$. Then, the partial function Γ_0 identified by the learner Ψ_0 is $(1, \omega|n)$-computable. On $\sigma \oplus \tau \in \mathbb{N}^{<\mathbb{N}}$ if $\Psi(\sigma) \neq i^\sigma_n$, then Ψ_1 guesses an index of the partial computable function $g \oplus h \mapsto h$. Otherwise, Ψ_1 guesses an index of partial computable function $g \oplus h \mapsto \Phi_{\Psi(\sigma)}(g)$.

We show that every $(n+1,1)$-computable function Γ can be represented as $\Gamma = \Gamma_1 \circ \Gamma_0$ for some $(n,1)$-computable function Γ_0 and $(2,1)$-computable function Γ_1. Assume that Γ is $(n+1,1)$-computable via a collection $\{\Delta_i\}_{i\leq n}$ of partial computable functions. For $g \in \mathbb{N}^{\mathbb{N}}$, if $\Gamma(g) = \Delta_i(g)$ for some $i < n$, then $\Gamma_0(g) = g \oplus \Delta_i(g)$. Otherwise, we set $\Gamma_0(g) = g \oplus \Delta_0(g)$. Then, clearly Γ_0 is $(n,1)$-computable via $\{\lambda g.g \oplus \Delta_i(g)\}_{i<n}$. For $g \oplus h \in \mathbb{N}^{\mathbb{N}}$, if $\Gamma(g) = \Delta_i(g)$ for some $i < n$, then $\Gamma_1(g \oplus h) = h$. Otherwise, we set $\Gamma_1(g \oplus h) = \Delta_n(g)$. Clearly, Γ_1 is $(2,1)$-computable. Note that, if $g \in \mathrm{dom}(\Gamma)$, then $\Gamma(g) = \Delta_i(g)$ for some $i \leq n$. If $\Gamma(g) = \Delta_i(g)$ for some $i < n$, then $\Gamma_1(\Gamma_0(g)) = \Gamma_1(g \oplus \Delta_i(g)) = \Delta_i(g)$. If $\Gamma(g) = \Delta_n(g)$, then $\Gamma_1(\Gamma_0(g)) = \Gamma_1(g \oplus \Delta_0(g)) = \Delta_n(g)$. Therefore, $\Gamma(g) = \Gamma_1 \circ \Gamma_0(g)$ for any $g \in \mathrm{dom}(\Gamma)$. By the similar way, it is easy to see that every $(n+1,\omega)$-computable function Γ can be represented as $\Gamma = \Gamma_1 \circ \Gamma_0$ for some (n,ω)-computable function Γ_0 and $(2,\omega)$-computable function Γ_1. □

Proof of Theorem 5. By Proposition 6, we have $[\mathfrak{C}_T]^1_{1|1} = [\mathfrak{C}_T]^1_{1|<\omega} = [\mathfrak{C}_T]^1_{1|\omega} = [\mathfrak{C}_T]^1_{<\omega|1} = [\mathfrak{C}_T]^1_{\omega|1}$; $[\mathfrak{C}_T]^1_{<\omega|<\omega} = [\mathfrak{C}_T]^1_{1|<\omega}$; and $[\mathfrak{C}_T]^\omega_{1|1} = [\mathfrak{C}_T]^\omega_{\beta|\gamma}$ for any $\beta,\gamma \in \{1,<\omega,\omega\}$. Moreover, by Proposition 6 and Proposition 7, $[\mathfrak{C}_T]^{<\omega}_{1|1} = [\mathfrak{C}_T]^{<\omega}_{\beta|\gamma}$ whenever $\langle\beta,\gamma\rangle \neq \langle\omega,\omega\rangle$. Therefore, by Proposition 9 and 11, we have just seven monoids, $[\mathfrak{C}_T]^1_1$, $[\mathfrak{C}_T]^1_{<\omega}$, $[\mathfrak{C}_T]^1_{\omega|<\omega}$, $[\mathfrak{C}_T]^{<\omega}_1$, $[\mathfrak{C}_T]^1_\omega$, $[\mathfrak{C}_T]^{<\omega}_\omega$, and $[\mathfrak{C}_T]^\omega_1$. □

2.3. *Degree Structures and Brouwer Algebras*

We will see some intuitionistic feature of our classes of nonuniformly computable functions.

Definition 12. Let $\mathcal{F}$ be a monoid consisting of partial functions $\Gamma :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ under composition. Then, $\mathcal{P}(\mathbb{N}^{\mathbb{N}})$ is preordered by the relation $P \leq_{\mathcal{F}} Q$ indicating the existence of a function $\Gamma \in \mathcal{F}$ from Q into P, that is, $P \leq_{\mathcal{F}} Q$ if and only if there is a partial function $\Gamma :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that $\Gamma \in \mathcal{F}$ and $\Gamma(g) \in P$ for every $g \in Q$. Let $\mathcal{D}/\mathcal{F}$ and $\mathcal{P}/\mathcal{F}$ denote the quotient sets $\mathcal{P}(\mathbb{N}^{\mathbb{N}})/\equiv_{\mathcal{F}}$ and $\Pi^0_1(2^{\mathbb{N}})/\equiv_{\mathcal{F}}$, respectively. Here, $\Pi^0_1(2^{\mathbb{N}})$ denotes the set of all nonempty Π^0_1 subsets of $2^{\mathbb{N}}$. For $P \in \mathcal{P}(\mathbb{N}^{\mathbb{N}})$, the equivalence class $\{Q \subseteq \mathbb{N}^{\mathbb{N}} : Q \equiv_{\mathcal{F}} P\} \in \mathcal{D}/\mathcal{F}$ is called *the $\mathcal{F}$-degree* of P.

Recall from Corollary 10 that $\mathcal{F} = [\mathfrak{C}_T]^\alpha_{\beta|\gamma}$ forms a monoid for every $\alpha,\beta,\gamma \in \{1,<\omega,\omega\}$.

Notation. If $\mathcal{F} = [\mathfrak{C}_T]^\alpha_{\beta|\gamma}$ for some $\alpha,\beta,\gamma \in \{1,<\omega,\omega\}$, we write $\leq^\alpha_{\beta|\gamma}$, $\mathcal{D}^\alpha_{\beta|\gamma}$, and $\mathcal{P}^\alpha_{\beta|\gamma}$ instead of $\leq_{\mathcal{F}}$, $\mathcal{D}/\mathcal{F}$ and $\mathcal{P}/\mathcal{F}$.

Remark. By Proposition 6 (4) and (5), the preorderings $\leq^1_1$ and $\leq^\omega_1$ are equivalent to the Medvedev reducibility [51] and the Muchnik reducibility [54], respectively.

We also introduce the truth-table versions of Definition 2.

Definition 13. Let D be a subset of Baire space $\mathbb{N}^{\mathbb{N}}$, and $\alpha,\beta,\gamma \leq \omega$ be ordinals. A function $\Gamma : D \to \mathbb{N}^{\mathbb{N}}$ is *$(\alpha,\beta|\gamma)$-truth-table* if there are a set $I \subseteq \mathbb{N}$ of cardinality α, and a collection $\{p(e,k) : e \in I \ \&\ k < \min\{\beta,\gamma\}\}$ of indices of *truth-table functionals* (i.e., $\mathrm{dom}(\Phi_{p(e,k)}) = \mathbb{N}^{\mathbb{N}}$) such that

1. (Popperian Condition) for any $e \in I$ and $\sigma \in \mathbb{N}^{<\mathbb{N}}$, there is $k < z$ such that $\Psi_e(\sigma) = p(e,k)$.

2. Γ is $(\alpha,\beta|\gamma)$-computable via the family $\{\Psi_e\}_{e\in I}$.

Here, we do not assume the uniform computability of the collection $\{p(e,k) : e \in I \;\&\; k < \min\{\beta,\gamma\}\}$. If $\gamma = \omega$, then we simply say that Γ is (α,β)-*truth-table* for $(\alpha,\beta|\gamma)$-truth-table function Γ. Let $[\mathfrak{C}_{tt}]^\alpha_\beta$ (resp. $[\mathfrak{C}_{tt}]^\alpha_{\beta|\gamma}$) denote the set of all (α,β)-truth-table (resp. $(\alpha,\beta|\gamma)$-truth-table) functions.

Remark. It is easily checked that the truth-table versions of Proposition 6, Proposition 9, Corollary 10 and Proposition 11 hold.

Notation. If $\mathcal{F} = [\mathfrak{C}_{tt}]^\alpha_{\beta|\gamma}$ for some $\alpha,\beta,\gamma \in \{1, <\omega, \omega\}$, we write $\leq^\alpha_{tt,\beta|\gamma}$, $\mathcal{D}^\alpha_{tt,\beta|\gamma}$, and $\mathcal{P}^\alpha_{tt,\beta|\gamma}$ instead of $\leq_\mathcal{F}$, $\mathcal{D}/\mathcal{F}$ and $\mathcal{P}/\mathcal{F}$.

Proposition 14. $\aleph_0 = \#[\mathfrak{C}_r]^1_1 = \#[\mathfrak{C}_r]^1_{<\omega} = \#[\mathfrak{C}_r]^1_{\omega|<\omega} = \#[\mathfrak{C}_r]^1_\omega < \#[\mathfrak{C}_r]^{<\omega}_1 = \#[\mathfrak{C}_r]^{<\omega}_\omega = \#[\mathfrak{C}_r]^\omega_1 = 2^{2^{\aleph_0}}$*, for each* $r \in \{tt, T\}$.

Proof. Every learner Ψ determines just one learnable function $\Gamma \in [\mathfrak{C}_T]^1_\omega$. Therefore, $[\mathfrak{C}_T]^1_\omega$ is countable. For non-uniform computability, we first see $\#[\mathfrak{C}_T]^\omega_1 \leq 2^{2^{\aleph_0}}$ since $\#(\mathbb{N}^\mathbb{N})^{\mathbb{N}^\mathbb{N}} = 2^{2^{\aleph_0}}$ by cardinal arithmetic. On the other hand, every function $\Gamma : \mathbb{N}^\mathbb{N} \to \{0^\mathbb{N}, 1^\mathbb{N}\}$ is $(<\omega, 1)$-truth-table via two constant truth-table functionals $\Gamma_0(f) = 0^\mathbb{N}$ and $\Gamma_1(f) = 1^\mathbb{N}$ for any $f \in \mathbb{N}^\mathbb{N}$. Therefore, $\#[\mathfrak{C}_{tt}]^{<\omega}_1 \geq 2^{2^{\aleph_0}}$. ◻

Proposition 15. *For each $\alpha,\beta,\gamma \in \{1, <\omega, \omega\}$, the order structures $\mathcal{D}^\alpha_{\beta|\gamma}$, $\mathcal{D}^\alpha_{tt,\beta|\gamma}$, $\mathcal{P}^\alpha_{\beta|\gamma}$, and $\mathcal{P}^\alpha_{tt,\beta|\gamma}$ form lattices with top and bottom elements.*

Proof. It is easy to see that the product $\otimes$ and the sum $\oplus$ form supremum and infimum operations in these structures. Moreover, every degree structure has top and bottom elements since it is coarser than $\mathcal{D}^1_1$, that has top and bottom elements. ◻

If a lattice $(L, \leq, \vee, \wedge)$ has the top element 1, the bottom element 0, and $\max\{c : c \wedge a \leq b\}$ (denoted by $a \to_L b$) exists for any $a, b \in L$, then $\mathcal{L} = (L, \leq, \vee, \wedge, \to_L, 0, 1)$ is called a *Heyting algebra*. An algebra $\mathcal{L} = (L, \leq, \vee, \wedge, \to, \bot, \top)$ is a *Brouwer algebra* if its dual $\mathcal{L}^{\rm op} = (L, \geq, \wedge, \vee, \leftarrow, \top, \bot)$ is a Heyting algebra. Recall that the Medvedev lattice $\mathcal{D}^1_1$ and the Muchnik lattice $\mathcal{D}^\omega_1$ form Brouwer algebras [51, 54].

Proposition 16. *The degree structures $\mathcal{D}^1_\omega$ and $\mathcal{D}^1_{tt,\omega}$ are Brouwerian.*

Proof. We just give a proof for $\mathcal{D}^1_\omega$, although it is straightforward to modify the proof for the truth-table version.

Set $B(P,Q) = \{R \subseteq \mathbb{N}^\mathbb{N} : P \leq^1_\omega Q \otimes R\}$. We need to a construct a function $\beta : \mathcal{P}(\mathbb{N}^\mathbb{N}) \times \mathcal{P}(\mathbb{N}^\mathbb{N}) \to \mathcal{P}(\mathbb{N}^\mathbb{N})$ such that $\beta(P,Q) = \min B(P,Q)$ for any $P, Q \subseteq \mathbb{N}^\mathbb{N}$. Let Λ_e denote the e-th $(1,\omega)$-computable function, i.e., $\Lambda_e(g) = \Phi_{\lim_n \Psi_e(g\upharpoonright n)}(g)$ for any $g \in \mathrm{dom}(\Lambda_e)$. Define β as follows.

$$\beta(P,Q) = \{e^\frown g \in \mathbb{N}^\mathbb{N} : (\forall f \in Q)\ \Lambda_e(f \oplus g) \in P\}.$$

It is easy to see that $\beta(P,Q) \in B(P,Q)$ for any $P, Q \subseteq \mathbb{N}^\mathbb{N}$. If $R \in B(P,Q)$, say $\Lambda_e : Q \otimes R \to P$, then clearly $e^\frown g \in \beta(P,Q)$ for any $g \in R$. Thus, $\beta(P,Q) \leq^1_1 R$. ◻

In contrast, we will show in Part II that *neither* $\mathcal{D}^1_{<\omega}$, *nor* $\mathcal{D}^1_{\omega|<\omega}$, *nor* $\mathcal{D}^{<\omega}_1$, *nor* $\mathcal{D}^{<\omega}_\omega$ form Brouwer algebras. In the meantime, the following modifications of $\mathcal{D}^1_{<\omega}$, $\mathcal{D}^1_{\omega|<\omega}$, $\mathcal{D}^{<\omega}_1$, and $\mathcal{D}^{<\omega}_\omega$ look more natural than our original definitions, from the viewpoint of constructive mathematics. Indeed, in Proposition 20, we will see that these modifications form Brouwer algebras.

Definition 17. Let D be a subset of Baire space $\mathbb{N}^\mathbb{N}$, and $\alpha,\beta,\gamma \leq \omega$ be ordinals, or $\mathtt{eff}$. We generalize the $(\alpha,\beta|\gamma)$-computability as follows. If $\alpha = \mathtt{eff}$, then we revise the term "for any $g \in D$, there is $e \in I$" to the term "there is a partial computable function $B_0 :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}$ such that, for any $g \in D$, there is $e < B_0(g)$". If $\beta = \mathtt{eff}$, then we revise the mind change condition as $\#\mathtt{mcl}_{\Psi_e}(g) < B_1(g)$, where B_1 is a partial computable function from $\mathbb{N}^\mathbb{N}$ to $\mathbb{N}$. If $\gamma = \mathtt{eff}$, then we revise the error condition as $\#\mathtt{indx}_{\Psi_e}(g) < B_2(g)$, where B_2 is a partial computable function from $\mathbb{N}^\mathbb{N}$ to $\mathbb{N}$. For new notions, $\leq^\alpha_{\beta|\gamma}$, $\mathcal{D}^\alpha_{\beta|\gamma}$, and $\mathcal{P}^\alpha_{\beta|\gamma}$ are also defined as the usual way.

Proposition 18. *Suppose that, if $\tau = \mathtt{eff}$, then let τ^* mean the symbol $< \omega$, and otherwise, set $\tau^* = \tau$. Then, every $(\alpha,\beta|\gamma)$-computable function with a compact domain is $(\alpha^*,\beta^*|\gamma^*)$-computable.*

Proof. By continuity of B_0, B_1, and B_2 in Definition 17, $\{B_i^{-1}(\{e\})\}_{e\in\mathbb{N}}$ for each $i < 3$ is an open cover of D. Hence, by compactness of D, we have the desired condition. □

Corollary 19. *$\mathcal{P}^1_{\mathtt{eff}} = \mathcal{P}^1_{<\omega}$; $\mathcal{P}^1_{\omega|\mathtt{eff}} = \mathcal{P}^1_{\omega|<\omega}$; $\mathcal{P}^{\mathtt{eff}}_1 = \mathcal{P}^{<\omega}_1$; and $\mathcal{P}^{\mathtt{eff}}_\omega = \mathcal{P}^{<\omega}_\omega$.* □

That is to say, for Π^0_1 subsets of Cantor space $2^\mathbb{N}$, no new reducibility notion is constructed from Definition 17. However, from the perspective of intuitionistic caluculus, our new notions in Definition 17 have nice features.

Proposition 20. *$\mathcal{D}^1_{\mathtt{eff}}$, $\mathcal{D}^1_{\omega|\mathtt{eff}}$, $\mathcal{D}^{\mathtt{eff}}_1$, and $\mathcal{D}^{\mathtt{eff}}_\omega$ are Brouwerian.*

Proof. Fix $\alpha,\beta,\gamma \in \{1, <\omega, \mathtt{eff}, \omega\}$, and set $B(P,Q) = \{R \subseteq \mathbb{N}^\mathbb{N} : P \leq^\alpha_{\beta|\gamma} Q \otimes R\}$. We need to construct a function $\beta : \mathcal{P}(\mathbb{N}^\mathbb{N}) \times \mathcal{P}(\mathbb{N}^\mathbb{N}) \to \mathcal{P}(\mathbb{N}^\mathbb{N})$ such that $\beta(P,Q) = \min B(P,Q)$ for any $P,Q \subseteq \mathbb{N}^\mathbb{N}$. Let Λ_e denote the e-th $(1,\omega)$-computable function, and Θ_e be the e-th partial computable function from $\mathbb{N}^\mathbb{N}$ to $\mathbb{N}$. Put $\mathrm{change}_e(g) = \#\{n \in \mathbb{N} : \Lambda_e(g \upharpoonright n+1) \neq \Lambda_e(g \upharpoonright n)\}$, and $\mathrm{error}_e(g) = \#\{\Lambda_e(g \upharpoonright n) : n \in \mathbb{N}\}$. Then,

$$\beta(P,Q) = \begin{cases} \{(e,d)^\frown g : (\forall f \in Q)\ \Lambda_e(f \oplus g) \in P \ \&\ \#\mathtt{mcl}_{\Lambda_e}(f \oplus g) < \Theta_d(f \oplus g)\}, \\ \hfill \text{if } (\alpha,\beta,\gamma) = (1,\mathtt{eff},\omega), \\ \{(e,d)^\frown g : (\forall f \in Q)\ \Lambda_e(f \oplus g) \in P \ \&\ \#\mathtt{indx}_{\Lambda_e}(f \oplus g) < \Theta_d(f \oplus g)\}, \\ \hfill \text{if } (\alpha,\beta,\gamma) = (1,\omega,\mathtt{eff}), \\ \{d^\frown g : (\forall f \in Q)(\exists e < \Theta(f \oplus g))\ \Phi_e(f \oplus g) \in P\}, \\ \hfill \text{if } (\alpha,\beta,\gamma) = (\mathtt{eff},1,\omega), \\ \{d^\frown g : (\forall f \in Q)(\exists e < \Theta(f \oplus g))\ \Lambda_e(f \oplus g) \in P\}, \\ \hfill \text{if } (\alpha,\beta,\gamma) = (\mathtt{eff},\omega,\omega), \end{cases}$$

It is easy to see that $\beta(P,Q) \in B(P,Q)$ for any $P,Q \subseteq \mathbb{N}^\mathbb{N}$. For the minimality, if $R \in B(P,Q)$, we have suitable d and e such that $(d,e)^\frown g \in \beta(P,Q)$ for any $g \in R$. Thus, $\beta(P,Q) \leq^1_1 R$. □

Remark. Unfortunately, neither $\mathcal{P}^1_{\mathtt{eff}}$ nor $\mathcal{P}^1_{\omega|\mathtt{eff}}$ nor $\mathcal{P}^{\mathtt{eff}}_1$ nor $\mathcal{P}^{\mathtt{eff}}_\omega$ form Brouwer algebra (see Part II).

2.4. Falsifiable Problems and Total Functions

In Part II, we will mainly pay attention to the behavior of nonuniform computability on Π^0_1 *subsets of Cantor space* $2^{\mathbb{N}}$. Such a restriction has an interesting feature by thinking of Π^0_1 sets as *falsifiable mass problems*. Consider a learner Ψ identifies a $(1,\omega)$-computable function $\Gamma : Q \to P$. On an observation $\sigma \in \mathbb{N}^{<\mathbb{N}}$ with $[\sigma] \cap Q \neq \emptyset$, a learner Ψ conjectures that e is a correct algorithm computing a solution of P from σ, that is, $\Phi_{\Psi(\sigma)}(f) = \Phi_e(f) \in P$ for any future observation $f \in Q \cap [\sigma]$. If Q is Π^0_1, Proposition 21 (3) suggests that we may assume that e is an index of a total computable function. Then, the learner Ψ can find mistakes of his hypothesis on P whenever P is also a Π^0_1 subset of the Baire space $\mathbb{N}^{\mathbb{N}}$. Therefore, restricting to Π^0_1 subsets is expected to be an analogy of *Popperian learning*. In this context, the usual Popperian learning on total computable functions could be regarded as a learning process on Π^0_1 singletons. We first see that, if we restrict our attention to Π^0_1 sets, then some reducibility notions collapse.

Proposition 21. *Let P be a Π^0_1 subset of $\mathbb{N}^{\mathbb{N}}$, and X be any subset of $\mathbb{N}^{\mathbb{N}}$.*

1. *$X \leq^1_{tt,1} P$ if and only if $X \leq^1_1 P$.*
2. *$X \leq^1_{tt,<\omega} P$ if and only if $X \leq^1_{<\omega} P$.*
3. *$X \leq^1_{tt,\omega} P$ if and only if $X \leq^1_\omega P$.*
4. *$P \leq^1_{tt,<\omega} X$ if and only if $P \leq^{<\omega}_{tt,\omega|<\omega} X$.*
5. *$P \leq^1_{tt,\omega} X$ if and only if $P \leq^{<\omega}_{tt,\omega} X$.*

Proof. (1) See Simpson [68].

(2,3) Assume that $X \leq^1_\omega P$ via a learner Ψ. From Ψ, we construct a Popperian learner $\Psi^* : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$, i.e., $\Psi(\sigma)$ is an index of truth-table functional for each $\sigma \in \mathbb{N}^{<\mathbb{N}}$. We may assume that $\Psi(\sigma)$ is defined, by Proposition 1. Let T_P be the corresponding computable tree for P. If $\sigma \notin T_P$, then $\Psi^*(\sigma)$ returns an index of the constant function $f \mapsto 0^{\mathbb{N}}$. If $\sigma \in T_P$, then let $\Psi^*(\sigma)$ be an index of the following computation procedure. Given $f \in \mathbb{N}^{\mathbb{N}}$, at stage $s \in \mathbb{N}$, if $\sigma \not\subset f$, then returns $0^{\mathbb{N}}$. If $f \upharpoonright s \in T_P$ extends σ, and $\Psi(f \upharpoonright t) = \sigma$ for any $|\sigma| \leq t \leq s$, then simulate the computation of $\Phi_{\Psi(\sigma)}(f \upharpoonright s)$. Otherwise, for the least such stage s, returns $\Phi_{\Psi(\sigma)}(f \upharpoonright s-1)^\frown 0^{\mathbb{N}}$. Clearly, $\Phi_{\Psi^*(\sigma)}(f)$ defines an element of $\mathbb{N}^{\mathbb{N}}$, for any $f \in \mathbb{N}^{\mathbb{N}}$. Moreover, Ψ^* agrees with Ψ on P, i.e., $\Phi_{\lim_n \Psi^*(f\upharpoonright n)}(f) = \Phi_{\lim_n \Psi(f\upharpoonright n)}(f)$ for any $f \in P$.

(4,5) Assume that $P \leq^{<\omega}_{tt,\omega|<\omega} X$ via n Popperian learners, $\{\Psi_i\}_{i<n}$. Given $g \in X$, on the first challenge, our leaner Δ guesses that $\Psi_0(g \upharpoonright 0)$ is a correct algorithm. As each Ψ_i is Popperian, and P is Π^0_1, the predicate $\Phi_{\Psi_0(g\upharpoonright 0)}(g) \in P$ is Π^0_1. Therefore, whenever $\Phi_{\Psi_0(g\upharpoonright 0)}(g) \in P$ is incorrect, the learner Δ is able to understand that his guess is refuted. If it happens, the learner goes to the next challenge. On the $(ns+i)$-th challenge, Δ guesses that $\Psi_i(g \upharpoonright s)$ is correct. By continuing this procedure, eventually Δ learns a collect algorithm to solve the problem P. Note that, if an (n,b,c)-computable function exists from X to P, then the learning procedure of Δ is stabilized before the (nc)-th challenge starts, i.e., Δ determines a $(1,nc)$-truth-table computable function. □

Corollary 22. *$\mathcal{P}^1_{tt,1} = \mathcal{P}^1_1$; $\mathcal{P}^1_{tt,<\omega} = \mathcal{P}^1_{tt,\omega|<\omega} = \mathcal{P}^{<\omega}_{tt,\omega|<\omega} = \mathcal{P}^1_{<\omega}$; and $\mathcal{P}^1_{tt,\omega} = \mathcal{P}^{<\omega}_{tt,\omega} = \mathcal{P}^1_\omega$. Hence, $\{\mathcal{P}^\alpha_{\beta|\gamma}, \mathcal{P}^\alpha_{tt,\beta|\gamma} : \alpha,\beta,\gamma \in \{1, <\omega, \omega\}\}$ consists of at most nine lattices: $\mathcal{P}^1_1$, $\mathcal{P}^{<\omega}_{tt,1}$, $\mathcal{P}^1_{<\omega}$, $\mathcal{P}^1_{\omega|<\omega}$, $\mathcal{P}^{<\omega}_1$, $\mathcal{P}^1_\omega$, $\mathcal{P}^{<\omega}_\omega$, $\mathcal{P}^\omega_{tt,1}$, and $\mathcal{P}^\omega_1$.* □

One can interpreted $\leq^1_1$ ($\leq^1_\omega$, resp.) as computable reducibility with no (finitely many, resp.) mind-changes. We see how $\leq^1_\omega$ behaves like a dynamical-approximation procedure.

Proposition 23. *For any Π^0_1 set $P \subseteq \mathbb{N}^\mathbb{N}$ and any set $Q \subseteq \mathbb{N}^\mathbb{N}$, $P \leq^\omega_1 Q$ if and only if*

$$(\exists\Psi)(\forall f \in Q)\ \Phi_{\liminf_n \Psi(f\upharpoonright n)}(f) \in P.$$

Here Ψ ranges over all learners (i.e., computable functions from $\mathbb{N}^{<\mathbb{N}}$ to $\mathbb{N}$).

Proof. The "only if" part is obvious. For the "if" part, we will inductively define $\Psi(\sigma)$ and $l(\sigma, e)$ for each $\sigma \in \mathbb{N}^{<\mathbb{N}}$ and $e \in \mathbb{N}$. Let T_P denote the corresponding tree for P. First, put $\Psi(\langle\rangle) = 0$ and $l(\langle\rangle, e) = 0$ for each e. Now assume that, for any $\tau \in \mathbb{N}^{<\mathbb{N}}$ with $|\tau| < |\sigma|$, we have already defined $\Psi(\tau)$, and $l(\tau, e)$ for each $e \in \mathbb{N}$. Then, we define $\Psi(\sigma)$ and $l(\sigma, e)$ for each e as follows:

$$\Psi(\sigma) = \begin{cases} \mu e < |\sigma|\ [\Phi_e(\sigma) \upharpoonright (l(\sigma^-, e) + 1) \in T_P] & \text{if such } e \text{ exists,} \\ |\sigma| & \text{otherwise} \end{cases}$$

$$l(\sigma, e) = \begin{cases} l(\sigma^-, e) + 1 & \text{if } e = \Psi(\sigma), \\ l(\sigma^-, e) & \text{otherwise.} \end{cases}$$

By our assumption $P \leq^\omega_1 Q$, $\liminf_n \Psi(f \upharpoonright n)$ exists for all $f \in P$. Thus, the desired condition $\Phi_{\liminf_n \Psi(f\upharpoonright n)}(f) \in Q$ holds. □

Remark. Recall that a subset of $2^\mathbb{N}$ is Π^0_1 if and only if it is the set of all infinite paths through a computable subtree of $2^{<\mathbb{N}}$. Thus, in our model of inductive inference, each learner tries to learn a program for an infinite branch of T from a given infinite branch of another tree T^*. Another model of *branch learning* has been studied by Kummer-Ott [47], and Ott-Stephan [59] in which each learner tries to learn a program for an infinite *computable* branch of T from the global information about T. They pointed out that the concept of branch learning is equivalent to learning winning strategies for closed computable Gale-Stewart games, since the class of Π^0_1 subsets of $2^\mathbb{N}$ correspond exactly to the class of winning strategies for such games (see also Cenzer-Remmel [20]). Case-Ott-Sharma-Stephan [17] explains the concept of branch learning by using a temperature controller. In their model, each learner tries to learn a program for an infinite computable branch of T from the global information about T *with an additional information about one infinite branch of T*, i.e., the learner may watch a human *master*. A k-wise variation for branch learning called *weak k-search problem* has been studied by Kaufmann-Kummer [44].

2.5. Learnability versus Piecewise Computability

Now we characterize our classes of nonuniformly computable functions using the concept of piecewise computability.

Definition 24. For a class Λ of subsets of Baire space $\mathbb{N}^\mathbb{N}$, we say that a collection $\{Q_i\}_{i\in I}$ is uniformly Λ if the set $\{(i,f)\in I\times\mathbb{N}^\mathbb{N} : f\in Q_i\}$ belongs to Λ. A partition or a cover $\{Q_i\}_{i\in I}$ of Q is (uniformly) Λ if there is a (uniform) Λ collection $\{Q_i^*\}_{i\in I}$ such that $Q_i = Q\cap Q_i^*$ for any $i\in I$. We say that $\{Q_i\}_{i\in I}$ is *a (uniform) Λ layer of Q* if there is a uniform Λ collection $\{Q_i^*\}_{i\in I}$ such that $Q_i^*\subseteq Q_{i+1}^*$ for each $i\in I$, $\{Q_i^*\}_{i\in I}$ covers Q, and $Q_i = Q\cap Q_i^*$. We also say that $\{Q_i\}_{i\in I}$ is *a (uniform) Λ d-layer of Q* if there is a (uniform) Λ layer $\{Q_i^*\}_{i\in I}$ of Q such that $Q_i = Q_i^*\setminus Q_{i-1}^*$ for any $i\in I$, where $Q_{-1}^* = \emptyset$.

Remark. The terminology "*layer*" comes from the concept of *layerwise computability* in algorithmic randomness theory (see Hoyrup-Rojas [36]).

Definition 25. Let $\mathcal{F}$ be a class of partial functions on $\mathbb{N}^\mathbb{N}$. For $X\in\omega\cup\{<\omega,\omega\}$ and $x\in\{\mathrm{p},\mathrm{c},\mathrm{d}\}$, a partial functions $\Gamma :\subseteq \mathbb{N}^\mathbb{N}\to\mathbb{N}^\mathbb{N}$ is of class $\mathrm{dec}_x^X[\Lambda]\mathcal{F}$ if there is a uniform Λ partition (if $x=\mathrm{p}$), uniform cover (if $x=\mathrm{c}$) or uniform d-layer (if $x=\mathrm{d}$), $\{Q_i\}_{i\in I}$, of $\mathrm{dom}(\Gamma)$ such that $\Gamma\upharpoonright Q_i$ is contained in $\mathcal{F}$ uniformly in $i\in I$, where $I=X$ if $X\in\omega\cup\{\omega\}$ and $I\in\omega$ if $X=<\omega$. If $\mathcal{F}$ is the class of all partial computable functions, we simply write $\mathrm{dec}_x^X[\Lambda]$ instead of $\mathrm{dec}_x^X[\Lambda]\mathcal{F}$. Moreover, if Λ is the class of all subsets of Baire space, then we write $\mathrm{dec}_x^X[-]$ and $\mathrm{dec}_x^X\mathcal{F}$ instead of $\mathrm{dec}_x^X[\Lambda]$ and $\mathrm{dec}_x^X[\Lambda]\mathcal{F}$, respectively. If we does not assume uniformity in the definition, we say that Γ is of $\underline{\mathrm{dec}}_x^X[\Lambda]\mathcal{F}$.

If $\Lambda\in\{\Sigma_n^0,\Pi_n^0,\Delta_n^0\}_{n\in\mathbb{N}}$, for every $X\in\{<\omega,\omega\}$, we have $\mathrm{dec}_\mathrm{p}^X[\Lambda]\subseteq\mathrm{dec}_\mathrm{c}^X[\Lambda]\subseteq\mathrm{dec}_\mathrm{d}^X[\Lambda]\subseteq\mathrm{dec}_\mathrm{p}^X[(\Lambda)_2]$. Here a set is $(\Lambda)_2$ if it is the difference of two Λ sets. Note that $\mathrm{dec}_p^\omega[\Pi_n^0]=\mathrm{dec}_c^\omega[\Sigma_{n+1}^0]$ holds for every $n\in\mathbb{N}$. Our seven concepts of nonuniform computability listed in Table 1 can be characterized as classes of piecewise computable functions.

Theorem 26. *Let k be any finite number.*

1. $[\mathfrak{C}_T]^1_k = \mathrm{dec}^k_\mathrm{d}[\Pi^0_1]$.
2. $[\mathfrak{C}_T]^1_{\omega|k} = \mathrm{dec}^k_x[\Delta^0_2] = \mathrm{dec}^k_\mathrm{c}[\Sigma^0_2]$ *for any* $x\in\{\mathrm{p},\mathrm{c},\mathrm{d}\}$.
3. $[\mathfrak{C}_T]^1_\omega = \mathrm{dec}^\omega_x[\Pi^0_1] = \mathrm{dec}^\omega_x[\Delta^0_2] = \mathrm{dec}^\omega_\mathrm{c}[\Sigma^0_2]$ *for any* $x\in\{\mathrm{p},\mathrm{c},\mathrm{d}\}$.
4. $[\mathfrak{C}_T]^k_1 = \mathrm{dec}^k_\mathrm{x}[-]$ *for any* $x\in\{\mathrm{p},\mathrm{c},\mathrm{d}\}$.
5. $[\mathfrak{C}_T]^k_\omega = \mathrm{dec}^k_y\mathrm{dec}^\omega_x[\Pi^0_1] = \mathrm{dec}^k_y\mathrm{dec}^\omega_x[\Delta^0_2] = \mathrm{dec}^k_y\mathrm{dec}^\omega_\mathrm{c}[\Sigma^0_2]$ *for any* $x,y\in\{\mathrm{p},\mathrm{c},\mathrm{d}\}$.
6. $[\mathfrak{C}_T]^\omega_1 = \mathrm{dec}^\omega_\mathrm{x}[-]$ *for any* $x\in\{\mathrm{p},\mathrm{c},\mathrm{d}\}$.

Proof. (1) Let $\Psi:\mathbb{N}^{<\mathbb{N}}\to\mathbb{N}$ be a learner witnessing $\Gamma\in[\mathfrak{C}_T]^1_k$. Then for each $m<k$, let $\mathtt{mc}_\Psi(\leq m)$ denote the set of all $g\in\mathbb{N}^\mathbb{N}$ such that $\#\mathtt{mcl}_\Psi(g)\leq m$. The sets $\mathtt{mc}_\Psi(<m)$ and $\mathtt{mc}_\Psi(=m)$ are also defined by the same manner. Then, it is easy to check that $\mathtt{mc}_\Psi(\leq m)$ and $\mathtt{mc}_\Psi(<m)$ are Π^0_1. For each $m<k$, consider the following computable procedure $\Phi_{e(m)}$: given $g\in\mathtt{mc}_\Psi(=m)$, look for the least $n\in\mathbb{N}$ such that $[g\upharpoonright n]$ is included in the open set $\mathtt{mc}_\Psi(\geq m)$, and then return $\Phi_{\Psi(g\upharpoonright n)}(g)$. It is not hard to see that Γ is decomposable into k many computable functions $\{\Phi_{e(m)}\}_{m<k}$ with Π^0_1 d-layered domains $\{\mathtt{mc}_\Psi(=m)\}_{m<k}$.

Conversely, assume that $\Gamma\in\mathrm{dec}^k_\mathrm{d}[\Pi^0_1]$ is given. Then, Γ is decomposed into computable functions $\{\Phi_{e(m)}\}_{m<k}$ with d-layered domains $\{Q_m\setminus Q_{m-1}\}_{m<k}$, where $\{Q_m\}_{m<k}$ computable increasing sequence $\{Q_m\}_{m<k}$ of Π^0_1 sets with $Q_{-1}=\emptyset$. For each $\sigma\in\mathbb{N}^{<\mathbb{N}}$, we compute the least $i(\sigma)$ such that $\sigma \in T_{Q_{i(\sigma)}}$, i.e., $\sigma \in T_{Q_{i(\sigma)}} \setminus T_{Q_{i(\sigma)-1}}$. Then, on $\sigma \in \mathbb{N}^{<\mathbb{N}}$, the learner Ψ guesses $\Psi(\sigma) = e(i(\sigma))$. By our assumption, for any $g \in \mathrm{dom}(\Gamma)$, we have $g \in Q_i$ for some $i \in \mathbb{N}$. Then, $\lim_n \Psi(g \upharpoonright n)$ converges to the least $e(i)$ such that $g \in Q_i$. Again, by our assumption, we have $\Phi_{\lim_n \Psi(g\upharpoonright n)}(g) = \Phi_{e(i)}(g) = \Gamma(g)$ for any $g \in \mathrm{dom}(\Gamma) \cap (Q_i \setminus Q_{i-1})$. Therefore, we have $\Gamma \in [\mathfrak{C}_T]^1_k$.

(2) Let $\Psi : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ be a learner witnessing $\Gamma \in [\mathfrak{C}_T]^1_{\omega|k}$. We define $\mathtt{reindex}_\Psi : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ reindexing $\Psi(\sigma)$ in order of occurrence. Put $\mathtt{reindex}_\Psi(\langle\rangle) = 0$. Fix $\sigma \in \mathbb{N}^{<\mathbb{N}}$, and assume that $\mathtt{reindex}_\Psi(\tau)$ has been already defined for each $\tau \subsetneq \sigma$. If $\Psi(\sigma) = \Psi(\tau)$ for some $\tau \subsetneq \sigma$, then we set $\mathtt{reindex}_\Psi(\sigma) = \mathtt{reindex}_\Psi(\tau)$ for such τ. If there is no such τ, then we set $\mathtt{reindex}_\Psi(\sigma) = \max\{\mathtt{reindex}_\Psi(\tau) : \tau \subsetneq \sigma\}+1$. Our assumption $\Gamma \in [\mathfrak{C}_T]^1_{\omega|k}$ implies that for every $g \in \mathrm{dom}(\Gamma)$, $\mathtt{reindex}_\Psi(g) = \lim_n \mathtt{reindex}_\Psi(g \upharpoonright n)$ converges to a value less than k. Hence, $R_m = \{g \in \mathbb{N}^{<\mathbb{N}} : \lim_n \mathtt{reindex}_\Psi(g \upharpoonright n) = m\}$ is Δ^0_2 in $\mathrm{dom}(\Gamma)$ uniformly in $m < k$. For each $m < k$, consider the following computable procedure $\Phi_{e(m)}$: given $g \in R_m$, look for the least $n \in \mathbb{N}$ such that $\mathtt{reindex}_\Psi(g \upharpoonright n) = m$, and then return $\Phi_{\Psi(g\upharpoonright n)}(g)$. It is not hard to see that Γ is decomposable into k many computable functions $\{\Phi_{e(m)}\}_{m<k}$ with Δ^0_2 domains $\{R_m\}_{m<k}$.

Conversely, assume that $\Gamma \in \mathrm{dec}^k_{\mathrm{c}}[\Sigma^0_2]$ is given. Then, Γ is decomposed into computable functions $\{\Phi_{e(m)}\}_{m<k}$ with Σ^0_2 domains $\{Q_m\}_{m<k}$. Then, there is a computable relation $R \subseteq \mathbb{N} \times \mathbb{N}^{<\mathbb{N}}$ such that $Q_m = \{g \in \mathrm{dom}(\Gamma) : (\exists s)(\forall t > s)\ R(m, g \upharpoonright t)\}$ for every $m \in \mathbb{N}$. We set $\Psi(\sigma) = e(\min(\{m : R(m,\sigma)\} \cup \{k-1\}))$. Since $\mathrm{dom}(\Gamma)$ is covered by $\{Q_m\}_{m<k}$, for any $g \in \mathrm{dom}(\Gamma)$, $\lim_n \Psi(g \upharpoonright n)$ converges to some value $e(m)$, where $g \in Q_m$. Moreover, the definition of Ψ ensures that $\#\{\Psi(\sigma) : \sigma \in \mathbb{N}^{<\mathbb{N}}\} \leq k$. Therefore, we have $\Gamma \in [\mathfrak{C}_T]^1_{\omega|k}$.

(3) It is straightforward to show the $[\mathfrak{C}_T]^1_\omega = \mathrm{dec}^\omega_d[\Pi^0_1]$ by the similar argument used in proof of (1). Here, we note that $\mathrm{dec}^\omega_p[\Pi^0_1] = \mathrm{dec}^\omega_c[\Sigma^0_2]$ as mentioned above.

(4) It is obvious from the definition.

(5) Combine (3) and (4).

(6) It is obvious from the definition. □

Remark. It is not hard to see that $\mathrm{dec}^{<\omega}_{\mathrm{p}}[\Pi^0_1]$ is exactly the class of all partial computable functions, because, given a finite Π^0_1 partition $\{Q_i\}_{i<k}$ and $g \in \mathrm{dom}(\Gamma)$, we can effectively find the unique piece containing g.

Proposition 27. *Let P and Q be subsets of $\mathbb{N}^\mathbb{N}$, where P is Π^0_n for $n \geq 2$. Let k be any finite number.*

1. *There is $\Gamma : Q \to P$ with $\Gamma \in [\mathfrak{C}_T]^k_1$ if and only if there is $\Gamma : Q \to P$ with $\Gamma \in \mathrm{dec}^k_{\mathrm{d}}[\Pi^0_n]$.*
2. *There is $\Gamma : Q \to P$ with $\Gamma \in [\mathfrak{C}_T]^{<\omega}_\omega$ if and only if there is $\Gamma : Q \to P$ with $\Gamma \in \mathrm{dec}^{<\omega}_{\mathrm{d}}[\Pi^0_n]\mathrm{dec}^\omega_{\mathrm{p}}[\Pi^0_1]$.*
3. *There is $\Gamma : Q \to P$ with $\Gamma \in [\mathfrak{C}_T]^\omega_1$ if and only if there is $\Gamma : Q \to P$ with $\Gamma \in \mathrm{dec}^\omega_{\mathrm{d}}[\Pi^0_n]$.*

Table 3: Seven Classes of Nonuniformly Computable Functions

$[\mathfrak{C}_T]^1_{<\omega}$	$\mathrm{dec}^{<\omega}_{\mathrm{d}}[\Pi^0_1]$	finite $(\Pi^0_1)_2$-piecewise computable
$[\mathfrak{C}_T]^1_{\omega\mid<\omega}$	$\mathrm{dec}^{<\omega}_{\mathrm{p}}[\Delta^0_2]$	finite Δ^0_2-piecewise computable
$[\mathfrak{C}_T]^1_{\omega}$	$\mathrm{dec}^{\omega}_{\mathrm{p}}[\Pi^0_1]$	Π^0_1-piecewise computable
$[\mathfrak{C}_T]^{<\omega}_1$	$\mathrm{dec}^{<\omega}_{\mathrm{p}}[-]$	finite piecewise computable
$[\mathfrak{C}_T]^{<\omega}_{\omega}$	$\mathrm{dec}^{<\omega}_{\mathrm{p}}\mathrm{dec}^{\omega}_{\mathrm{p}}[\Pi^0_1]$	finite piecewise Π^0_1-piecewise computable
$[\mathfrak{C}_T]^{\omega}_1$	$\mathrm{dec}^{\omega}_{\mathrm{p}}[-]$	countably computable

Hence, $\mathcal{P}^{<\omega}_1 = \mathcal{P}/\mathrm{dec}^{<\omega}_{\mathrm{d}}[\Pi^0_2]$, $\mathcal{P}^{<\omega}_{\omega} = \mathcal{P}/\mathrm{dec}^{<\omega}_{\mathrm{d}}[\Pi^0_2]\mathrm{dec}^{\omega}_{\mathrm{p}}[\Pi^0_1]$, *and* $\mathcal{P}^{\omega}_1 = \mathcal{P}/\mathrm{dec}^{\omega}_{\mathrm{d}}[\Pi^0_2]$. *Here, recall from Definition 12 that* $\mathcal{P}/\mathcal{F}$ *denotes the* $\mathcal{F}$*-degree structure of nonempty* Π^0_1 *subsets of Cantor space.*

Proof. We can show the assertions (1) and (3) by the same argument. To see the assertion (3), we assume that $P \leq^{\omega}_1 Q$. Every partial computable function Φ_e can be assumed to have a Π^0_2 domain D_e. Then, $Q_e = \bigcup_{d\leq e}(D_d \cap \Phi_d^{-1}[P])$ is Π^0_n, and $\{Q_e\}_{e\in\mathbb{N}}$ forms a Π^0_n layer. Moreover, it is not hard to see that Φ_e maps every element of $Q_e \setminus Q_{e-1}$ into P.

For (2), we assume that $P \leq^{<\omega}_{\omega} Q$ is witnessed by two functions $\Gamma \in \mathrm{dec}^2_{\mathrm{p}}\mathrm{dec}^{\omega}_{\mathrm{p}}[\Pi^0_1]$ by Theorem 26. Then there is a collection of partial computable functions $\{\Gamma^i_n\}_{i<2,n\in\mathbb{N}}$ and a partition $\{E_i\}_{i<2}$ of Q and collections $\{Q^i_n\}_{n\in\mathbb{N}}$ of pairwise disjoint Π^0_1 sets that covers E_i and Γ agrees with Γ^i_n on the domain $E_i \cap Q^i_n$ for every $i < 2$ and $n \in \mathbb{N}$. Then, $E^*_1 = \bigcup_{n\in\mathbb{N}}(Q^0_n \cap (\Gamma^0_n)^{-1}[\mathbb{N}^{\mathbb{N}} \setminus P])$ is Σ^0_n and included in E_1. Thus, $\{E^*_0, E^*_1\}$ forms a Π^0_n d-layer, where $E^*_0 = \mathbb{N}^{\mathbb{N}} \setminus E^*_1$. It is not hard to see that Γ agrees with Γ^i_n on the domain $Q \cap E^*_i \cap Q^i_n$ for every $i < 2$ and $n \in \mathbb{N}$. □

3. Strange Set Constructions

3.1. Medvedev's Semantics for Intuitionism

To introduce useful set constructions, let us return back to Medvedev's original idea. To formulate semantics for the intuitionistic propositional calculus (IPC), Kolmogorov tried to interpret each proposition as a problem. Medvedev [51] formalized his idea by interpreting each proposition p as a mass problem $[\![p]\!] \subseteq \mathbb{N}^{\mathbb{N}}$. Under the interpretation:

1. A *proof* π is a dynamical process represented by an infinite sequence of natural numbers, i.e., $\pi \in \mathbb{N}^{\mathbb{N}}$.
2. $[\![p]\!]$ is the set of all proofs of a proposition p, i.e., $[\![p]\!] \subseteq \mathbb{N}^{\mathbb{N}}$.
3. A proposition p is *provable* if p has a computable proof, i.e., $[\![p]\!] \subseteq \mathbb{N}^{\mathbb{N}}$ contains a computable element.

To prove the disjunction $p_0 \vee p_1$, we need to algorithmically decide which part is valid, i.e., we first declare one part to be valid and then construct a witness for this part. Consequently, $p_0 \vee p_1$ is provable under that interpretation if and only if we can algorithmically construct an element of $[\![p_0 \vee p_1]\!] = [\![p_0]\!] \oplus [\![p_1]\!] = \{\langle i\rangle^\frown f : i < 2 \ \& \ f \in$

$[\![p_i]\!]\}$. Generally, let Form denote the all propositional formulas. Medvedev's idea is defining a mass-problem-interpretation of IPC by a function $[\![\cdot]\!] : \mathsf{Form} \to \mathcal{P}(\mathbb{N}^{\mathbb{N}})$ as in Definition 28.

Definition 28. We say that a function $[\![\cdot]\!] : \mathsf{Form} \to \mathcal{P}(\mathbb{N}^{\mathbb{N}})$ is *a Medvedev interpretation* if it satisfies the following six conditions.

1. $[\![\top]\!]$ contains a computable element.
2. $[\![\bot]\!] = \emptyset$.
3. $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \otimes [\![\psi]\!] = \{f \oplus g : f \in [\![\varphi]\!] \;\&\; g \in [\![\psi]\!]\}$.
4. $[\![\varphi \vee \psi]\!] = [\![\varphi]\!] \oplus [\![\psi]\!] = \{\langle 0\rangle^\frown f : f \in [\![\varphi]\!]\} \cup \{\langle 1\rangle^\frown g : g \in [\![\psi]\!]\}$.
5. $[\![\varphi \to \psi]\!] = [\![\varphi]\!] \boldsymbol{\to} [\![\psi]\!] = \{e^\frown g \mid \Phi_e(g \oplus *) : [\![\varphi]\!] \to [\![\psi]\!]\}$.
6. $[\![\neg\varphi]\!] = [\![\varphi \to \bot]\!]$.

Here, $\Phi(g \oplus *)$ denotes the partial function $\lambda f.\Phi(g \oplus f) :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, and recall that Φ_e is the e-th partial computable function on $\mathbb{N}^{\mathbb{N}}$. Arithmetical quantifications can also be interpreted as follows.

7. $[\![\exists n \varphi(n)]\!] = \bigoplus_{n \in \mathbb{N}} [\![\varphi(n)]\!]$.
8. $[\![\forall n \varphi(n)]\!] = \bigotimes_{n \in \mathbb{N}} [\![\varphi(n)]\!]$.

As mentioned in Section 2.3, Medvedev [51] showed that the quotient algebra $\mathcal{D}_1^1$ called the Medvedev lattice is Brouwerian under Medvedev's interpretation (Definition 28). Following him, Muchnik [54] showed that $\mathcal{D}_1^\omega$ called the Muchnik lattice is Brouwerian. Usually, the Medvedev reducibility is written as $\leq_M$ or $\leq_s$ rather than $\leq_1^1$, and the Muchnik reducibility is written by $\leq_w$ rather than $\leq_1^\omega$.

Remark.

1. Both of the Medvedev lattice $\mathcal{D}_1^1$ and the Muchnik lattice $\mathcal{D}_1^\omega$ provide sound and complete semantics for *Jankov's Logic* $\mathsf{KC} = \mathsf{IPC} + \neg p \vee \neg\neg p$, the intuitionistic propositional logic with *the weak law of excluded middle*, which is also called *De Morgan logic*. The Medvedev lattice and the Muchnik lattice are extensively studied from the aspect of Intermediate Logic. See Sorbi-Terwijn [76] and Hinman [35].
2. Forty years after the pioneering work by Muchnik, the Muchnik reducibility become useful in the context of Reverse Mathematics (see Simpson [71]). The reason is that the Muchnik reducibility $\leq_1^\omega$ is strongly associated with the provability relation in RCA, *the recursive comprehension axiom*. Then, the Muchnik degrees of Π^0_1 subsets of $2^{\mathbb{N}}$ might be seen as instances of WKL, *the weak König's lemma*. For example, by using a result of Binns and Simpson [8] for the Muchnik degrees of Π^0_1 subsets of $2^{\mathbb{N}}$, Mummert [55] obtains an embedding theorem about the Lindenbaum algebra between RCA_0 and WKL_0.
3. For more basic results about the Medvedev and Muchnik degrees of Π^0_1 subsets of $2^{\mathbb{N}}$, see Simpson [68–70, 72]. There are lots of research on the algebraic structure of the Medvedev degrees of Π^0_1 subsets of $2^{\mathbb{N}}$, such as density [19], embeddability of distributive lattices [8], join-reducibility [7], meet-irreducibility [2], noncuppability [18], decidability [22], and undecidability [66]. The structure of Weihrauch degrees, an extension of the Medvedev degrees, has also been widely studied as a computable-analysistic approach to (Constructive) Reverse Mathematics (see [11–13]).

3.2. Disjunction Operations Based on Learning Theory

Hayashi [30, 31] introduced *Limit Computable Mathematics* (LCM), an extended constructive mathematics based on *Learning Theory*. Like the BHK-interpretation for intuitionistic logic, there is a *limit-BHK interpratation* for Limit Computable Mathematics. We introduce three mass-problem-interpretations $[\![\cdot]\!]^i_{\mathsf{LCM}} : \mathsf{Form} \to \mathcal{P}(\mathbb{N}^\mathbb{N})$ of LCM based on the limit-BHK interpretation. To formulate a mass-problem-style interpretation of LCM, imagine the following *dynamic* proof models.

The one-tape model is defined as follows: When a verifier Ψ tries to prove that "P_0 or P_1", a tape Λ is given. At each stage, Ψ declares 0 or 1, and writes one letter on the tape Λ.

- **Intuitionism**: Ψ does not change his declaration, say $i \in \{0, 1\}$, and the infinite word written on the tape Λ witnesses the validity of P_i.
- **LCM**: the sequence of declarations of Ψ converges, say $i \in \{0, 1\}$, and the infinite word written on the tape Λ witnesses the validity of P_i.
- **Classical**: any declaration of Ψ is nonsense, and the infinite word written on the tape Λ witnesses the validity of P_0 or P_1.

The two-tape model is follows: When a verifier Ψ tries to prove "P_0 or P_1", two tapes Λ_0 and Λ_1 are given. At each stage, Ψ declares 0 or 1, say i, and he writes one letter on the tape Λ_i.

- **Intuitionism**: For either $i < 2$, the word written on Λ_{1-i} is empty, and the infinite word written on Λ_i witnesses the validity of P_i.
- **LCM**: For either $i < 2$, the word written on Λ_{1-i} is finite, and the infinite word written on Λ_i witnesses the validity of P_i.
- **Classical**: For either $i < 2$, the infinite word written on Λ_i witnesses the validity of P_i.

The backtrack-tape model is follows: When a verifier Ψ tries to prove that "P_0 or P_1", a cell $\square$, and two infinite tapes Λ, Δ are given. The cell $\square$ is called *the declaration*, Λ is called *the working tape*, and Δ is called *the record tape*. At each stage, the verifier Ψ works as follows.

1. If no letter is written on the declaration $\square$, then Ψ declares 0 or 1 and this is written on the declaration $\square$ and the record tape Δ.
2. When some letter is written on the declaration $\square$, the verifier Ψ chooses one letter k from $\mathbb{N} \cup \{\sharp\}$, and his choice k is written on the record tape Δ.
   (a) In the case $k \neq \sharp$, it expresses that Ψ writes the letter k on the working tape Λ.
   (b) In the case $k = \sharp$, it expresses that Ψ erases all letters from the declaration $\square$ and the working tape Λ.

- **Intuitionism**: Ψ does not choose $\sharp$, hence he does not change his declaration, say i, and the infinite word written on the tape Λ witnesses the validity of P_i.

- **LCM**: Ψ chooses $\sharp$ at most finitely often, hence the sequence of declarations of Ψ converges, say i, and the infinite word written on the tape Λ witnesses the validity of P_i.
- **Classical**: No classical counterpart.

To give formal definitions of these dynamic proof models, we introduce some auxiliary definitions.

Definition 29 (Notations for One/Two-Tape Models)**.** Let $I \subseteq \mathbb{N}$ be a set of indices of working tapes. A pair $(x_0, x_1) \in I \times \mathbb{N}$ indicates the instruction to write the letter $x_1 \in \mathbb{N}$ on the x_0-th tape. Then every string $\sigma = (i(t), n(t))_{t<s} \in (I \times \mathbb{N})^{<\mathbb{N}}$ can be think of as the *record* of the process that obeys the sequence of instructions $(i(0), n(0)), (i(1), n(1)), \dots, (i(s-1), n(s-1))$. Fix $\sigma \in (I \times \mathbb{N})^{<\mathbb{N}}$, and $i \in I$. Then *the i-th projection of σ* is inductively defined as follows.

$$\mathtt{pr}_i(\langle\rangle) = \langle\rangle, \qquad \mathtt{pr}_i(\sigma) = \begin{cases} \mathtt{pr}_i(\sigma^-)^\frown n, & \text{if } \sigma = \sigma^{-\frown}\langle(i, n)\rangle, \\ \mathtt{pr}_i(\sigma^-), & \text{otherwise.} \end{cases}$$

The string $\mathtt{pr}_i(\sigma)$ represents the word written on the i-th tape reconstructed from the record σ. Moreover, *the number of times of mind-changes of (the process reconstructed from a record) $\sigma \in (I \times \mathbb{N})^{<\mathbb{N}}$* is given by

$$\mathtt{mc}(\sigma) = \#\{n < |\sigma| - 1 : (\sigma(n))_0 \neq (\sigma(n+1))_0\}.$$

Here, for $x = (x_0, x_1) \in I \times \mathbb{N}$, the first (second, resp.) coordinate x_0 (x_1, resp.) is denoted by $(x)_0$ ($(x)_1$, resp.). Furthermore, for $f \in (I \times \mathbb{N})^{\mathbb{N}}$, we define $\mathtt{pr}_i(f) = \bigcup_{n\in\mathbb{N}} \mathtt{pr}_i(f \upharpoonright n)$ for each $i \in I$, and $\mathtt{mc}(f) = \lim_n \mathtt{mc}(f \upharpoonright n)$, where if the limit does not exist, we write $\mathtt{mc}(f) = \infty$.

Definition 30 (Notations for Backtrack-Tape Models)**.** For any set X and string $\sigma \in X^{<\mathbb{N}}$, *the n-th shift $\sigma^{\leftharpoonup n}$* is defined as $\sigma^{\leftharpoonup n}(m) = \sigma(n + m)$ for each $m < |\sigma| - n$. The *tail of σ* is defined by

$$\mathtt{tail}(\sigma) = \sigma^{\leftharpoonup n}, \text{ for } n = \min\{m \in \mathbb{N} : \sigma(k) \neq \sharp \text{ for all } k \geq m\}.$$

Intuitively, the symbol $\sharp$ indicates the instruction to erase all letters written on the working tape. Hence, the string $\mathtt{tail}(\sigma)$ extracts the remaining data from the record σ after the latest erasing. Furthermore, for $f \in X^{\mathbb{N}}$, we define $f^{\leftharpoonup n} = \bigcup_{m\geq n}(f \upharpoonright m)^{\leftharpoonup n}$, and $\mathtt{tail}(f) = \lim_m \mathtt{tail}(f \upharpoonright m)$ if the limit exists. Here, note that $\lim_m \mathtt{tail}(f \upharpoonright m)$ exists if and only if f contains only finitely many $\sharp$'s.

Example 31. We consider two functions $\sigma \in (2 \times \mathbb{N})^{<\mathbb{N}}$ and $\tau \in (\mathbb{N} \cup \{\sharp\})^{<\mathbb{N}}$.

1. If $\sigma = \langle(1, 3), (1, 1), (0, 4), (0, 15), (1, 9), (0, 26), (0, 5)\rangle$, then the projections of σ are $\mathtt{pr}_0(\sigma) = \langle 4, 15, 26, 5\rangle$, and $\mathtt{pr}_1(\sigma) = \langle 3, 1, 9\rangle$. Moreover, $\mathtt{mc}(\sigma) = 3$.
2. If $\tau = \langle 0, 2, 7, 18, 28, \sharp, 1, 8, 2, 8, 45, 9, \sharp, 0, 4, 52, 35, 3, 6\rangle$, then the tail of τ is $\mathtt{tail}(\tau) = \tau^{\leftharpoonup 13} = \langle 0, 4, 52, 35, 3, 6\rangle$.

Definition 32 (One-Tape Disjunctions)**.** Let P_0 and P_1 be subsets of Baire space $\mathbb{N}^\mathbb{N}$.

1. $[\![P_0 \vee P_1]\!]^1_{\mathsf{Int}} = \bigcup_{i<2}(\{i^\mathbb{N}\} \otimes P_i)$.
2. $[\![P_0 \vee P_1]\!]^1_{\mathsf{LCM}} = \bigcup_{i<2}(\{f \in 2^\mathbb{N} : (\forall^\infty n)\ f(n) = i\} \otimes P_i)$.
3. $[\![P_0 \vee P_1]\!]^1_{\mathsf{CL}} = \bigcup_{i<2}(2^\mathbb{N} \otimes P_i)$.

Here, $i^\mathbb{N}$ denotes the infinite sequence consisting of i's, i.e., $i^\mathbb{N} = \langle i, i, i, \dots, i, i, i, \dots\rangle$.

Definition 33 (Two-Tape Disjunctions)**.** Let P_0 and P_1 be subsets of Baire space $\mathbb{N}^\mathbb{N}$.

1. $[\![P_0 \vee P_1]\!]^2_{\mathsf{Int}} = \{f \in (2 \times \mathbb{N})^\mathbb{N} : ((\exists i < 2)\ \mathtt{pr}_i(f) \in P_i)\ \&\ \mathtt{mc}(f) = 0\}$.
2. $[\![P_0 \vee P_1]\!]^2_{\mathsf{LCM}} = \{f \in (2 \times \mathbb{N})^\mathbb{N} : ((\exists i < 2)\ \mathtt{pr}_i(f) \in P_i)\ \&\ \mathtt{mc}(f) < \infty\}$.
3. $[\![P_0 \vee P_1]\!]^2_{\mathsf{CL}} = \{f \in (2 \times \mathbb{N})^\mathbb{N} : (\exists i < 2)\ \mathtt{pr}_i(f) \in P_i\}$.

Definition 34 (Backtrack Disjunctions)**.** Let P_0 and P_1 be subsets of Baire space $\mathbb{N}^\mathbb{N}$.

1. $[\![P_0 \vee P_1]\!]^3_{\mathsf{Int}} = \{f \in (\mathbb{N} \cup \{\sharp\})^\mathbb{N} : \mathtt{tail}(f)^{\leftharpoonup 1} \in P_{\mathtt{tail}(f;0)}\ \&\ (\forall n)\ f(n) \neq \sharp\}$.
2. $[\![P_0 \vee P_1]\!]^3_{\mathsf{LCM}} = \{f \in (\mathbb{N} \cup \{\sharp\})^\mathbb{N} : \mathtt{tail}(f)^{\leftharpoonup 1} \in P_{\mathtt{tail}(f;0)}\ \&\ (\forall^\infty n)\ f(n) \neq \sharp\}$.

In Definition 34, for example, the string $\tau = \langle\sharp\rangle^\frown\langle i\rangle^\frown\sigma$ represents the record that a verifier Ψ erased all letters from tapes (this action is indicated by $\sharp$), declared that P_i is valid, and wrote the word σ on the working tape. That is to say, $\mathtt{tail}(\tau; 0) = i$ is the current declaration of the verifier and $\mathtt{tail}(\tau)^{\leftharpoonup 1} = \sigma$ is the current word written on the working tape.

Remark. Note that we always have to choose a new symbol $\sharp$ which has not been already used, since we may need to distinguish the new $\sharp$ from other symbols and other $\sharp$'s used in other disjunctions. Formally, we can assume that all objects in our paper are elements of $\mathbb{N}^\mathbb{N}$, subsets of $\mathbb{N}^\mathbb{N}$, or (partial) functions on $\mathbb{N}^\mathbb{N}$ by setting $0^\bullet = \sharp$, $(n+1)^\bullet = n$, and $f^\bullet(n) = f(n^\bullet)$ for every $n \in \mathbb{N}$. For instance, $[\![P_0 \vee P_1]\!]^3_{\mathsf{LCM}}$ is always interpreted as the set $[\![P_0 \vee P_1]\!]^{3\bullet}_{\mathsf{LCM}}$ of all $f \in \mathbb{N}^\mathbb{N}$ such that $f^\bullet \in [\![P_0 \vee P_1]\!]^3_{\mathsf{LCM}}$, and then $[\![Q \vee [\![P_0 \vee P_1]\!]^3_{\mathsf{LCM}}]\!]^3_{\mathsf{LCM}}$ is interpreted as $[\![Q \vee [\![P_0 \vee P_1]\!]^{3\bullet}_{\mathsf{LCM}}]\!]^{3\bullet}_{\mathsf{LCM}}$ of all $f \in \mathbb{N}^\mathbb{N}$ such that $f^\bullet \in [\![P_0 \vee P_1]\!]^3_{\mathsf{LCM}}$. Then, note that outer $\sharp$'s are automatically distinguished from inner $\sharp$'s contained in $f \in [\![Q \vee [\![P_0 \vee P_1]\!]^{3\bullet}_{\mathsf{LCM}}]\!]^{3\bullet}_{\mathsf{LCM}}$. Hereafter, $[\![P_0 \vee P_1]\!]^3_{\mathsf{LCM}}$ is identified with $[\![P_0 \vee P_1]\!]^{3\bullet}_{\mathsf{LCM}}$.

Notation. Hereafter, we frequently use the notation $\mathtt{write}(i, \sigma)$ for any $i \in \mathbb{N}$ and $\sigma \in \mathbb{N}^{<\mathbb{N}}$.

$$\mathtt{write}(i, \sigma) = i^{|\sigma|} \oplus \sigma = \langle (i, \sigma(0)), (i, \sigma(1)), (i, \sigma(2)), \dots, (i, \sigma(|\sigma| - 1))\rangle.$$

This string indicates the *instruction to write the string σ on the i-th tape* in the one/two-tape model. We also use the notation $\mathtt{write}(i, f) = \bigcup_{n\in\mathbb{N}} \mathtt{write}(i, f \upharpoonright n) = i^\mathbb{N} \oplus f$ for any $f \in \mathbb{N}^\mathbb{N}$.

Proposition 35. *Let P and Q be subsets of Baire space $\mathbb{N}^\mathbb{N}$.*

1. $[\![P \vee P]\!]^1_X \equiv^1_1 P$ *for each* $X \in \{\mathsf{Int}, \mathsf{LCM}, \mathsf{CL}\}$.
2. $[\![P \vee Q]\!]^i_{\mathsf{CL}} \leq^1_1 [\![P \vee Q]\!]^i_{\mathsf{LCM}} \leq^1_1 [\![P \vee Q]\!]^i_{\mathsf{Int}}$ *for each* $i \in \{1, 2, 3\}$ *(except for* CL *if* $i = 3$*).*

3. $[\![P\vee Q]\!]^i_X \leq^1_1 [\![P\vee Q]\!]^j_X$ *for each* $j\leq i$ *and* $X\in\{\mathsf{Int},\mathsf{LCM},\mathsf{CL}\}$.
4. $P\oplus Q\equiv^1_1 [\![P\vee Q]\!]^i_{\mathsf{Int}}$ *for each* $i\in\{1,2,3\}$.
5. $P\cup Q\equiv^1_1 [\![P\vee Q]\!]^1_{\mathsf{CL}}$.

Proof. (1) The reduction $f\oplus g\mapsto g$ witnesses $P\leq^1_1 [\![P\vee P]\!]^1_X$, and the reduction $f\mapsto \mathtt{write}(0,f)$ witnesses $[\![P\vee P]\!]^1_X\leq^1_1 P$, for each $X\in\{\mathsf{Int},\mathsf{LCM},\mathsf{CL}\}$. Intuitively, $\mathtt{write}(0,f)$ indicates the instruction, in the one-tape model, to declare "P_0 is correct" at each stage and to write the infinite word f on the tape Λ.

(2) Clearly, $[\![P\vee Q]\!]^i_{\mathsf{CL}}\supseteq[\![P\vee Q]\!]^i_{\mathsf{LCM}}\supseteq[\![P\vee Q]\!]^i_{\mathsf{Int}}$ for each $i\in\{1,2,3\}$ (except for CL if $i=3$).

(3) Fix $X\in\{\mathsf{Int},\mathsf{LCM},\mathsf{CL}\}$. We inductively construct a computable function Ξ witnessing $[\![P\vee Q]\!]^2_X\leq^1_1[\![P\vee Q]\!]^1_X$. First set $\Xi(\langle\rangle)=\langle\rangle$, and assume that $\Xi(\sigma\oplus\tau)$ has been already defined for every strings σ and τ of length s. Then we now define $\Xi(\sigma\oplus\tau)$ for each strings σ and τ of length $s+1$. We inductively assume that $\mathtt{pr}_i(\Xi(\sigma^-\oplus\tau^-))\subseteq\tau^-$ for each $i<2$ (recall that σ^- denotes the immediate predecessor of σ). For $p=|\mathtt{pr}_{\sigma(s)}(\Xi(\sigma^-\oplus\tau^-))|$, we put $\Xi(\sigma\oplus\tau)=\Xi(\sigma^-\oplus\tau^-)^\frown\mathtt{write}(\sigma(s),\tau^{\leftarrow p})$. Intuitively, this indicates the instruction to add some tail $\tau(p),\tau(p+1),\dots,\tau(s)$ to the word $\tau(0),\tau(1),\dots,\tau(p-1)$ written on the $\sigma(s)$-tape. Then, we can inductively ensure the following condition.

$$\mathtt{pr}_{\sigma(s)}(\Xi(\sigma\oplus\tau))=\mathtt{pr}_{\sigma(s)}(\Xi(\sigma^-\oplus\tau^-))^\frown(\tau^{\leftarrow p})=(\tau^-\upharpoonright p)^\frown\tau^{\leftarrow p}=\tau.$$

Finally, we set $\Xi(f\oplus g)=\bigcup_{n\in\mathbb{N}}\Xi((f\upharpoonright n)\oplus(g\upharpoonright n))$, for any $f,g\in\mathbb{N}^\mathbb{N}$. Therefore, for any $f\oplus g\in[\![P\vee Q]\!]^1_X$ and each $i<2$, if $f(n)=i$ for infinitely many $n\in\mathbb{N}$, then $\mathtt{pr}_i(\Xi(f\oplus g))$ is total, and $\mathtt{pr}_i(\Xi(f\oplus g))=g$. By definition, $\mathtt{pr}_i(\Xi(f\oplus g))=g\in P_i$ for some $i<2$. Hence, $\Xi(f\oplus g)\in[\![P\vee Q]\!]^2_X$.

Fix $X\in\{\mathsf{Int},\mathsf{LCM}\}$. We inductively construct a computable function Ξ witnessing $[\![P\vee Q]\!]^3_X\leq^1_1[\![P\vee Q]\!]^2_X$. First set $\Xi(\langle(i,n)\rangle)=\langle i,n\rangle$ for each $(i,n)\in 2\times\mathbb{N}$. Fix $\sigma=\sigma^{--\frown}\langle(i,m),(j,n)\rangle\in(2\times\mathbb{N})^{<\mathbb{N}}$, and assume that $\Xi(\sigma^-)$ has been already defined. Then, let us define $\Xi(\sigma)$ as follows:

$$\Xi(\sigma^{--\frown}\langle(i,m),(j,n)\rangle)=\begin{cases}\Xi(\sigma^-)^\frown\langle n\rangle & \text{if } j=i;\\ \Xi(\sigma^-)^\frown\langle\sharp,j\rangle^\frown\mathtt{pr}_j(\sigma) & \text{otherwise.}\end{cases}$$

Finally set $\Xi(f)=\bigcup_n\Xi(f\upharpoonright n)$, for any $f\in(2\times\mathbb{N})^\mathbb{N}$. It is easy to see that $\mathtt{tail}(f)$ is defined for any $f\in[\![P\vee Q]\!]^2_X$, since $\#\{k\in\mathbb{N}:\Xi(f;k)=\sharp\}=\mathtt{mc}(f)$. Therefore, $\mathtt{tail}^{\leftarrow 1}(\Xi(f))\in P_{\mathtt{tail}(\Xi(f);0)}$. If $X=\mathsf{Int}$, then no $\sharp$ occurs in $\Xi(f)$.

(4) By definition, $[\![P\vee Q]\!]^3_{\mathsf{Int}}=P\oplus Q$. (5) The reduction $f\oplus g\mapsto g$ witnesses $P\cup Q\leq^1_1[\![P\vee Q]\!]^1_{\mathsf{CL}}$, and the reduction $f\mapsto\mathtt{write}(0,f)=0^\mathbb{N}\oplus f$ witnesses $[\![P\vee Q]\!]^1_{\mathsf{CL}}\leq^1_1 P\cup Q$. ◻

Definition 36. For each proof model, there are variations of LCM disjunctions, for any *bound of mind changes*. Let P_0,P_1 be any subsets of Baire space $\mathbb{N}^\mathbb{N}$, and n be any natural number.

1. *The one-tape* LCM *disjunction of* P_0 *and* P_1 *with mind-changes-bound* n is defined as follows.

$$[\![P_0\vee P_1]\!]^1_{\mathsf{LCM}[n]}=[\![P_0\vee P_1]\!]^1_{\mathsf{LCM}}\cap\{f\in 2^\mathbb{N}:\#\{n\in\mathbb{N}:f(n+1)\neq f(n)\}<n\}\otimes 2^\mathbb{N}.$$

2. *The two-tape* LCM *disjunction of* P_0 *and* P_1 *with mind-changes-bound* n is defined as follows.

$$[\![P_0 \vee P_1]\!]^2_{\mathsf{LCM}[n]} = [\![P_0 \vee P_1]\!]^2_{\mathsf{LCM}} \cap \{f \in (2 \times \mathbb{N})^{\mathbb{N}} : \mathtt{mc}(f) < n\}.$$

3. *The backtrack-tape* LCM *disjunction of* P_0 *and* P_1 *with mind-changes-bound* n is defined as follows.

$$[\![P_0 \vee P_1]\!]^3_{\mathsf{LCM}[n]} = [\![P_0 \vee P_1]\!]^3_{\mathsf{LCM}} \cap \{f \in (\mathbb{N} \cup \{\sharp\})^{\mathbb{N}} : \#\{k \in \mathbb{N} : f(k) = \sharp\} < n\}.$$

Proposition 37. *Let* P, Q *be subsets of Baire space* $\mathbb{N}^{\mathbb{N}}$.

1. $P \oplus Q \equiv^1_1 [\![P \vee Q]\!]^i_{\mathsf{LCM}[1]}$ *for each* $i \in \{1, 2, 3\}$.
2. $[\![P \vee P]\!]^2_{\mathsf{LCM}[2]} \equiv^1_1 [\![P \vee P]\!]^3_{\mathsf{LCM}[2]}$. *Indeed,* $[\![\bigvee_{i<n} P_i]\!]^2_{\mathsf{LCM}[n]} \equiv^1_1 [\![P \vee P]\!]^3_{\mathsf{LCM}[n]}$, *where* $P_i = P$ *for each* $i < n$. *Here, for each collection* $\{P_i\}_{i<k}$ *of subsets of Baire space,* $[\![\bigvee_{i<k} P_i]\!]^2_{\mathsf{LCM}[n]}$ *is defined as follows.*

$$\{f \in (k \times \mathbb{N})^{\mathbb{N}} : ((\exists i < k)\ \mathtt{pr}_i(f) \in P_i)\ \&\ \mathtt{mc}(f) < n\}.$$

Proof. (1) Clearly $[\![P \vee Q]\!]^i_{\mathsf{LCM}[1]} = [\![P \vee Q]\!]^i_{\mathsf{Int}}$ for each $i \in \{1, 2, 3\}$. By Proposition 35 (4), we have $P \oplus Q \equiv^1_1 [\![P \vee Q]\!]^i_{\mathsf{Int}}$.

(2) The reduction $\Xi : h \mapsto h^*$ in the proof of Proposition 35 (3) also witnesses $[\![P \vee P]\!]^3_{\mathsf{LCM}[n]} \leq^1_1 [\![\bigvee_{i<n} P_i]\!]^2_{\mathsf{LCM}[n]}$. We inductively define a computable function Ξ^* witnessing $[\![\bigvee_{i<n} P_i]\!]^2_{\mathsf{LCM}[n]} \leq^1_1 [\![P \vee P]\!]^3_{\mathsf{LCM}[n]}$. Put $\Xi^*(\langle\rangle) = \langle\rangle$, and fix $\sigma = \sigma^{-\frown}\langle k\rangle \in (\mathbb{N} \cup \{\sharp\})^{<\mathbb{N}}$. Assume that $\Xi^*(\sigma^-)$ has been already defined. Then, $\Xi^*(\sigma)$ is defined as follows.

$$\mathtt{count}(\sigma) = \#\{m < |\sigma| : \sigma(m) = \sharp\},$$
$$\Xi^*(\sigma^{-\frown}\langle k\rangle) = \begin{cases} \Xi^*(\sigma^-)^\frown\langle(\mathtt{count}(\sigma), k)\rangle & \text{if } k \neq \sharp, \\ \Xi^*(\sigma^-) & \text{otherwise.} \end{cases}$$

For any $g \in [\![P \vee P]\!]^3_{\mathsf{LCM}[n]}$, we have $\mathtt{count}(g \upharpoonright s) < n$ for any $s \in \mathbb{N}$, and hence $\mathtt{mc}(\Xi^*(g)) < n$, since g contains at most n many $\sharp$'s. Moreover, $\mathtt{pr}_{\lim_s \mathtt{count}(g \upharpoonright s)}(\Xi^*(g)) = \mathtt{tail}(g)^{\leftarrow 1} \in P$. □

Proposition 38. *Let* P_0, P_1, Q_0, *and* Q_1 *be subsets of Baire space* $\mathbb{N}^{\mathbb{N}}$, *and fix* $i \in \{2, 3\}$ *and* $X \in \{\mathsf{Int}, \mathsf{LCM}, \mathsf{CL}\} \cup \{\mathsf{LCM}[n] : n \in \mathbb{N}\}$. *If* $P_0 \leq^1_1 Q_0$ *and* $P_1 \leq^1_1 Q_1$, *then* $[\![P_0 \vee P_1]\!]^i_X \leq^1_1 [\![Q_0 \vee Q_1]\!]^i_X$. *Hence, the operator* $\mathbf{D}^i_X : \mathcal{D}^1_1 \times \mathcal{D}^1_1 \to \mathcal{D}^1_1$ *introduced by* $\mathbf{D}^i_X(\deg^1_1(P), \deg^1_1(Q)) = \deg^1_1([\![P \vee Q]\!]^i_X)$ *is well-defined. Here,* $\deg^1_1(P)$ *denotes the equivalent class* $\{R \subseteq \mathbb{N}^{\mathbb{N}} : R \equiv^1_1 P\}$.

Proof. We first consider the two-tape model. Assume that $P_0 \leq^1_1 Q_0$ and $P_1 \leq^1_1 Q_1$ via computable functions Γ_0 and Γ_1, respectively. We construct a computable function Δ witnessing $[\![P_0 \vee P_1]\!]^2_X \leq^1_1 [\![Q_0 \vee Q_1]\!]^2_X$. Set $\Delta(\langle\rangle) = \langle\rangle$. Fix $\sigma \in (2 \times \mathbb{N})^{<\mathbb{N}}$ and assume that $\Delta(\sigma^-)$ has been already defined. For each $i < 2$, we define $\mathtt{new}\Gamma_i(\mathtt{pr}_i(\sigma)) \in \mathbb{N}^{<\mathbb{N}}$ by

$P \cup Q$						$P \oplus Q$
$\equiv$						$\equiv$
$[\![P \vee Q]\!]^1_{\mathsf{CL}}$	$\leq (\equiv)$	$[\![P \vee Q]\!]^1_{\mathsf{LCM}}$	$\leq (\equiv)$	$[\![P \vee Q]\!]^1_{\mathsf{LCM[2]}}$	$\leq (\equiv)$	$[\![P \vee Q]\!]^1_{\mathsf{Int}}$
$\geq$		$\geq$		$\geq$		$\equiv$
$[\![P \vee Q]\!]^2_{\mathsf{CL}}$	$\leq$	$[\![P \vee Q]\!]^2_{\mathsf{LCM}}$	$\leq$	$[\![P \vee Q]\!]^2_{\mathsf{LCM[2]}}$	$\leq$	$[\![P \vee Q]\!]^2_{\mathsf{Int}}$
		$\geq$		$\geq (\equiv)$		$\equiv$
		$[\![P \vee Q]\!]^3_{\mathsf{LCM}}$	$\leq$	$[\![P \vee Q]\!]^3_{\mathsf{LCM[2]}}$	$\leq$	$[\![P \vee Q]\!]^3_{\mathsf{Int}}$

Table 4: Degrees of difficulty of disjunctions, where $\leq$ and $\equiv$ denote the Medvedev reducibility and equivalence, and $(\equiv)$ denotes the Medvedev equivalence when $P = Q$

the unique string such that $\Gamma_i(\mathtt{pr}_i(\sigma)) = \Gamma_i(\mathtt{pr}_i(\sigma^-))^\frown\mathtt{new}\Gamma_i(\mathtt{pr}_i(\sigma))$. Then we define $\Delta(\sigma)$ as follows.

$$\Delta(\sigma) = \Delta(\sigma^-)^\frown\mathtt{write}(0, \mathtt{new}\Gamma_0(\mathtt{pr}_0(\sigma)))^\frown\mathtt{write}(1, \mathtt{new}\Gamma_1(\mathtt{pr}_1(\sigma))).$$

Note that $\mathtt{new}\Gamma_i(\mathtt{pr}_i(\sigma)) = \langle\rangle$ for some $i < 2$, since $\mathtt{pr}_i(\sigma) = \mathtt{pr}_i(\sigma^-)$ for either $i < 2$. Therefore, $\mathtt{mc}(\Delta(g)) = \mathtt{mc}(g)$ for any $g \in \mathbb{N}^\mathbb{N}$. Furthermore, for any $g \in \mathbb{N}^\mathbb{N}$, we have $\mathtt{pr}_i(\Delta(g)) = \Gamma_i(\mathtt{pr}_i(g))$ for each $i < 2$. Thus, $\Delta(g) \in [\![P_0 \vee P_1]\!]^2_X$ for any $g \in [\![Q_0 \vee Q_1]\!]^2_X$.

Next we consider the backtrack-tape model. Assume that $P_0 \leq^1_1 Q_0$ and $P_1 \leq^1_1 Q_1$ via computable functions Γ_0 and Γ_1, respectively. We construct a computable function Θ witnessing $[\![P_0 \vee P_1]\!]^3_X \leq^1_1 [\![Q_0 \vee Q_1]\!]^3_X$. Set $\Theta(\langle\rangle) = \langle\rangle$. Fix $\sigma \in (\mathbb{N} \cup \{\sharp\})^{<\mathbb{N}}$ and assume that $\Theta(\tau)$ has been already defined for each $\tau \subsetneq \sigma$. If $\sigma = \sigma^{--\frown}\langle m, n\rangle$ for some $m, n \in \mathbb{N}$, then we have $\Gamma_{\mathtt{tail}(\sigma;0)}(\mathtt{tail}(\sigma)^{\leftharpoonup 1}) = \Gamma_{\mathtt{tail}(\sigma;0)}(\mathtt{tail}(\sigma)^{\leftharpoonup 1})^\frown\eta$ for some $\eta \in \mathbb{N}^{<\mathbb{N}}$, and we define $\Theta(\sigma) = \Theta(\sigma^-)^\frown\eta$. If $\sigma = \sigma^{--\frown}\langle\sharp, i\rangle$ for some $i < 2$, i.e., $\mathtt{tail}(\sigma; 0) = i$, then define $\Theta(\sigma) = \Theta(\sigma^-)^\frown\langle\sharp, i\rangle$. Otherwise, we set $\Theta(\sigma) = \Theta(\sigma^-)$. Note that $\#\{n \in \mathbb{N} : \Theta(g; n) = \sharp\} = \#\{n \in \mathbb{N} : g(n) = \sharp\}$ for any $g \in \mathbb{N}^\mathbb{N}$. Furthermore, $\mathtt{tail}(\Theta(g); 0) = \mathtt{tail}(g; 0)$, and $\mathtt{tail}(\Theta(g))^{\leftharpoonup 1} = \Gamma_{\mathtt{tail}(g;0)}(\mathtt{tail}(g)^{\leftharpoonup 1})$ for any $g \in [\![Q_0 \vee Q_1]\!]^3_X$. Hence, $\Theta(g) \in [\![P_0 \vee P_1]\!]^3_X$ for any $g \in [\![Q_0 \vee Q_1]\!]^3_X$. □

Remark. Though the original limit-BHK interpretation of the disjunctive notion seems to be a one-tape notion, we will observe that the two-tape notions and the backtrack notions exhibit amazing and fascinating behaviors as operations on the subsets of Baire space. While the one-tape models are almost static, the two-tape models can be understood as learning proof models with *bounded-errors*, and the backtrack tape models can be understood as learning proof models with no predetermined bound for errors. In Part II, we adopt the two-tape notions except for the classical one-tape disjunction $\cup$, since the two-tape notions (the bounded-errors learning models) are useful to clarify differences among the classes $[\mathfrak{C}_T]^1_1, [\mathfrak{C}_T]^1_{<\omega}, [\mathfrak{C}_T]^1_{\omega|<\omega}, [\mathfrak{C}_T]^{<\omega}_1$ of bounded-errors functions. In Part II, we also adopt dynamic generalizations of the backtrack tape models since such models turn out to be a strong tool to establish many theorems.

4. Galois Connection

4.1. Decomposing Disjunction by Piecewise Computable Functions

The main theorem in this section (Theorem 40) states that our degree structures $\mathcal{D}^{\alpha}_{\beta|\gamma}$ (Definition 12) are completely characterized by the disjunction operations (Definitions 32, 33, and 34).

Proposition 39 (Untangling)**.** *Let P, Q be subsets of Baire space $\mathbb{N}^{\mathbb{N}}$.*

1. *There is a $(1, n|2)$-truth-table function $\Gamma : [\![P \vee Q]\!]^1_{\mathsf{LCM}[n]} \to P \oplus Q$.*
2. *There is a $(1, n|2)$-computable function $\Gamma : [\![P \vee Q]\!]^2_{\mathsf{LCM}[n]} \to P \oplus Q$.*
3. *There is a $(1, n)$-computable function $\Gamma : [\![P \vee Q]\!]^3_{\mathsf{LCM}[n]} \to P \oplus Q$.*
4. *There is a $(1, \omega|2)$-truth-table function $\Gamma : [\![P \vee Q]\!]^1_{\mathsf{LCM}} \to P \oplus Q$.*
5. *There is a $(1, \omega|2)$-computable function $\Gamma : [\![P \vee Q]\!]^2_{\mathsf{LCM}} \to P \oplus Q$.*
6. *There is a $(1, \omega)$-computable function $\Gamma : [\![P \vee Q]\!]^3_{\mathsf{LCM}} \to P \oplus Q$.*
7. *There is a $(2, 1)$-truth-table function $\Gamma : [\![P \vee Q]\!]^1_{\mathsf{CL}} \to P \oplus Q$.*
8. *There is a $(2, 1)$-computable function $\Gamma : [\![P \vee Q]\!]^2_{\mathsf{CL}} \to P \oplus Q$.*

Proof. For the items (1), (4), and (7), we consider the truth-table functionals $\Delta_0 : f \oplus g \mapsto 0^\frown g$ and $\Delta_1 : f \oplus g \mapsto 1^\frown g$. By the definition of $[\![P \vee Q]\!]^1_{\mathsf{CL}}$, obviously $\Delta_0(f \oplus g) \in P \oplus Q$ or $\Delta_1(f \oplus g) \in P \oplus Q$ for any $f \oplus g \in [\![P \vee Q]\!]^1_{\mathsf{CL}}$. Let e_0 and e_1 be indices of Δ_0 and Δ_1, respectively. On $\sigma \oplus \tau \in (2 \times \mathbb{N})^{<\mathbb{N}}$, we set $\Psi(\sigma \oplus \tau) = e_{\sigma(|\sigma|-1)}$. Note that the partial function Γ identified by the learner Ψ is $(1, n|2)$-truth-table on $[\![P \vee Q]\!]^1_{\mathsf{LCM[n]}}$, and $(1, \omega|2)$-truth-table on $[\![P \vee Q]\!]^1_{\mathsf{LCM}}$. Moreover, clearly $\Gamma(f \oplus g) = (\lim_s f(s))^\frown g \in P \oplus Q$ for every $f \oplus g \in [\![P \vee Q]\!]^1_{\mathsf{LCM}}$.

For the items (2), (5), and (8), we consider the partial computable functions $\Delta_0 : f \mapsto 0^\frown \mathtt{pr}_0(f)$ and $\Delta_1 : f \mapsto 1^\frown \mathtt{pr}_1(f)$. By the definition of $[\![P \vee Q]\!]^2_{\mathsf{CL}}$, obviously $\Delta_0(f) \in P \oplus Q$ or $\Delta_1(f) \in P \oplus Q$ for any $f \in [\![P \vee Q]\!]^2_{\mathsf{CL}}$. Let e_0 and e_1 be indices of Δ_0 and Δ_1, respectively. On $\sigma \in (2 \times \mathbb{N})^{<\mathbb{N}}$, we set $\Psi(\sigma) = e_{(\sigma(|\sigma|-1))_0}$. Note that the partial function Γ identified by the learner Ψ is $(1, n|2)$-computable on $[\![P \vee Q]\!]^2_{\mathsf{LCM[n]}}$, and $(1, \omega|2)$-computable on $[\![P \vee Q]\!]^2_{\mathsf{LCM}}$. Moreover, clearly $\Gamma(f) \in P \oplus Q$ for every $f \in [\![P \vee Q]\!]^2_{\mathsf{LCM}}$.

For the items (3) and (6), on $\sigma \in (\mathbb{N} \cup \{\sharp\})^{<\mathbb{N}}$, $\Psi(\sigma)$ guesses an index of the partial computable function $g \mapsto g^{\leftharpoondown t(\sigma)}$, where $t(\sigma) = \max\{n : \sigma(n) = \sharp\} + 1$ if such n exists; otherwise, $t(\sigma) = 0$. Note that the partial function Γ identified by the learner Ψ is $(1, n)$-computable on $[\![P \vee Q]\!]^3_{\mathsf{LCM[n]}}$, and $(1, \omega)$-computable on $[\![P \vee Q]\!]^3_{\mathsf{LCM}}$. Moreover, clearly $\Gamma(f) \in P \oplus Q$ for every $f \in [\![P \vee Q]\!]^3_{\mathsf{LCM}}$. □

Notation. One can iterate two-tape disjunction operations as $[\![\bigvee^{(1)} P]\!]^2_X = P$, and $[\![\bigvee^{(n+1)} P]\!]^2_X = [\![P \vee [\![\bigvee^{(n)} P]\!]^2_X]\!]^2_X$. Then, for instance, $[\![\bigvee^{(n)} P]\!]^2_{\mathsf{LCM}}$ can be identified with the following subset of Baire space.

$$\{f \in (n \times \mathbb{N})^{\mathbb{N}} : ((\exists i < n)\ \mathtt{pr}_i(f) \in P)\ \&\ \mathtt{mc}(f) < \infty\}.$$

As in the proof of Proposition 38, we use the notation $\mathtt{new}\Gamma(\sigma)$ for any function $\Gamma : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ and $\sigma \in \mathbb{N}^{<\mathbb{N}}$ in the proof of the next theorem. Here, $\mathtt{new}\Gamma(\sigma)$ is the unique string that satisfies the following condition.

$$\Gamma(\sigma) = \Gamma(\sigma^-)^\frown \mathtt{new}\Gamma(\sigma).$$

Theorem 40. *Let P and Q be any subsets of Baire space $\mathbb{N}^{\mathbb{N}}$.*

1. $P \leq^1_{<\omega} Q$ *if and only if* $[\![P \vee P]\!]^3_{\mathsf{LCM}[m]} \leq^1_1 Q$ *for some* $m \in \mathbb{N}$.
2. $P \leq^1_{\omega|<\omega} Q$ *if and only if* $[\![\bigvee^{(m)} P]\!]^2_{\mathsf{LCM}} \leq^1_1 Q$ *for some* $m \in \mathbb{N}$.
3. $P \leq^1_{\omega} Q$ *if and only if* $[\![P \vee P]\!]^3_{\mathsf{LCM}} \leq^1_1 Q$.
4. $P \leq^{<\omega}_1 Q$ *if and only if* $[\![\bigvee^{(m)} P]\!]^2_{\mathsf{CL}} \leq^1_1 Q$ *for some* $m \in \mathbb{N}$.
5. $P \leq^{<\omega}_{\omega} Q$ *if and only if* $[\![\bigvee^{(m)}[\![P \vee P]\!]^3_{\mathsf{LCM}}]\!]^2_{\mathsf{CL}} \leq^1_1 Q$.
6. $P \leq^{\omega}_1 Q$ *if and only if* $\bigcup_{m\in\mathbb{N}}[\![\bigvee^{(m)} P]\!]^2_{\mathsf{CL}} \leq^1_1 Q$.

Proof. The "if" parts of all items follow from Proposition 39. We show the "only if" part for every item.

(1) Assume that $P \leq^1_{<\omega} Q$ via a learner Ψ with mind-change-bound n. We need to construct a computable function Δ witnessing $[\![P \vee P]\!]^3_{\mathsf{LCM}[n]} \leq^1_1 Q$. For any $g \in Q$, by uniformly computable procedure, we can enumerate all elements of $\mathtt{mcl}_\Psi(g)$ as $m^g_0, m^g_1, \dots, m^g_{k-1}$, where $k < n$. Then, we define $\Delta(g)$ as follows.

$$\Delta(g) = 0^\frown\Phi_{\Psi(\langle\rangle)}(g \upharpoonright m^g_0)^\frown\sharp^\frown 0^\frown\left(\prod_{j<k-1}\Phi_{\Psi(g\upharpoonright m^g_i+1)}(g \upharpoonright m^g_i)^\frown\sharp^\frown 0\right)^\frown\Phi_{\Psi(g\upharpoonright m^g_{k-1}+1)}(g).$$

It is easy to see that Δ is computable. Note that $\mathtt{tail}(\Delta(g)) = \Phi_{\Psi(g\upharpoonright m^g_{k-1}+1)}(g) \in P$, since $P \leq^1_{<\omega} Q$ via Ψ, and $\lim_s \Psi(g \upharpoonright s)$ converges to $\Psi(g \upharpoonright m^g_{k-1} + 1)$. Furthermore, $\sharp$ occurs k times in $\Delta(g)$, and $k < n$ because of mind-change-bound n. Thus, $\Delta(g) \in [\![P\vee P]\!]^3_{\mathsf{LCM}[n]}$ for any $g \in Q$, as desired.

(2) Assume that $P \leq^1_{\omega|<\omega} Q$ via a leaner Ψ, where $\#\mathtt{indx}_\Psi(g) < n$ for any $g \in Q$. We need to construct a computable function Δ witnessing $[\![\bigvee^{(n)} P]\!]^2_{\mathsf{LCM}} \leq^1_1 Q$. We again use the function $\mathtt{reindex}_\Psi : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ defined in the proof of Theorem 26 (2). Fix $\sigma \in \mathbb{N}^{<\mathbb{N}}$. Pick the greatest substring $\tau \subsetneq \sigma$ such that $\Psi(\tau) = \Psi(\sigma)$. Then, define $\mathtt{new}^*\Phi_{\Psi(\sigma)}(\sigma)$ by the unique η such that $\Phi_{\Psi(\sigma)}(\sigma) = \Phi_{\Psi(\sigma)}(\tau)^\frown\eta$. Here, if there is no such τ, then we define $\mathtt{new}^*\Phi_{\Psi(\sigma)}(\sigma) = \Phi_{\Psi(\sigma)}(\sigma)$. Assume that $\Delta(\sigma^-)$ has been already defined. Then, we define $\Delta(\sigma)$ as follows.

$$\Delta(\sigma) = \Delta(\sigma^-)^\frown\mathtt{write}(\mathtt{reindex}_\Psi(\sigma), \mathtt{new}^*\Phi_{\Psi(\sigma)}(\sigma)).$$

Fix $g \in Q$. Note that $\mathtt{reindex}_\Psi(g \upharpoonright s) < n$ for each $s \in \mathbb{N}$, since $\#\mathtt{indx}_\Psi(g) < n$. Thus, we have $\Delta(g) \in (n \times \mathbb{N})^{\mathbb{N}}$. Moreover, $\mathtt{mc}(\Delta(g)) < \infty$, since Ψ is a learner converging on Q. Thus, $\lim_s \Psi(g \upharpoonright s)$ and hence $\lim_s \mathtt{reindex}_\Psi(g \upharpoonright s)$ converge. Therefore, $\mathtt{pr}_{\lim_s \mathtt{reindex}_\Psi(g\upharpoonright s)}(\Delta(g)) = \Phi_{\lim_s \Psi(g\upharpoonright s)}(g) \in P$. Hence, $[\![\bigvee^{(n)} P]\!]^2_{\mathsf{LCM}} \leq^1_1 Q$.

(3) By similar argument used in proof of (1).

(4) Assume that $P \leq^{<\omega}_1 Q$ via a finite collection $\{\Phi_e\}_{e<n}$ of partial computable functions. We need to construct a computable function Δ witnessing $[\![\bigvee^{(n)} P]\!]^2_{\mathsf{CL}} \leq^1_1 Q$. Assume that $\Delta(\sigma^-)$ is already defined. Define $\Delta(\sigma)$ as follows.

$$\Delta(\sigma) = \Delta(\sigma^-)^\frown\prod_{e<n}\mathtt{write}(e, \mathtt{new}\Phi_e(\sigma)).$$

Note that $\mathtt{pr}_e(\Delta(g)) \in P$ if $\Phi_e(g) \in P$. Thus, for any $g \in Q$, we have $\mathtt{pr}_e(\Delta(g)) \in P$ for some $e < n$. In other words, $[\![\bigvee^{(m)} P]\!]^2_{\mathsf{CL}} \leq^1_1 Q$ via Δ.

(5) Assume that $P \leq^{<\omega}_{\omega}$ via a team $\{\Psi_i\}_{i<n}$ of learners. We construct a computable function Δ. We first set $\Delta(\langle\rangle) = \langle\rangle$. Fix $\sigma \in \mathbb{N}^{<\mathbb{N}}$, and assume that $\Delta(\sigma^-)$ has been already defined. We define $\eta^\sigma_i \in \mathbb{N}^{<\mathbb{N}}$ for each $i < n$ as follows. Fix $i < n$. If $\Psi_i(\sigma) = \Psi_i(\sigma^-)$, put $\mathtt{new}^{**}\Phi_{\Psi_i(\sigma)}(\sigma) = \mathtt{new}\Phi_{\Psi_i(\sigma)}(\sigma)$. If $\Psi_i(\sigma) \neq \Psi_i(\sigma^-)$, put $\mathtt{new}^{**}\Phi_{\Psi_i(\sigma)}(\sigma) = \sharp^\frown\Phi_{\Psi_i(\sigma)}(\sigma)$. Then, we define $\Delta(\sigma)$ as follows.

$$\Delta(\sigma) = \Delta(\sigma^-)^\frown \prod_{i<n} \mathtt{write}(i, \mathtt{new}^{**}\Phi_{\Psi_i(\sigma)}(\sigma)).$$

Pick $g \in Q$. Then, by our assumption, $\Phi_{\lim_n \Psi_i(g\upharpoonright n)}(g) \in P$ for some $i < b$. Then $\mathtt{tail}(\mathtt{pr}_i(\Delta(g)))$ converges, and $\mathtt{tail}(\mathtt{pr}_i(\Delta(g)))^{\leftarrow 1} = \Phi_{\lim_n \Psi_i(g\upharpoonright n)}(g) \in P$. Thus, $\Delta(g) \in [\![\bigvee^{(m)}[\![P \vee P]\!]^3_{\mathsf{LCM}}]\!]^2_{\mathsf{CL}}$.

(6) Assume that $P \leq^\omega_1 Q$. We need to construct a computable function Δ witnessing $\bigcup_{m\in\mathbb{N}}[\![\bigvee^{(m)} P]\!]^2_{\mathsf{CL}} \leq^1_1 Q$. Assume that $\Delta(\sigma^-)$ has been already defined. Define $\Delta(\sigma)$ as follows.

$$\Delta(\sigma) = \Delta(\sigma^-)^\frown \left(\prod_{e<|\sigma|} \mathtt{write}(e, \mathtt{new}\Phi_e(\sigma))\right)^\frown(\mathtt{write}(|\sigma|, \Phi_{|\sigma|}(\sigma))).$$

Note that $\mathtt{pr}_e(\Delta(g)) = \Phi_e(g)$. Thus, for any $g \in Q$, we have $\mathtt{pr}_e(\Delta(g)) \in P$ for some $e \in \mathbb{N}$. In other words, $\bigcup_{m\in\mathbb{N}}[\![\bigvee^{(m)} P]\!]^2_{\mathsf{CL}} \leq^1_1 Q$ via Δ. □

Remark. Given an operation $O : \mathcal{P}(\mathbb{N}^\mathbb{N}) \times \mathcal{P}(\mathbb{N}^\mathbb{N}) \to \mathcal{P}(\mathbb{N}^\mathbb{N})$, one can introduce the reducibility notion $\leq_O$ by defining $P \leq_O Q$ as $O^{(n)}(P) \leq^1_1 Q$ for some $n \in \mathbb{N}$, where $O^{(1)}(P) = P$ and $O^{(n+1)}(P) = O(P, O^{(n)}(P))$. Then, Theorem 40 indicates that our reducibility notions induced by seven monoids in Theorem 5 are also induced from corresponding disjunction operations.

4.2. Galois Connection between Degree Structures

Remark. For degree structures $\mathcal{D}_u$ and $\mathcal{D}_r$ on $\mathcal{P}(\mathbb{N}^\mathbb{N})$, each operator $O : \mathcal{P}(\mathbb{N}^\mathbb{N}) \to \mathcal{P}(\mathbb{N}^\mathbb{N})$ induces the new operator $O_{ur} : \mathcal{D}_u \to \mathcal{D}_r$ defined by $O_{ur}(\deg_u(P)) = \deg_r(O(P))$ for any $P \subseteq \mathbb{N}^\mathbb{N}$. We identify O with O_{ur} whenever O_{ur} is well-defined. Recall that every partially ordered set can be viewed as a category. Sorbi [75] showed that $\widehat{\mathrm{Deg}} : \mathcal{D}^\omega_1 \to \mathcal{D}^1_1$ is left-adjoint to $\mathrm{id} : \mathcal{D}^1_1 \to \mathcal{D}^\omega_1$, and $\mathrm{id} \circ \widehat{\mathrm{Deg}} : \mathcal{D}^\omega_1 \to \mathcal{D}^\omega_1$ is identity, where $\widehat{\mathrm{Deg}}(P)$ denotes the Turing upward closure of $P \subseteq \mathbb{N}^\mathbb{N}$.

Definition 41.

1. $\mathbb{V}^1_{\mathtt{eff}}(P) = \bigoplus_{m\in\mathbb{N}}[\![P \vee P]\!]^3_{\mathsf{LCM}[m]}$.
2. $\mathbb{V}^1_{\omega|\mathtt{eff}}(P) = \bigoplus_{m\in\mathbb{N}}[\![\bigvee^{(m)} P]\!]^2_{\mathsf{LCM}}$.
3. $\mathbb{V}^1_\omega(P) = [\![P \vee P]\!]^3_{\mathsf{LCM}}$.
4. $\mathbb{V}^{\mathtt{eff}}_1(P) = \bigoplus_{m\in\mathbb{N}}[\![\bigvee^{(m)} P]\!]^2_{\mathsf{CL}}$.
5. $\mathbb{V}^{\mathtt{eff}}_\omega(P) = \bigoplus_{m\in\mathbb{N}}[\![\bigvee^{(m)}[\![P \vee P]\!]^3_{\mathsf{LCM}}]\!]^2_{\mathsf{CL}}$.

6. $\mathbb{V}^{\omega}_1(P) = \bigcup_{m\in\mathbb{N}}[\![\bigvee^{(m)} P]\!]^2_{\mathsf{CL}}$.

Corollary 42.

1. $\mathbb{V}^1_{\mathtt{eff}} : \mathcal{D}^1_{\mathtt{eff}} \to \mathcal{D}^1_1$ *is left-adjoint to* $\mathrm{id}_{\mathcal{P}(\mathbb{N}^\mathbb{N})} : \mathcal{D}^1_1 \to \mathcal{D}^1_{\mathtt{eff}}$*, and* $\mathrm{id}_{\mathcal{P}(\mathbb{N}^\mathbb{N})} \circ \mathbb{V}^1_{\mathtt{eff}}$ *is the identity on* $\mathcal{D}^1_{\mathtt{eff}}$*.*
2. $\mathbb{V}^1_{\omega|\mathtt{eff}} : \mathcal{D}^1_{\omega|\mathtt{eff}} \to \mathcal{D}^1_1$ *is left-adjoint to* $\mathrm{id}_{\mathcal{P}(\mathbb{N}^\mathbb{N})} : \mathcal{D}^1_1 \to \mathcal{D}^1_{\omega|\mathtt{eff}}$*, and* $\mathrm{id}_{\mathcal{P}(\mathbb{N}^\mathbb{N})} \circ \mathbb{V}^1_{\omega|\mathtt{eff}}$ *is the identity on* $\mathcal{D}^1_{\omega|\mathtt{eff}}$*.*
3. $\mathbb{V}^1_{\omega} : \mathcal{D}^1_{\omega} \to \mathcal{D}^1_1$ *is left-adjoint to* $\mathrm{id}_{\mathcal{P}(\mathbb{N}^\mathbb{N})} : \mathcal{D}^1_1 \to \mathcal{D}^1_{\omega}$*, and* $\mathrm{id}_{\mathcal{P}(\mathbb{N}^\mathbb{N})} \circ \mathbb{V}^1_{\omega}$ *is the identity on* $\mathcal{D}^1_{\omega}$*.*
4. $\mathbb{V}^{\mathtt{eff}}_1 : \mathcal{D}^{\mathtt{eff}}_1 \to \mathcal{D}^1_1$ *is left-adjoint to* $\mathrm{id}_{\mathcal{P}(\mathbb{N}^\mathbb{N})} : \mathcal{D}^1_1 \to \mathcal{D}^{\mathtt{eff}}_1$*, and* $\mathrm{id}_{\mathcal{P}(\mathbb{N}^\mathbb{N})} \circ \mathbb{V}^{\mathtt{eff}}_1$ *is the identity on* $\mathcal{D}^{\mathtt{eff}}_1$*.*
5. $\mathbb{V}^{\mathtt{eff}}_{\omega} : \mathcal{D}^{\mathtt{eff}}_{\omega} \to \mathcal{D}^1_1$ *is left-adjoint to* $\mathrm{id}_{\mathcal{P}(\mathbb{N}^\mathbb{N})} : \mathcal{D}^1_1 \to \mathcal{D}^{\mathtt{eff}}_{\omega}$*, and* $\mathrm{id}_{\mathcal{P}(\mathbb{N}^\mathbb{N})} \circ \mathbb{V}^{\mathtt{eff}}_{\omega}$ *is the identity on* $\mathcal{D}^{\mathtt{eff}}_{\omega}$*.*
6. $\mathbb{V}^{\omega}_1 : \mathcal{D}^{\omega}_1 \to \mathcal{D}^1_1$ *is left-adjoint to* $\mathrm{id}_{\mathcal{P}(\mathbb{N}^\mathbb{N})} : \mathcal{D}^1_1 \to \mathcal{D}^{\omega}_1$*, and* $\mathrm{id}_{\mathcal{P}(\mathbb{N}^\mathbb{N})} \circ \mathbb{V}^{\omega}_1$ *is the identity on* $\mathcal{D}^{\omega}_1$*.*

Proof. By Theorem 26. □

4.3. Σ^0_2 Decompositions

In computability theory, we sometimes encounter conditional branching given by a Σ^0_2 formula $S \equiv \exists n\tilde{S}(n)$. That is, if S is true, one chooses a procedure p_1, and if S is false, one chooses another procedure p_2. Thus, one may define *the computability with a* Σ^0_2 *conditional branching* as the class $\mathrm{dec}^2_{\mathrm{d}}[\Pi^0_2]$. However, even if we know that S is true, we have no algorithm to find a witness of S since $\tilde{S}(n)$ is Π^0_1, while we sometimes require a witness of S. This observation motivates us to study a missing interesting subclass of the nonuniformly computable functions.

Proposition 43. $\mathrm{dec}^{<\omega}_{\mathrm{d}}[\Pi^0_2]\mathrm{dec}^{\omega}_{\mathrm{p}}[\Pi^0_1]$ *is the smallest monoid including* $\mathrm{dec}^2_{\mathrm{d}}[\Pi^0_2]$ *and* $\mathrm{dec}^{\omega}_{\mathrm{p}}[\Pi^0_1]$.

Proof. It suffices to show that every $\Gamma \in \mathrm{dec}^2_{\mathrm{d}}[\Pi^0_2]\mathrm{dec}^{\omega}_{\mathrm{p}}[\Pi^0_1]$ is the composition of some $\Gamma_0 \in \mathrm{dec}^2_{\mathrm{d}}[\Pi^0_2]$ and $\Gamma_1 \in \mathrm{dec}^{\omega}_{\mathrm{p}}[\Pi^0_1]$. For every $\Gamma \in \mathrm{dec}^2_{\mathrm{d}}[\Pi^0_2]\mathrm{dec}^{\omega}_{\mathrm{p}}[\Pi^0_1]$, there exist a Π^0_2 d-layer $\{D_0, D_1\}$ and Π^0_1 partitions $\{\{P^0_n\}_{n\in\mathbb{N}}, \{P^1_n\}_{n\in\mathbb{N}}\}$ such that $\Gamma^i_n = \Gamma \upharpoonright D_i \cap P^i_n$ is computable uniformly in $i < 2$ and $n \in \mathbb{N}$, where $\{P^i_n\}_{n\in\mathbb{N}}$ is a partition of D_i for every $i \in \{0,1\}$. Let $\Gamma_0 : D_0 \cup D_1 \to D_0 \oplus D_1$ be the union of two computable homeomorphisms $D_0 \simeq 0^\frown D_0$ and $D_1 \simeq 1^\frown D_1$. For instance, put $\Gamma_0(g) = i^\frown g$ for $g \in D_i$. Then $\Gamma_0 \in \mathrm{dec}^2_{\mathrm{d}}[\Pi^0_2]$ since $\{D_0, D_1\}$ is a Π^0_2 d-layer. Define $\Gamma_1(i^\frown g) = \Gamma^i_n(g)$ for any $i < 2$ and $g \in i^\frown P^i_n$. Then, $\Gamma_1 \in \mathrm{dec}^{\omega}_{\mathrm{p}}[\Pi^0_1]$, since $\{\Gamma^i_n\}_{i<2,n\in\mathbb{N}}$ is uniformly computable, and $\{P^i_n\}_{i<2,n\in\mathbb{N}}$ is uniformly Π^0_1. Clearly we have $\Gamma^i_n \upharpoonright D_i \cap P^i_n = \Gamma_1 \circ \Gamma_0 \upharpoonright D_i \cap P^i_n$ for any $i < 2$ and $n \in \mathbb{N}$. Hence, $\Gamma = \Gamma_1 \circ \Gamma_0$. □

The following concept of *hyperconcatenation* (Definition 45) plays a key role in many proofs in Part II. In the next section, we will see that the hyperconcatenation can be defined as *infinitary disjunction along an ill-founded tree* or *iterated concatenation along an ill-founded tree*. Before defining the notion of hyperconcatenation, we introduce some auxiliary notations.

Definition 44. For any strings $\sigma \in (\mathbb{N} \cup \{\mathtt{pass}\})^{<\mathbb{N}}$ and $\tau \in (\mathbb{N} \cup \{\sharp, \mathtt{pass}\})^{<\mathbb{N}}$, the *content of* σ, $\mathtt{content}(\sigma) \in \mathbb{N}^{<\mathbb{N}}$, and the *walk of* τ, $\mathtt{walk}(\tau) \in (\mathbb{N} \cup \{\mathtt{pass}\})^{<\mathbb{N}}$, are inductively defined as follows.

$$\mathtt{content}(\langle\rangle) = \langle\rangle, \quad \mathtt{content}(\sigma) = \begin{cases} \mathtt{content}(\sigma^-)^\frown \sigma(|\sigma| - 1) & \text{if } \sigma(|\sigma| - 1) \neq \mathtt{pass}, \\ \mathtt{content}(\sigma^-) & \text{otherwise.} \end{cases}$$

$$\mathtt{walk}(\tau \upharpoonright 1) = \langle\rangle, \quad \mathtt{walk}(\tau) = \begin{cases} \mathtt{walk}(\tau^-)^\frown v & \text{if } \tau(|\tau| - 2) = \sharp \ \& \ \tau(|\tau| - 1) = v \neq \sharp, \\ \mathtt{walk}(\tau^-) & \text{otherwise.} \end{cases}$$

Then, the *content* of $f \in (\mathbb{N} \cup \{\mathtt{pass}\})^{\mathbb{N}}$ and the *walk* of $g \in (\mathbb{N} \cup \{\sharp, \mathtt{pass}\})^{\mathbb{N}}$ are defined by $\mathtt{content}(f) = \bigcup_{n \in \mathbb{N}} \mathtt{content}(f \upharpoonright n)$ and $\mathtt{walk}(g) = \bigcup_{n \in \mathbb{N}} \mathtt{walk}(g \upharpoonright n)$, respectively.

The walk produces a sequence by extracting only the immediate successors $r \in \mathbb{N} \cup \{\mathtt{pass}\}$ of $\sharp$'s, but it may contain the symbol $\mathtt{pass}$. Then, the content removes all symbols $\mathtt{pass}$ from this sequence. For instance, let $\tau \in (\mathbb{N} \cup \{\sharp, \mathtt{pass}\})^{<\mathbb{N}}$ be the following sequence.

$$\tau = \langle 1, 6, \sharp, 1, 8, 0, \sharp, \sharp, \sharp, 3, 3, 9, \sharp, \mathtt{pass}, 8, \sharp, 8, \sharp, \sharp, \mathtt{pass}, 7, \dots \rangle$$

Then, $\mathtt{walk}(\tau) = \langle 1, 3, \mathtt{pass}, 8, \mathtt{pass}, \dots \rangle$, and its content is $\mathtt{content} \circ \mathtt{walk}(\tau) = \langle 1, 3, 8, \dots \rangle$. Now we introduce the concept of the hyperconcatenation.

Definition 45 (Hyperconcatenation)**.** Let P and Q be *any* subsets of Baire space $\mathbb{N}^{\mathbb{N}}$. The *hyperconcatenation* $[\![Q \vee P]\!]^{\blacktriangledown}_{\Sigma^0_2}$ and the *non-Lipschitz hyperconcatenation* $[\![Q \vee P]\!]_{\Sigma^0_2}$ of Q and P are defined as follows.

$$[\![Q \vee P]\!]^{\blacktriangledown}_{\Sigma^0_2} = \{g \in (\mathbb{N} \cup \{\sharp\})^{\mathbb{N}} : \mathtt{walk}(g) \in Q \text{ or } \mathtt{tail}(g)^{\leftarrow 1} \in P\},$$

$$[\![Q \vee P]\!]_{\Sigma^0_2} = \{g \in (\mathbb{N} \cup \{\sharp, \mathtt{pass}\})^{\mathbb{N}} : \mathtt{content} \circ \mathtt{walk}(g) \in Q \text{ or } \mathtt{tail}(g)^{\leftarrow 1} \in P\}.$$

Theorem 46 (As the Law of Excluded Middle)**.** *The implications* $(b^+) \to (a) \to (a^-) \leftrightarrow (b^-)$ *hold for any* $P, Q, R \subseteq \mathbb{N}^{\mathbb{N}}$*:*

- *(a)* $[\![Q \vee P]\!]^{\blacktriangledown}_{\Sigma^0_2} \leq^1_1 R$.
- *(a⁻)* $[\![Q \vee P]\!]_{\Sigma^0_2} \leq^1_1 R$.
- *(b⁺) There is a* Σ^0_2 *sentence* $\varphi \equiv \exists v \theta(v)$ *with a uniform sequence* $\{\Gamma_i\}_{i \in \mathbb{N}}, \Delta$ *of partial computable functions on* $\mathbb{N}^{\mathbb{N}}$ *such that*
  - *if* $g \in R$ *satisfies* $\theta(v)$*, then* $\Gamma_v(g; u) \downarrow$ *for any* $u \in \mathbb{N}$*, and* $\Gamma_v(g) \in P$*.*
  - *if* $g \in R$ *satisfies* $\neg\theta(v)$*, then* $\Delta(g; u) \downarrow$ *for any* $u \leq v$*, and* $[\Delta(g) \upharpoonright v + 1]$ *intersects with* Q*.*
  - *if* $g \in R$ *satisfies* $\neg\exists v \theta(v)$*, then* $\Delta(g; u) \downarrow$ *for any* $u \in \mathbb{N}$*, and* $\Delta(g) \in Q$*.*
- *(b⁻) There is a* Σ^0_2 *sentence* $\varphi \equiv \exists v \theta(v)$ *with a uniform sequence* $\{\Gamma_i\}_{i \in \mathbb{N}}, \Delta$ *of partial computable functions on* $\mathbb{N}^{\mathbb{N}}$ *such that*
  - *if* $g \in R$ *satisfies* $\theta(v)$*, then* $\Gamma_v(g; u) \downarrow$ *for any* $u \in \mathbb{N}$*, and* $\Gamma_v(g) \in P$*.*

- *if $g \in R$ satisfies $\neg\exists v\theta(v)$, then $\Delta(g;u)\downarrow$ for any $u \in \mathbb{N}$, and $\Delta(g) \in Q$.*

Proof. (b$^+$)→(a): Assume that $S_i = \{g \in \mathbb{N}^\mathbb{N} : \Theta(g;i)\uparrow\}$ for some computable function Θ, and that $P \leq^1_1 R \cap S_i$ via Γ_i and $Q \leq^1_1 R \setminus \bigcup_{i\in\mathbb{N}} S_i$ via Δ. For a string $\sigma \in \mathbb{N}^{<\mathbb{N}}$, define $d(\sigma)$ and $t(\sigma;i)$ as follows:

$$\begin{aligned} d(\sigma) &= \max\{d \in \mathbb{N} : (\forall i < d)\ \Theta(\sigma;i)\downarrow\};\\ t(\sigma;i) &= \min\{t \in \mathbb{N} : \Theta(\sigma \upharpoonright t;i)\downarrow\},\ \text{ for any } i < d(\sigma). \end{aligned}$$

Then let us define $\Lambda(\sigma) = \prod_{i<d(\sigma)} (\Gamma_i(\sigma \upharpoonright t(\sigma;i))^\frown\sharp^\frown\Delta(\sigma;i))^\frown\Gamma_{d(\sigma)}(\sigma)$.

(a$^-$)→(b$^-$): Assume that $[\![Q \vee P]\!]_{\Sigma^0_2} \leq^1_1 R$ via a computable function Φ. Set $S_v = \{g \in \mathbb{N}^\mathbb{N} : (\forall n \geq v)\ \Phi(g;n) \neq \sharp\}$. For a string $\sigma \in \mathbb{N}^{<\mathbb{N}}$, we first computes the following $\mathtt{count}(\sigma)$ and $\mathtt{mcl}_\sharp(\sigma,n)$ for each $n \in \mathbb{N}$:

$$\begin{aligned} \mathtt{count}(\sigma) &= \#\{m < |\sigma| : \Phi(\sigma;m) = \sharp\},\\ \mathtt{mcl}_\sharp(\sigma,n) &= \min\{m \leq |\sigma| : \mathtt{count}(\sigma \upharpoonright m) > n\},\ \text{ if such } m \text{ exists.} \end{aligned}$$

Then set $\Gamma_v(\sigma) = \Phi(\sigma)^{\leftarrow\mathtt{mcl}_\sharp(\sigma,\mathtt{count}(\sigma\upharpoonright v))+1}$; and set $\Delta(\sigma) = \lambda n.\Phi(\sigma,\mathtt{mcl}_\sharp(\sigma,n))$. Note that if $g \in R \cap S_k$ for some $k \in \mathbb{N}$, then $\Gamma_k(g) \in P$; otherwise, $\Delta(g) \in Q$. Therefore, $P \leq^1_1 R \cap S_v$ via Γ_v and $Q \leq^1_1 R \setminus S$ via Δ.

(b$^-$)→(a$^-$): For each $\sigma \in \mathbb{N}^{<\mathbb{N}}$, let $v(\sigma)$ be the least v such that $R(u,v,\sigma)$ holds for all $u < |\sigma|$, where $\varphi(g) \equiv (\exists v)(\forall u)R(u,v,g \upharpoonright u)$. We inductively define a computable function Φ as follows. We first set $\Phi(\langle\rangle) = \langle\rangle$. Assume that $\Phi(\sigma^-)$ has been already defined.

$$\Phi(\sigma) = \begin{cases} \Phi(\sigma^-)^\frown\gamma, & \text{if } v(\sigma) = v(\sigma^-)\ \&\ \Gamma_{v(\sigma)}(\sigma) = \mathtt{tail}^+(\Phi(\sigma^-))^\frown\gamma,\\ \Phi(\sigma^-)^\frown\langle\sharp,\delta(0)\rangle, & \text{if } v(\sigma) \neq v(\sigma^-)\ \&\ \Delta(\sigma) = \mathtt{content}\circ\mathtt{walk}(\Phi(\sigma^-))^\frown\delta,\\ \Phi(\sigma^-)^\frown\langle\sharp,\mathtt{pass}\rangle, & \text{if } v(\sigma) \neq v(\sigma^-)\ \&\ \Delta(\sigma) = \mathtt{content}\circ\mathtt{walk}(\Phi(\sigma^-)). \end{cases}$$

For any $g \in \mathbb{N}^\mathbb{N}$, if $\varphi(g) \equiv (\exists v)(\forall u)R(u,v,g \upharpoonright u)$, then for the least such $v \in \mathbb{N}$, we have $\mathtt{tail}^+(\Phi(g)) = \Gamma_v(g)$. Otherwise, we have $\mathtt{content}\circ\mathtt{walk}(\Phi(g)) = \Delta(g)$. Hence, $\Phi(g) \in [\![Q \vee P]\!]_{\Sigma^0_2}$, for any $g \in R$. □

Definition 47. Let $\{S_n\}_{n\in\mathbb{N}}$ be an increasing sequence of subsets of $\mathbb{N}^\mathbb{N}$. We say that a partial function $\Gamma :\subseteq \mathbb{N} \to \mathbb{N}$ is *computable along* $\{S_n\}_{n\in\mathbb{N}}$ if $\Gamma \upharpoonright \mathrm{dom}(\Gamma) \setminus \bigcup_n S_n$ and $\Gamma \upharpoonright \mathrm{dom}(\Gamma) \cap S_n \setminus S_{n-1}$ is computable uniformly in $n \in \mathbb{N}$, where $S_{-1} = \emptyset$. Moreover, we also say that a partial function $\Gamma :\subseteq \mathbb{N} \to \mathbb{N}$ is *computable strictly along* $\{S_n\}_{n\in\mathbb{N}}$ if there is a uniform sequence of computable functions $\{\Gamma_n\}_{n\in\mathbb{N}}$ and Δ such that $\Gamma \upharpoonright \mathrm{dom}(\Gamma) \setminus \bigcup_n S_n = \Delta \upharpoonright \mathrm{dom}(\Gamma) \setminus \bigcup_n S_n$ and $\Gamma \upharpoonright \mathrm{dom}(\Gamma) \cap S_n \setminus S_{n-1} = \Gamma_n \upharpoonright \mathrm{dom}(\Gamma) \cap S_n \setminus S_{n-1}$ and $\Delta(g) \upharpoonright n$ is defined for any $g \in \mathrm{dom}(\Gamma) \setminus S_n$.

Remark. Theorem 46 implies that there is a function $\Gamma : [\![Q \vee P]\!]_{\Sigma^0_2} \to P \oplus Q$ ($\Gamma : [\![Q \vee P]\!]^\blacktriangledown_{\Sigma^0_2} \to P \oplus Q$) such that Γ is computable (strictly) along sequences of Π^0_1 sets.

Corollary 48. $\mathrm{dec}^{<\omega}_{\mathrm{d}}[\Pi^0_2]\mathrm{dec}^{\omega}_{\mathrm{p}}[\Pi^0_1]$ *is the smallest monoid containing all functions computable (strictly) along sequences of* Π^0_1 *sets.*

Proof. Let $\mathcal{S}$ be the class of all functions computable (strictly) along sequences of Π^0_1 sets. Then, clearly, we have $\mathrm{dec}^2_{\mathrm{d}}[\Pi^0_2] \cup \Gamma_1 \in \mathrm{dec}^{\omega}_{\mathrm{p}}[\Pi^0_1] \subseteq \mathcal{S} \subseteq \mathrm{dec}^{<\omega}_{\mathrm{d}}[\Pi^0_2]\mathrm{dec}^{\omega}_{\mathrm{p}}[\Pi^0_1]$. Thus, the desired condition follows from Proposition 43. □

Remark. It is easy to see that the hyperconcatenation operations are non-commutative as follows. For instance, if f and g are Turing incomparable, then $[\![\{g\} \vee \{f\}]\!]^{\blacktriangledown}_{\Sigma^0_2} \not\leq^1_1 [\![\{f\} \vee \{g\}]\!]^{\blacktriangledown}_{\Sigma^0_2}$. Otherwise, we have a witness Γ of the reduction, and then $\mathtt{walk} \circ \Gamma(\tilde{f}) \leq g$ for any $\tilde{f}$ with $\mathtt{walk}(\tilde{f}) \leq f$. This is because for any n, $\tilde{g} = (\tilde{f} \upharpoonright n)\sharp ig \in [\![\{f\} \vee \{g\}]\!]^{\blacktriangledown}_{\Sigma^0_2}$ for a suitable i, and it is Turing equivalent to g. Hence, $\Gamma(\tilde{g})$ cannot have f as a tail, since $f \not\leq_T g$ Therefore, $\mathtt{walk} \circ \Gamma(\tilde{g}) = g$.

Thus, given σ_n with $\mathtt{walk}(\sigma_n)^\frown i \prec f$, concatenate a sufficiently long initial segment τ_n of $\sharp ig$ to force $\mathtt{walk} \circ \Gamma(\sigma_n{}^\frown\tau_n) \succeq g \upharpoonright n$. Now, consider the closed subspace $C_f = \{h \in (\mathbb{N} \cup \{\sharp\})^{\mathbb{N}} : \mathtt{walk}(h) \leq f\}$ that is f-computably homeomorphic to $\mathbb{N}^{\mathbb{N}}$. If we can extend $\sigma_n{}^\frown\tau_n$ to some string ρ extendible in C_f that forces $\Phi_n(\rho; k) \neq g(k)$ for some $k \in \omega$, then go to the next step. If not, there exists k such that $\Phi_n(h; k)$ is undefined for any $h \in C_f$ extending $\sigma_n{}^\frown\tau_n$, since otherwise, given k, one can f-computably find $\rho_k \succeq \sigma_n{}^\frown\tau_n$ in C_f such that $\Phi_n(\rho_k; k)$ converges, but then it must be equal to $g(k)$, and this contradicts our assumption $g \not\leq_T f$.

Consequently, one can extend $\sigma_n{}^\frown\tau_n$ to some string σ_{n+1} which forces not to compute g via the n-th Turing functional, that is, $\Phi_n(h) \neq g$ for every $h \in C_f$ extending σ_{n+1}. Finally, put $\hat{f} = \bigcup_n \sigma_n$. By our construction, we have $g \not\leq_T \hat{f}$, and $g = \mathtt{walk} \circ \Gamma(\hat{f}) \leq_T \hat{f}$, a contradiction.

5. Going Deeper and Deeper

5.1. Falsifiable Mass Problems

We are mostly interested in local degree structures such as Turing degrees of c.e. subsets of $\mathbb{N}$ and Medvedev degrees of Π^0_1 subsets of $2^{\mathbb{N}}$. In such cases, the straightforward two-tape (backtrack) notions in Definitions 33 and 34 are hard to use, since, for instance, $[\![P \vee Q]\!]^2_{\mathsf{LCM}[2]}$ may not belong to Π^0_1 even if P and Q are Π^0_1. This observation prompts us to define *consistent* two-tape disjunctions.

Let $\{T_i\}_{i\in I}$ be a sequence of trees $T_i \subseteq \mathbb{N}^{<\mathbb{N}}$. Then, *the consistency set* $\mathrm{Con}(T_i)_{i\in I}$ *for* $\{T_i\}_{i\in I}$ is defined as follows.

$$\mathrm{Con}(T_i)_{i\in I} = \{f \in (I \times \mathbb{N})^{\mathbb{N}} : (\forall i \in I)(\forall n \in \mathbb{N})\ \mathtt{pr}_i(f \upharpoonright n) \in T_i\}.$$

The notion of consistency sets has a relationship with consistent learning (see also Remark below Proposition 54). The consistency sets are useful to reduce the complexity of our disjunctions to be Π^0_1. We now introduce the following consistent modifications of our disjunctive notions.

Definition 49. Let P_0 and P_1 denote Π^0_1 subsets of $\mathbb{N}^{\mathbb{N}}$.

$$P_0 \triangledown_{\omega} P_1 = [\![P_0 \vee P_1]\!]^2_{\mathsf{LCM}} \cap \mathrm{Con}(T_{P_0}, T_{P_1}).$$
$$P_0 \triangledown_{n} P_1 = [\![P_0 \vee P_1]\!]^2_{\mathsf{LCM}[n]} \cap \mathrm{Con}(T_{P_0}, T_{P_1}).$$
$$P_0 \triangledown_{\infty} P_1 = [\![P_0 \vee P_1]\!]^2_{\mathsf{CL}} \cap \mathrm{Con}(T_{P_0}, T_{P_1}).$$

Here T_{P_0} and T_{P_1} are corresponding (computable) trees for P_0 and P_1, respectively (where recall from Section 1.3 that such a tree is assumed to be uniquely determined when an index of P_i is given).

Remark. Obviously, Definition 49 depends on our choice of indices (hence, corresponding trees) of given Π^0_1 sets, that is, the operations in Definition 49 is defined on subtrees of $\mathbb{N}^{<\mathbb{N}}$ rather than subsets of $\mathbb{N}^\mathbb{N}$. However, Proposition 50 indicates that it does not really matter what we chose, if we only focus on the degree-theoretic behavior. We will frequently use index-dependent definitions (e.g., Definitions 49 and 52) in order to simplify our notations, but in each case, one can easily ensure that it cause no problems at all (e.g., Propositions 50 and 54).

Proposition 50. *Let P and Q be Π^0_1 subsets of $\mathbb{N}^\mathbb{N}$.*

1. $P\triangledown_n Q \equiv^1_1 [\![P \vee Q]\!]^2_{\mathsf{LCM}[n]}$ *for each $n \in \mathbb{N}$.*
2. $P\triangledown_\omega Q \equiv^1_1 [\![P \vee Q]\!]^2_{\mathsf{LCM}}$.
3. $P\triangledown_\infty Q \equiv^1_1 [\![P \vee Q]\!]^2_{\mathsf{CL}}$.

Proof. For each item, clearly $P\triangledown_* Q \geq^1_1 [\![P \vee Q]\!]^2_*$. Thus, it suffices to construct a computable functional Φ witnessing $P\triangledown_* Q \leq^1_1 [\![P \vee Q]\!]^2_*$. Let T_0 and T_1 denote the corresponding computable trees for P and Q respectively. Set $\Phi(\langle\rangle) = \langle\rangle$. Fix $\sigma \in (2 \times \mathbb{N})^{<\mathbb{N}}$. Assume that $\Phi(\sigma^-)$ has already been defined, and $\sigma = \sigma^{-\frown}\langle(i,k)\rangle$ for some $i < 2$ and $k \in \mathbb{N}$. Then,

$$\Phi(\sigma) = \begin{cases} \Phi(\sigma^-)^\frown\langle(i,k)\rangle & \text{if } \mathtt{pr}_i(\sigma) \in T_i, \\ \Phi(\sigma^-) & \text{if } \mathtt{pr}_i(\sigma) \notin T_i, \end{cases}$$

Clearly, Φ is a computable function, since T_i is computable for each $i < 2$. For any $g \in (2\times\mathbb{N})^\mathbb{N}$, clearly $\mathtt{mc}(\Phi(g)) \leq \mathtt{mc}(g)$. Fix $g \in [\![P\vee Q]\!]^2_*$, where $* \in \{\mathsf{LCM}, \mathsf{LCM}[n], \mathsf{CL}\}$. Then $\mathtt{pr}_i(g) \in P_i$ for some $i < 2$, where $P_0 = P$ and $P_1 = Q$. Therefore, $\Phi(g)$ is total, and $\mathtt{pr}_i(\Phi(g)) \in P_i$ for such $i < 2$. □

Proposition 51. *Let P and Q be Π^0_1 subsets of $\mathbb{N}^\mathbb{N}$.*

1. *$P\triangledown_n Q$ is Π^0_1, for any $n \in \mathbb{N}$.*
2. *$P\triangledown_\omega Q$ is Σ^0_2.*
3. *$P\triangledown_\infty Q$ is Π^0_1.*

Proof. Let T_0 and T_1 denote corresponding computable trees for P and Q respectively. We consider the following computable tree:

$$T_{P,Q,n} = \{\sigma \in (2 \times \mathbb{N})^{<\mathbb{N}} : (\forall i < 2)\ \mathtt{pr}_i(\sigma) \in T_i\ \&\ \mathtt{mc}(\sigma) < n\}.$$

Note that $T_{P,Q,n}$ is uniformly computable in n, since $\mathtt{pr}_i(\sigma)$ and $\mathtt{mc}(\sigma)$ are computable uniformly in $\sigma \in \mathbb{N}^{<\mathbb{N}}$. Clearly, $P\triangledown_n Q \subseteq [T_{P,Q,n}]$. Moreover, for any $g \in [T_{P,Q,n}]$, $\mathtt{pr}_i(g)$ is total for some $i < 2$. Then, $\mathtt{pr}_i(g) \in [T_i]$ for such i, and $\mathtt{mc}(g) \leq n$, since the relation $\mathtt{mc}(f) \leq n$ is equivalent to $(\forall k)\ \mathtt{mc}(f \upharpoonright k) \leq n$. Thus, $g \in P\triangledown_n Q$. Consequently, $P\triangledown_n Q = [T_{P,Q,n}]$ is Π^0_1. Hence, $P\triangledown_\omega Q = \bigcup_n [T_{P,Q,n}]$ is Σ^0_2. The items (3) also follows from the similar argument. □

Definition 52. The *concatenation* of trees $T_0, T_1 \subseteq \mathbb{N}^{<\mathbb{N}}$ is defined as

$$T_0{}^\frown T_1 = \{\sigma^\frown\langle\sharp\rangle^\frown\tau : \sigma \in T_0 \ \& \ \tau \in T_1\}.$$

One can introduce the concatenation of Π^0_1 sets $P_0, P_1 \subseteq \mathbb{N}^\mathbb{N}$ by the set $[T_{P_0}{}^\frown T_{P_1}]$ for corresponding computable trees T_{P_0} and T_{P_1} of P_0 and P_1. Here, this definition is also index-dependent (recall Remark below Definition 49).

However, we adopt the following *conservative* version as our definition of the concatenation, which is easier to handle in many proofs. Let L_P denote the set of all leaves of the corresponding computable tree for a nonempty Π^0_1 set P. Then *the (conservative) concatenation of P and Q* is defined as follows.

$$P^\frown Q = P \cup \bigcup_{\rho\in L_P} \rho^\frown Q.$$

The commutative (conservative) concatenation of P and Q is defined by $P\triangledown Q = (P^\frown Q)\oplus(Q^\frown P)$.

Remark. On the study of Wadge degrees of finite level of Borel hierarchy, Duparc [26] introduced various operators such as $P^{\longrightarrow}Q = P \cup \bigcup_{\rho\in\mathbb{N}^{<\mathbb{N}}} \rho^\frown\langle\sharp\rangle^\frown Q$. The following proposition indicates that our non-commutative concatenation is essentially same as Duparc's operation $P^{\longrightarrow}Q$.

Proposition 53. *Let P, Q be Π^0_1 subsets of Baire space $\mathbb{N}^\mathbb{N}$. Then, the concatenation $P^\frown Q$ is $(1,1)$-equivalent to the set $P^{\rightarrow}Q := [T_P{}^\frown T_Q]$.*

Proof. To see $P^{\rightarrow}Q \leq^1_1 P^\frown Q$, we inductively define a total computable function $\mathtt{cut} : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$. First set $\mathtt{cut}(\langle\rangle) = \langle\rangle$, and fix $\sigma = \sigma^{-\frown}\langle n\rangle \in \mathbb{N}^{<\mathbb{N}}$. We assume that $\mathtt{cut}(\sigma^-)$ has been already defined. If $\sigma = \sigma^{-\frown}\langle n\rangle \in L_P$, then we set $\mathtt{cut}(\sigma) = \mathtt{cut}(\sigma^-)^\frown\langle n, \sharp\rangle$. Otherwise, we set $\mathtt{cut}(\sigma) = \mathtt{cut}(\sigma^-)^\frown\langle n\rangle$. Then, $\mathtt{cut}$ is computable, since P is Π^0_1 and then T_P is computable. Moreover, we can see the following.

$$\mathtt{cut}(f) = \begin{cases} f & \text{if } f \in P, \\ (f \upharpoonright k)^\frown\langle\sharp\rangle^\frown f^{\leftarrow k} & \text{if } (\exists k \in \mathbb{N}) \ f \upharpoonright k \in L_P. \end{cases}$$

Clearly, $P^{\rightarrow}Q \leq^1_1 P^\frown Q$ via the computable function $\mathtt{cut}$.

Conversely, we consider the computable function $\mathtt{leaf} : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ which maps σ to the least leaf of L_P extending σ. Then, we inductively define a computable function Γ witnessing $P^\frown Q \leq^1_1 P^{\rightarrow}Q$ as follows. First set $\Gamma(\langle\rangle) = \langle\rangle$, and fix $\sigma = \sigma^{-\frown}\langle n\rangle \in (\mathbb{N} \cup \{\sharp\})^{<\mathbb{N}}$. We assume that $\Gamma(\sigma^-)$ has been already defined. If $n \neq \sharp$, then we set $\Gamma(\sigma) = \Gamma(\sigma^-)^\frown\langle n\rangle$. If $n = \sharp$, then we set $\Gamma(\sigma) = \mathtt{leaf}(\Gamma(\sigma^-))$. It is easy to see that $P^\frown Q \leq^1_1 P^{\rightarrow}Q$ via Γ. □

Remark. Inspired by our method used in Part II, Cenzer-Kihara-Weber-Wu [18] explicitly employed the concept of the (non-commutative) concatenation to show that $\mathsf{CPA}^\frown\mathsf{CPA}$ has a greatest Medvedev degree of Π^0_1 subsets of $2^\mathbb{N}$ with no tree-immune. Here, a Π^0_1 set $P \subseteq 2^\mathbb{N}$ is *tree-immune* if the Π^0_1 tree $\{\sigma \in 2^{<\mathbb{N}} : P \cap [\sigma] \neq \emptyset\}$ includes no infinite computable subtree, and CPA is the set of all *complete consistent extensions of Peano Arithmetic*. Note that CPA is a *Medvedev complete* Π^0_1 subset of $2^\mathbb{N}$.

Proposition 54. *Let P, Q be Π^0_1 subsets of $\mathbb{N}^\mathbb{N}$.*

1. $P \triangledown P \equiv^1_1 P^\frown P$.
2. $P \triangledown Q \equiv^1_1 [\![P \vee Q]\!]^2_{\mathsf{LCM}[2]}$.

Proof. (1) $P\triangledown P = (P^\frown P) \oplus (P^\frown P) \equiv^1_1 P^\frown P$. (2) By Proposition 50 (1), we have $P\triangledown_2 Q \equiv^1_1 [\![P \vee Q]\!]^2_{\mathsf{LCM}[2]}$. Then, $P\triangledown_2 Q \leq^1_1 P\triangledown Q$ is witnessed by the following reduction Δ.

$$\Delta(f) = \begin{cases} \mathtt{write}(f(0), f^{\leftarrow 1}), & \text{if } f^{\leftarrow 1} \in [T_{\sigma(0)}], \\ \mathtt{write}(f(0), f^{\leftarrow 1} \upharpoonright k)^\frown \mathtt{write}(1 - f(0), f^{\leftarrow k+1}), & \text{if } (\exists k \in \mathbb{N})\ f^{\leftarrow 1} \upharpoonright k \in L_{\sigma(0)}. \end{cases}$$

Here, T_0 and T_1 are the corresponding computable trees for P and Q respectively, and L_i is the set of all leaves of T_i for each $i < 2$. Clearly, Δ is computable. Fix $\langle i\rangle^\frown g \in P\triangledown Q$. Obviously, $\mathtt{mc}(\langle i\rangle^\frown g) < 2$. If $g \in [T_i]$ then $\mathtt{pr}_i(\Delta(\langle i\rangle^\frown g)) = g \in [T_i]$, and if $g = \sigma^\frown h$ for some $\sigma \in L_i$ and $h \in [T_{1-i}]$ then $\mathtt{pr}_i(\Delta(\langle i\rangle^\frown \sigma^\frown h)) = h \in [T_{1-i}]$. Hence, $\Delta(\langle i\rangle^\frown g) \in P\triangledown_2 Q$.

To see $P\triangledown Q \leq^1_1 P\triangledown_2 Q$, it suffices to construct a computable functional Γ witnessing $(P^\rightarrow Q) \oplus (Q^\rightarrow P) \leq^1_1 P\triangledown_2 Q$ by Proposition 53. Set $\Gamma(\langle\rangle) = \langle\rangle$, and $\Gamma(\langle(i,n)\rangle) = \langle i, n\rangle$ for any $i < 2$ and $n \in \mathbb{N}$. Fix $\sigma = \sigma^{---\frown}\langle(i,m),(j,n)\rangle \in (2 \times \mathbb{N})^{<\mathbb{N}}$, and assume that $\Gamma(\sigma^-)$ is already defined. If $i \neq j$, then set $\Gamma(\sigma) = \Gamma(\sigma^-)^\frown\langle\sharp, n\rangle$. Otherwise, set $\Gamma(\sigma) = \Gamma(\sigma^-)^\frown\langle n\rangle$. Clearly Γ is computable. Fix $g \in P\triangledown_2 Q$. If $\mathtt{mc}(g) = 0$, then $\Gamma(g) = \langle i\rangle^\frown \mathtt{pr}_i(g) \in P \oplus Q \subseteq (P^\rightarrow Q) \oplus (Q^\rightarrow P)$, where $i = (g(0))_0$. If $\mathtt{mc}(g) = 1$, then $\mathtt{pr}_i(g)$ is a finite string, where $i = (g(0))_0$. In this case, we can easily see $\Gamma(g) = \langle i\rangle^\frown \mathtt{pr}_i(g)^\frown\langle\sharp\rangle^\frown \mathtt{pr}_{1-i}(g) \in (P^\rightarrow Q) \oplus (Q^\rightarrow P)$. □

In the case of $P\triangledown P$, we use the non-commutative concatenation $P^\frown P$ to simplify our proof without mentioning.

Remark. These disjunctions have some connection with *consistent conservative Popperian learning* (see [37]).

- The term "*consistent*" means: the scientist should modify his hypothesis whenever it was found to be refuted.
- The term "*conservative*" means: the scientist changes his hypothesis only when it was found to be refuted.
- The term "*Popperian*" means: the scientist can test whether his hypothesis is currently consistent or refuted.

The notion of *Popperian learning* is introduced by Case and Ngo-Manguelle [16] based on Gold's theory of "identification in the limit" [29]. A learner (a scientist) is a computable function $\Psi : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$, and a natural phenomenon is a computable function $f : \mathbb{N} \to \mathbb{N}$. Then the formula $\Psi(f \upharpoonright n) = e$ means the following situation: the scientist Ψ predicts that a rule generating the phenomenon f can be explained by a word (a formula, or an algorithm) e (i.e., $f = \Phi_e$) when he observes $f(0), \dots f(n-1)$. We say that Ψ *learns* f if $\Phi_{\lim_n \Psi(f\upharpoonright n)} = f$. The learner Ψ is *Popperian* if $\Phi_{\Psi(\sigma)}$ is total for each $\sigma \in \mathbb{N}^{<\mathbb{N}}$. The learner Ψ is *consistent* at $\sigma \in \mathbb{N}^{<\mathbb{N}}$ if $\Phi_{\Psi(\sigma)} \upharpoonright |\sigma| = \sigma$. The learner

Table 5: Hierarchy of Consistent Disjunctions

$P \oplus Q$	$[\![P \vee Q]\!]^1_{\mathsf{Int}}$	Intuitionistic disujunction (= $P\triangledown_1 Q$)
$P \cup Q$	$[\![P \vee Q]\!]^1_{\mathsf{CL}}$	Classical one-tape disjunction
$P\triangledown Q$	$[\![P \vee Q]\!]^2_{\mathsf{LCM}[2]}$	Commutative concatenation ($\equiv P^\frown Q$ if $P = Q$)
$P\triangledown_n Q$	$[\![P \vee Q]\!]^2_{\mathsf{LCM}[n]}$	LCM disjunction with mind-changes-bound n
$P\triangledown_\omega Q$	$[\![P \vee Q]\!]^2_{\mathsf{LCM}}$	LCM disjunction
$P\triangledown_\infty Q$	$[\![P \vee Q]\!]^2_{\mathsf{CL}}$	Classical disjunction

Ψ is *conservative* if, for any $\sigma \in \mathbb{N}^{<\mathbb{N}}$, $\Psi(\sigma) = \Psi(\sigma^-)$ whenever $\Phi_{\Psi(\sigma^-)} \upharpoonright |\sigma| = \sigma$. Note that, for every Popperian learner Ψ, he can algorithmically determine whether Ψ is consistent at σ or not, for a given $\sigma \in \mathbb{N}^{<\mathbb{N}}$. The terminology "*Popperian*" derives from Popper's falsifiabillity principle in philosophy of science.

The complexity Π^0_1 reflects the concept of Popperian learning. The consistency set $\mathrm{Con}(T_i)_{i\in I}$ restricts our learning process to be consistent. Additionally, the non-commutative concatenation $P^\frown Q$ of P and Q restricts our learning process to be conservative, since it represents the following situation: a choice on the first hypothesis P is refuted if, and only if, the scientist proposes the second (refutable) hypothesis Q and start verifying it.

Proposition 55. *For Π^0_1 sets $P, Q \subseteq \mathbb{N}^\mathbb{N}$ and $n \in \mathbb{N}$,*

$$[\![P \vee Q]\!]^2_{\mathsf{LCM}} \leq^1_1 [\![P \vee Q]\!]^2_{\mathsf{LCM}[n+2]} \leq^1_1 [\![P \vee Q]\!]^1_{\mathsf{CL}} \leq^1_1 [\![P \vee Q]\!]^1_{\mathsf{Int}}.$$

Proof. It suffices to show $P\triangledown Q \leq^1_1 P \cup Q$, since $[\![P \vee Q]\!]^1_{\mathsf{CL}} \equiv^1_1 P \cup Q$ by Proposition 35 (5) and $[\![P \vee Q]\!]^2_{\mathsf{LCM}[2]} \equiv^1_1 P\triangledown Q$ by Proposition 54 (2). Indeed, we can show that $(P^\frown Q) \otimes (Q^\frown P) \leq^1_1 P \cup Q$. We construct a computable functional Φ witnessing $P^\frown Q \leq^1_1 P \cup Q$. If $\sigma \in T_P$, then set $\Phi(\sigma) = \sigma$. If $\sigma \notin T_P$, then pick a unique $\rho \subseteq \sigma$ such that $\rho \in L_P$, and set $\Phi(\sigma) = \rho^\frown\sigma$ for such ρ, where L_P is the set of all leaves of T_P. Clearly Φ is computable, and note that $\Phi(\sigma) \subseteq \Phi(\tau)$ whenever $\sigma \subseteq \tau$. If $g \in P$, then $\Phi(g) = g \in P$. If $g \in Q \setminus P$, then there is a unique $\rho \subset g$ such that $\rho \in L_P$, and $\Phi(g) = \rho^\frown g \in P^\frown Q$. Thus, $P^\frown Q \leq^1_1 P \cup Q$ via Φ. □

Remark. Our notation $\triangledown$ is inspired by the sequential disjunction [39] in Computability Logic [38]. One may also compare $\triangledown_\omega$ and $\triangledown_\infty$ with the toggling disjunction and the parallel disjunction [40].

5.2. Compactified Infinitaly Disjunctions

This subsection is concerned with a trick to represent *infinitary* disjunctive notions as effective compact sets.

Definition 56. Fix a collection $\{P_i\}_{i\in I}$ of subsets of Baire space $\mathbb{N}^\mathbb{N}$.

1. $[\![\bigvee_{i\in I} P_i]\!]_{\mathsf{Int}} = \{f \in (I \times \mathbb{N})^\mathbb{N} : ((\exists i \in I)\ \mathtt{pr}_i(f) \in P_i)\ \&\ \mathtt{mc}(f) = 0\}$.

2. $[\![\bigvee_{i\in I} P_i]\!]_{\mathsf{LCM}} = \{f \in (I \times \mathbb{N})^{\mathbb{N}} : ((\exists i \in I)\ \mathtt{pr}_i(f) \in P_i)\ \&\ \mathtt{mc}(f) < \infty\}$.
3. $[\![\bigvee_{i\in I} P_i]\!]_{\mathsf{CL}} = \{f \in (I \times \mathbb{N})^{\mathbb{N}} : (\exists i \in I)\ \mathtt{pr}_i(f) \in P_i\}$.

Proposition 57. *Let $\{P_n\}_{n\in\mathbb{N}}$ be an infinite collection of subsets of Baire space $\mathbb{N}^{\mathbb{N}}$.*

1. $[\![\bigvee_{n\in\mathbb{N}} P_n]\!]_{\mathsf{Int}} \equiv^1_1 \bigoplus_{n\in\mathbb{N}} P_n$*, where* $\bigoplus_{n\in\mathbb{N}} P_n = \{\langle n\rangle^\frown f : f \in P_n\}$.
2. $[\![\bigvee_{i,n} P_{i,n}]\!]_{\mathsf{LCM}} \equiv^1_1 [\![P_0 \vee P_1]\!]^3_{\mathsf{LCM}}$*, where $P_{i,n} = P_i$ for each $i < 2$ and $n \in \mathbb{N}$.*

Proof. (1) $[\![\bigvee_{n\in\mathbb{N}} P_n]\!]_{\mathsf{Int}} \geq^1_1 \bigoplus_{n\in\mathbb{N}} P_n$ is witnessed by $f \mapsto (f(0))_0{}^\frown\mathtt{pr}_{(f(0))_0}(f)$, and $[\![\bigvee_{n\in\mathbb{N}} P_n]\!]_{\mathsf{Int}} \leq^1_1 \bigoplus_{n\in\mathbb{N}} P_n$ is witnessed by $f \mapsto \mathtt{write}(f(0), f^{\leftharpoonup 1})$, where recall that $\mathtt{write}(f(0), f^{\leftharpoonup 1}) = (f(0))^{\mathbb{N}} \oplus (\lambda n.f(n+1))$ indicates the instruction to writing the infinite word $f^{\leftharpoonup 1}$ on the $f(0)$-th tape.

(2) We first construct a computable function Ξ witnessing $[\![\bigvee_{i,n} P_{i,n}]\!]_{\mathsf{LCM}} \geq^1_1 [\![P_0 \vee P_1]\!]^3_{\mathsf{LCM}}$. For $((i,n),v) \in (2\times\mathbb{N})\times\mathbb{N}$, we first set $\Xi(\langle((i,n),v)\rangle) = \langle((i,n),v)\rangle$. For each string $\sigma = \sigma^{--\frown}\langle((i,n),v),((j,m),w)\rangle \in ((2\times\mathbb{N})\times\mathbb{N})^{<\mathbb{N}}$, inductively assume that $\Xi(\sigma^-)$ has been already defined. If $(i,n) = (j,m)$, then we set $\Xi(\sigma) = \Xi(\sigma^-)^\frown\langle w\rangle$. Otherwise, we set $\Xi(\sigma) = \Xi(\sigma^-)^\frown\langle\sharp, j, w\rangle$. For any $f \in [\![\bigvee_{i,n} P_{i,n}]\!]_{\mathsf{LCM}}$, the backtrack symbol $\sharp$ occurs in $\Xi(f)$ finitely often, since $\mathtt{mc}(f) < \infty$. Therefore, $\mathtt{tail}(\Xi(f))$ converges, and $\mathtt{tail}(\Xi(f))^{\leftharpoonup 1} = \mathtt{pr}_{i,m}(f) \in P_i$ for some $i < 2$ and $m \in \mathbb{N}$. Thus, $\Xi(f) \in [\![P_0 \vee P_1]\!]^3_{\mathsf{LCM}}$.

We next construct a computable function Ξ^* witnessing $[\![\bigvee_{i,n} P_{i,n}]\!]_{\mathsf{LCM}} \leq^1_1 [\![P_0 \vee P_1]\!]^3_{\mathsf{LCM}}$. Set $\Xi^*(\langle\rangle) = \langle\rangle$. For $\sigma = \sigma^{--\frown}\langle v,w\rangle \in (\mathbb{N}\cup\{\sharp\})^{<\mathbb{N}}$, inductively assume that $\Xi^*(\sigma^-)$ has been already defined. To define $\Xi^*(\sigma)$, recall the definition $\mathtt{count}(\sigma) = \#\{n < |\sigma| : \sigma(n) = \sharp\}$. Then $\Xi^*(\sigma)$ is defined as follows.

$$\Xi^*(\sigma) = \begin{cases} \Xi^*(\sigma^-)^\frown\langle((\mathtt{tail}(\sigma;0),\mathtt{count}(\sigma)),w)\rangle, & \text{if } v \neq \sharp \text{ and } w \neq \sharp,\\ \Xi^*(\sigma^-), & \text{otherwise}\end{cases}$$

For any $f \in [\![P_0 \vee P_1]\!]^3_{\mathsf{LCM}}$, we have $\mathtt{mc}(\Xi^*(f)) < \infty$, since $\mathtt{count}(f) = \#\{k \in \mathbb{N} : f(k) = \sharp\}$ is finite. Therefore, we have $\mathtt{pr}_{(\mathtt{tail}(f;0),\mathtt{count}(f))}(\Xi^*(f)) = \mathtt{tail}(f)^{\leftharpoonup 1} \in P_{\mathtt{tail}(f;0)}$. Thus, $\Xi^*(f) \in [\![\bigvee_{i,n} P_{i,n}]\!]_{\mathsf{LCM}}$. □

We again use the consistent modifications of infinitary models, $\left[\bigtriangledown_\omega\right]_{n\in\mathbb{N}} P_n = [\![\bigvee_{n\in\mathbb{N}} P_n]\!]_{\mathsf{LCM}} \cap \mathrm{Con}(T_{P_n})_{n\in\mathbb{N}}$, and $[\bigtriangledown_\infty]_{n\in\mathbb{N}} P_n = [\![\bigvee_{n\in\mathbb{N}} P_n]\!]_{\mathsf{CL}} \cap \mathrm{Con}(T_{P_n})_{n\in\mathbb{N}}$.

Proposition 58. *Let $\{P_n\}_{n\in\mathbb{N}}$ be a computable collection of Π^0_1 subsets of Baire space $\mathbb{N}^{\mathbb{N}}$.*

1. $[\![\bigvee_{n\in\mathbb{N}} P_n]\!]_{\mathsf{LCM}} \equiv^1_1 \left[\bigtriangledown_\omega\right]_{n\in\mathbb{N}} P_n$.
2. $[\![\bigvee_{n\in\mathbb{N}} P_n]\!]_{\mathsf{CL}} \equiv^1_1 [\bigtriangledown_\infty]_{n\in\mathbb{N}} P_n$.

Proof. As in the proof of Proposition 50. □

However, the problem is that our models of infinitary disjunctions are not compact. A modification of infinitary sum was introduced by Binns-Simpson [8] to embed a free Boolean algebra into the Muchnik lattice of Π^0_1 subsets of Cantor space, and such a variation was called *a recursive meet*. An important feature of their modification is that it is a Π^0_1 subset of the compact space $2^{\mathbb{N}}$.

Definition 59 (Binns-Simpson [8]). Let P and $\{Q_n\}_{n\in\mathbb{N}}$ be computable collection of Π^0_1 subsets of $2^\mathbb{N}$, and let ρ_n denote the length-lexicographically n-th leaf of the corresponding computable tree of P. Then, we define the *infinitary concatenation* and *recursive meet* as follows:

$$P^\frown\{Q_i\}_{i\in\mathbb{N}} = P \cup \bigcup_n \rho_n{}^\frown Q_n, \qquad \bigoplus \overrightarrow{_{i\in\mathbb{N}}} Q_i = \mathsf{CPA}^\frown\{Q_i\}_{i\in\mathbb{N}}.$$

Here, recall that CPA is a Medvedev complete set, which consists of all *complete consistent extensions of Peano Arithmetic*. The Medvedev completeness of CPA ensures that for any nonempty Π^0_1 subset $P \subseteq 2^\mathbb{N}$, a computable function $\Phi : \mathsf{CPA} \to P$ exists. Of course, these definitions are also index-dependent (recall Remark below Definition 49).

Proposition 60. *For any computable sequence $\{P_n\}_{n\in\mathbb{N}}$ of nonempty Π^0_1 subsets of $2^\mathbb{N}$, $\bigoplus \overrightarrow{_{n\in\mathbb{N}}} P_n \equiv^1_{<\omega} \bigoplus_{n\in\mathbb{N}} P_n$.*

Proof. The condition $\bigoplus \overrightarrow{_{n\in\mathbb{N}}} P_n \leq^1_1 \bigoplus_{n\in\mathbb{N}} P_n$ is witnessed by a computable function $n^\frown g \mapsto \rho_n{}^\frown g$. We will construct a learner witnessing $\bigoplus \overrightarrow{_{n\in\mathbb{N}}} P_n \geq^1_{<\omega} \bigoplus_{n\in\mathbb{N}} P_n$. Fix a computable function $\Phi_e : \mathsf{CPA} \to 0^\frown P_0$. Such Φ_e exists, since every nonempty Π^0_1 subset of $2^\mathbb{N}$ is $(1,1)$-reducible to CPA. We also fix a partial computable function $\Phi_{i(n)} : \rho_n{}^\frown g \mapsto n^\frown g$, for each $n \in \mathbb{N}$. For $\sigma \in 2^{<\mathbb{N}}$, if $\sigma \in T_{\mathsf{CPA}}$ then set $\Psi(\sigma) = e$. If $\sigma \notin T_{\mathsf{CPA}}$, then $\rho_n \subseteq \sigma$ for some n. For such n, we set $\Psi(\sigma) = i(n)$. The function Γ identified by the learner Ψ is clearly $(1,2)$-computable, and $\Gamma(g) \in \bigoplus_{n\in\mathbb{N}} P_n$ for any $g \in \bigoplus \overrightarrow{_{n\in\mathbb{N}}} P_n$. □

5.3. Infinitary Disjunctions along well-Founded Trees

One can consider a computational learning process with *transfinite mind-changes*, i.e., a model represented by transfinitely iterated concatenations. We use Kleene's $\mathcal{O}$ to deal with computable ordinals in a uniformly computable way.

Definition 61 (Transfinite Mind-Changes). Let $(\mathcal{O}, \leq_\mathcal{O})$ denote *Kleene's system of ordinal notations* (see Rogers [63]). Then for each $a \in \mathcal{O}$ we introduce the *a-th derivative* of $P \subseteq \mathbb{N}^\mathbb{N}$ as follows.

$$P^a = \begin{cases} P \\ [\![P \vee P^b]\!]^2_{\mathsf{LCM}[2]} \\ \bigoplus_{n\in\mathbb{N}} P^{\Phi_e(n)} \end{cases} \quad P^{a+} = \begin{cases} P & \text{if } a = 0, \\ [\![P \vee P^{b+}]\!]^2_{\mathsf{LCM}[2]} & \text{if } a = 2^b, \\ [\![P \vee \bigoplus_{n\in\mathbb{N}} P^{\Phi_e(n)+}]\!]^2_{\mathsf{LCM}[2]} & \text{if } a = 3 \cdot 5^e. \end{cases}$$

Here, we require $\Phi_e(n) <_\mathcal{O} \Phi_e(n+1)$ for every $3 \cdot 5^e \in \mathcal{O}$ in the definition of $\mathcal{O}$. In particular, this implies that $P^{(\Phi_e(m))} \leq^1_1 P^{(\Phi_e(n))}$ whenever $n \leq m$. Additionally, we may require that $\Phi_e(n) < \Phi_e(n+1)$ as a natural number by padding. If P is a nonempty Π^0_1 subset of $2^\mathbb{N}$, we also define another derivative $P^{(a)}$ as follows.

$$P^{(a)} = \begin{cases} P & \text{if } a = 0, \\ P^\frown P^{(b)} & \text{if } a = 2^b, \\ P^\frown\{P^{(\Phi_e(n))}\}_{n\in\mathbb{N}} & \text{if } a = 3 \cdot 5^e. \end{cases}$$

Proposition 62. *For any nonempty Π^0_1 set $P \subseteq 2^{\mathbb{N}}$ and any notation $a \in O$, the a-th derivative $P^{(a)}$ is a Π^0_1 subset of $2^{\mathbb{N}}$.*

Proof. Fix $a \in O$. By our definition, obviously $P^{(a)}$ is a subset of $2^{\mathbb{N}}$. We inductively assume that $\{P^{(b)} : b <_O a\}$ is uniformly Π^0_1. For $a = 2^b$, we can easily compute a Π^0_1 index of $P^{(a)} = P^\frown P^{(b)}$ is from a Π^0_1 index of $P^{(b)}$. For $a = 3 \cdot 5^e$, we can also easily compute a Π^0_1 index of $P^a = P^\frown\{P^{(\Phi_e(n))}\}_{n\in\mathbb{N}}$ from a computable sequence of Π^0_1 indices of $\{P^{(\Phi_e(n))}\}_{n\in\mathbb{N}}$. Thus, $\{P^{(b)} : b \leq_O a\}$ is uniformly Π^0_1. □

Proposition 63. *For any nonempty Π^0_1 set $P \subseteq 2^{\mathbb{N}}$ and any notation $a \in O$, the condition $P^{a+} \leq^1_1 P^{(a)} \leq^1_1 P^a$ holds.*

Proof. Clearly $P^\frown P^{(b)}$ is $(1,1)$-equivalent to $[\![P \vee P^{(b)}]\!]^2_{\mathsf{LCM}[2]}$, since $P^{(b)}$ is Π^0_1 by Proposition 62, where the $(1,1)$-equivalence follows by Proposition 37 and 54. It is easy to see that $[\![P \vee \bigoplus_{n\in\mathbb{N}} P^{(\Phi_e(n))}]\!]^2_{\mathsf{LCM}[2]} \leq^1_1 P^\frown\{P^{(\Phi_e(n))}\}_{n\in\mathbb{N}} \leq^1_1 \bigoplus_{n\in\mathbb{N}} P^{(\Phi_e(n))}$ holds. For successor steps, it suffices to show that $P^\frown P^{(b)} \leq_W (P^{(b)\frown}P)$. If $|b|_O$ is a finite ordinal, it is clear. If $|b|_O$ is an infinite ordinal, say $b = 3 \cdot 5^e$, then $P^{(b)} \leq^1_1 P^{(b)\frown}P$ holds, since $\Phi_e(n) + 1 \leq_O \Phi_e(n+1)$. □

Notation. Every $a \in O$ is often identified with the corresponding well-founded tree T_a consisting of all finite nonempty $<_O$-decreasing sequences $\langle a_0, a_1, a_2, \dots\rangle$, where $a_0 = a$ and for every $i \in \mathbb{N}$, either $2^{a_{i+1}} = a_i$ or $a_{i+1} = \Phi_e(n)$ holds for some $n \in \mathbb{N}$ and e with $3 \cdot 5^e = a_i$. Our padding assumption $\Phi_e(n) < \Phi_e(n+1)$ implies that T_a is computable.

Definition 61 immediately induces associated piecewise computability notions. For a notation $a \in O$, a collection $\{S_\kappa\}_{\kappa\in T_a}$ of Σ^0_1 subsets of $X \subseteq \mathbb{N}^{\mathbb{N}}$ is *a-indexed* if $S_{\langle a\rangle} = X$ and the mapping $\kappa \mapsto S_\kappa$ is an order preserving homomorphism from the tree $(T_a, \subseteq)$ onto the ordered set $(\{S_\kappa\}_{\kappa\in T_a}, \supseteq)$, where $O(\leq a) = \{b : b \leq_O a\}$. It is *strictly a-indexed* if it is a-indexed and $S_\kappa = \bigcup_{n\in\mathbb{N}} S_{\kappa^\frown\Phi_e(n)}$ whenever $\kappa = \kappa^{-\frown}3 \cdot 5^e$. A partial function $\Gamma :\subseteq \omega^\omega \to \omega^\omega$ is said to be *(strictly) a-indexed Π^0_1 d-layerwise computable* if there are a (strictly) a-indexed collection of Σ^0_1 subsets $\{S_\kappa\}_{\kappa\in T_a}$ of the domain of Γ and a uniformly computable collection $\{\Gamma_\kappa\}_{\kappa\in T_a}$ of partial computable functions such that Γ agrees with Γ_κ on the domain $S_\kappa \setminus \bigcup_{\lambda\supsetneq\kappa} S_\lambda$.

It is easy to see that these notions are subclasses of $\mathrm{dec}^\omega_{\mathrm{p}}[\Pi^0_1]$. If the order type $|a|_O$ of $\{b : b <_O a\}$ is ω, the strict a-indexed Π^0_1 d-layerwise computability realizes the class $[\mathfrak{C}_T]^1_{\mathtt{eff}}$. Obviously, a strict a-indexed Π^0_1 d-layerwise computable function $\Gamma : P^a \to P$ and an a-indexed Π^0_1 d-layerwise computable function $\Gamma^* : P^{a+} \to P$ exist.

Remark. Obviously, a-indexed Π^0_1 d-layerwise computability can be viewed as the effective version of discontinuity level $\leq_O a$ in the sense of Hertling [33] and Hemmerling [32]. Here, a partial function $\Gamma :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ shall be said to be *of effective discontinuity level $\leq_O a$* if there is a computable collection $\{\Gamma_b\}_{b\leq_O a}$ of partial computable functions with uniform Σ^0_1 domains $\{S_b\}_{b\leq_O a}$ such that for every $x \in \mathrm{dom}(\Gamma)$, $\Gamma(x) = \Gamma_b(x)$ for a unique $b \leq_O a$ with $x \in S_b \setminus \bigcup_{c<_O b} S_c$.

Note that Hemmerling [32] studied its boldface version in the context of levels of subhierarchy (see Małek [50]) of the Baire one star functions $\mathcal{B}^*_1$ (see O'Malley [58]), whose original definition seems to be a boldface version of the Blum-Blum locking [9] in learning theory. Then, the boldface version of the learnability with mind-change 1 seems to be interpreted as the Baire one double star functions $\mathcal{B}_1^{**}$ (see Pawlak [61]).

Indeed, the notion of the discontinuity level is a useful tool to analyze the Baire hierarchy of the Borel measurable functions. For instance, Solecki [74, Theorem 3.1] used a transfinite derivation process in the proof of his dichotomy theorem for the Baire one functions, and Semmes [65, Lemma 4.3.3] introduced a high level analog of a transfinite derivation process in the proof of his decomposition theorem for the $\mathbf{\Lambda}_{2,3}$ functions (a subclass of the Baire two functions).

See also de Brecht [23] for a systematic study on the levels of discontinuity.

Definition 64 (see Freivalds-Smith [28] and Luo-Schulte [49])**.** Let $\Psi : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ be a learner. We say that $c : \mathbb{N}^{<\mathbb{N}} \to \mathcal{O}$ is a *mind-change counter* for Ψ if, for any $\sigma \in \mathbb{N}^{<\mathbb{N}}$, $c(\sigma) <_{\mathcal{O}} c(\sigma^-)$ whenever $\Psi(\sigma) \neq \Psi(\sigma^-)$. A learner Ψ is *a-bounded* if there is a computable mind-change counter $c : \mathbb{N}^{<\mathbb{N}} \to \mathcal{O}$ for Ψ such that $c(\langle\rangle) \leq_{\mathcal{O}} a$.

Remark. The computational power of a-bounded learnability is very closely related to Ershov's mind-change hierarchy (Ershov hierarchy [27]) of Δ^0_2 subsets of $\mathbb{N}$, or the effective version of the Hausdorff difference hierarchy of $\mathbf{\Delta}^0_2$ subsets of $\mathbb{N}^{\mathbb{N}}$ (for Ershov hierarchy, see also Stephan-Yang-Yu [77]).

Proposition 65. *For a notation $a \in \mathcal{O}$, a partial function $\Gamma :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is of effective discontinuity level $\leq_{\mathcal{O}} a$ if and only if it is learnable via an a-bounded learner.*

Proof. The desired equivalence is obtained from an interpretation between S_b and the Σ^0_1 set generated by the c.e. set $\{\sigma \in \mathbb{N}^{<\mathbb{N}} : c(\sigma) \leq b\}$. □

5.4. Infinitary Disjunctions along any Graphs

In the classical proof process, a verifier Ψ on "P_0 or P_1" may change his mind infinitely often. In the backtrack-tape model, this situation means that Ψ chooses the backtrack symbol $\sharp$ infinitely many often. Then the word on Λ is eventually finite, and it verifies neither P_0 nor P_1. Therefore, in the model, if Ψ succeeds to verify "P_0 or P_1" then the backtrack symbol $\sharp$ occurs on the record Δ at most finitely often. Consequently, in the backtrack-tape model, classical verification coincides with LCM verification. However, we would like to cover the case that unbounded or infinitely many mind-changes occur. This may be archived by regarding the backtrack-tape model as a kind of infinitary tape model.

The dynamic-tape model: Assume that a directed graph (V, E) is given, where V can be infinite, $E \subseteq V \times V$, and an *initial vertex* $\varepsilon \in V$ is chosen. For any $v \in V$, let $\mathrm{adj}(v) = \{w \in V : (v, w) \in E\}$. When a verifier Ψ tries to prove that "$\bigvee_{v \in V} P_v$", infinite tapes $\square$, and Λ_v for $v \in V$ are given. The tape $\square$ is called *the declaration*, Λ_v is called *the working tape* for each $v \in V$. First the letter ε is written on $\square$, and no word is written on Λ_v for $v \in V$. At each stage s, assume that $v[s]$ is written on $\square$. Then the verifier Ψ executes one or the other of two following actions.

1. Ψ declares some $w \in \mathrm{adj}(v[s])$, erases all words on $\square$, and writes w on $\square$; or
2. Ψ writes a letter $k \in \mathbb{N}$ on the working tape $\Lambda_{v[s]}$.

Assume that a verifier Ψ tries to prove that "P_0 or P_1".

- **Intuitionism**: Consider $V = \{\varepsilon, 0, 1\}$, $E = \{(\varepsilon, 0), (\varepsilon, 1)\}$, and $P_\varepsilon = \emptyset$.
- **LCM with ordinal-bounded mind-changes**: For a computable well-founded tree $V = T \subseteq \mathbb{N}^{<\mathbb{N}}$, consider the following.

$$E = E(T) = \{(\sigma, \tau) \in T \times T : (\exists i \in \mathbb{N})\ \tau = \sigma^\frown i\},\ P_\sigma = \begin{cases} P_0, & \text{if } |\sigma| \text{ is even,} \\ P_1, & \text{if } |\sigma| \text{ is odd,} \end{cases}$$

- **LCM**: Consider $V = \mathbb{N}$; $E = \{(n, n+1) : n \in \mathbb{N}\}$; $P_{2n} = P_0$ for any $n \in \mathbb{N}$; and $P_{2n+1} = P_1$ for any $n \in \mathbb{N}$. Moreover, the word written on the declaration $\square$ must converge.
- (V, E)**-relaxed Classical**: $(V, E) = (V_0, V_1, E)$ is a given directed bipartite graph, and $P_\tau = P_i$ for any $\tau \in V_i$ and $i < 2$.

Definition 66 (Dynamic Disjunctions). Let $G = (V, E)$ be a directed graph, and let $\{P_v\}_{v \in V}$ be a collection of subsets of Baire space. For $E \subseteq V^2$, put $\overline{E} = E \cup \{\langle v, v\rangle : v \in V\}$. We define the *dynamic disjunction of* $\{P_v\}_{v \in V}$ *along the graph* (V, E) as follows.

$$\left[\!\!\left[\bigvee_{v \in (V,E)} P_v \right]\!\!\right] = \left\{ f \in (V \times \mathbb{N})^{\mathbb{N}} : (\forall n \in \mathbb{N})\ (\langle (f(n))_0, (f(n+1))_0\rangle \in \overline{E})\ \&\ (\exists v \in V)\ \mathtt{pr}_v(f) \in P_v \right\}.$$

Moreover, if $\{P_v\}_{v \in V}$ is a computable sequence of Π^0_1 subsets of $\mathbb{N}^{\mathbb{N}}$, and T_{P_v} be the corresponding tree for P_v, we also define its consistent versions.

1. $\bigtriangledown_{v \in (V,E)} P_v = [\![\bigvee_{v \in (V,E)} P_v]\!] \cap \mathrm{Con}(T_{P_v})_{v \in V}$.
2. $\blacktriangledown_{v \in (V,E)} P_v = \{f \in (V \times \mathbb{N})^{\mathbb{N}} : (\forall n \in \mathbb{N})\ (\langle (f(n))_0, (f(n+1))_0\rangle \in \overline{E})\} \cap \mathrm{Con}(T_{P_v})_{v \in V}$.

Here, recall that, for $x = (x_0, x_1)$, the first coordinate x_0 is denoted by $(x)_0$. If $P_v = P$ for any $v \in V$, then we simply write $\bigtriangledown_{v \in V} P$ and $\blacktriangledown_{v \in V} P$ for $\bigtriangledown_{v \in (V,E)} P_v$ and $\blacktriangledown_{v \in (V,E)} P_v$ respectively.

As our dynamic-tape model is an infinitary-tape model, this model may be natural to be regarded as expressing a proof process of an infinitary disjunction $\bigvee_{v \in V} P_v$. Therefore, we refer the model with (V, E) as *an infinitary disjunction along* (V, E). Later we will introduce a more complicated model. It will be called *the nested-tape model*. We first see an upper and lower bound of the degrees of difficulty of these disjunctive notions, and a relationship among various models we have introduced. Let $\widehat{\mathrm{Deg}}(P)$ denote the *Turing upward closure* of P, i.e., $\widehat{\mathrm{Deg}}(P) = \{g : (\exists f \leq_T g)\ f \in P\}$, and $[(V, E)]$ denote the set of all infinite paths through a graph (V, E), i.e., $[(V, E)] = \{p \in V^{\mathbb{N}} : (p(n), p(n+1)) \in E\}$.

Proposition 67. *Let* (V, E) *be a computable directed graph, and* $\{P_v\}_{v \in V}$ *be a computable sequence of* Π^0_1 *subsets of* $\mathbb{N}^{\mathbb{N}}$.

1. $\widehat{\mathrm{Deg}}\left(\bigoplus_{v \in V} P_v\right) \leq^1_1 \bigtriangledown_{v \in (V,E)} P_v \leq^1_1 \bigoplus_{v \in V} P_v$.
2. $\widehat{\mathrm{Deg}}\left([(V, E)] \oplus \bigoplus_{v \in V} P_v\right) \leq^1_1 \blacktriangledown_{v \in (V,E)} P_v \leq^1_1 [(V, E)] \oplus \bigoplus_{v \in V} P_v$.

Proof. (1) $\bigtriangledown_{v\in(V,E)} P_v \leq^1_1 \bigoplus_{v\in V} P_v$ is witnessed by $v^\frown f \mapsto \mathtt{write}(v,f) = v^{\mathbb{N}} \oplus f$. For any $f \in \bigtriangledown_{v\in(V,E)} P_v$, we have $\mathtt{pr}_v(f) \in P_v$ for some $v \in V$. Thus, we have $\mathtt{pr}_v(f) \leq_T f$, since $\mathtt{pr}_v$ is partially computable, and $f \in \mathrm{dom}(\mathtt{pr}_v)$. Hence, $f \in \widehat{\mathrm{Deg}}(P_v)$.

(2) Fix $f \in [(V,E)] \oplus \bigoplus_{v\in V} P_v$. If $f(0) = 1$, then we can show the desired condition as in (1). If f is of the form $f = 0^\frown g$, we have $\lambda n.\langle g(n), 0\rangle \in \blacktriangledown_{v\in(V,E)} P_v$ since $g \in [(V,E)]$. Hence, $\blacktriangledown_{v\in(V,E)} P_v \leq^1_1 [(V,E)] \oplus \bigoplus_{v\in V} P_v$. To see $\widehat{\mathrm{Deg}}\left([(V,E)] \oplus \bigoplus_{v\in V} P_v\right) \leq^1_1 \blacktriangledown_{v\in(V,E)} P_v$, we inductively define a partial computable function $\mathtt{walk} :\subseteq (V \times \mathbb{N})^{\mathbb{N}} \to V^{\mathbb{N}}$ as follows. Set $\mathtt{walk}(\langle\rangle) = \langle\rangle$, and fix $\sigma = \sigma^{--\frown}\langle (u,m),(v,n)\rangle \in (V\times\mathbb{N})^{<\mathbb{N}}$. Assume that $\mathtt{walk}(\sigma^-)$ has been already defined. Then, $\mathtt{walk}(\sigma)$ is defined as follows.

$$\mathtt{walk}(\sigma^{--\frown}\langle (u,m),(v,n)\rangle) = \begin{cases} \mathtt{walk}(\sigma^-)^\frown\langle v\rangle & \text{if } v \neq u, \\ \mathtt{walk}(\sigma^-) & \text{otherwise.} \end{cases}$$

The notation $\mathtt{walk}$ has already been introduced in Definition 45 with a slightly different definition, but these two notions are essentially equivalent. Therefore, we may use the same notation.

For any $f \in \blacktriangledown_{v\in(V,E)} P_v$, if $\mathtt{pr}_v(f)$ is total for some $v \in V$, then the desired condition follows as in (1). Otherwise, $\mathtt{mc}(f) = \infty$, i.e., there are infinitely many $n \in \mathbb{N}$ such that $(f(n+1))_0 \neq (f(n))_0$. In this case, $\mathtt{walk}(f) = \bigcup_{s\in\mathbb{N}} \mathtt{walk}(f \upharpoonright s)$ is an infinite path through the graph (V,E). In other words, the condition $f \in \blacktriangledown_{v\in(V,E)} P_v$ ensures that $\mathtt{pr}_v(f)$ is total and belongs to P_v for some $v \in V$, or otherwise $\mathtt{walk}(f)$ is total and belongs to $[(V,E)]$. Consequently, $f \in \widehat{\mathrm{Deg}}\left([(V,E)] \oplus \bigoplus_{v\in V} P_v\right)$, since $\mathtt{pr}_v$ and $\mathtt{walk}$ are partial computable. □

Proposition 68. *Let P, P_0, P_1, P_v, for $v \in V$, be Π^0_1 subsets of $\mathbb{N}^{\mathbb{N}}$, uniformly.*

1. $\bigtriangledown_{v\in(T,E(T))} P_v = \blacktriangledown_{v\in(T,E(T))} P_v$ *for any well-founded tree* $T \subseteq \mathbb{N}^{<\mathbb{N}}$.
2. $P_0 \oplus P_1 \equiv^1_1 \bigtriangledown_{v\in(V_1,E_1)} P_v \equiv^1_1 \blacktriangledown_{v\in(V_1,E_1)} P_v$, *where* $V_1 = \{\varepsilon, 0, 1\}$, $E_1 = \{(\varepsilon,0),(\varepsilon,1)\}$, *and* $P_\varepsilon = \emptyset$.
3. $P_0 \triangledown P_1 \equiv^1_1 \bigtriangledown_{v\in(V_2,E_2)} P_v \equiv^1_1 \blacktriangledown_{v\in(V_1,E_1)} P_v$, *where* $V_2 = \{\varepsilon,0,1,01,10\}$, $E_2 = \{(\varepsilon,0),(\varepsilon,1),(0,01),(1,10)\}$, $P_\varepsilon = \emptyset$, $P_{01} = P_1$, *and* $P_{10} = P_0$.
4. $P^{a+} \equiv^1_1 \bigtriangledown_{v\in(T_a,E(T_a))} P$ *for every* $a \in O$, *where recall the definition of* P^{a+} *and* T_a *in Definition 61 and the notation below Proposition 63.*
5. $[\![P_0 \vee P_1]\!]^2_{\mathsf{CL}} \equiv^1_1 \bigtriangledown_{v\in(\{0,1\},\{0,1\}^2)} P_v$.
6. $[\![P_0 \vee P_1]\!]^3_{\mathsf{LCM}} \equiv^1_1 \bigtriangledown_{v\in(\mathbb{N},S)} P_v$, *where* $S = \{(n,n+1) : n \in \mathbb{N}\}$; $P_{2n} = P_0$ *and* $P_{2n+1} = P_1$ *for any* $n \in \mathbb{N}$.
7. $[\![P \vee P]\!]^3_{\mathsf{LCM}} \equiv^1_1 [\![\bigvee_{n\in\mathbb{N}} P]\!]_{\mathsf{LCM}} \equiv^1_1 \bigtriangledown_{v\in(\mathbb{N},S)} P$.
8. $\widehat{\mathrm{Deg}}\left(\bigoplus_{v\in\mathbb{N}} P_v\right) \equiv^1_1 [\![\bigvee_{v\in\mathbb{N}} P_v]\!]_{\mathsf{CL}} \equiv^1_1 \bigtriangledown_{v\in(\mathbb{N},\mathbb{N}^2)} P_v \equiv^1_1 [\bigtriangledown_\infty]_{v\in\mathbb{N}} P_v$.

Proof. (1) By Definition, $\bigtriangledown_v P_v \subseteq \blacktriangledown_v P_v$. On the other hand, any $f \in \blacktriangledown_{v\in(T,E(T))} P_v$ can pass at most finitely many vertices since $(T,E(T))$ has no infinite path. In other words, the set $\{(f(n))_0 : n \in \mathbb{N}\}$ is finite. By Pigeon Hole Principle, there is a vertex $v \in T$ such that $(f(n))_0 = v$ occurs for infinitely many $n \in \mathbb{N}$. Then, $\mathtt{pr}_v(f)$ must be infinite. Therefore, $\mathtt{pr}_v(f) \in [T_{P_v}] = P_v$ since $f \in \mathrm{Con}(T_{P_v})_{v\in V}$. Hence, $f \in \bigtriangledown_{v\in(T,E(T))} P_v$.

(2) The condition $\bigtriangledown_{v\in(V_1,E_1)} P_v \leq^1_1 P_0 \oplus P_1$ follows from Proposition 67 (1). For any $f \in \bigtriangledown_{v\in(V_1,E_1)} P_v$, there is $i < 2$ such that $(f(n))_0 = i$ for any $n \in \mathbb{N}$. Thus, $i^\frown f \in P_0 \oplus P_1$.

The $(1,1)$-equivalence of $\bigtriangledown_{v\in(V_1,E_1)} P_v$ and $\blacktriangledown_{v\in(V_1,E_1)}P_v$ follows from the item (1) since (V_1,E_1) is finite.

(3) Clearly, $P_0\triangledown_2 P_1 \subseteq \bigtriangledown_{v\in(V_2,E_2)} P_v$. Thus, by Proposition 54 (2), $P_0\triangledown P_1 \geq^1_1 \bigtriangledown_{v\in(V_2,E_2)} P_v$. For $f\in \bigtriangledown_{v\in(V_2,E_2)}P_v$, if $|(f(0))_0|=1$ then $\Phi(f)=f\in P_0\triangledown_2P_1$. If $|(f(0))_0|=2$, say $(f(0))_0=\langle i,j\rangle$, then $\Phi(f)=\mathtt{write}(j,\mathtt{pr}_j(f))\in P_0\triangledown_1P_1$. Hence, $P_0\triangledown P_1\leq^1_1 \bigtriangledown_{v\in(V_2,E_2)} P_v$ via the computable function Φ. The $(1,1)$-equivalence of $\bigtriangledown_{v\in(V_1,E_1)} P_v$ and $\blacktriangledown_{v\in(V_1,E_1)}P_v$ follows from the item (1) since (V_1,E_1) is finite.

(4) If σ is extendible to an element of $\bigtriangledown_{v\in(T_a,E(T_a))} P$, there is a unique $\kappa\in T_a$ such that σ can be represented as $\bigsqcap_{i\leq|\kappa|}\mathtt{write}(\kappa\upharpoonright i,\mathtt{cut}(\sigma;i))$ for some sequence $\mathtt{cut}(\sigma)\in(T_P)^{|\kappa|}$. Conversely, if σ is extendible to an element of P^{a+}, there is a unique $\kappa\in T_a$ such that σ can be represented as $(\bigsqcap_{i<|\kappa|-1}\kappa^*(i)^\frown\mathtt{cut}(\sigma;i)^\frown\sharp)^\frown\kappa^*(|\kappa|-1)$ for some sequence $\mathtt{cut}(\sigma)\in(T_P)^{|\kappa|}$, where $\kappa^*(i)$ indicates the location of $\kappa(i)$ in the tree T_P. The procedures to interchange these cuts are the desired $(1,1)$-reductions.

(5) It is easy to see that $\bigtriangledown_{v\in(\{0,1\},\{0,1\}^2)} P_v = P_0\triangledown_\infty P_1$. Moreover, $[\![P_0\vee P_1]\!]^2_{\mathsf{CL}} \equiv^1_1 P_0\triangledown_\infty P_1$ by Proposition 50.

(6) For each $\sigma=\tau^\frown\langle(i,m),(j,n)\rangle\in(\mathbb{N}\times\mathbb{N})^{<\mathbb{N}}$, we inductively define a computable function $\Xi(\sigma)$ as follows. If $i=j$, then we set $\Xi(\sigma)=\Xi(\tau^\frown\langle(i,m)\rangle)^\frown\langle n\rangle$. Otherwise, we set $\Xi(\sigma)=\Xi(\tau^\frown\langle(i,m)\rangle)^\frown\langle\sharp,j,n\rangle$. Then, $[\![P_0\vee P_1]\!]^3_{\mathsf{LCM}}\leq^1_1 \bigtriangledown_{v\in(\mathbb{N},S)}P_v$ is witnessed by Ξ. Conversely, to see $\bigtriangledown_{v\in(\mathbb{N},S)}P_v\leq^1_1[\![P_0\vee P_1]\!]^3_{\mathsf{LCM}}$, we again inductively define another computable function $\Xi^*(\sigma)$, for each $\sigma\in(\mathbb{N}\cup\{\sharp\})$. Set $\Xi^*(\langle\rangle)=\langle\rangle$, fix $\sigma=\sigma^{--\frown}\langle j,k\rangle\in(\mathbb{N}\cup\{\sharp\})^{<\mathbb{N}}$, and assume that $\Xi^*(\sigma^-)$ has been already defined. For $w\geq v{+}2$, we consider the instruction $\mathtt{move}(v,w)=\langle(v{+}1,0),(v{+}2,0),\dots,(w{-}2,0),(w{-}1,0)\rangle\in(V{\times}\mathbb{N})^{w-v-1}$ to move from the tape Λ_v to the tape Λ_w in the dynamic tape model. If $w<v{+}2$, then we assume that $\mathtt{move}(v,w)$ is the empty string. Put $p(\sigma)=2{\cdot}\mathtt{count}(\sigma){+}\mathtt{tail}(\sigma;0)$, where recall that $\mathtt{count}(\sigma)=\#\{n<|\sigma|:\sigma(n)=\sharp\}$. If $j\neq\sharp$ and $k\neq\sharp$, then we define $\Xi^*(\sigma)=\Xi^*(\sigma^-)^\frown\mathtt{move}(p(\sigma^*),p(\sigma))^\frown\langle(p(\sigma),k)\rangle$, where σ^* is the last string $\Xi^*(\sigma^*)\supsetneq\Xi^*((\sigma^*)^-)$. Otherwise, we set $\Xi^*(\sigma)=\Xi^*(\sigma^-)$. Then, we have $\langle(\Xi^*(f;n))_0,(\Xi^*(f;n+1))_0\rangle\in\overline{S}$ for any $f\in[\![P_0\vee P_1]\!]^3_{\mathsf{LCM}}$. It is easy to verify that $\Xi^*(f)\in\bigtriangledown_{v\in(\mathbb{N},S)}P_v$.

(7) The $(1,1)$-equivalence of $[\![P\vee P]\!]^3_{\mathsf{LCM}}$ and $[\![\bigvee_{n\in\mathbb{N}}P]\!]_{\mathsf{LCM}}$ follows from Proposition 57 (2). Thus, the desired condition follows from (5).

(8) Clearly, $[\![\bigvee_{v\in\mathbb{N}}P_v]\!]_{\mathsf{CL}}\cap\mathrm{Con}(T_{P_v})_{v\in\mathbb{N}}=\bigtriangledown_{v\in(\mathbb{N},\mathbb{N}^2)}P_v$. Thus, the equivalence $[\![\bigvee_{v\in\mathbb{N}}P_v]\!]_{\mathsf{CL}}\equiv^1_1\bigtriangledown_{v\in(\mathbb{N},\mathbb{N}^2)}P_v\equiv^1_1[\bigtriangledown_\infty]_{v\in\mathbb{N}}P_v$ follows from Proposition 50 and 58. $\widehat{\mathrm{Deg}}(\bigcup_{v\in\mathbb{N}}P_v)\leq^1_1\bigtriangledown_{v\in(\mathbb{N},\mathbb{N}^2)}P_v$ follows from Proposition 67 (1). We may assume that $\Phi_e(\langle\rangle)=\langle\rangle$ for each index $e\in\mathbb{N}$. We inductively define a computable function Γ witnessing $\bigtriangledown_{v\in(\mathbb{N},\mathbb{N}^2)}P_v\leq^1_1\widehat{\mathrm{Deg}}(\bigcup_{v\in\mathbb{N}}P_v)$. For each $\sigma\in\mathbb{N}^{<\mathbb{N}}$ and $e\in\mathbb{N}$, we also inductively define two parameters $\mathtt{act}_e(\sigma)\in\mathbb{N}$ and $\mathtt{rq}_e(\sigma)\in\mathbb{N}\cup\{-1\}$. Here, $\mathtt{act}_e(\sigma)$ will represent the last stage at which the e-th strategy acts along σ, and $\mathtt{rq}_e(\sigma)\geq 0$ will indicate that the e-th strategy *requires attention*. First we set $\mathtt{act}_e(\langle\rangle)=0$ and $\mathtt{rq}_e(\langle\rangle)=-1$ for each $e\in\mathbb{N}$. Inductively we assume that $\Gamma(\sigma^-)$, $\mathtt{act}_e(\sigma^-)$, and $\mathtt{rq}_e(\sigma^-)$ is already defined. Calculate $r=\min\{\mathtt{rq}_e(\sigma^-):e<|\sigma|\ \&\ \mathtt{rq}_e(\sigma^-)>0\}$, and pick the least e such that $\mathtt{rq}_e(\sigma^-)=r$ if such r and e exist. In this case, we say that *e acts*. If there is no such e, we set $\Gamma(\sigma)=\Gamma(\sigma^-)$, $\mathtt{act}_e(\sigma)=\mathtt{act}_e(\sigma^-)$, and $\mathtt{rq}_e(\sigma)=\mathtt{rq}_e(\sigma^-)$. If there is such e, put $\sigma^*=(\Phi_e(\sigma))^{\leftarrow|\Phi_e(\sigma\upharpoonright|\mathtt{act}_e(\sigma)|)|}$, i.e., $\Phi_e(\sigma)=(\Phi_e(\sigma\upharpoonright|\mathtt{act}_e(\sigma)|)|)^\frown\sigma^*$. Then we set $\Gamma(\sigma)=\Gamma(\sigma^-)^\frown\mathtt{write}(e,\sigma^*)$. Then, put $\mathtt{rq}_e(\sigma)=-1$ and $\mathtt{act}_e(\sigma)=|\sigma|$. For each $e^*\in\mathbb{N}\setminus\{e\}$, set $\mathtt{act}_{e^*}(\sigma)=\mathtt{act}_{e^*}(\sigma^-)$. Moreover, if $e^*\leq|\sigma|$, $\mathtt{rq}_{e^*}(\sigma^-)=-1$,

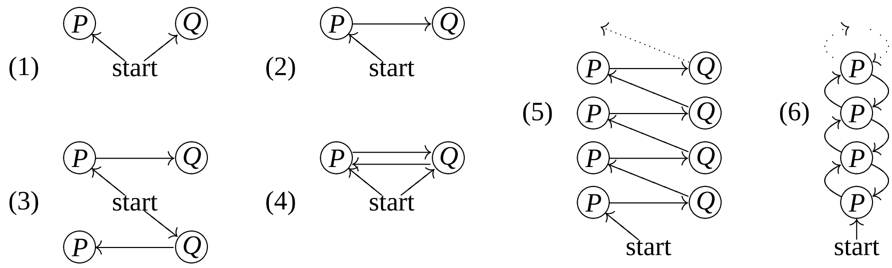

Figure 1: The dynamical representations of disjunction operations: (1) $[\![P \vee Q]\!]_{\mathsf{Int}}$ $(P \oplus Q)$; (2) $P^\frown Q$; (3) $[\![P \vee Q]\!]^2_{\mathsf{LCM}[2]}$ $(P \triangledown Q)$; (4) $[\![P \vee Q]\!]^2_{\mathsf{CL}}$ $(P \triangledown_\infty Q)$; (5) $[\![P \vee Q]\!]^3_{\mathsf{LCM}}$; (6) $\widehat{\mathrm{Deg}}(P)$, the Turing upward closure of P.

and $|\Phi_{e^*}(\sigma \upharpoonright |\mathtt{act}_{e^*}(\sigma)|)| < |\Phi_{e^*}(\sigma)|$, then declare $\mathtt{rq}_{e^*}(\sigma) = |\sigma|$. Otherwise, put $\mathtt{rq}_{e^*}(\sigma) = \mathtt{rq}_{e^*}(\sigma^-)$. Fix $g \in \mathbb{N}^\mathbb{N}$. We claim that $\Phi_e(g)$ act infinitely often whenever $\Phi_e(g)$ is total. Our construction ensures that only finitely many e's require attentions along $g \upharpoonright s$ for each $s \in \mathbb{N}$. Therefore, for $R = \{e \in \mathbb{N} : \mathtt{rq}_e(g \upharpoonright s) > 0\}$, if $e \in R$, then the strategy e acts by stage $s + \#R$, i.e., $\mathtt{act}_e(g \upharpoonright s + \#R) \geq s$. Assume that e act at stage $t \in \mathbb{N}$. Then the algorithm $\Gamma(g \upharpoonright t)$ writes the new information $(g \upharpoonright t)^*$ of $\Phi_e(g)$ on the e-th tape, i.e., $\mathtt{pr}_e(\Gamma(g \upharpoonright t)) = \Phi_e(g \upharpoonright t)$. Thus, eventually, we have $\mathtt{pr}_e(\Gamma(g)) = \Phi_e(g)$. For any $g \in \widehat{\mathrm{Deg}}(\bigcup_{v \in \mathbb{N}} P_v)$, there is an index $e \in \mathbb{N}$ such that $\Phi_e(g) \in P_v$ for some $v \in \mathbb{N}$. Consequently, $\Gamma(g) \in \bigtriangledown_{v \in (\mathbb{N}, \mathbb{N}^2)} P_v$. □

Proposition 69. *Let (V, E) be a computable directed graph, and $\{P_v\}_{v \in V}$ be a computable sequence of Π^0_1 subsets of $2^\mathbb{N}$. Then we have the following.*

1. $\bigtriangledown_{v \in (V,E)} P_v$ *is* Σ^0_3.
2. $\blacktriangledown_{v \in (V,E)} P_v$ *is* Π^0_1.

Proof. Clearly, $\mathrm{Con}(T_{P_v})_{v \in V}$ is Π^0_1. Moreover, the relation $\langle (f(n))_0, (f(n+1))_0 \rangle \in \overline{E}$ is computable, uniformly in $f \in (\mathbb{N} \times \mathbb{N})^\mathbb{N}$ and $n \in \mathbb{N}$. Thus, $\blacktriangledown_{v \in (V,E)} P_v$ is Π^0_1. The relation $\mathtt{pr}_v(f) \in P_v$ is Π^0_2 in $v \in V$ and $f \in \mathbb{N}^\mathbb{N}$, since it is equivalent to the following formula.

$$(\forall n \in \mathbb{N})(\exists m \in \mathbb{N})\ |\mathtt{pr}_v(f \upharpoonright m)| > n \ \&\ \mathtt{pr}_v(f \upharpoonright m) \in T_{P_v}.$$

Therefore, $\bigtriangledown_{v \in (V,E)} P_v$ is Σ^0_3. □

5.5. Infinitary Disjunctions along ill-Founded Trees

To study $(<\omega, \omega)$-degrees, the team-learning proof model of P is expected to be useful. However, the model may be far from Π^0_1 whenever P is Π^0_1. To break out of the dilemma, the following minor modification of consistent dynamic disjunction is helpful. For any tree $T_P \subseteq \mathbb{N}^{<\mathbb{N}}$ and $i \in \mathbb{N}$, we let $T_P{}^\frown\langle i \rangle$ denote the tree $T_P \cup \bigcup_{\rho \in L_P} \rho^\frown\langle i \rangle$, and $T_P{}^\frown T_Q$ denote the tree $T_P \cup \bigcup_{\rho \in L_P} \rho^\frown T_Q$. In other words, $T_P{}^\frown T_Q$ is a corresponding tree of $P^\frown Q$.

Definition 70. Let V be a subtree of $\mathbb{N}^{<\mathbb{N}}$, $\{P_\sigma\}_{\sigma\in V}$ be a computable sequence of Π^0_1 subsets of $2^{\mathbb{N}}$, and $T_\sigma \subseteq 2^{<\mathbb{N}}$ be the corresponding tree of P_σ for each $\sigma \in V$. Then *the concatenation of* $\{P_\sigma\}_{\sigma\in V}$ *along the tree* V is defined as follows.

$$\blacktriangledown_{\sigma\in V} P_\sigma = \left[\bigcup_{\tau\in V}\left(\bigsqcap_{i<|\tau|} T_{\tau\upharpoonright i}{}^\frown\langle\tau(i)\rangle\right)^\frown T_\tau\right].$$

We assume that T_σ is the full binary tree $2^{<\mathbb{N}}$ for each $\sigma \notin V$. Each $\alpha \in 2^{<\mathbb{N}}$ is uniquely represented as

$$\alpha = \rho_0{}^\frown\langle\tau(0)\rangle^\frown\rho_1{}^\frown\langle\tau(1)\rangle^\frown \dots {}^\frown\langle\tau(|\tau|-2)\rangle^\frown\rho_{|\tau|-1}{}^\frown\langle\tau(|\tau|-1)\rangle^\frown\beta,$$

where $\tau \in 2^{<\mathbb{N}}$, $\rho(i) \in T_{\tau\upharpoonright i}$ for each $i < |\tau|$, and $\beta \in T_\tau$. For such τ and β, we set $\mathtt{walk}(\alpha) = \tau$, and $\mathtt{cut}(\alpha) = \langle\rho_0,\rho_1,\dots,\rho_{|\tau|-1},\beta\rangle$. We also define $\mathtt{tail}^{\mathtt{cut}}(\alpha) = \mathtt{cut}(\alpha;|\mathtt{walk}(\alpha)|) = \beta$. Hence, each $\alpha \in 2^{<\mathbb{N}}$ is represented as

$$\alpha = \left(\bigsqcap_{i<|\mathtt{walk}(\alpha)|} \mathtt{cut}(\alpha;i)^\frown\langle\mathtt{walk}(\alpha;i)\rangle\right)^\frown\mathtt{cut}(\alpha;|\mathtt{walk}(\alpha)|).$$

Then the set $\blacktriangledown_{\sigma\in V} P_\sigma$ is characterized as follows.

$$\blacktriangledown_{\sigma\in V} P_\sigma = \left[\left\{\alpha \in 2^{<\mathbb{N}} : \mathtt{walk}(\alpha) \in V \;\&\; (\forall i \le |\mathtt{walk}(\alpha)|)\ \mathtt{cut}(\alpha;i) \in T_{\mathtt{walk}(\alpha)\upharpoonright i}\right\}\right].$$

Remark. The notation walk has already been introduced in Definition 45 and the proof of Proposition 67. The meanings of the symbol walk in Definitions 45 and 70 are formally different, but the ideas behind these definitions are the same. Thus, there is no confusion in using the same notation.

Proposition 71. *Let* V *be a computable subtree of* $2^{<\mathbb{N}}$*, and* $\{P_\sigma\}_{\sigma\in V}$ *be a computable sequence of* Π^0_1 *subsets of* $2^{\mathbb{N}}$*. Then* $\blacktriangledown_{\sigma\in V} P_\sigma$ *is* Π^0_1 *subset of* $2^{\mathbb{N}}$*. Moreover,* $\blacktriangledown_{\sigma\in V} P_\sigma$ *is* $(1,1)$*-equivalent to* $\blacktriangledown_{\sigma\in(V,E(V))} P_\sigma$ *in the sense of Definition 66.*

Proof. Note that walk, cut, and $\mathtt{tail}^{\mathtt{cut}}$ are total computable on $\mathbb{N}^{<\mathbb{N}}$. Therefore, it is Π^0_1. Then,

$$\Phi(\alpha) = \bigsqcap_{i\le|\mathtt{walk}(\alpha)|} \mathtt{write}(\mathtt{walk}(\alpha)\upharpoonright i, \mathtt{cut}(\alpha;i))$$

witnesses $\blacktriangledown_{\sigma\in V} P_\sigma \ge^1_1 \blacktriangledown_{\sigma\in(V,E(V))} P_\sigma$.

Conversely, to see $\blacktriangledown_{\sigma\in V} P_\sigma \le^1_1 \blacktriangledown_{\sigma\in(V,E(V))} P_\sigma$, we inductively define a computable function Ξ. Set $\Phi(\langle\rangle)$. Fix $\alpha = \alpha^{--\frown}\langle(\sigma,m),(\tau,n)\rangle \in (V\times 2)^{<\mathbb{N}}$, and assume that $\Phi(\alpha^-)$ has been already defined. If $\sigma = \tau$, then set $\Xi(\alpha) = \Xi(\alpha^-)^\frown\langle n\rangle$. If $\sigma \ne \tau$, say $\tau = \sigma^\frown\langle i\rangle$, then we first calculate the least leaf $\mathtt{leaf}(\Xi(\alpha^-))$ of T_{P_σ} extending $\Xi(\alpha^-)$. Then we set $\Xi(\alpha) = \mathtt{leaf}(\Xi(\alpha^-)^\frown\langle i,n\rangle$. Note that, for each $\alpha = \alpha^{-\frown}\langle(\tau,n)\rangle \in (V\times 2)^{<\mathbb{N}}$, we have $\mathtt{walk}(\Xi(\alpha)) = (\alpha(|\alpha|-1))_0 = \tau$, and $\mathtt{tail}^{\mathtt{cut}}(\Xi(\alpha)) = \mathtt{pr}_{\mathtt{walk}(\alpha)}(\alpha)$. Thus, Ξ witnesses $\blacktriangledown_{\sigma\in V} P_\sigma \le^1_1 \blacktriangledown_{\sigma\in(V,E(V))} P_\sigma$. □

Definition 72 (Hyperconcatenation)**.** For Π^0_1 sets $P, Q \subseteq 2^{\mathbb{N}}$, the *hyperconcatenation of P and Q* is defined by

$$Q\blacktriangledown P = \blacktriangledown_{\sigma\in T_Q} P_\sigma = \{g \in 2^{\mathbb{N}} : (\forall n)\ \mathtt{walk}(g \upharpoonright n) \in T_Q \ \&\ (\forall n \leq |\mathtt{walk}(g)|)\ \mathtt{cut}(g; n) \in T_P\},$$

where T_Q denotes the corresponding tree for Q, and $P_\sigma = P$ for any $\sigma \in T_Q$.

Remark. For every $g \in Q\blacktriangledown P$, if $\mathtt{walk}(g)$ is total, then $\mathtt{walk}(g) \in Q$, or otherwise $\mathtt{tail}^{\mathtt{cut}}(g) \in P$. Therefore, the hyperconcatenation $Q\blacktriangledown P$ in the sense of Definition 72 can be seen as a consistent conservative version of the hyperconcatenation $[\![Q \vee P]\!]^{\blacktriangledown}_{\Sigma^0_2}$ in the sense of Definition 45.

To see the learnability feature of hyperconcatenation, we introduce new learnability notions.

Definition 73. Let Ψ be a learner.

1. Ψ is *confident* (see also [37]) if $\lim_s \Psi(f \upharpoonright s)$ converges for every $f \in \mathbb{N}^{\mathbb{N}}$.
2. Ψ is *eventually-Popperian* if, for every $f \in \mathbb{N}^{\mathbb{N}}$, $\Phi_{\lim_s \Psi(f\upharpoonright s)}(f)$ is total whenever $\lim_s \Psi(f \upharpoonright s)$ converges.
3. Ψ is *eventually-Lipschitz* if there is a constant $c \in \mathbb{N}$ such that, for every $f \in \mathbb{N}^{\mathbb{N}}$, $|\Phi_{\lim_s \Psi(f\upharpoonright s)}(f \upharpoonright l + c)| \geq l$ for any $l \in \mathbb{N}$, whenever $\lim_s \Psi(f \upharpoonright s)$ converges.

Proposition 74.

1. *For any set $X, Y \subseteq \mathbb{N}^{\mathbb{N}}$, if $X \leq^{<\omega}_{tt,\omega} Y$, then $X \leq^{<\omega}_{\omega} Y$ via a team of eventually-Popperian learners.*
2. *For any Σ^0_2 set $S \subseteq 2^{\mathbb{N}}$ and any set $R \subseteq \mathbb{N}^{\mathbb{N}}$, if $R \leq^1_\omega S$, then it can be witnessed by an eventually-Popperian learner. Moreover, if S is Π^0_1, then it can be witnessed by a confident eventually-Popperian learner.*
3. *For any Π^0_1 set $P \subseteq 2^{\mathbb{N}}$ and any set $Q \subseteq \mathbb{N}^{\mathbb{N}}$, if $P \leq^{<\omega}_1 Q$ then $P \leq^{<\omega}_\omega Q$ by a team of confident learners.*

Proof. (1) Straightforward from the definition.

(2) Fix a computable increasing sequence $\{T_i\}_{i\in\omega}$ of infinite computable trees such that $S = \bigcup_i [T_i]$. By padding, there is a computable function $p : \mathbb{N}^2 \to \mathbb{N}$ such that $\Phi_{p(e,n)}$ corresponds exactly to Φ_e, and $p(e, n + 1) > p(e, n)$ for any index e and n. Assume that $R \leq^1_\omega S$ via a learner Ψ. We need to construct a eventually-Popperian learner Δ witnessing $R \leq^1_\omega S$. At each stage s, we define a value of $\Delta(\sigma)$ for each $\sigma \in 2^s$. For a given $\sigma \in 2^s$, we compute $q(\sigma) = \min(\{i < s : (\forall\tau \in 2^s)\ \tau \supseteq \sigma \ \rightarrow \tau \in T_i\} \cup \{s\})$, and put $\Delta(\sigma) = p(\Psi(\sigma), q(\sigma))$. If $f \notin S$, then $\lim_n q(f \upharpoonright n)$ diverges. Therefore, $\lim_n \Delta(f \upharpoonright n)$ diverges. On the other hand, if $f \in S$, then $\lim_n q(f \upharpoonright n)$ converges to some q. Then $\Phi_{\lim_n \Delta(f\upharpoonright n)}(f) = \Phi_{p(\lim_n \Psi(f\upharpoonright n),q)}(f) = \Phi_{\lim_n \Psi(f\upharpoonright n)}(f) \in R$. Consequently, Δ is eventually-Popperian, and witnesses $R \leq^1_\omega S$. If S is Π^0_1, then we modify Δ by setting $\Delta(\sigma)$ to be a fixed index of a total computable function $g \mapsto 0^\omega$, whenever σ extends a leaf of T_S. Then, Δ is also confident.

(3) If $P \leq^{<\omega}_1 Q$ via n many computable functions $\{\Phi_i\}_{i<n}$, then each learner Ψ_i for each $i < n$ guesses an index of Φ_i. Note that Ψ_i does not change his mind. In particular, Ψ_i is confident. ◻

Proposition 75. *Let V be a computable subtree of $\mathbb{N}^{<\mathbb{N}}$, and $\{P_\sigma\}_{\sigma\in V}$ be a computable collection of Π^0_1 subsets of $\mathbb{N}^\mathbb{N}$. Then $[(V,E)]\oplus\bigoplus_{\sigma\in\mathbb{N}^{<\mathbb{N}}}P_\sigma\leq^{<\omega}_\omega\blacktriangledown_{\sigma\in V}P_\sigma$ by a team of a confident learner and an eventually-Popperian learner.*

Proof. We consider two learners: a learner Ψ_0 who guesses an index of $\alpha\mapsto 0^\frown\mathtt{walk}(\alpha)$, and a learner Ψ_1 who guesses an index of $\alpha\mapsto\langle 1,\mathtt{walk}(\alpha)\rangle^\frown\mathtt{tail}^{\mathtt{cut}}(\alpha)$. As $f\mapsto 0^\frown\mathtt{walk}(f)$ is partial computable, Ψ_0 does not change his mind. In particular, Ψ_0 is confident. On $f\in\mathbb{N}^\mathbb{N}$, the learner Ψ_1 changes his mind whenever $\mathtt{walk}(f\upharpoonright n+1)$ properly extends $\mathtt{walk}(f\upharpoonright n)$. If $\lim_{n\in\mathbb{N}}\Psi_1(f\upharpoonright n)$ converges, then $\mathtt{walk}(f)$ must be partial. Thus, $\mathtt{tail}^{\mathtt{cut}}(f)$ must be total. Then, $\langle 1,\mathtt{walk}(f)\rangle^\frown\mathtt{tail}^{\mathtt{cut}}(f)$ is total. Therefore, Ψ_1 is eventually-Popperian. □

Proposition 76. *Let P_0,P_1,Q_0,Q_1 be Π^0_1 subsets of $2^\mathbb{N}$ such that $Q_0\leq^1_\omega Q_1$ via an eventually Lipschitz learner and that $P_0\leq^1_1 P_1$. Then, $Q_0\blacktriangledown P_0\leq^1_\omega Q_1\blacktriangledown P_1$.*

Proof. For any partial computable function Φ, without loss of generality, we may assume $|\Phi(\sigma)|\leq|\Phi(\sigma^-)|+1$ for any string $\sigma\in\mathbb{N}^{<\mathbb{N}}$. For given indices i and j, we effectively construct a computable function $\Phi_{\mathtt{hyp}(i,j)}$ as follows. Put $\Phi_{\mathtt{hyp}(i,j)}(\langle\rangle)=\langle\rangle$, and assume that $\Phi_{\mathtt{hyp}(i,j)}(\sigma^-)$ has been already defined. Note that, either $|\mathtt{walk}(\sigma)|=|\mathtt{walk}(\sigma^-)|+1$ or $|\mathtt{tail}^{\mathtt{cut}}(\sigma)|=|\mathtt{tail}^{\mathtt{cut}}(\sigma^-)|+1$ is satisfied. Here, the notation $\mathtt{tail}^{\mathtt{cut}}$ is used in referring to decomposing $Q_1\blacktriangledown P_1$. If the former is the case (i.e., $|\mathtt{walk}(\sigma)|=|\mathtt{walk}(\sigma^-)|+1$), then we extend $\mathtt{tail}^{\mathtt{cut}}(\Phi_{\mathtt{hyp}(i,j)}(\sigma^-))$ to $\mathtt{leaf}\circ\mathtt{tail}^{\mathtt{cut}}(\Phi_{\mathtt{hyp}(i,j)}(\sigma^-))$, the least leaf of T_{P_0} extending it, and then, concatenate the bit $\Phi_i(\mathtt{walk}(\sigma);|\mathtt{walk}(\sigma)|-c)$ to it. Formally, for a string $\tau\in\mathbb{N}^{<\mathbb{N}}$ with $\Phi_{\mathtt{hyp}(i,j)}(\sigma^-)=\tau^\frown\mathtt{tail}^{\mathtt{cut}}(\Phi_{\mathtt{hyp}(i,j)}(\sigma^-))$, we define

$$\Phi_{\mathtt{hyp}(i,j)}(\sigma)=\tau^\frown\mathtt{leaf}\circ\mathtt{tail}^{\mathtt{cut}}(\Phi_{\mathtt{hyp}(i,j)}(\sigma^-))^\frown\langle\Phi_i(\mathtt{walk}(\sigma);|\mathtt{walk}(\sigma)|-c)\rangle.$$

Here, we fix some string $\rho\in T_{Q_0}$ of length c, and we set $\Phi_i(\sigma;k-c)=\sigma(k)$ for each $k<c$. If $\Phi_i(\mathtt{walk}(\sigma);|\mathtt{walk}(\sigma)|-c)$ is undefined, then $\Phi_{\mathtt{hyp}(i,j)}(\tau)$ is undefined for any $\tau\supseteq\sigma$. If the former is not the case (then, $|\mathtt{tail}^{\mathtt{cut}}(\sigma)|=|\mathtt{tail}^{\mathtt{cut}}(\sigma^-)|+1$), then we concatenate the new values of $\Phi_j(\mathtt{tail}^{\mathtt{cut}}(\sigma))$ to $\Phi_{\mathtt{hyp}(i,j)}(\sigma^-)$ if it belongs to T_{P_0}. Formally, if $\Phi_j(\mathtt{tail}^{\mathtt{cut}}(\sigma^-))\subsetneq\Phi_j(\mathtt{tail}^{\mathtt{cut}}(\sigma))\in T_{P_0}$, say $\Phi_j(\mathtt{tail}^{\mathtt{cut}}(\sigma))=\Phi_j(\mathtt{tail}^{\mathtt{cut}}(\sigma^-))^\frown\rho$, then we define $\Phi_{\mathtt{hyp}(i,j)}(\sigma)=\Phi_{\mathtt{hyp}(i,j)}(\sigma^-)^\frown\rho$. Otherwise, we set $\Phi_{\mathtt{hyp}(i,j)}(\sigma)=\Phi_{\mathtt{hyp}(i,j)}(\sigma^-)$.

Now assume that $P_0\leq^1_\omega P_1$ via a computable function Φ_e, and $Q_0\leq^\omega Q_1$ via an eventually Lipschitz learner Ψ with a constant c. We construct a learner Δ witnessing $Q_0\blacktriangledown P_0\leq^1_\omega Q_1\blacktriangledown P_1$. At first the learner Δ guesses the index $\Delta(\langle\rangle)=\mathtt{hyp}(\Psi(\langle\rangle),e)$. Fix $\sigma\in\mathbb{N}^{<\mathbb{N}}$, and assume that $\Delta(\sigma^-)$ has been already defined. If $\Psi(\mathtt{walk}(\sigma))\neq\Psi(\mathtt{walk}(\sigma^-))$, then Δ also changes his mind as $\Delta(\langle\rangle)=\mathtt{hyp}(\Psi(\mathtt{walk}(\sigma)),e)$. Assume not. In the case $|\mathtt{walk}(\sigma)|>|\mathtt{walk}(\sigma^-)|$, if either $|\mathtt{walk}(\sigma)|<c$ or $\mathtt{walk}(\sigma)\notin T^{ext}_{Q_1}$ is witnessed, the learner Δ changes his mind (this situation occurs only finitely often). Otherwise, the learner Δ keeps his previous guess, i.e., $\Delta(\sigma)=\Delta(\sigma^-)$. In this way, it is not hard to see that we may construct a learner Δ witnessing $Q_0\blacktriangledown P_0\leq^1_\omega Q_1\blacktriangledown P_1$. □

5.6. Nested Infinitary Disjunctions along ill-Founded Trees

In Part II, we employ finite iterations of the hyperconcatenation $\blacktriangledown$ to show that some (local) degree structures are not Brouwerian. Beyond this, it is important to see that

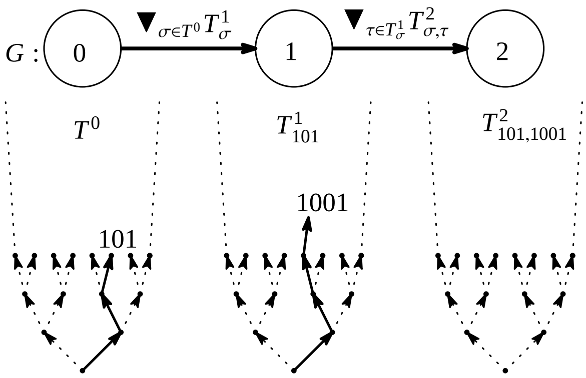

Figure 2: An example nested tape model when G is a linear order of length 3: $\langle 012\rangle$ is written on $\Lambda_\Box$; $\langle 101\rangle$ is written on Λ^0; $\langle 1001\rangle$ is written on Λ^1_{101}; then $\Lambda_\Box$, Λ^0, Λ^1_{101}, and $\Lambda^2_{101,1001}$ are available.

one can iterate the hyperconcatenation $\blacktriangledown$ along any directed graph (V,E), though the iteration of $\blacktriangledown$ does not represented by our previous dynamic proof model. Therefore, we introduce a new model called *the nested disjunction model*.

The nested tape model: As an example, first we consider the nested disjunction $T^* = \blacktriangledown_{\sigma\in T^0}\blacktriangledown_{\tau\in T^1_\sigma}[T^2_{\sigma,\tau}]$ along the graph $G = (\{0,1,2\},\{(0,1),(1,2)\})$ with the initial vertex $\varepsilon = 0$, where $T = \{T^0\}\cup\{T^1_\sigma\}_{\sigma\in\mathbb{N}^{<\mathbb{N}}}\cup\{T^2_{\sigma,\tau}\}_{\langle\sigma,\tau\rangle\in(\mathbb{N}^{<\mathbb{N}})^2}$ is a given collection of subtrees of $\mathbb{N}^{<\mathbb{N}}$. The nested tape model for T^* consists of a collection $\{\Lambda_\Box\}\cup\{\Lambda^0\}\cup\{\Lambda^1_\sigma\}_{\sigma\in\mathbb{N}^{<\mathbb{N}}}\cup\{\Lambda^2_{\sigma,\tau}\}_{\langle\sigma,\tau\rangle\in(\mathbb{N}^{<\mathbb{N}})^2}$ of infinite tapes.

Generally, a *nested system* (G,T,Λ) consists of a graph $G = (V,E)$ with the initial vertex ε, a collection $T = \{T^v_\sigma\}_{v\in V,\sigma\in(\mathbb{N}^{<\mathbb{N}})^{<\mathbb{N}}}$ of (ill-founded) trees, and a collection $\Lambda = \{\Lambda_\Box\}\cup\{\Lambda^v_\sigma\}_{v\in V,\sigma\in(\mathbb{N}^{<\mathbb{N}})^{<\mathbb{N}}}$ of infinite tapes. A verifier Ψ is only allowed to write a letter on tapes which are *available*. Assume that a word $\mathtt{pr}[v,\sigma]$ is written on Λ^v_σ for each $v\in V$ and $\sigma\in(\mathbb{N}^{<\mathbb{N}})^{<\mathbb{N}}$. Then, the availability conditions are given as follows.

- $\Lambda_\Box$ and $\Lambda^\varepsilon_{\langle\rangle}$ are available at each stage.
- If a finite word $v = \langle v[0], v[1],\dots,v[l]\rangle$ is written on the tape $\Lambda_\Box$, then the following tapes are available.

$$\Lambda^{v[1]}_{\mathtt{pr}[v[0],\langle\rangle]},\Lambda^{v[2]}_{\mathtt{pr}[v[1],\mathtt{pr}[v[0],\langle\rangle]]},\dots,\Lambda^{v[i]}_{\mathtt{pr}[v[i-1],\mathtt{pr}[v[i-2],\dots,\mathtt{pr}[v[1],\mathtt{pr}[v[0],\langle\rangle]]]]}.$$

Here, on the tape $\Lambda_\Box$, the verifier Ψ is only allowed to write a path starting from the initial vertex ε within the graph $G = (V,E)$.

Example 77. On the nested tape model for T^*, let $\alpha\in((I\cup\{\Box\})\times\mathbb{N})^{<\mathbb{N}}$ be the record of a proof process of Ψ by some stage, i.e., $\mathtt{pr}_\Box(\alpha)$ and $\mathtt{pr}_{(v,\sigma)}(\alpha)$, for each $(v,\sigma)\in I^{<\mathbb{N}}$, represent the words written on $\Lambda_\Box$ and Λ^v_σ, respectively. Here, I denotes $V\times(\mathbb{N}^{<\mathbb{N}})^{<\mathbb{N}}$. If the letter 1 representing the vertex $1\in V$ has been written on $\Lambda_\Box$ (i.e., $\mathtt{pr}_\Box(\alpha)\supseteq\langle 01\rangle$), then the three tapes $\Lambda_\Box$, Λ^0, and Λ^1_p are available, where $p = \mathtt{pr}_0(\alpha)$.

The verifier Ψ *succeeds* if he eventually writes a correct solution on some tape from Λ (i.e., some solution $f \in [T^v_\sigma]$ is eventually written on Λ^v_σ for some $(v,\sigma) \in V \times (\mathbb{N}^{<\mathbb{N}})^{<\mathbb{N}}$, or otherwise, some infinite path though G is written on $\Lambda_\Box$). For each $u, v \in V$ and $(v,\sigma) \in V \times (\mathbb{N}^{<\mathbb{N}})^{<\mathbb{N}}$, the tuple $\langle T^v_\sigma, \Lambda^v_\sigma, T^u_{\sigma,\tau}, \Lambda^u_{\sigma,\tau}\rangle_{\tau \in T^v_\sigma}$ is called *the* (σ, v, u)*-component of* (G, T, Λ). The (σ, v, u)-component of our nested system consists of an infinite disjunction along an ill-founded tree, $\blacktriangledown_{\tau \in T^v_\sigma}[T^u_{\sigma,\tau}]$. In other words, on the (σ, v, u)-component of the system (I, Λ, T, G), the set Λ^v_σ plays the role of the declaration $\Box$, and $\Lambda^u_{\sigma,\tau}$ plays the role of the working tape for each $\tau \in T^v_\sigma$, as in the dynamic tape model.

Definition 78. Fix a directed graph $G = (V, E)$, and we denotes $V \times (\mathbb{N}^{<\mathbb{N}})^{<\mathbb{N}}$ by I. Assume that a collection $\{T_{(v,\sigma)}\}_{(v,\sigma)\in I}$ of subtrees of $\mathbb{N}^{<\mathbb{N}}$ are given. For $\alpha \in ((I \cup \{\Box\}) \times \mathbb{N})^{<\mathbb{N}}$, we inductively define *the n-th available index along* α, $p(\alpha, n) \in I$, for each $n \leq |\mathtt{pr}_\Box(\alpha)|$, as follows.

$$p(\alpha, 0) = (\varepsilon, \langle\rangle), \quad p(\alpha, i+1) = (\mathtt{pr}_\Box(\alpha)(i), (p(\alpha, i))_1{}^\frown\langle \mathtt{pr}_{p(\alpha,i)}(\alpha)\rangle).$$

Then we define the set of all indices of *available tapes along* α by $A(\alpha) = \{p(\alpha, n) : n \leq |\mathtt{pr}_\Box(\alpha)|\}$. The set $S(\alpha)$ of *successors of* α is defined as follows:

$$S(\alpha) = \{(p, n) \in (I \cup \{\Box\}) \times \mathbb{N} : p \in A(\alpha) \;\&\; \mathtt{pr}_p(\alpha)^\frown n \in T_p\} \\ \cup \{(\Box, v) : (\mathtt{pr}_\Box(\alpha)(|\mathtt{pr}_\Box(\alpha)| - 1), v) \in E\}.$$

Then *the nested infinitary disjunction* $\mathbf{W}_{\sigma\in I}[T_\sigma] \subseteq ((I \cup \{\Box\}) \times \mathbb{N})^{\mathbb{N}}$ *of* $\{T^v_\sigma\}_{(v,\sigma)\in I}$ is defined by

$$\mathbf{W}_{\sigma\in I}[T_\sigma] = \{f \in ((I \cup \{\Box\}) \times \mathbb{N})^{\mathbb{N}} : (\forall n \in \mathbb{N})\ f(n) \in S(f \upharpoonright n)\}.$$

We can also define $\mathbb{W}_{\sigma\in I}[T_\sigma] = \{f \in \mathbf{W}_{\sigma\in I}[T_\sigma] : |\mathtt{pr}_\Box(f)| < \infty\}$.

Proposition 79. *Assume that* $G = (V, E)$ *is a computable directed graph, and* $\{T_\sigma\}_{\sigma\in I}$ *is a computable collection of computable subtrees of* $\mathbb{N}^{<\mathbb{N}}$*, where* $I = V \times (\mathbb{N}^{<\mathbb{N}})^{<\mathbb{N}}$*. Then,* $\mathbf{W}_{\sigma\in I}[T_\sigma]$ *is* Π^0_1*. Moreover, if* G *and* T_σ *are subtrees of* $2^{<\mathbb{N}}$ *for each* $\sigma \in I$*, then* $\mathbf{W}_{\sigma\in I}[T_\sigma]$ *is* $(1,1)$*-equivalent to a* Π^0_1 *subset of* $2^{\mathbb{N}}$.

Proof. Note that $\alpha \mapsto A(\alpha)$ is computable. Therefore, $\alpha \mapsto S(\alpha)$ is also computable. Thus, $\mathbf{W}_{\sigma\in I}[T_\sigma]$ is Π^0_1.

Assume that $G = (V, E(V))$ and T_σ are subtrees of $2^{<\mathbb{N}}$ for each $\sigma \in I$. Fix new symbols $+, -$ which does not belong to $\mathbb{N}$. To construct a Π^0_1 subset of $(\{+,-\} \cup 2)^{\mathbb{N}}$ which is $(1,1)$-equivalent to $\mathbf{W}_{\sigma\in I}[T_\sigma]$, we inductively define a computable function $\mathrm{head} : (\{+,-\} \cup 2)^{<\mathbb{N}} \to \mathbb{Z}$. Fix $\alpha = \alpha^{-\frown}\langle w\rangle \in (\{+,-\} \cup 2)^{<\mathbb{N}}$. Put $\mathrm{head}(\langle\rangle) = 0$, Put $\mathrm{head}(\alpha) = \mathrm{head}(\alpha^-) + 1$ if $w = +$; put $\mathrm{head}(\alpha) = \mathrm{head}(\alpha^-)$ if $w \notin \{+.-\}$; and put $\mathrm{head}(\alpha) = \mathrm{head}(\alpha^-) - 1$ if $w = -$. If $\alpha = \alpha^{--\frown}\langle +, +\rangle$ and $\mathrm{head}(\alpha) = \max\{\mathrm{head}(\beta) : \beta \subsetneq \alpha\} + 2$, or if $\mathrm{head}(\alpha) = -1$, then we say that α is *overflowing*. If α has an overflowing initial segment $\beta \subseteq \alpha$, then we also say that α is overflowing. Let Rule denote the set of all non-overflowing strings $\alpha \in (\{+,-\} \cup 2)^{<\mathbb{N}}$ which has neither $\langle +, -\rangle$ nor $\langle -, +\rangle$ as substrings. Note that Rule is computable.

We now inductively define $\tilde{\mathtt{pr}}_\Box$, $\tilde{p}$, and $\tilde{\mathtt{pr}}_\sigma$ for each $\sigma \in V$. Put $\tilde{\mathtt{pr}}_\Box(\langle\rangle)$, and $\tilde{p} = \langle\langle\rangle\rangle$. Fix $\alpha = \alpha^{-\frown}w \in$ Rule. Assume that $\tilde{\mathtt{pr}}_\Box(\alpha^-)$, and $\tilde{p}(\alpha^-)$ have been already defined. If $w \in \{+,-\}$, then $\tilde{\mathtt{pr}}_\square(\alpha) = \tilde{\mathtt{pr}}_\square(\alpha^-)$ and $\tilde{p}(\alpha) = \tilde{p}(\alpha^-)$. Assume $w \notin \{+,-\}$. Then, if $\mathrm{head}(\alpha) > \max\{\mathrm{head}(\beta) : \beta \subsetneq \alpha\}$, then we define $\tilde{\mathtt{pr}}_\square(\alpha) = \tilde{\mathtt{pr}}_\square(\alpha^-)^\frown w$. Otherwise, set $\tilde{\mathtt{pr}}_\square(\alpha) = \tilde{\mathtt{pr}}_\square(\alpha^-)$. If $\tilde{\mathtt{pr}}_\square(\alpha) \neq \tilde{\mathtt{pr}}_\square(\alpha^-)$, then $\tilde{p}(\alpha) = \tilde{p}(\alpha^-)^\frown\langle\langle\rangle\rangle$. Otherwise, define $\tilde{p}(\alpha) \in (2^{<\mathbb{N}})^{|V(\alpha)|}$ as follows.

$$(\tilde{p}(\alpha))(n) = \begin{cases} (\tilde{p}(\alpha^-))(n), & \text{if } n < \mathrm{head}(\alpha); \\ (\tilde{p}(\alpha^-))(n)^\frown w, & \text{if } n = \mathrm{head}(\alpha); \\ \langle\rangle, & \text{if } h(\alpha) < n \le |\tilde{\mathtt{pr}}_\square(\alpha)|. \end{cases}$$

Then, for each $\sigma \in V$, we define $\tilde{\mathtt{pr}}_\sigma(\alpha) = (\tilde{p}(\beta))(|\sigma|)$ for the greatest $\beta \subseteq \alpha$ such that $\sigma \subseteq \tilde{p}(\beta)$. Set $\mathrm{Rule}_\nabla = \{f \in (\{+,-\} \cup 2)^{\mathbb{N}} : (\forall n \in \mathbb{N})\ f \upharpoonright n \in \mathrm{Rule}\}$. Note that any $g \in \mathrm{Rule}_\nabla$ has no infinite $\{+,-\}$-sequence; otherwise $g \upharpoonright s$ for some $s \in \mathbb{N}$ is overflowing or has a substring $\langle +,-\rangle$ or $\langle -,+\rangle$, and hence $g \upharpoonright s$ must go against Rule. Then P is defined as follows.

$$P = \{f \in \mathrm{Rule}_\nabla : (\forall n \in \mathbb{N})\ (\tilde{\mathtt{pr}}_\square(f \upharpoonright n) \in V \ \&\ (\forall \sigma \in I)\ \tilde{\mathtt{pr}}_\sigma(f \upharpoonright n) \in T_\sigma)\}.$$

Clearly, P is computably bounded, and Π^0_1. It remains to show that $P \equiv^1_1 \mathbf{W}_{\sigma\in I}[T_\sigma]$. We first inductively define a computable function Φ witnessing $P \geq^1_1 \mathbf{W}_{\sigma\in I}[T_\sigma]$. Set $\Phi(\langle\rangle) = \langle\rangle$, fix $\alpha = \alpha^{-\frown} w \in \mathrm{Rule}$, and assume that $\Phi(\alpha^-)$ has been already defined. If $w \in \{+.-\}$, then set $\Phi(\alpha) = \Phi(\alpha^-)$. Assume $w \notin \{+,-\}$. If $\mathrm{head}(\alpha) > \max\{\mathrm{head}(\beta) : \beta \subsetneq \alpha\}$, then we set $\Phi(\alpha) = \Phi(\alpha^-)^\frown\langle(\square, w)\rangle$. Otherwise, we set $\Phi(\alpha) = \Phi(\alpha^-)^\frown\langle((\tilde{\mathtt{pr}}_\square(\alpha), \tilde{p}(\alpha) \upharpoonright h(\alpha)), w)\rangle$. It is not hard to check $P \geq^1_1 \mathbf{W}_{\sigma\in I}[T_\sigma]$ via Φ.

To prove $P \geq^1_1 \mathbf{W}_{\sigma\in I}[T_\sigma]$, we first define a computable function head^*. Firstly put $\mathrm{head}^*(\langle\rangle) = 0$. Fix $\alpha = \alpha^{-\frown}\langle(\sigma, w)\rangle \in ((I \cup \{\square\}) \cup \mathbb{N})^{<\mathbb{N}}$. If $\sigma = \square$, then we set $\mathrm{head}^*(\alpha) = |\mathtt{pr}_\square(\alpha)|$. If $\sigma \in I$, then we set $\mathrm{head}^*(\alpha) = |(\sigma)_1|$. Set $\Phi(\langle\rangle) = \langle\rangle$, and assume that $\Phi(\alpha^-)$ has already been defined. Put $d = \mathrm{head}^*(\alpha) - \mathrm{head}^*(\alpha^-)$. If $d \geq 0$, then $\Phi(\alpha) = \Phi(\alpha^-)^\frown +^{d\ \frown} w$. If $d < 0$, then $\Phi(\alpha) = \Phi(\alpha^-)^\frown -^{-d\ \frown} w$. It is not hard to check $P \leq^1_1 \mathbf{W}_{\sigma\in I}[T_\sigma]$ via Φ. □

If T^v_σ only depends on $v \in V$, i.e., $T^v_\sigma = T_v$, then the nested system (I, Λ, T, G) can be viewed as the iteration of the hyperconcatenation $\blacktriangledown$ along the graph G. In this case, we write $\mathbf{W}_{v\in(V,E)}P_v$ for this notion.

Proposition 80. *Let (V, E) be a computable directed graph, and $\{P_v\}_{v\in V}$ be a computable collection of Π^0_1 subsets of $\mathbb{N}^{\mathbb{N}}$. Then, $\mathbf{W}_{v\in(V,E)}P_v \leq^1_1 \blacktriangledown_{v\in(V,E)}P_v$.*

Proof. We inductively define a computable function Φ which witnesses the condition $\mathbf{W}_{v\in(V,E)}P_v \leq^1_1 \blacktriangledown_{v\in(V,E)}P_v$. Set $\Phi(\langle\rangle) = \langle\rangle$. Fix $\alpha = \alpha^{--\frown}\langle(u,i),(v,j)\rangle \in (V \times \mathbb{N})^{<\mathbb{N}}$. Assume that $\Phi(\alpha^-)$ has already been defined, and $\Phi(\alpha^-)$ is of the form $\Phi(\alpha^-) = \beta^\frown\langle(\sigma, k)\rangle$ for some $\beta \in ((I \cup \{\square\}) \times \mathbb{N})^{<\mathbb{N}}$, $\sigma \in I \cup \{\square\}$, and $k \in \mathbb{N}$. If $v = u$, then we set $\Phi(\alpha) = \Phi(\alpha^-)^\frown\langle(\sigma, j)\rangle$. If $v \neq u$, then we set $\Phi(\alpha) = \Phi(\alpha^-)^\frown\langle(\square, v), ((v, (\sigma)_1{}^\frown\mathtt{pr}_u(\alpha)), j)\rangle$. Fix $g \in \blacktriangledown_{v\in(V,E)}P_v$. By induction, we can show $\mathtt{pr}_{v[n]}(g \upharpoonright n+1) = \mathtt{pr}_{\sigma[n]}(\Phi(g \upharpoonright n+1))$, where $g(n) = (v[n], j)$ and $\Phi(g \upharpoonright n+1) = \beta^\frown\langle(\sigma[n], j)\rangle$. Then, $(\sigma[n])_1 = (\sigma[n]^-)_1{}^\frown\mathtt{pr}_{\sigma[n]^-}(\Phi(g \upharpoonright n+1))$, by our definition of Φ. Therefore, $\sigma[n]$ is available whenever $\sigma[n]^-$ is available. By induction, $\sigma[n]$ is available at $g \upharpoonright n$, for any $n \in \mathbb{N}$.

Moreover, $\mathtt{pr}_{\sigma[n]}(\Phi(g)) = \mathtt{pr}_{v[n]}(g) \in T_{v[n]} = T_{\sigma[n]}$, and $\mathtt{pr}_{\square}(\Phi(g)) = \mathrm{walk}(g)$. Here $\mathrm{walk}(g)$ is inductively defined as follows. Set $\mathrm{walk}(g \upharpoonright 1) = (g(0))_0$. If $(g(n+1))_0 = (g(n))_0$, then $\mathrm{walk}(g \upharpoonright n+1) = \mathrm{walk}(g \upharpoonright n)$. If $(g(n+1))_0 \neq (g(n))_0$, then $\mathrm{walk}(g \upharpoonright n+1) = \mathrm{walk}(g \upharpoonright n)^\frown (g(n+1))_0$. Note that $\langle \mathrm{walk}(g;n), \mathrm{walk}(\mathrm{g};\mathrm{n}+1)\rangle \in E$ for each $n < |\mathrm{walk}(g)| - 1$. Thus, $\Phi(g;s) \in S(\Phi(g) \upharpoonright s)$ for any $s \in \mathbb{N}$. Consequently, $\Phi(g) \in \mathbf{W}_{v\in(V,E)} P_v$. □

If $G = (V,E)$ is linearly ordered, then we have no choice of the next vertex at each stage. In this case, to simplify our argument, we assume that only $\{\Lambda_\sigma\}_{\langle v,\sigma\rangle\in I}$ is given, i.e., the (v,σ)-th tape Λ^v_σ does not depend on the vertex $v \in V$, and. Moreover, if $T_\sigma = T_\tau$ for any $\sigma,\tau \in I$, then we only require $\{\Lambda_{|\sigma|}\}_{\langle v,\sigma\rangle\in I}$. We will use the simplest depth n nested system. The system (G,T,Λ) is *an $\mathbb{N}^{<n}$-nested system* if $G = (n,S)$ and $T_\sigma = T_\tau$ for any $\sigma,\tau \in I$. This system is equivalent to the n-th iteration of $\blacktriangledown$. Let $P^{\blacktriangledown(0)} = P$, and $P^{\blacktriangledown(n+1)} = P\blacktriangledown P^{\blacktriangledown(n)}$. We also write $\blacktriangledown P$ for $\bigcup_{n\in\mathbb{N}} P^{\blacktriangledown(n)}$.

Proposition 81. *Let $G = (n+2,S)$, where $n+2 = \{m \in \mathbb{N} : m < n+2\}$ and $S = \{(m,m+1) : m \leq n\}$, and $\{P^v_\sigma\}_{\langle v,\sigma\rangle\in I}$ be a computable collection of Π^0_1 subsets of $\mathbb{N}^\mathbb{N}$. Let T^v_σ denote the corresponding tree of P^v_σ for each $\langle v,\sigma\rangle \in I$. Then $\mathbf{W}_{\langle v,\sigma\rangle\in I} P^v_\sigma$ is $(1,1)$-equivalent to the following set:*

$$Q = \blacktriangledown_{\sigma(0)\in T^0_{\langle\rangle}} \left(\blacktriangledown_{\sigma(1)\in T^1_{\sigma(0)}} \left(\dots \left(\blacktriangledown_{\sigma(n)\in T^n_{\sigma(0),\dots,\sigma(n-1)}} P^{n+1}_{\sigma(0),\dots,\sigma(n)} \right) \dots \right) \right).$$

In particular, $\mathbf{W}_{v\in(n,S)} P = P^{\blacktriangledown(n)}$ for any Π^0_1 subset of $\mathbb{N}^\mathbb{N}$.

Proof. Straightforward. □

Remark. We may introduce a transfinite iteration $P^{\blacktriangledown(a)}$ of hyperconcatenation as in Definition 61, or equivalently, as a nested infinitary disjunction $\mathbf{W}_{\sigma\in(T_a,E(T_a))} P$ along the well-founded tree T_a. Recall from Corollary 48 that the hyperconcatenation $\blacktriangledown$ induces $\mathrm{dec}^{<\omega}_{\mathrm{d}}[\Pi^0_2]\mathrm{dec}^{\omega}_{\mathrm{p}}[\Pi^0_1]$. The induced piecewise computability concept becomes the a-indexed version of $\mathrm{dec}^{<\omega}_{\mathrm{d}}[\Pi^0_2]\mathrm{dec}^{\omega}_{\mathrm{p}}[\Pi^0_1]$.

Remark. We may introduce the "*nested nested*" model, the "*nested nested nested*" model, and so on. Let $Q\mathbf{w}P$ be $\mathbf{W}_{v\in(T_Q,E(T_Q))} P_v$, where $P_v = P$ for each $v \in T_Q$. Then, for example, the nested nested model can be introduced as the iteration of $\mathbf{w}$ along any directed graph (V,E). Therefore, *inside the Muchnik degree* of any Π^0_1 set $P \subseteq 2^\mathbb{N}$, one may iterate this procedure as "*nested nested nested … nested nested …*" Actually one may iterate "*nested nested nested … nested nested …*" along any directed graph, for example, along the corresponding tree of P. If we call it a "*hypernested*" model, then, of course, we may introduce models which are "*hypernested hypernested*", and "*hypernested hypernested hypernested*", and so on. By iterating this notion along the corresponding tree of P, we obtain a "*hyperhypernested*" model. Iterating this procedure, of course, we have the iteration of "*hyper*" along the correspoding tree of P.

In Part II, we show that the concatenation $P \mapsto P^\frown P$ *always* decreases the Medvedev degree, and the hyperconcatenation $P \mapsto P\blacktriangledown P$ *always* decreases the $(1,\omega)$-degree on nontrivial Π^0_1 subsets of $2^\mathbb{N}$, while these operations preserve the Muchnik degree. This observation reveals to us that there are a fine structure, a deep hierarchy, and a morass inside each Muchnik degree (or equivalently, each Turing upward closure) of a Π^0_1 subset of $2^\mathbb{N}$.

6. Weihrauch Degrees and Wadge Games

6.1. Weihrauch Degrees and Constructive Principles

6.1.1. Basic Notation

We can also give a characterization of our nonuniformly computable functions in the context of the Weihrauch degrees which is a generalization of the Medvedev degrees. Then, our results could be translated into the results on the Weihrauch degrees. A partial function $P :\subseteq \mathbb{N}^\mathbb{N} \to \mathcal{P}(\mathbb{N}^\mathbb{N})$ is called *a multi-valued function*. Then P is also written as $P : \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$. One can think of each multi-valued function P as a collection $\{P(x)\}_{x\in\mathrm{dom}(P)}$ of mass problems $P(x) \subseteq \mathbb{N}^\mathbb{N}$, or a Π_2*-theorem* $(\forall x \in \mathrm{dom}(P))(\exists y)\ y \in P(x)$.

Definition 82 ([11–14])**.** Let $P :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ and $Q :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ be multi-valued partial functions.

1. A single-valued function $q :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is said to be *a realizer of* Q if $q(x) \in Q(x)$ for any $x \in \mathrm{dom}(Q)$.
2. We say that P *is Weihrauch reducible to* Q (written $P \leq_W Q$) if there are partial computable functions H, K such that $K(x, q \circ H(x)) \in Q(x)$ for any $x \in \mathrm{dom}(P)$ and any realizer q of Q.

Remark. If we think of the values $P(x)$ and $Q(x)$ as *relativized mass problems* P^x and Q^x, then $P \leq_W Q$ can be represented as the existence of partial computable functions $\Phi, \Delta :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ satisfying $\Phi^x : Q^{\Delta(x)} \to P^x$ for any $x \in \mathrm{dom}(Q)$, where Φ^x is the x-computable function mapping $y \in \mathbb{N}^\mathbb{N}$ to $\Phi(x \oplus y)$.

Indeed, Brattka-Gherardi [13] introduced the following embedding of the Medvedev degrees into the Weihrauch degrees. For any subset P of Baire space $\mathbb{N}^\mathbb{N}$, we define $\iota(P) : \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ by $\iota(P)(x) = P$ for any $x \in \mathbb{N}^\mathbb{N}$. Then, the map ι provides an embedding of the Medvedev degrees into the Weihrauch degrees, i.e., $P \leq^1_1 Q$ if and only if $\iota(P) \leq_W \iota(Q)$. See also Higuchi-Pauly [34].

Definition 83 ([11–14, 60, 82])**.** Let $P, Q :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ be partial multi-valued functions.

1. (Pairing) $\langle P, Q\rangle(x) = P(x) \times Q(x)$.
2. (Product) $(P \times Q)(\langle x, y\rangle) = P(x) \times Q(y)$.
3. (Coproduct) $(P \coprod Q)(0, x) = \{0\} \times P(x)$; and $(P \coprod Q)(1, x) = \{1\} \times Q(x)$.
4. (Composition) $(P \circ Q)(x) = \bigcup\{P(y) : y \in Q(x)\}$, where $x \in \mathrm{dom}(P \circ Q)$ if $x \in \mathrm{dom}(Q)$ and $Q(x) \subseteq \mathrm{dom}(P)$.
5. (Parallelization) $\widehat{P}(\langle x_i : i \in \mathbb{N}\rangle) = \prod_{i\in\mathbb{N}} P(x_i)$.

Note that (2), (3) and (5) in Definition 83 are operations on the Weihrauch degrees [12, 13, 60], while neither (1) nor (4) is an operation on the Weihrauch degrees.

Thus, the degrees of difficulty of Π^0_1 sets has also studied under the name of *closed choice* in the context of Weihrauch degrees. Let X be a computable metric space (for definition, see Weihrauch [82]). Then, $\mathcal{A}_-(X)$ denotes the hyperspace of closed subsets of X with the upper Fell representation ψ_- (see [11]). For example, P is a computable point in the hyperspace $\mathcal{A}_-(\mathbb{N}^\mathbb{N})$ (resp. $\mathcal{A}_-(2^\mathbb{N})$) if and only if P is a Π^0_1 subset of Baire space $\mathbb{N}^\mathbb{N}$ (resp. of Cantor space $2^\mathbb{N}$). The closed choice function represents a problem to find an element of a given closed set (i.e., a set Π^0_1 relative to some oracle α).

Definition 84 (Closed Choice [11–14])**.** Let X be a computable metric space. Then, the *closed choice* operation of X is defined as the following partial function.

$$\mathsf{C}_X :\subseteq \mathcal{A}_-(X) \rightrightarrows X, \quad A \mapsto A$$

Here, $\mathrm{dom}(\mathsf{C}_X) = \{A \in \mathcal{A}_-(X) : A \neq \emptyset\}$.

The Medvedev reducibility can be interpreted as a computability of a *constant* multi-valued function.

Definition 85 (Reducibility Problem)**.** Let P and Q be subsets of $\mathbb{N}^\mathbb{N}$. Then, the *reducibility problem of P to Q* is defined as the following constant multi-valued function.

$$P/Q :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}, \quad x \mapsto P, \quad \mathrm{dom}(P/Q) = Q.$$

Clearly, P is Medvedev reducible to Q if and only if P/Q has a computable realizer, that is, P/Q is Weihrauch reducible to the identity $\mathrm{id}_{\mathbb{N}^\mathbb{N}} : x \mapsto x$.

6.1.2. Principles of Omniscience

Definition 86. A formula is *tame* if it is well-formed formula constructed from symbols $\{\top, \bot, \land, \lor, \neg, \forall n, \exists n\}_{n\in\mathbb{N}}$ and one variable symbol $\mathbf{V}(n)$ with a number parameter $n \in \mathbb{N}$. For any tame formula A and $p \in \mathbb{N}^\mathbb{N}$, let $A[\mathbf{V}/p]$ denote the new formula obtained from A by replacing $\mathbf{V}(n)$ with $\top$ if $p(n) = 0$ and $\mathbf{V}(n)$ with $\bot$ if $p(n) \neq 0$. Then, let TameForm denote the class of formulas of the form $A \longrightarrow B$ for some tame formulas A and B.

Example 87. The following formulas are contained in TameForm.

1. Σ^0_1-LEM : $\top \longrightarrow \exists n\mathbf{V}(n) \lor \neg\exists n\mathbf{V}(n)$.
2. Σ^0_2-LEM : $\top \longrightarrow \exists m\forall n\mathbf{V}(\langle m, n\rangle) \lor \neg\exists m\forall n\mathbf{V}(\langle m, n\rangle)$.
3. Σ^0_2-DNE : $\neg\neg\exists m\forall n\mathbf{V}(\langle m, n\rangle) \longrightarrow \exists m\forall n\mathbf{V}(\langle m, n\rangle)$.
4. Σ^0_3-DNE : $\neg\neg\exists k\forall m\exists n\mathbf{V}(\langle k, m, n\rangle) \longrightarrow \exists k\forall m\exists n\mathbf{V}(\langle k, m, n\rangle)$.
5. Σ^0_1-LLPO : $\neg(\exists n\mathbf{V}(\langle 0, n\rangle) \land \exists n\mathbf{V}(\langle 1, n\rangle)) \longrightarrow \neg\exists n\mathbf{V}(\langle 0, n\rangle) \lor \neg\exists n\mathbf{V}(\langle 1, n\rangle)$.
6. Σ^0_2-LLPO : $\neg(\exists m\forall n\mathbf{V}(\langle 0, m, n\rangle) \land \exists m\forall n\mathbf{V}(\langle 1, m, n\rangle)) \longrightarrow \neg\exists m\forall n\mathbf{V}(\langle 0, m, n\rangle) \lor \neg\exists m\forall n\mathbf{V}(\langle 1, m, n\rangle)$.

Remark. The symbols LEM, DNE, LLPO express the *law of excluded middle*, the *double negation elimination*, and the *lessor limited principle of omniscience* (i.e., *de Morgan's law*), respectively.

Definition 88. Given any $A \longrightarrow B \in$ TameForm, we define a partial multivalued function $\mathbb{F}_{A\longrightarrow B} :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ as follows:

$$\mathrm{dom}(\mathbb{F}_{A\longrightarrow B}) = \{p \oplus q \in \mathbb{N}^\mathbb{N} : q \in [\![A[\mathbf{V}/p]]\!]\},$$
$$\mathbb{F}_{A\longrightarrow B}(p \oplus q) = [\![B[\mathbf{V}/p]]\!],$$

where $[\![\cdot]\!] : \mathsf{Form} \to \mathcal{P}(\mathbb{N}^\mathbb{N})$ is a unique Medvedev interpretation in Definition 28 with $[\![\top]\!] = \mathbb{N}^\mathbb{N}$.

One can easily see that either $[\![\neg\varphi]\!] = \mathbb{N}^{\mathbb{N}}$ or $[\![\neg\varphi]\!] = \emptyset$ holds for every arithmetical sentence φ in any Medvedev interpretation. Therefore, for every principle $A \longrightarrow B$ in Example 87, its domain is $\{p \oplus q \in \mathbb{N}^{\mathbb{N}} : [\![A[\mathbf{V}/p]]\!] \neq \emptyset\}$, that is, we need not to use the information on q. This observation immediately implies the following proposition.

Proposition 89. *The induced function $\mathbb{F}_{A\longrightarrow B}$ from a principle $A \longrightarrow B$ in Example 87 is Weihrauch equivalent to the following associated partial multi-valued function $A \longrightarrow B$ on Baire space.*

$$\Sigma^0_1\text{-}\mathsf{LEM} : \mathbb{N}^{\mathbb{N}} \to 2, \qquad \Sigma^0_1\text{-}\mathsf{LEM}(p) = \begin{cases} 0, & \textit{if } (\exists n \in \mathbb{N})\ p(n) = 0, \\ 1, & \textit{otherwise.} \end{cases}$$

$$\Sigma^0_2\text{-}\mathsf{LEM} : \mathbb{N}^{\mathbb{N}} \rightrightarrows 2 \times \mathbb{N}, \qquad \Sigma^0_2\text{-}\mathsf{LEM}(p) \ni \begin{cases} (0, s), & \textit{if } (\forall m \in \mathbb{N})(\exists n > m)\ p(n) = 0, \\ (1, s), & \textit{if } (\forall n > s)\ p(n) \neq 0. \end{cases}$$

$$\Sigma^0_2\text{-}\mathsf{DNE} :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}, \qquad \Sigma^0_2\text{-}\mathsf{DNE}(p) = \{m \in \mathbb{N} : (\forall n > m)\ p(n) \neq 0\}.$$

$$\Sigma^0_3\text{-}\mathsf{DNE} :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}, \qquad \Sigma^0_3\text{-}\mathsf{DNE}(p) = \{k : (\forall m \in \mathbb{N})(\exists n \geq m)\ p(\langle k, n\rangle) = 0\}.$$

$$\Sigma^0_1\text{-}\mathsf{LLPO} :\subseteq (\mathbb{N}^{\mathbb{N}})^2 \rightrightarrows 2, \quad \Sigma^0_1\text{-}\mathsf{LLPO}(p_0, p_1) \ni \begin{cases} 0, & \textit{if } (\forall n \in \mathbb{N})\ p_0(n) = 0, \\ 1, & \textit{if } (\forall n \in \mathbb{N})\ p_1(n) = 0. \end{cases}$$

$$\Sigma^0_2\text{-}\mathsf{LLPO} :\subseteq (\mathbb{N}^{\mathbb{N}})^2 \rightrightarrows 2, \quad \Sigma^0_2\text{-}\mathsf{LLPO}(p_0, p_1) \ni \begin{cases} 0, & \textit{if } (\forall m)(\exists n > m)\ p_0(n) = 0, \\ 1, & \textit{if } (\forall m)(\exists n > m)\ p_1(n) = 0. \end{cases}$$

Here, their domains are given as follows.

$$\mathrm{dom}(\Sigma^0_2\text{-}\mathsf{DNE}) = \{p \in \mathbb{N}^{\mathbb{N}} : (\exists m \in \mathbb{N})(\forall n > m)\ p(n) \neq 0\}.$$
$$\mathrm{dom}(\Sigma^0_3\text{-}\mathsf{DNE}) = \{p \in \mathbb{N}^{\mathbb{N}} : (\exists k \in \mathbb{N})(\forall m \in \mathbb{N})(\exists n \geq m)\ p(\langle k, n\rangle) = 0\}.$$
$$\mathrm{dom}(\Sigma^0_1\text{-}\mathsf{LLPO}) = \{(p_0, p_1) \in (\mathbb{N}^{\mathbb{N}})^2 : (\exists i < 2)(\forall n \in \mathbb{N})\ p_i(n) = 0\}.$$
$$\mathrm{dom}(\Sigma^0_2\text{-}\mathsf{LLPO}) = \{(p_0, p_1) \in (\mathbb{N}^{\mathbb{N}})^2 : (\exists i < 2)(\forall m)(\exists n > m)\ p_i(n) = 0\}.$$

Remark.

1. The single-valued function Σ^0_1-LEM is usually called *the limited principle of omniscience* (LPO). Brattka-de Brecht-Pauly [11] showed that a single-valued partial function $f :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is $(1, \omega)$-computable if and only if f is Weihrauch reducible to the closed choice $\mathsf{C}_{\mathbb{N}}$ for the discrete space $\mathbb{N}$. Here, in their term, the $(1, \omega)$-computability is called *the computability with finitely many mind changes*.
2. Σ^0_2-LLPO is Weihrauch equivalent to the *jump* LLPO′ *of* LLPO in the sense of Brattka-Gherardi-Marcone [14]. They also showed that LLPO′ is Weihrauch equivalent to the Borzano-Weierstrass Theorem BWT_2 for the discrete space $\{0, 1\}$. Brattka-Gherardi-Marcone [14] also pointed out that the n-th jump of LLPO and LPO correspond to Σ^0_{n+1}-LLPO (that is, the lessor limited principle of omniscience for Σ^0_{n+1}-formulas) and Σ^0_{n+1}-LEM (the law of excluded middle for Σ^0_{n+1}-formulas), respectively.
3. The study of arithmetical hierarchy of semiclassical principles such as Σ^0_n-LEM, Σ^0_n-LLPO, and Σ^0_n-DNE was initiated by Akama-Berardi-Hayashi-Kohlenbach [1]. In particular, on the study of the second level of arithmetical hierarchy for

semiclassical principles, see also Berardi [4] and Toftdal [79]. The relationship between the learnability and Σ^0_2-DNE has been also studied by Nakata-Hayashi [57] in the context of a realizability interpretation of limit computable mathematics.

Definition 90 (Unique variant [14])**.** Let $P : X \rightrightarrows Y$ be a multi-valued function. Then Unique$P : X \rightrightarrows Y$ is defined as the restriction of P up to dom(Unique$P$) $= \{x \in \mathrm{dom}(P) : \#P(x) = 1\}$.

Definition 91. We define the partial multi-valued function Δ^0_2-LEM as follows.

$$\Delta^0_2\text{-LEM} :\subseteq \mathbb{N}^2 \times \mathbb{N}^{\mathbb{N}} \to 2, \qquad \Delta^0_2\text{-LEM}(i, j, p) = \begin{cases} 0, & \text{if } p \in \mathrm{Tot}_i, \\ 1, & \text{otherwise.} \end{cases}$$

Here, $\mathrm{dom}(\Delta^0_2\text{-LEM}) = \{(i, j, p) \in \mathbb{N}^2 \times \mathbb{N}^{\mathbb{N}} : \mathrm{Tot}_i = \mathbb{N}^{\mathbb{N}} \setminus \mathrm{Tot}_j\}$, where Tot_e denotes the set of all oracles $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $\Phi_e(\alpha; n)$ converges for all inputs $n \in \mathbb{N}$.

Proposition 92. Δ^0_2-LEM *is Weihrauch reducible to* UniqueΣ^0_2-LLPO.

Proof. To see Δ^0_2-LEM $\leq_W$ UniqueΣ^0_2-LLPO, given $(e_0, e_1, p) \in \mathbb{N}^2 \times \mathbb{N}^{\mathbb{N}}$, define $H(e_0, e_1, p)$ to be a pair (x_0, x_1), where $x_i(s) = 0$ if and only if the computation $\Phi_{e_i,s+1}(p)$ at stage $s+1$ properly extends $\Phi_{e_i,s}(p)$ at the previous stage. Then x_i contains infinitely many 0's if and only if p is contained in Tot_{e_i}. Note that, whenever (e_0, e_1, p) is contained in the domain of Δ^0_2-LEM, $H(e_0, e_1, p)$ is also contained in the domain of UniqueΣ^0_2-LLPO, since $\mathrm{Tot}_{e_0} = \mathbb{N}^{\mathbb{N}} \setminus \mathrm{Tot}_{e_1}$. Therefore, Unique$\Sigma^0_2$-LLPO $\circ H(e_0, e_1, p) = \Delta^0_2$-LEM$(e_0, e_1, p)$. □

Theorem 93. *Let $f :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be a single-valued partial function.*

1. *f is $(1, 2)$-computable if and only if $f \leq_W \Sigma^0_1$-LEM.*
2. *f is $(1, \omega|2)$-computable if and only if $f \leq_W \Delta^0_2$-LEM.*
3. *f is $(1, \omega)$-computable if and only if $f \leq_W \Sigma^0_2$-DNE.*

Proof. (1) Let f be a $(1, 2)$-computable function. By Theorem 26 (1), we have $f \in \mathrm{dec}^2_{\mathrm{d}}[\Pi^0_1]$. Then, there is a Σ^0_1 set $S \subseteq \mathbb{N}^{\mathbb{N}}$ such that $f_0 = f \upharpoonright S$ and $f_1 = f \upharpoonright \mathbb{N}^{\mathbb{N}} \setminus S$ is computable. Put $U = \{p \in \mathbb{N}^{\mathbb{N}} : (\exists n)\ p(n) = 0\}$. Note that Σ^0_1-LEM is the characteristic function $\mathbf{1}_U$ of U. By Σ^0_1 completeness of U, we can find a Wadge reduction (indeed, a computable function) H such that $\mathbf{1}_S = \mathbf{1}_U \circ H$. Put $K(x, i) = f_i(x)$ for every $i < 2$ and $x \in \mathbb{N}^{\mathbb{N}}$. Then, for every $x \in \mathrm{dom}(f)$,

$$K(x, \mathbf{1}_U \circ H(x)) = K(x, \mathbf{1}_S(x)) = \begin{cases} K(x, 0) = f_0(x) & \text{if } x \in S, \\ K(x, 1) = f_1(x) & \text{if } x \notin S. \end{cases}$$

Conversely, we have Σ^0_1-LEM $= \mathbf{1}_U \in \mathrm{dec}^2_{\mathrm{d}}[\Pi^0_1]$ since U is Σ^0_1. This implies that $H \circ \langle id, \mathbf{1}_U \circ H\rangle \in \mathrm{dec}^2_{\mathrm{d}}[\Pi^0_1]$ for every partial computable functions H and K.

(2) Let f be a $(1, \omega|2)$-computable function. By Theorem 26 (2), we have $f \in \mathrm{dec}^2_{\mathrm{d}}[\Delta^0_2]$. Then, there are Π^0_2 sets $P_0, P_1 \subseteq \mathbb{N}^{\mathbb{N}}$ with $P_0 = \mathbb{N}^{\mathbb{N}} \setminus P_1$ such that $f \upharpoonright P_0$ and $f \upharpoonright P_1$ are computable. Then, we can find indices i, j such that $P_0 = \mathrm{Tot}_i$ and

$P_1 = \mathrm{Tot}_j$. Let H be the function sending $p \in \mathbb{N}^\mathbb{N}$ to (i, j, p). Put $K(x, i) = f_i(x)$ for every $i < 2$ and $x \in \mathbb{N}^\mathbb{N}$. It is not hard to see that $K(x, \Delta^0_2\text{-LEM} \circ H(x)) = f(x)$ for every $x \in \mathrm{dom}(f)$.

We show the converse implication. By Proposition 92, we have $f \leq_W \Delta^0_2\text{-LEM} \leq_W \mathsf{Unique}\Sigma^0_2\text{-LLPO}$. Assume that $f \leq_W \mathsf{Unique}\Sigma^0_2\text{-LLPO}$ via partial computable functions $K :\subseteq \mathbb{N}^\mathbb{N} \times 2 \to \mathbb{N}^\mathbb{N}$ and $H :\subseteq \mathbb{N}^\mathbb{N} \to (\mathbb{N}^\mathbb{N})^2$. Let e_i be an index of $\lambda x.K(x, i)$ for each $i < 2$. We first compute $h(\sigma, i) = \#\{n < |H_i(\sigma)| : H_i(\sigma; n) = 0\}$, where $H(\sigma) = \langle H_i(\sigma)\rangle_{i<2}$. Then let $c(\sigma)$ be the least $i < 2$ such that $h(\sigma, k) \leq h(\sigma, i)$ for any $k < 2$. Let us consider a learner $\Psi : \mathbb{N}^{<\mathbb{N}} \to \{e_i\}_{i<m}$ defined by $\Psi(\sigma) = e_{c(\sigma)}$. For any $x \in \mathrm{dom}(f)$, we have $H(x) \in \mathrm{dom}(\mathsf{Unique}\Sigma^0_2\text{-LLPO})$, and then $\lim_n h(x \upharpoonright n, i) = \infty$ for just one $i < 2$. Then, $\lim_n c(x \upharpoonright n)$ also converges to such $i < 2$. Moreover, for any $x \in \mathrm{dom}(f)$, $\mathsf{Unique}\Sigma^0_2\text{-LLPO}(H(x)) = \{i\}$ if and only if $\lim_n h(x \upharpoonright n, i) = \infty$. We fix a realizer U of $\mathsf{Unique}\Sigma^0_2\text{-LLPO}$, i.e., $U(x) \in \mathsf{Unique}\Sigma^0_2\text{-LLPO}(x)$ for any $x \in \mathrm{dom}(\mathsf{Unique}\Sigma^0_2\text{-LLPO})$. Then, $\lim_n c(x \upharpoonright n) = U \circ H(x)$ for any $x \in \mathrm{dom}(f)$ Therefore, the limit $\lim_n \Psi(x \upharpoonright n)$ converges to $e_{U\circ H(x)}$, and $\#\mathtt{indx}_\Psi(x) \leq \#\{e_i : i < 2\} \leq 2$. Thus, $\Phi_{\lim_n \Psi(x\upharpoonright n)}(x) = \Phi_{e_{U\circ H(x)}}(x) = K(x, U \circ H(x)) = f(x)$ for any $x \in \mathrm{dom}(f)$. Hence, f is $(1, \omega|2)$-computable.

(3) Clearly, Σ^0_2-DNE is Weihrauch equivalent to the closed choice $\mathsf{C}_\mathbb{N}$ for discrete space $\mathbb{N}$. Therefore, the desired condition follows from Brattka-Brecht-Pauly [11]. □

In particular, for instance, $P \leq^1_{<\omega} Q$ if and only if $P/Q \leq_W f_0 \circ \cdots \circ f_n$ for some $f_0, \dots, f_n \leq_W \Sigma^0_1\text{-LEM}$. One can apply this idea to any non-constructive principle.

Definition 94. Let $\Theta :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ be a partial multi-valued function. A partial *multi-valued* function $F :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ is Θ-*computable* if F is Weihrauch reducible to Θ. By $\mathfrak{C}_\Theta$, we denote the least class containing all *multi-valued* Θ-computable functions and closed under composition (in the sense of Definition 83 (4)). Then, for subsets P, Q of $\mathbb{N}^\mathbb{N}$, we write $P \leq_\Theta Q$ if $P/Q \leq_W F$ for some $F \in \mathfrak{C}_\Theta$.

Theorem 95. *Let P be a Π^0_2 subset of $\mathbb{N}^\mathbb{N}$, and Q be any subset of $\mathbb{N}^\mathbb{N}$.*

1. *$P \leq^{<\omega}_1 Q$ if and only if $P \leq_{\Sigma^0_2\text{-LLPO}} Q$.*
2. *$P \leq^{\omega}_1 Q$ if and only if $P \leq_{\Sigma^0_3\text{-DNE}} Q$.*

Proof. (1) If $P \leq^{<\omega}_1 Q$ via two algorithms, we have a function $f : Q \to P$ with $f \in \mathrm{dec}^2_\mathrm{d}[\Pi^0_2]$ by Proposition 27 (3). Then, $f_0 = f \upharpoonright Q_0$ and $f_1 = f \upharpoonright \mathbb{N}^\mathbb{N} \setminus Q_0$ are computable for some Π^0_2 set $Q_0 \subseteq \mathbb{N}^\mathbb{N}$. Since f_1 is computable, we can extend the domain of f_0 to a Π^0_2 set Q^+ including $\mathbb{N}^\mathbb{N} \setminus Q_0$. Then $Q_1 = Q^+ \cap f_1^{-1}[P]$ is Π^0_2 since P is Π^0_2 and f_1 is computable. It is easy to see that $Q_0 \cup Q_1$ includes Q. Since Q_0 and Q_1 are Π^0_2, they are (computably) Wadge reducible to the Π^0_2 complete set $U = \{x \in \mathbb{N}^\mathbb{N} : (\exists^\infty n)\ x(n) = 0\}$. That is, for every $i < 2$, there is a computable functions H_i such that $\mathbf{1}_{Q_i} = \mathbf{1}_U \circ H_i$. Let H be a computable function sending $x \in \mathbb{N}^\mathbb{N}$ to the pair $(H_0(x), H_1(x))$, and put $K(x, i) = f_i(x)$. We can easily see that

$$x \in Q_i \leftrightarrow \mathbf{1}_{Q_i}(x) = 1 \leftrightarrow \mathbf{1}_U(H_i(x)) = 1 \leftrightarrow i \in \Sigma^0_2\text{-LLPO}(H(x)).$$

Thus, for every realizer $G :\subseteq \mathbb{N}^\mathbb{N} \to 2$ of Σ^0_2-LLPO, we have $K(x, G \circ H(x)) = f_{G\circ H(x)}(x) \in P$.

Assume that the reducibility problem P/Q is Weihrauch reducible to Σ^0_2-LLPO. Then, there are computable functions $H : \mathbb{N}^\mathbb{N} \to (\mathbb{N}^\mathbb{N})^2$ and $K : \mathbb{N}^\mathbb{N} \times 2 \to \mathbb{N}^\mathbb{N}$ such that $K(x, G \circ H(x)) \in P$ for any realizer G of Σ^0_2-LLPO and any element $x \in Q$. Then $K(x, i) \in P$ for some $i < 2$, since $G \circ H(x) < 2$. Set $\Phi_{e(i)}(x) = K(x, i)$ for each $i < 2$. Then $P \leq^{<\omega}_1 Q$ via $\{\Phi_{e(i)}\}_{i<2}$.

(2) Assume that $P \leq^\omega_1 Q$. It suffices to show that $P/Q \leq_W \Sigma^0_3$-DNE. Note that the condition $\Phi_e(x)$ is total and belongs to P is Π^0_2, uniformly in $e \in \mathbb{N}$ and $x \in \mathbb{N}^\mathbb{N}$. Thus, there is a computable function $H : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ satisfying that $H(x; e, n) = 0$ for infinitely many $n \in \mathbb{N}$ if and only if $\Phi_e(x)$ is total and belongs to P. By our assumption, there is $e \in \mathbb{N}$ such that $H(x; e, n) = 0$ for infinitely many $n \in \mathbb{N}$, for any $x \in Q$. Therefore, $H(x) \in \mathrm{dom}(\Sigma^0_3$-DNE$)$ for any $x \in Q$, and, for any realizer G of Σ^0_3-DNE, $G \circ H(x)$ chooses $e < b$ such that $\Phi_e(x) \in P$. Then, for a computable function $K : \mathbb{N}^\mathbb{N} \times \mathbb{N} \to \mathbb{N}^\mathbb{N}$ mapping (x, e) to $\Phi_e(x)$, we have $K(x, G \circ H(x)) = \Phi_e(x) \in P$.

If $P/Q \leq_W \Sigma^0_3$-DNE, then there are computable functions $H : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ and $K : \mathbb{N}^\mathbb{N} \times \mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that $K(x, G \circ H(x)) \in P$ for any realizer G of Σ^0_3-DNE and any element $x \in Q$. Then $K(x, i) \in P$ for some $i \in \mathbb{N}$, since $G \circ H(x) < m$. Set $\Phi_{e(i)}(x) = K(x, i)$ for each $i \in \mathbb{N}$. Then $P \leq^\omega_1 Q$ via $\{\Phi_{e(i)}\}_{i\in\mathbb{N}}$. □

Theorem 96. *Let P and Q be Π^0_1 subsets of $\mathbb{N}^\mathbb{N}$. Then, $P \leq^{<\omega}_{tt,1} Q$ if and only if $P \leq_{\Sigma^0_1\text{-LLPO}} Q$.*

Proof. We assume that $P \leq^{<\omega}_{tt,1} Q$ via two truth-table functionals f_0 and f_1. Note that $f^{-1}(P)$ is Π^0_1 whenever f is total computable, and P is Π^0_1. Then, for $Q_i = Q \cap \Theta_i^{-1}(P)$, the domain Q is covered by $Q_0 \cup Q_1$. By Π^0_1 completeness of $U = \{x : (\forall n)\ x(n) \neq 0\}$, for every $i < 2$, we have a computable function H_i such that $\mathbf{1}_{Q_i} = \mathbf{1}_U \circ H_i$. As in the proof of Theorem 95 (2), we set $H : x \mapsto (H_0(x), H_1(x))$ and $K : (x, i) \mapsto f_i(x)$. Then, it is not hard to see that the condition $P \leq_{\Sigma^0_1\text{-LLPO}} Q$ is witnessed by H and K

If $P/Q \leq_W \Sigma^0_1$-LLPO, then there are computable functions $H : \mathbb{N}^\mathbb{N} \to (\mathbb{N}^\mathbb{N})^2$ and $K : \mathbb{N}^\mathbb{N} \times 2 \to \mathbb{N}^\mathbb{N}$ such that $K(x, G \circ H(x)) \in P$ for any realizer G of Σ^0_1-LLPO and any element $x \in Q$. Then $K(x, i) \in P$ for some $i < 2$, since $G \circ H(x) < 2$. For $U = \{x : (\forall n)\ x(n) \neq 0\}$, define $D_i = H_i^{-1}[U]$, where $H(x) = (H_0(x), H_1(x))$. The computability of H_i implies that D_i is Π^0_1. Define $f_i : D_i \to \mathbb{N}^\mathbb{N}$ by $f_i(x) = K(x, i)$ on D_i. Since D_i is Π^0_1, f_i has a total computable extension $\Phi_{e(i)}$. Therefore, $P \leq^{<\omega}_{tt,1} Q$ via $\{\Phi_{e(i)}\}_{i<2}$. □

Recall from Remark after Theorem 40 that $\leq_{\Sigma^0_2}$ is the reducibility relation induced by the disjunction operation $[\![\cdot \vee \cdot]\!]_{\Sigma^0_2}$.

Theorem 97. *Let P and Q be any subsets of $\mathbb{N}^\mathbb{N}$. Then, $P \leq_{\Sigma^0_2} Q$ if and only if $P \leq_{\Sigma^0_2\text{-LEM}} Q$.*

Proof. Assume that there are two computable functions $H : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ and $K : \mathbb{N}^\mathbb{N} \times 2 \times \mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that $K(x, G \circ H(x)) \in P$ for any $x \in Q$ and any realizer $G : \mathbb{N}^\mathbb{N} \to 2 \times \mathbb{N}$ of Σ^0_2-LEM. Then the Σ^0_2 sentence $(\exists v)\theta(v, x)$ is given by $(\exists v)(\forall n > v)H(x; n) \neq 0$. We also define $\Delta(x) = K(x, \langle 0, 0\rangle)$, and $\Gamma_v(x) = K(x, \langle 1, v\rangle)$, for any $x \in \mathbb{N}^\mathbb{N}$. Fix $x \in Q$. If $\theta(v, x)$ is true, then there is a realizer G of Σ^0_2-LEM mapping $H(x)$ to $(1, v)$. Therefore,

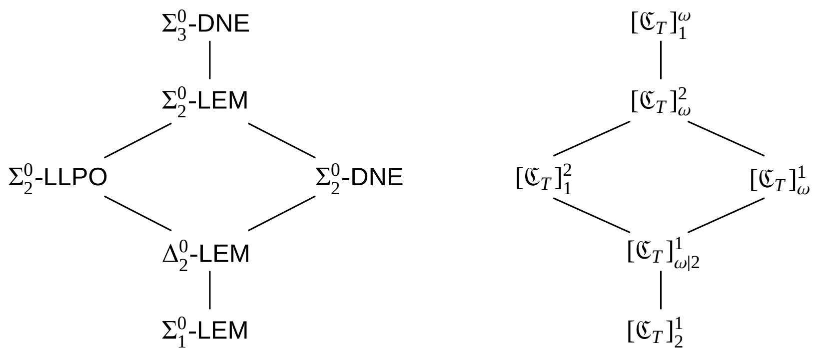

Figure 3: Constructive principles, and nonuniform computability.

$\Gamma_v(x) = K(x, \langle 1, v\rangle) = K(x, G \circ H(x)) \in P$. If $(\forall v)\neg\theta(v, x)$ is true, then there is a realizer G of Σ^0_2-LEM mapping $H(x)$ to $(0, 0)$. Therefore, $\Delta(x) = K(x, \langle 0, 0\rangle) = K(x, G \circ H(x)) \in P$. Hence, by Theorem 46, we obtain $[\![P \vee P]\!]_{\Sigma^0_2} \leq^1_1 Q$.

Conversely, we assume that $[\![P \vee P]\!]_{\Sigma^0_2} \leq^1_1 Q$. Then, there are computable collection Δ, $\{\Gamma_v\}_{v\in}$ of computable functions, and a Σ^0_2 sentence $\exists v\theta(v, x)$, as in Theorem 46. By analyzing the proof of Theorem 46, we may assume that this Σ^0_2 sentence has an additional property that, if $\theta(v, x)$ is true and $v \leq u$, then $\theta(u, x)$ is also true. For any $x \in \mathbb{N}^\mathbb{N}$, put $K(x, \langle 0, n\rangle) = \Delta(x)$ for each $n \in \mathbb{N}$, and $K(x, \langle 1, v\rangle) = \Gamma_v(x)$. From the Σ^0_2 sentence $\exists v\theta(v, x)$, we can easily construct a computable function $H : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ satisfying that $\theta(v, x)$ is true if and only if $H(x; n) \neq 0$ for any $n > v$. Fix $x \in Q$. If $\exists v\theta(v, x)$ is true, then any realizer G of Σ^0_2-LEM maps $H(x)$ to some $(1, v)$ witnessing $\theta(v, x)$. Then, $K(x, G \circ H(x)) = \Gamma_v(x) \in P$. If $\forall v\neg\theta(v, x)$ is true, then any realizer G of Σ^0_2-LEM maps $H(x)$ to $(0, s)$ for some $s \in \mathbb{N}$. Then, $K(x, G \circ H(x)) = \Delta(x) \in P$. □

Corollary 98. *Let P and Q be subsets of $\mathbb{N}^\mathbb{N}$, where P is Π^0_2. Then, $P \leq^{<\omega}_\omega Q$ if and only if $P \leq_{\Sigma^0_2\text{-LEM}} Q$.*

Proof. By Proposition 27 (2) and Theorem 97. □

6.2. Duality between Dynamic Operations and Nonconstructive Principles

We now interpret our results in Section 4 in context of the Weihrauch degrees.

Definition 99 ([14, 48])**.** Let $F, G :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ be any multi-valued functions. Then, $F \star G = \max_{\leq_W}\{F^* \circ G^* : F^* \leq_W F \;\&\; G^* \leq_W G\}$. □

If multi-valued functions $C, D :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ satisfy the condition

$$D \circ E \leq_W F \iff E \leq_W C \star F$$

for any multi-valued functions $E, F :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$, then we may think of D as the *inverse* of C. One could think of our disjunction operators as inverse operators of various constructive principles.

Definition 100. Fix $x \in \mathbb{N}^{\mathbb{N}}$.

1. $\triangledown(x) = \{y \in (\mathbb{N} \cup \{\sharp\})^{\mathbb{N}} : \#\{n \in \mathbb{N} : y(n) = \sharp\} \leq 1 \ \& \ \mathtt{tail}(y) = x\}$.
2. $\triangledown_\omega(x) = \{y \in (2 \times \mathbb{N})^{\mathbb{N}} : (\exists i < 2)\ \mathtt{pr}_i(y) = x \ \& \ \mathtt{mc}(y) < \infty\}$.
3. $\triangledown_\infty(x) = \{y \in (2 \times \mathbb{N})^{\mathbb{N}} : (\exists i < 2)\ \mathtt{pr}_i(y) = x\}$.
4. $\widehat{\deg}_T(x) = \{y \in \mathbb{N}^{\mathbb{N}} : x \leq_T y\}$.

The n-th iteration of $\triangledown$ ($\triangledown_\omega$ and $\triangledown_\infty$) is denoted by $\triangledown^{(n)}$ ($\triangledown_\omega^{(n)}$ and $\triangledown_\infty^{(n)}$). Here, recall from Remark below Definition 34 that the symbol $\sharp$ is supposed to be updated each time. For instance, $\triangledown^{(2)}$ refers to two special symbols $\sharp_0$ and $\sharp_1$, and then $\triangledown^{(n)}(x)$ can be identified with the set of all sequences y such that y contains at most n many $\sharp$'s and $\mathtt{tail}(y) = x$. More precisely, given a partial multi-valued function E, every element of $\triangledown^{(n)} \circ E(x)$ is of the form $\sigma_1\sharp\sigma_2\sharp\dots\sharp\sigma_n\sharp y$ with $y \in E(x)$. Thus, $\triangledown^{(n)} \circ \Sigma^0_1\text{-}\mathsf{LEM}^n(x)$ has a computable realizer, and indeed, $\triangledown^{(n)} \circ E$ has a computable realizer for every $E \leq_W \Sigma^0_1\text{-}\mathsf{LEM}^n(x)$. We will see more general results in Proposition 101.

A multi-valued function $P :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ is *Popperian* if there is a computable function $r :\subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ satisfying $\Sigma^0_1\text{-}\mathsf{LEM} \circ r(x,y) = \mathbf{1}_{P(x)}(y)$, for any $x \in \mathrm{dom}(P)$ and $y \in \mathbb{N}^{\mathbb{N}}$, where $\mathbf{1}_{P(x)}$ denotes the characteristic function of $P(x)$. In other words, P is Popperian if and only if the condition $y \in P(x)$ is Π^0_1, uniformly in $x \in \mathrm{dom}(P)$ and $y \in \mathbb{N}^{\mathbb{N}}$. Every Popperian multi-valued function is clearly Weihrauch reducible to the closed choice $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$ of Baire space $\mathbb{N}^{\mathbb{N}}$.

Proposition 101. *Let $E, F : \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ be any multi-valued functions.*

1. $\triangledown^{(n)} \circ E \leq_W F$ *if and only if* $E \leq_W \Sigma^0_1\text{-}\mathsf{LEM}^n \star F$.
2. $\triangledown_\omega^{(n)} \circ E \leq_W F$ *if and only if* $E \leq_W \mathsf{Unique}\Sigma^0_2\text{-}\mathsf{LLPO}_n \star F$.
3. $\triangledown \circ E \leq_W F$ *if and only if* $E \leq_W \Sigma^0_2\text{-}\mathsf{DNE} \star F$, *where* $\triangledown = \bigcup_{n\in\mathbb{N}} \triangledown^{(n)}$.

Moreover, if E is Popperian, then we also have the following conditions.

4. $\triangledown_\infty^{(n)} \circ E \leq_W F$ *if and only if* $E \leq_W \Sigma^0_2\text{-}\mathsf{LLPO}_n \star F$.
5. $\triangledown_\infty^{(n)} \circ \triangledown \circ E \leq_W F$ *if and only if* $E \leq_W (\Sigma^0_2)_2\text{-}\mathsf{LLPO}_n \star F$.
6. $\widehat{\deg}_T \circ E \leq_W F$ *if and only if* $E \leq_W \Sigma^0_3\text{-}\mathsf{DNE} \star F$.

Proof. (1) Assume that there are partial computable functions $H :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ and $K :\subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that $K(x, f \circ H(x)) \in \triangledown^{(n)} \circ E(x)$ for any $x \in \mathrm{dom}(\triangledown^{(n)} \circ E)$ and any realizer f of F. Then, for any realizer f of F, we have the following condition for any $x \in \mathrm{dom}(E)$.

$$K \circ (\mathrm{id} \times f) \circ \langle \mathrm{id}, H\rangle(x) = K(x, f \circ H(x)) \in \triangledown^{(n)} \circ E(x) = \bigtriangledown^1_n E(x).$$

Note that $H^* = \langle \mathrm{id}, H\rangle : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ is computable, and $F^* = K \circ (\mathrm{id} \times F) : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ is Weihrauch reducible to F. As in the proof of Theorem 26, we can construct an $(1, n)$-computable function $\gamma : \bigtriangledown^1_n E(x) \to E(x)$, uniformly in $x \in \mathrm{dom}(E)$. Therefore, by Theorem 93, we have a function $\gamma \leq_W \Sigma^0_1\text{-}\mathsf{LCM}^n$ satisfying $\gamma \circ f^* \circ H^*(x) \in E(x)$ for any $x \in \mathrm{dom}(E)$ and any realizer f^* of F^*. Consequently, $E \leq_W \Sigma^0_1\text{-}\mathsf{LEM}^n \star F$.

Conversely, we assume that $E \leq_W S^* \circ F^*$ for some $S^* \leq_W \Sigma^0_1\text{-}\mathsf{LEM}^n$ and $F^* \leq_W F$. Then there are computable functions H^*, K^* such that $K^*(x, H^* \circ f(x)) \in F^*(x)$ for any realizer f of F. From any single valued function $f : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$, we can effectively obtain $f^*(x) = K^*(x, H^* \circ f(x))$. Assume that $S^* \leq_W \Sigma^0_1\text{-LEM}^n$ via $\tilde{H}$ and $\tilde{K}$, and $E \leq_W S^* \circ F^*$ via H and K. We consider $H_f(x) = \tilde{H} \circ f^* \circ H(x)$ and $K_f(x,i) = K(x, \tilde{K}(f^* \circ H(x), i))$. Then, we have the following condition for any $x \in \mathrm{dom}(E)$.

$$K_f(x, \Sigma^0_1\text{-LCM}^n \circ H_f(x)) \in E(x).$$

By calculating $H_f(x) = \tilde{H} \circ f^* \circ H(x)$, we can approximate $i(f;x) = \Sigma^0_1\text{-LEM}^n \circ H_f(x)$ uniformly in f. Therefore, we can construct F^+_f to show $\triangledown^{(n)} \circ E \leq_W F$ by the following way. Set $F^+_f(\langle\rangle) = \langle\rangle$, fix $\sigma \in \mathbb{N}^{<\mathbb{N}}$, and assume that $F^+_f(\sigma^-)$ has been already defined. If $i(f;\sigma) \neq i(f;\sigma^-)$, we put $F^+_f(\sigma) = F^+_f(\sigma^-)^\frown\sharp^\frown K_f(\sigma, i(f;\sigma))$. Otherwise, F^+_f continues the approximation of $K_f(\sigma, i(f;\sigma))$. It is not hard to see that $F^+_f(x) \in \triangledown^{(n)} \circ E(x)$ for any $x \in \mathrm{dom}(E)$ and any realizer f of F. Then, F^+_f is Weihrauch reducible to $\langle K_f, H_f\rangle$, and $\langle K_f, H_f\rangle$ is Weihrauch reducible to f. Moreover, these reductions do not depend on f. Hence, $\triangledown^{(n)} \circ E \leq_W F$.

(2,3) By the same argument as in the proof of the item (1).

(4) Assume that $E :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ is Popperian, and there are partial computable functions $H :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ and $K :\subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that $K(x, f \circ H(x)) \in \triangledown^{(n)}_\infty \circ E(x)$ for any $x \in \mathrm{dom}(\triangledown^{(n)}_\infty \circ E)$ and any realizer f of F. Then, for any realizer f of F, we have the following condition for any $x \in \mathrm{dom}(E)$.

$$K \circ (\mathrm{id} \times f) \circ \langle \mathrm{id}, H\rangle(x) = K(x, f \circ H(x)) \in \triangledown^{(n)}_\infty \circ E(x) = [\bigtriangledown_\infty]^1_n E(x).$$

As in the proof of Theorem 26, we can construct an $(n,1)$-computable function $\gamma : [\bigtriangledown]^1_n E(x) \to E(x)$, uniformly in $x \in \mathrm{dom}(E)$. Here, note that $E(x)$ is a $\Pi^0_1(x)$ subset of Baire space, uniformly in x. Therefore, by relativizing Theorem 95, we have a function $\gamma \leq_W \Sigma^0_2\text{-LLPO}^n$ satisfying $\gamma \circ (\mathrm{id} \times f) \circ \langle \mathrm{id}, H\rangle(x) \in E(x)$ for any $x \in \mathrm{dom}(E)$ and any realizer f of F. Consequently, $E \leq_W \Sigma^0_1\text{-LLPO}^n \star F$.

(5,6) By the same argument as in the proof of the item (4). □

6.3. Borel Measurability, and Backtrack Games

Berardi-Coquand-Hayashi [5] showed that a 1-*backtrack Tarski game* provides a semantics of positive arithmetical fragment of Limit Computable Mathematics (i.e., Δ^0_2-*mathematics*, in the sense of Kleene realizability). A positive arithmetical formula A is true in the Limit Realizability Interpretation if and only if the $\exists$-player has a computable winning strategy in the 1-*backtracking game* $\mathsf{bck}(\mathcal{G}(A))$ associated with the Tarski game for A (for notations, see [5]).

Meanwhile, Van Wesep [80] introduced *backtrack game* to study the Wadge degrees, and Andretta [3] used this game to characterize the $\boldsymbol{\Delta}^0_2$-*measurable functions* (also called the first level Borel functions) on Baire space $\mathbb{N}^\mathbb{N}$. Motto Ros [52] and Semmes [65] studied more general games to study the Baire hierarchy of Borel measurable functions. The hierarchy of Borel measurable functions are deeply studied in descriptive set theory [45]. We consider the following notions for a function f on Baire space $\mathbb{N}^\mathbb{N}$ and a countable ordinal $\xi < \omega_1$.

1. f is a *Borel function at level* ξ (or a $\boldsymbol{\Sigma}^0_{\xi+1,\xi+1}$ function; see [41, 42, 53, 65]) if the preimage $f^{-1}(A)$ is $\boldsymbol{\Sigma}^0_{\xi+1}$ for every $\boldsymbol{\Sigma}^0_{\xi+1}$ set $A \subseteq \mathbb{N}^\mathbb{N}$.

2. f is $\mathbf{\Sigma}^0_{\xi+1}$*-measurable* (or equivalently, of *Baire class* ξ; see for instance, Kechris [45]) if the preimage $f^{-1}(A)$ is $\mathbf{\Sigma}^0_{\xi+1}$ for every open set $A \subseteq \mathbb{N}^\mathbb{N}$.

Clearly, every level ξ Borel function on Baire space $\mathbb{N}^\mathbb{N}$ is $\mathbf{\Sigma}^0_{\xi+1}$-measurable. The effective hierarchy of Borel measurable functions is studied by Brattka [10] and developed by many researchers (see [23, 46]). Every effective $\Sigma^0_{\xi+1}$ measurable function maps each point x to a point computable in the ξ-th Turing jump $x^{(\xi)}$ uniformly. Therefore, the class of (effectively) Σ^0_ξ-measurable functions does not closed under composition, whereas the class of the level ξ Borel functions must be closed under composition. Our results (Theorem 26) suggest that our notions of piecewise computability behave more like effective versions of the level ξ Borel functions rather than effectively Σ^0_ξ-measurable functions.

Recall from Definition 25 that $\mathrm{dec}^\omega_\mathrm{p}[\Gamma]\mathcal{F}$ denotes the class of Γ-piecewise $\mathcal{F}$ functions. If $\mathcal{F}$ is the class of all partial continuous functions on Baire space, we abbreviate it as $\mathbf{dec}^\omega_\mathrm{p}[\Gamma]$. Jayne-Rogers [43] proved that $\mathbf{dec}^\omega_\mathrm{p}[\mathbf{\Pi}^0_1]$ is exactly the class of the first level Borel functions, and Semmes [65] showed that f is $\mathbf{dec}^\omega_\mathrm{p}[\mathbf{\Pi}^0_2]$ is exactly the class of the second level Borel functions.

As shown in Theorem 26 and Proposition 27, $\mathrm{dec}^\omega_\mathrm{p}[\Pi^0_1]$ is exactly the class of the learnable functions, and the degree structure $\mathcal{P}/\mathrm{dec}^\omega_\mathrm{p}[\Pi^0_2]$ is exactly the degree structure $\mathcal{P}^\omega_1$ induced from nonuniform computability. Actually, our dynamic models directly fit into the backtrack and multitape game characterization of subclasses of Borel measurable functions. We now introduce various games based on *the Wadge game*, *the backtrack game*, and *the multitape game*,

Definition 102 (see also Motto Ros [52] and Semmes [65])**.** Fix a partial function f on $\mathbb{N}^\mathbb{N}$, and a set X which has no intersection with $\mathbb{N}$. The set X may contain `pass`, `back`$\sharp$, $(\mathtt{move}, i)$ for each $i \in \mathbb{N}$. Then, we introduce various two-players games on f as follows. At every round $n \in \mathbb{N}$, Player I chooses an element $x_n \in \mathbb{N}$, and Player II chooses an element $y_n \in \mathbb{N} \cup X$.

$$\begin{array}{lccccccc} \text{I:} & x_0 & & x_1 & & x_2 & & \dots \\ \text{II:} & & y_0 & & y_1 & & y_2 & \dots \end{array}$$

A pair of infinite sequences $\langle x, y\rangle \in \mathbb{N}^\mathbb{N} \times (\mathbb{N} \cup X)^\mathbb{N}$ is called *a play*. Fix a play $\langle x, y\rangle$, where $x = \langle x_n\rangle_{n\in\mathbb{N}}$ and $y = \langle y_n\rangle_{n\in\mathbb{N}}$. Player I constructs an input $x \in \mathrm{dom}(f)$ step by step, and Player II try to write a collect output $f(x)$ on some tape, where there may be infinitely many tapes $\{\Lambda_i\}_{i\in\mathbb{N}}$. Here, Player II can select a special symbol contained in X at each step.

- $(\mathtt{move}, i)$ indicates the instruction to move the head on the i-th tape Λ_i.
- `pass` indicates that Player II writes no letter at this step.
- `back`$\sharp$ indicates the instruction to delete all words on the tape under the head.

Formally, we define the following notions. For each $i \in \mathbb{N}$, *the i-th content* of the play y of Player II is a function $\mathtt{content}_i : (\mathbb{N} \cup X)^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ which is inductively defined as follows. Set $\mathtt{content}_i(\langle\rangle) = \langle\rangle$ and $\mathtt{tape}(\langle\rangle) = 0$. Assume that $\mathtt{content}_i(y \upharpoonright n)$ and $\mathtt{tape}(y \upharpoonright n)$ have been already defined for each $i \in \mathbb{N}$.

$$\mathtt{content}_i(y \upharpoonright n+1) = \begin{cases} \mathtt{content}_i(y \upharpoonright n)^\frown\langle y_n\rangle & \text{if } y_n \in \mathbb{N} \ \&\ i = \mathtt{tape}(y \upharpoonright n),\\ \langle\rangle & \text{if } y_n = \mathtt{back}\sharp \ \&\ i = \mathtt{tape}(y \upharpoonright n),\\ \mathtt{content}_i(y \upharpoonright n) & \text{otherwise.} \end{cases}$$

$$\mathtt{tape}(y \upharpoonright n+1) = \begin{cases} i & \text{if } y_n = (\mathtt{move}, i),\\ \mathtt{tape}(y \upharpoonright n) & \text{otherwise.} \end{cases}$$

Then, for each $i \in \mathbb{N}$, we define $\mathtt{content}_i(y) = \lim_{n\in\mathbb{N}} \mathtt{content}_i(y \upharpoonright n)$ for any $y \in (\mathbb{N} \cup X)^{\mathbb{N}}$. We consider the following special *rules* for this game.

- Player I *violates the basic rule* if $x \notin \mathrm{dom}(f)$.
- Player II *violates the basic rule* if either $y_n \in \{\mathtt{pass}, (\mathtt{move}, i) : i \in \mathbb{N}\}$ for almost all $n \in \mathbb{N}$, or $y_n = \mathtt{back}\sharp$ for infinitely many $n \in \mathbb{N}$.
- Player II *violates the rule m* if y contains at least m many $\mathtt{back}\sharp$'s.
- Player II *violate the rule* $*$ if $y_n \in \{(\mathtt{move}, i) : i \in \mathbb{N}\}$ for infinitely many $n \in \mathbb{N}$.

We say that Player II *wins* (resp. *is winnable*) on the play $\langle x, y\rangle \in \mathbb{N}^{\mathbb{N}} \times (\mathbb{N} \cup X)^{\mathbb{N}}$ of the game $G(f, X)$ if either Player II does not violate the basic rule, and $f(x) = \mathtt{content}_i(y)$ for the least $i \in \mathbb{N}$ with $\mathtt{content}_i(y)$ being total (resp. for some $i \in \mathbb{N}$), or Player I violates the basic rule. We also say that Player II *wins* (resp. *is winnable*) on the play $\langle x, y\rangle$ of the game $G_m(f, X)$ if Player II wins (resp. is winnable) the game $G(f, X)$ and does not violate the rule m, and that Player II *wins* (resp. *is winnable*) the game $G_*(f, X)$ if Player II wins (resp. is winnable) the game $G(f, X)$ and does not violate the rule $*$.

A *strategy* of Player II is a function $\psi : \mathbb{N}^{<\mathbb{N}} \to (\mathbb{N} \cup X)^{<\mathbb{N}}$ such that $|\psi(\sigma)| = |\sigma|$ for each $\sigma \in \omega^{<\omega}$, and $\psi(\sigma) \subseteq \psi(\tau)$ whenever $\sigma \subseteq \tau$. A strategy ψ of Player II is *winning* (resp. *winnable*) in the game G if Player II wins (resp. is winnable) the game G on the play $\langle x, \bigcup_{n\in\mathbb{N}} \psi(x \upharpoonright n)\rangle$ for any $x \in \mathbb{N}^{\mathbb{N}}$.

We write P, B, and M_α for $\{\mathtt{pass}\}$, $\{\mathtt{back}\sharp\}$, and $\{(\mathtt{move}, i) : i < \alpha\}$, respectively, for each $\alpha \leq \omega$. Then, for $\mathsf{S}, \mathsf{T}, \mathsf{U} \in \{\mathsf{P}, \mathsf{B}, \mathsf{M}_\alpha\}_{\alpha\leq\omega}$, the union $\mathsf{S} \cup \mathsf{T} \cup \mathsf{U}$ is denoted by STU.

Remark. The games $G(f, \mathsf{P})$, $G(f, \mathsf{PB})$, and $G(f, \mathsf{PM}_\omega)$ are essentially same as *the Wadge game*, *the backtrack game*, and *the multitape game*, respectively. See also Motto Ros [52] and Semmes [65].

Let f be a partial function on Baire space $\mathbb{N}^{\mathbb{N}}$.

1. (Wadge [81]) f is continuous if and only if Player II has a winning strategy in the game $G(f, \mathsf{P})$.
2. (Andretta [3]) f is $\mathbf{\Delta}^0_2$ if and only if Player II has a winning strategy in the game $G(f, \mathsf{PB})$.
3. (Andretta, Semmes [64]) f is $\mathbf{\Pi}^0_2$-piecewise continuous if and only if Player II has a winning strategy in the game $G(f, \mathsf{PM}_\omega)$.

Theorem 103 (Game representation)**.** *Let f be a partial function on Baire space $\mathbb{N}^{\mathbb{N}}$.*

1. *f is $(1,1)$-computable if and only if Player II has a computable winning strategy in the game $G(f,\mathsf{P})$.*
2. *f is $(1,m)$-computable if and only if Player II has a computable winning strategy in the game $G_m(f,\mathsf{PB})$.*
3. *f is $(1,\omega|m)$-computable if and only if Player II has a computable winning strategy in the game $G_*(f,\mathsf{PM}_m)$.*
4. *f is $(1,\omega)$-computable if and only if Player II has a computable winning strategy in the game $G(f,\mathsf{PB})$.*
5. *f is $(m,1)$-computable if and only if Player II has a computable winnable strategy in the game $G(f,\mathsf{PM}_m)$.*
6. *f is (m,ω)-computable if and only if Player II has a computable winnable strategy in the game $G(f,\mathsf{PBM}_m)$.*
7. *f is $(\omega,1)$-computable if and only if Player II has a computable winnable strategy in the game $G(f,\mathsf{PM}_\omega)$.*

Proof. (2,4) We need to construct a winning strategy $\psi:\mathbb{N}^{<\mathbb{N}}\to(\mathbb{N}\cup\{\mathtt{pass},\mathtt{back}\sharp\})^{<\mathbb{N}}$ from a given partial $(1,\omega)$-computable function $f:\subseteq\mathbb{N}^{\mathbb{N}}\to\mathbb{N}^{\mathbb{N}}$. Assume that f is $(1,\omega)$-computable via a learner Ψ. We inductively define a strategy $\psi:\mathbb{N}^{<\mathbb{N}}\to(\mathbb{N}\cup\{\mathtt{pass},\mathtt{back}\sharp\})^{<\mathbb{N}}$ and an auxiliary parameter $\mathtt{backlog}:\mathbb{N}^{<\mathbb{N}}\to(\mathbb{N}\cup\{\mathtt{back}\sharp\})^{<\mathbb{N}}$. Set $\psi(\langle\rangle)=\mathtt{backlog}(\langle\rangle)=\langle\rangle$, and assume that $\psi(\sigma^-)$ and $\mathtt{backlog}(\sigma^-)$ have been already defined. Then, define $\psi(\sigma)$ and $\mathtt{backlog}(\sigma)$ as follows:

$$\psi(\sigma)=\begin{cases}\psi(\sigma^-)^\frown\mathtt{pass} & \text{if } \mathtt{backlog}(\sigma^-)=\langle\rangle,\\ \psi(\sigma^-)^\frown(\mathtt{backlog}(\sigma^-)(0)) & \text{if } \mathtt{backlog}(\sigma^-)\neq\langle\rangle,\end{cases}$$

$$\mathtt{backlog}(\sigma)=\begin{cases}\mathtt{backlog}(\sigma^-)^{\leftarrow 1\frown}\mathtt{new}\Phi_{\Psi(\sigma)}(\sigma) & \text{if } \Psi(\sigma)=\Psi(\sigma^-),\\ \mathtt{backlog}(\sigma^-)^{\leftarrow 1\frown}\mathtt{back}\sharp^\frown\Phi_{\Psi(\sigma)}(\sigma) & \text{if } \Psi(\sigma)\neq\Psi(\sigma^-).\end{cases}$$

Here, recall the notation $\mathtt{new}\Phi_{\Psi(\sigma)}(\sigma)$ defined before Theorem 40. Note that $\{n\in\mathbb{N}:(\bigcup_k\psi(x\upharpoonright k))(n)=\mathtt{back}\sharp\}=\mathtt{mcl}_\Psi(x)$ for any $x\in\mathrm{dom}(f)$. It is easy to see that ψ is a computable winning strategy in the game $G(f,\mathsf{PB})$.

Assume that a computable winning strategy ψ^* in the game $G(f,\mathsf{PB})$ is given. We consider the computable function $\psi(\sigma)=\mathtt{content}_0(\psi^*(\sigma))$. Then $\{n\in\mathbb{N}:\psi(x\upharpoonright n+1)\not\supseteq\psi(x\upharpoonright n)\}$ is finite, for any $x\in\mathrm{dom}(f)$, since $\bigcup_{n\in\mathbb{N}}\psi(x\upharpoonright n)$ contains finitely many $\mathtt{back}\sharp$'s. Moreover, $f(x)=\lim_n\psi(x\upharpoonright n)$. Thus, by Proposition 3, f is $(1,\omega)$-computable.

(3) Assume that f is $(1,\omega|<\omega)$-computable via a learner Ψ. We inductively define a strategy $\psi:\mathbb{N}^{<\mathbb{N}}\to(\mathbb{N}\cup\{\mathtt{pass},\mathtt{back}\sharp\})^{<\mathbb{N}}$ and an auxiliary parameter $\mathtt{backlog}:\mathbb{N}^{<\mathbb{N}}\to(\mathbb{N}\cup\{\mathtt{back}\sharp\})^{<\mathbb{N}}$. Set $\psi(\langle\rangle)=\mathtt{backlog}(\langle\rangle)=\langle\rangle$, and assume that $\psi(\sigma^-)$ and $\mathtt{backlog}(\sigma^-)$ have been already defined. Then, define $\psi(\sigma)$ and $\mathtt{backlog}(\sigma)$ as follows:

$$\psi(\sigma)=\begin{cases}\psi(\sigma^-)^\frown\mathtt{pass} & \text{if } \mathtt{backlog}(\sigma^-)=\langle\rangle,\\ \psi(\sigma^-)^\frown(\mathtt{backlog}(\sigma^-)(0)) & \text{if } \mathtt{backlog}(\sigma^-)\neq\langle\rangle,\end{cases}$$

$$\mathtt{backlog}(\sigma)=\mathtt{backlog}(\sigma^-)^{\leftarrow 1\frown}(\mathtt{move},\Psi(\sigma))^\frown\mathtt{new}^*\Phi_{\Psi(\sigma)}(\sigma)$$

Here, recall the notation $\mathtt{new}^*\Phi_{\Psi(\sigma)}(\sigma)$ defined in the proof of Theorem 40 (2). Note that $\{n \in \mathbb{N} : (\bigcup_k \psi(x \upharpoonright k))(n) = \mathtt{back}\sharp\} = \{n \in \mathbb{N} : \Psi(x \upharpoonright n+1) \neq \Psi(x \upharpoonright n)\}$ for any $x \in \mathrm{dom}(f)$. It is easy to see that ψ is a computable winning strategy in the game $G(f, \mathsf{PM}_m)$. Moreover, since $\#\mathtt{indx}_\Psi(x)$ is finite, $\psi(x) = \bigcup_n \psi(x \upharpoonright n)$ contains $(\mathtt{move}, i)$ for only finitely many different i's. Therefore, ψ does not violate the rule $*$. Hence, ψ is a winning strategy in the game $G_*(f, \mathsf{PM}_m)$.

Assume that a computable winning strategy ψ^* in the game $G_*(f, \mathsf{PM}_m)$ is given. Let $e(i)$ be an index of a partial computable function $x \mapsto \mathtt{content}_i \circ \psi^*(x)$ for each $i < m$. Since ψ^* does not violate the rule $*$, there is a unique $i < m$ such that $\Phi_{e(i)} = \mathtt{content}_i \circ \psi^*(x)$ is total, for any $x \in \mathrm{dom}(f)$. We inductively define a learner Ψ. The learner Ψ first guesses $\Psi(\langle\rangle) = e(0)$. Set $\Psi(\sigma) = \Psi(\sigma^-)$ when there is no $i < m$ such that $|\Phi_{e(i)}(\sigma)| > |\Phi_{e(i)}(\sigma^-)|$. Otherwise, for the least such $i < m$, the learner guesses $\Psi(\sigma) = e(i)$. Clearly, $\#\{\Psi(x \upharpoonright n) : n \in \mathbb{N}\} < m$ for any $x \in \mathbb{N}^\mathbb{N}$. It is easy to check that, for any $x \in \mathrm{dom}(f)$, $\lim_n \Psi(x \upharpoonright n)$ converges to $e(i)$ for the unique $i < m$ ensuring the totality of $\mathtt{content}_i \circ \psi^*(x)$, and, for such $i < m$, we have $\Phi_{\lim_n \Psi(x \upharpoonright n)}(x) = \mathtt{content}_i \circ \psi^*(x) = f(x)$. Consequently, f is $(1, \omega | m)$-computable.

(5,7) For a given collection $\{\Phi_i\}_{i \in I}$ of partial computable functions, we can easily construct a strategy $\psi : \mathbb{N}^{<\mathbb{N}} \to (\mathbb{N} \cup \{\mathtt{pass}, (\mathtt{move}, i) : i \in I\})$ ensuring $\mathtt{content}_i \circ \psi(x) = \Phi_i(x)$ for any $x \in \mathbb{N}^\mathbb{N}$. Therefore, f is nonuniformly computable via $\{\Phi_i\}_{i \in I}$, then ψ is winnable in $G(f, \mathsf{PM}_I)$. Conversely, if a winnable strategy $\psi : \mathbb{N}^{<\mathbb{N}} \to (\mathbb{N} \cup \{\mathtt{pass}, (\mathtt{move}, i) : i \in I\})$ of the game $G(f, \mathsf{PM}_I)$ is given. Then we consider the partial computable function Γ_i computing $\Gamma_i(x) = \mathtt{content}_i \circ \psi(x)$ for any $x \in \mathbb{N}^\mathbb{N}$. It is easy to see that f is nonuniformly computable via $\{\Gamma_i\}_{i \in I}$.

(6) By combining the proofs of the items (3) and (4), it is not hard to see the equivalence of the (m, ω)-computability of f and the computable winnability in the game $G(f, \mathsf{PBM}_m)$. □

Remark. We may introduce more general multitape games based on our dynamic tape models, and nested (nested nested, nested nested nested, etc.) tape models.

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